POISSON COHOMOLOGY OF HOLOMORPHIC TORIC POISSON MANIFOLDS

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Abstract. A holomorphic toric Poisson manifold is a nonsingular toric variety equipped with a holomorphic Poisson structure, which is invariant under the torus action. In this paper, we compute the Poisson cohomology for holomorphic toric Poisson structures on $\mathbb{C}\mathbf{P}^n$, with the stand Poisson structure on $\mathbb{C}\mathbf{P}^n$ as a special case. Two conjectures are proposed, one for the holomorphic multi-vector fields on nonsingular toric varieties, and the other for the Poisson cohomology of holomorphic toric Poisson manifolds.

1. introduction

Holomorphic Poisson manifolds have attracted the interest of many mathematicians recently. The algebraic geometry of the Poisson brackets on projective spaces was studied by Bondal [2] and Polishchuk [20]. In [?Hitchin06, ?Hitchin11], Hitchin revealed the connections of holomorphic Poisson structures with generalized complex geometry and mathematical physics. The deformations of holomorphic Poisson structures appeared in the work of [?Hitchin12] and [?Kim14]. The standard Poisson structures on affine spaces and flag varieties were studied by Brown, Goodear and Yakimov [?B-G-Y06, ?G-Y09]. Laurent-Gengoux, Stiénon and Xu [?L-S-X08] described the Poisson cohomology of holomorphic Poisson manifolds using Lie algebroids. In various situations, the Poisson cohomology of holomorphic Poisson manifolds were computed [?Hong-Xu11, ?Mayansky15, ?C-F-P16].

This paper is devoted to the study of the Poisson geometry of toric varieties, especially, the Poisson cohomology of holomorphic toric Poisson manifolds. A holomorphic toric Poisson manifold is a nonsingular toric variety X, equipped with a holomorphic Poisson structure π, which is invariant under the torus action (Notice that real toric Poisson structures were studied in [6]). Holomorphic toric Poisson manifold is a special case of the "T-Poisson manifold" in the sense of [?Lu-Mouquin15].

The main results of this paper are as following.

- In the case of $X = \mathbb{C}\mathbf{P}^n$, we proved that $H^i(X, \wedge^j \mathcal{T}_X) = 0$ for all $i > 0$ and $0 \le j \le n$.
- The space of holomorphic vector fields and multi-vector fields on $X = \mathbb{C}\mathbf{P}^n$ are described by considering $X = \mathbb{C}\mathbf{P}^n$ as a toric variety.
- For any holomorphic toric Poisson structure π on $X = \mathbb{C}\mathbf{P}^n$, we give an algorithm for the Poisson cohomology groups. As a special case, we compute the Poisson cohomology of standard Poisson structures on $X = \mathbb{C}\mathbf{P}^n$ in some situations.

Key words and phrases. holomorphic Poisson manifolds, Poisson cohomology, toric variety, standard Poisson structure.

Two conjectures are proposed at the end of this paper , one for the holomorphic multi-vector fields on nonsingular toric varieties, and the other for the Poisson cohomology of holomorphic toric Poisson manifolds. We expect that these conjectures could stimulate meaningful research on related topics. And it would also be interesting to explore the relations of our results with [?B-G-Y06, ?G-Y09].

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2. Preliminary

2.1. **Poisson cohomology of holomorphic Poisson manifolds.**

Definition 2.1. A holomorphic Poisson manifold is a complex manifold X equipped with a holomorphic bivector field π such that $[\pi,\pi]=0$, where $[\cdot,\cdot]$ is the Schouten bracket.

The Poisson cohomology of a holomorphic Poisson manifold is defined in the following way:

Definition 2.2. Let (X,π) be a holomorphic Poisson manifold of dimension n. The Poisson cohomology $H^\bullet_\pi(X)$ is the cohomology group of the complex of sheaves:

$$\mathcal{O}_X \xrightarrow{d_\pi} T_X \xrightarrow{d_\pi} \ldots.. \xrightarrow{d_\pi} \wedge^{i-1}T_X \xrightarrow{d_\pi} \wedge^{i}T_X \xrightarrow{d_\pi} \wedge^{i+1}T_X \xrightarrow{d_\pi} \ldots\ldots \xrightarrow{d_\pi} \wedge^{n}T_X, \tag{2.1}$$

where $d_\pi=[\pi,\cdot]$.

