Lunch seminar

The discrete Pompeiu problem on the plain

2016.12.07.

joint work with M. Laczkovich and Cs. Vincze

Dimitrie Pompeiu (1873-1954) was a Romanian mathematician. As one of the students of Henri Poincaré he obtained a Ph.D. degree in mathematics in 1905 at the Université de Paris (Sorbonne).

His contributions were mainly connected to the fields of mathematical analysis, especially the theory of complex functions. In one of his article[1] he posed a question of integral geometry.

[1] *Sur certains systémes d'équations linéaires et sur une propriété intégrale des fonctions de plusieurs variables* (Comptes Rendus de l'Académie des Sciences. Série I. Mathématique 188 (1929) pp. 1138-1139)

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The classical Pompeiu problem

Question

Let f be a continuous function defined on the plain, and let K be a closed set of positive Lebesgue measure. Suppose that

$$\int_{\sigma(K)} f(x,y)\,d\lambda_x d\lambda_y = 0 \tag{1}$$

for every rigid motion σ of the plain, where λ denotes the Lebesque measure. Is it true that $f \equiv 0$?

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We say that the set K has the **Pompeiu's property** if the answer of this question is affirmative (YES).

The continuous case

1. $K =$ square:
   - Pompeiu showed that the square has the Pompeiu property.

[2] L. Brown, B. M. Schreiber and B. A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** (1973), 125-154.

[3] Chalakov, Sur un problème de D. Pompeiu, *Annaire Univ. Sofia Fac. Phys. Math.*, Livre 1, **40** (1944), 1-44.

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2. $K =$ disk:
   - Chakalov[3] showed infinitely many linearly independent solutions of the form $\sin(ax + by)$ for appropriately chosen constants a, b.
   - Williams[4] proved that if K is simply-connected and has a sufficiently smooth boundary ∂K then there is a function $f \not\equiv 0$.

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Another type of problem - The discrete case

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Another type of problem - The discrete case

- 70th Putman Mathematical Competition problem (2009): Let f be a real-valued function on the plain such that for every square $ABCD$ in the plain, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plain?

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- Similar question is whether $f \equiv 0$ whenever $f(A) + f(B) + f(C) + f(D) = 0$ holds for every **unit** square $ABCD$. Katz, Krebs, Shaheen[5] shown a nice elementary proof.

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- Groote, Duerinckx[6] investigated the following version: Given a nonempty finite set $H \subset \mathbb{R}^2$ and a function $f : \mathbb{R}^2 \to \mathbb{R}^d$ such that the **arithmetic mean** of f at the elements of any **similar** copy of H is constant. Does it follow that f is constant on $\mathbb{R}^2$?

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More abstractly

Let G be the transformation group of $\mathbb{C}$ and $a_1, \ldots, a_n \in \mathbb{C}$ be given nonzero numbers.

Definition

The finite set $H = \{d_1, \ldots, d_n\} \subset \mathbb{C}$ has the *discrete Pompeiu property* w.r.t. G if for every function $f : \mathbb{C} \to \mathbb{C}$ the equation

$$\sum_{i=1}^{n} f(\sigma(d_i)) = 0 \qquad (2)$$

holds for all $\sigma \in G$ implies that $f \equiv 0$.

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We focus on the similarity group ($\mathcal{S}$), the translation group ($\mathcal{T}$) and the isometry group ($\mathcal{I}$) of $\mathbb{C}$.

Similarity case

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Let $d_1, d_2, \ldots, d_n \in \mathbb{C}$ be the points of H. Then the equation (2) can be written in the form

$$f(x + d_1 y) + f(x + d_2 y) + \ldots + f(x + d_n y) = 0 \tag{3}$$

holds for every $x, y \in \mathbb{C},\ y \neq 0$.

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$$a_1 f(x + d_1 y) + a_2 f(x + d_2 y) + \ldots + a_n f(x + d_n y) = 0 \tag{4}$$

for every $x, y \in \mathbb{C},\ y \neq 0$ with constant $a_1, \ldots, a_n \in \mathbb{C}$.