Lemma 2.3. [?L-S-X08] *The Poisson cohomology of a holomorphic Poisson manifold (X,π) is isomorphic to the total cohomology of the double complex*

$$\begin{array}{ccccccc}
\ldots\ldots & & \ldots\ldots & & \ldots\ldots & & \\
d_\pi\uparrow & & d_\pi\uparrow & & d_\pi\uparrow & & \\
\Omega^{0,0}(X,T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X,T^{2,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X,T^{2,0}X) & \xrightarrow{\bar\partial} & \ldots\ldots \\
d_\pi\uparrow & & d_\pi\uparrow & & d_\pi\uparrow & & \\
\Omega^{0,0}(X,T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X,T^{1,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X,T^{1,0}X) & \xrightarrow{\bar\partial} & \ldots\ldots \\
d_\pi\uparrow & & d_\pi\uparrow & & d_\pi\uparrow & & \\
\Omega^{0,0}(X,T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,1}(X,T^{0,0}X) & \xrightarrow{\bar\partial} & \Omega^{0,2}(X,T^{0,0}X) & \xrightarrow{\bar\partial} & \ldots\ldots
\end{array}$$

Lemma 2.4. *Let (X,π) be a holomorphic Poisson manifold. If all the higher cohomology groups $H^i(X,\wedge^j T_X)$ vanish for $i>0$, then the Poisson cohomology $H^\bullet_\pi(X)$ is isomorphic to the cohomology of the complex*

$$H^0(X,\mathcal{O}_X) \xrightarrow{d_\pi} H^0(X,T_X) \xrightarrow{d_\pi} H^0(X,\wedge^2 T_X) \xrightarrow{d_\pi} \ldots \xrightarrow{d_\pi} H^0(X,\wedge^n T_X), \tag{2.2}$$

where $d_\pi=[\pi,\cdot]$.

2.2. **Holomorphic toric Poisson structures.** Let us recall some classical knowledge of toric varieties. One may consult [7], [9] and [19].

Definition 2.5. A toric variety is an irreducible variety X such that

(1) $(\mathbb{C}^*)^n$ is a Zariski open set of X, and
(2) the action of $(\mathbb{C}^*)^n$ on it extends to an action of $(\mathbb{C}^*)^n$ on X.

Example 2.6. Let $X = \mathbb{C}\mathbf{P}^n$ and let $[z_0, z_1, \ldots, z_n]$ be homogenous coordinates on it. The map

$$(\mathbb{C}^*)^n \to \mathbb{C}\mathbf{P}^n$$

defined by $(t_1, t_2, ..., t_n) \mapsto [1, t_1, t_2, ..., t_n]$ allows us to identify $(\mathbb{C}^*)^n$ with the Zariski open subset $\{[z_0, z_1, \ldots, z_n] \in \mathbb{C}\mathbf{P}^n \mid z_i \neq 0,\ 0 \leq i \leq n\}$ of $\mathbb{C}\mathbf{P}^n$. The $(\mathbb{C}^*)^n$ action on $\mathbb{C}\mathbf{P}^n$ given by

$$(t_1, \ldots, t_n).[z_0, z_1, \ldots, z_n] = [z_0, t_1 z_1, \ldots, t_n z_n]$$

shows that $X = \mathbb{C}\mathbf{P}^n$ is a toric variety.

A toric veriety can be described by a Lattice $N \cong \mathbb{Z}^n$ and a fan Δ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. Let $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_Z \mathbb{R}$. The canonical $\mathbb{Z}$-bilinear pairing

$$\langle , \rangle : M \times N \to \mathbb{Z}$$

extending to the field $\mathbb{R}$ of real numbers gives a $\mathbb{R}$-bilinear pairing $\langle , \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$.

Let $T_N = Hom_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Then $T_N \cong (\mathbb{C}^*)^n$. Moreover, we have $M \cong Hom(T_N, \mathbb{C}^*)$ and $N \cong Hom(\mathbb{C}^*, T_N)$.

Each element m in M gives rise to a character $\chi^m \in Hom(T_N, \mathbb{C}^*)$, given by

$$\chi^m(t) = \langle t, m \rangle \quad \text{for} \quad t \in T_N.$$

And each element a in N gives rise to a one-parameter subgroup $\gamma_a \in Hom(\mathbb{C}^*, T_N)$ given by

$$\gamma_a(\lambda)(m) = \lambda^{\langle a, m \rangle} \quad \text{for} \quad \lambda \in \mathbb{C}^* \quad \text{and} \quad m \in M.$$

Choose a $\mathbb{Z}$-basis $\{e_1, ... e_n\}$ of N and let $\{e_1^*, ... e_n^*\}$ be the dual basis of M. Let $t_i = \chi(e_i^*)$. Then we have an isomorphism

$$T_N \cong (\mathbb{C}^*)^n : t \longleftrightarrow (t_1, t_2, ... t_n),$$

where $t_1, t_2, ... t_n \in \mathbb{C}^*$ can be seen as the coordinates on T_N. For $m = \sum_{i=1}^n m_i e_i^*$, we have $\chi^m = t_1^{m_1} t_2^{m_2} ... t_n^{m_n}$, which is a Laurent monomial on T_N. For $a = \sum_{i=1}^n a_i e_i$, the one-parameter subgroup γ_a can be written as $\gamma_a(\lambda) = (\lambda^{a_1}, ... \lambda^{a_n})$.

Definition 2.7. A subset σ of $N_{\mathbb{R}}$ is called a rational polyhedral cone (with apex at the origin O), if there there exist a finite number of elements $e_1, e_2, ..., e_s$ in N such that

$$\begin{aligned} \sigma &= \mathbb{R}_{\geq 0} e_1 + ... \mathbb{R}_{\geq 0} e_s \\ &= \{a_1 e_1 + ... + a_s e_s \mid a_i \in \mathbb{R}, a_i \geq 0 \text{ for all } 0 \leq i \leq s\}, \end{aligned}$$

where we denote by $R_{\geq 0}$ the set of nonnegative real numbers.

(1) σ is strongly convex if $\sigma \cap (-\sigma) = O$.
(2) The dimension of σ is the dimension of the smallest subspace of $N_{\mathbb{R}}$ containing σ.

In this paper, a cone is always a rational polyhedral cone.

For a cone $\sigma \in N_{\mathbb{R}}$, its dual cone in $M_{\mathbb{R}}$ is defined to be

$$\sigma^{\vee} = \{x \in M_{\mathbb{R}} \mid \langle x, y\rangle \geq 0 \ \text{ for all } y \in \sigma\}.$$

A face of σ is a subset of σ, with the form $m^{\perp} \cap \sigma = \{x \in \sigma \mid \langle x, m\rangle = 0\}$ for an element $m \in \sigma^{\vee}$.

Definition 2.8. A fan in N is a nonempty collection Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:

(1) Every face of any $\sigma \in \Delta$ is contained in Δ.
(2) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The union $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ is called the support of Δ.

For a fan Δ, the set of one dimensional cones in Δ is denoted by $\Delta(1)$. The primitive element of $\alpha \in \Delta(1)$ is the unique generator of $\alpha \cap N$, denoted by $n(\alpha)$.

Let $S_{\sigma} = \sigma^{\vee} \cap M$. For a strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$, the semigroup algebra

$$\mathbb{C}[S_{\sigma}] = \oplus_{m \in S_{\sigma}} \chi^m$$

is a finitely generated commutative $\mathbb{C}$-algebra. The corresponding affine variety $U_{\sigma} = Spec(\mathbb{C}[S_{\sigma}])$ is a n-dimensional toric variety. If τ is a face of σ, then U_{τ} can be regarded as a Zariski open set of U_{σ}. Especially, $U_O = Spec(\mathbb{C}[M]) = T_N \cong (\mathbb{C}^*)^n$ is a Zariski open set of U_{σ}. This leads to the following definition.