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Theorem (K.- Varga, '14)

$$\exists f \not\equiv 0 \text{ solution of } (4) \iff$$
$$\iff \exists \text{ automorphism } \phi \text{ of } \mathbb{C} \text{ which is a solution of } (4) \iff$$
$$\iff \phi \text{ satisfies } \sum_i a_i = 0 \text{ and } \sum_i a_i \phi(b_i) = 0.$$

Translations

Theorem

Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

[7] D. Zeilberger, Pompeiu's problem in discrete space, *Proc. Nat. Acad. Sci. USA* **75** (1978), no. 8, 3555-3556.

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Let G be a torsion free Abelian group. No finite set $H \subset G, |H| \geq 2$ has the discrete Pompeiu property w.r.t. G.

Let G_H be the subgroup of G generated by H. Then G_H is a finitely generated torsion free Abelian group, and thus

$$G_H \cong \mathbb{Z}^n \text{ for some finite } n.$$

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For every finite set $H \subset \mathbb{Z}^n$ there is a nonzero function $f : \mathbb{Z}^n \to \mathbb{C}$ such that $\sum_{i=1}^{n} f(\sigma(d_i)) = 0$ for any translate σ of $\mathbb{Z}^n$.

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Easily, we can find such a function on every coset $G : G_H$.

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Isometries

Proposition

Let E be a finite set in the plain. If there exists an isometry σ such that $|E \cap \sigma(E)| = |E| - 1$, then E has the discrete Pompeiu property w.r.t. isometries.

Corollary

Every 2- and 3-element set has the discrete Pompeiu property w.r.t. isometries.

Recent results

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Theorem

Let D be an n-tuple of collinear points in the plain with pairwise commensurable distances. Then D has the discrete Pompeiu property w.r.t. $\mathcal{I}$.

Open Questions

Question

Do the following sets have the discrete Pompeiu property

1. *Symmetric trapezoid,*
2. *4 points in a line,*
3. *Pentagon, regular n-gon?*

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   $g \neq 0$ and $g(x+y) = g(x) \cdot g(y)$ for every $x, y \in G$.

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3. *Variety V on G*:
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   subspace of $\mathbb{C}^G$.
4. *Spectral analysis holds in G*: every $V \neq \{0\}$ on G contains an exponential function.

The *torsion free rank* $r_0(G)$ of G is the cardinality of a maximal independent system of elements of infinite order.

[8] M. Laczkovich and G. Székelyhidi, Harmonic analysis on discrete Abelian groups, *Proc. Am. Math. Soc.* **133** (2004), no. 6, 1581-1586.

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Theorem (Laczkovich, Székelyhidi[8])

Spectral analysis holds on every Abelian group G iff

$$r_0(G) < \mathfrak{c}.$$

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Theorem (Laczkovich, Székelyhidi[8])

Spectral analysis holds on every Abelian group G iff

$$r_0(G) < \mathfrak{c}.$$

Every function f satisfying

$$\sum_{i=1}^{n} f(x + d_i y) = 0,$$

where $x, y \in \mathbb{C}$, $|y| = 1$ and $H = \{0 = d_1, d_2, \ldots, d_n\}$, constructs a variety V on the additive group of $\mathbb{C}$.

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Method of the proof

Theorem

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Let G be a finitely generated additive subgroup of G. The statement can be written in the following form:

$$f(x) + f(x + s_1 y) + f(x + s_2 y) + f(x + (s_1 + s_2)y) = 0 \qquad (5)$$

holds for every $x, y \in G$ and $|y| = 1$. Assume that $f(0) \neq 0$.