Theorem 2.9. *Given a lattice $N \cong \mathbb{Z}^n$ and a fan Δ in $N_{\mathbb{R}} \cong R^n$, there exists a toric variety X_{Δ}, obtained from the affine variety $U_{\sigma}, \sigma \in \Delta$, by gluing together U_{σ} and U_{τ} along their common open subset $U_{\sigma \cap \tau}$ for all $\sigma, \tau \in \Delta$.*

For the toric variety X_{Δ}, $U_O = T_N$ is the algebraic torus embedding in it. There is a T_N-action on X_{Δ}, which extends the T_N action on itself.

A cone σ is called nonsingular if σ can be written as

$$\sigma = \mathbb{R}_{\geq 0} e_1 + ... \mathbb{R}_{\geq 0} e_s,$$

where $\{e_1, e_2, ..., e_s\}$ is a subset of a $\mathbb{Z}$-basis of N.

Theorem 2.10. *Let X_{Δ} be the toric variety associated with a fan Δ in $N_{\mathbb{R}}$. Then*

(1) *X_{Δ} is compact $\Longleftrightarrow |\Delta| = N_{\mathbb{R}}$.*
(2) *X_{Δ} is nonsingular $\Longleftrightarrow$ each $\sigma \in \Delta$ is nonsingular.*

For a nonsingular toric variety X_{Δ}, the action map $T_N \times X_{\Delta} \to X_{\Delta}$ is a holomorphic map. Let $N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$. Then $Lie(T_N) \cong N_{\mathbb{C}}$. The infinitesimal action of Lie algebra $Lie(T_N)$ on X_{Δ} induces a map

$$\rho : N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C} \to \mathfrak{X}(X_{\Delta}) \tag{2.3}$$

by identifying $Lie(T_N)$ with $N_{\mathbb{C}}$. The image of ρ are holomorphic vector fields on X_{Δ}. For any $a \in N_{\mathbb{C}}$ and $m \in M$, we have

$$\rho(a)(\chi^m) = \langle a, m\rangle \chi^m, \tag{2.4}$$

where χ^m is considered as a rational function on X_Δ, $\langle a, m\rangle$ is defined by the $\mathbb{C}$-linear extension of the pair $\langle , \rangle : M \times N \to \mathbb{Z}$. By abuse of notation, we denote the induced map $\wedge^k N_{\mathbb{C}} \to \mathfrak{X}^k(X_\Delta)$ also by ρ.

Example 2.11. As we have shown in Example 2.6, $X = \mathbb{C}\mathbf{P}^n$ is a toric variety. We will associate the toric variety $X = \mathbb{C}\mathbf{P}^n$ with a fan Δ in $N_{\mathbb{R}}$, where $N = \mathbb{Z}^n$. Let $e_0 = (-1, -1, ..., -1), e_1 = (1, 0, , .., 0), ..., e_n = (0, ..., 0, 1)$ be vectors in $N \subset N_{\mathbb{R}} = \mathbb{R}^n$. Choose the $\mathbb{Z}$-basis $\{e_1, e_2, ..., e_n\}$ of N and let $\{e_1^*, e_2^*, ..., e_n^*\}$ be the dual basis of M. Let $t_i = \chi(e_i^*)$. Then there is an isomorphism $T_N \cong (\mathbb{C}^*)^n : t \longleftrightarrow (t_1, t_2, ...t_n)$. For $m = \sum_{i=1}^n m_i e_i^*$, we have $\chi^m = t_1^{m_1} t_2^{m_2} ... t_n^{m_n}$. which is a Laurent monomial on T_N.