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$$g(x + y) = g(x)g(y). \qquad (6)$$

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holds for every $x,y\in G$ and $|y|=1$. Assume that $f(0)\neq 0$. There exists an exponential function $g\not\equiv 0$ in V which satisfies

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Thus, we get

$$(1+g(s_1y))(1+g(s_2y))=0. \qquad (7)$$

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i.e $g(s_1 y) = -1$ or $g(s_2 y) = -1$ $(\forall y \in G, |y| = 1)$.

$$
\begin{aligned}
a &= x, \quad & b &= x + s_1 y, \\
d &= x + s_2 y, \quad & c &= x + (s_1 + s_2) y,
\end{aligned}
$$

$$a = x, \quad b = x + s_1 y,$$
$$d = x + s_2 y, \quad c = x + (s_1 + s_2) y,$$

Then g at the points a, b, c, d are

$$\text{either } g(a), -g(a), -g(d), g(d)$$
$$\text{or } g(a), g(b), -g(b), -g(a),$$

respectively.

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Let A and B be such that $\mathbb{C} \setminus \emptyset = \mathbb{C}^* = A \cup^* B$ and $A = -B$

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respectively.
Let A and B be such that $\mathbb{C} \setminus \emptyset = \mathbb{C}^* = A \cup^* B$ and $A = -B$
We define $h \colon \mathbb{C} \to \{-1, 1\}$ as follows:

$$h(x) = \begin{cases} 1, & \text{if } g(x) \in A \\ -1, & \text{if } g(x) \in B. \end{cases}$$

Euclidean Ramsey theory

Color the points of the plain $\mathbb{R}^2$ with 2 colors.

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[9] L. E. Shader, All right triangles are Ramsey in E^2!, *Journ. Comb. Theory (A)* **20** (1976), 385-389.

[10] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.* **2** (1951), 172.

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Color the points of the plain $\mathbb{R}^2$ with 2 colors.
Then we have a 2-coloring h of any finitely generated subgroup $(G, +)$ of the plain such that two-two sides of every parallelogram have the same color.

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Theorem (Shader[9])

For every 2-coloring of the plain and every parallelogram H, there is a congruent copy $\sigma(H)$ such that at least three of its vertices has the same color.

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For any given parallelogram H, there is a finite witness[10] set R.
Thus, if the generator set of G contains R, we get a contradiction. □

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Coloring of the plain

Question (Hadwiger-Nelson's problem[11])

What is the chromatic number χ of the plain?

[11] https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem

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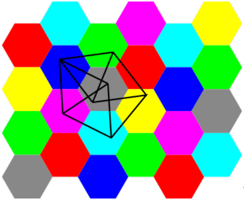
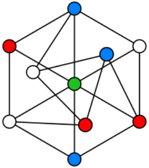
Theorem

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$



$\chi(\mathbb{R}^2) \leq 7$



$\chi(\mathbb{R}^2) \geq 4$

[11] https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem

Application

We denote the set of forbidden distance by $\mathcal{FD}$. Previously, $\mathcal{FD} = \{1\}$ and $4 \geq \chi_{\mathcal{FD}}(\mathbb{R}^2) \geq 7$.

Application

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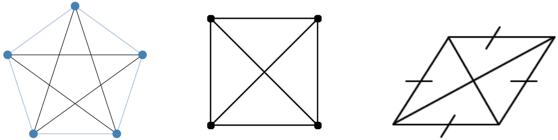
Proposition

If

1. *(a.)* $\mathcal{FD}_1 = \{1, \frac{\sqrt{5}+1}{2}\}$
2. *(b.)* $\mathcal{FD}_2 = \{1, \sqrt{2}\}$
3. *(c.)* $\mathcal{FD}_3 = \{1, \sqrt{3}\}$

then $\chi_{\mathcal{FD}_i}(\mathbb{R}^2) > 4$.



Thank you for your kind attention.

References

[1.] G. Kiss and A. Varga, Existence of nontrivial solutions of linear functional equation, *Aequationes Math.* (2014), **88**, no. 1, 151-162.

[2.] G. Kiss, M. Laczkovich and Cs. Vincze, The discrete Pompeiu problem on the plain, submitted to *Monatshefte für Mathematik*, https://arxiv.org/pdf/1612.00284v1.pdf