Let the fan Δ be the collection of the cones of the following form:

$$\sigma = \sum_{s=1}^{k} \mathbb{R}_{\geq 0} e_{i_s}, \quad \{i_1, i_2, ..., i_k\} \subsetneq \{0, 1, ..., n\}.$$

By gluing together U_σ and U_τ along their common open subset $U_{\sigma\cap\tau}$ for all $\sigma, \tau \in \Delta$, we get that $X_\Delta = \mathbb{C}\mathbf{P}^n$ as a toric variety.

Let

$$\sigma_i = \sum_{s=1}^{n} \mathbb{R}_{\geq 0} e_{i_s}, \quad \{i_1, i_2, ..., i_n\} = \{0, 1, ..., n\} \backslash \{i\}.$$

Then we may identify U_{σ_i} with the affine open set $U_i = \{[z_0, z_1, ..., z_n] \in \mathbb{C}\mathbf{P}^n \mid z_i \neq 0\}$. And $(t_1, t_2, ..., t_n)$ can be identified with the affine coordinates on $U_0 = \{[z_0, z_1, ..., z_n] \in \mathbb{C}\mathbf{P}^n \mid z_0 \neq 0\}$, i.e., $t_i = \frac{z_i}{z_0}$. For $m = \sum_{i=1}^n m_i e_i^*$, the rational function χ^m on $X_\Delta = \mathbb{C}\mathbf{P}^n$ can be written as

$$\chi^m = t_1^{m_1} t_2^{m_2} ... t_n^{m_n} = z_0^{m_0} z_1^{m_1} ... z_n^{m_n}, \tag{2.5}$$

where $m_0 = -\sum_{i=1}^n m_i$.

Definition 2.12. Let X be a nonsingular toric variety. If a holomorphic Poisson structure π on X is invariant under the torus action, then π is called a holomorphic toric Poisson structure on X, and X is called a holomorphic toric Poisson manifold.

Proposition 2.13. *Let X_Δ be a nonsingular toric variety associated with a fan Δ in $N_{\mathbb{R}}$. Then the set of holomorphic toric Poisson structures on X coincide with $\rho(\wedge^2 N_{\mathbb{C}})$.*

Suppose $e_1, e_2, \ldots, e_n$ is a basis of $N \subset N_{\mathbb{C}}$. Then $v_i = \rho(e_i)$ $(i = 1, 2, \ldots, n)$ are holomorphic vector fields on X_Δ. The Proposition 2.13 can be state in an equivalent way:

Proposition 2.14. *Let X_Δ be a nonsingular toric variety associated with a fan Δ in $N_{\mathbb{R}}$. Suppose $e_1, e_2, \ldots, e_n$ is a basis of $N \subset N_{\mathbb{C}}$, $v_i = \rho(e_i)$ $(i = 1, 2, \ldots, n)$. Then π is a holomorphic toric Poisson structure on X_Δ if and only if π can be written as*

$$\pi = \sum_{1\leq i<j\leq n} a_{ij} v_i \wedge v_j,$$

where a_{ij} $(1 \leq i < j \leq n)$ are complex constants.

Proof. $\Leftarrow$: Suppose $\pi = \sum_{1\leq i<j\leq n} a_{ij} v_i \wedge v_j$ with a_{ij} $(1 \leq i < j \leq n)$ being complex constants. Since T_N is abelian, we have that $[v_i, v_j] = 0$ for all $1 \leq i < j \leq n$, which imply $[\pi, \pi] = 0$. Obviously, π is holomorphic and T_N-invariant. Hence $\pi = \sum_{1\leq i<j\leq n} a_{ij} v_i \wedge v_j$ is a holomorphic toric Poisson structure on X_Δ.

$\Rightarrow$: Suppose π is a holomorphic toric Poisson structure on X_Δ. Then the restriction of π on $T_N \subset X_\Delta$ is a holomorphic toric Poisson structure on T_N, which is denoted by $\tilde{\pi}$. Any T_N-invariant holomorphic bi-vector field on $T_N \subset X_\Delta$ can be written as

$$\sum_{1 \le i < j \le n} a_{ij} \tilde{v}_i \wedge \tilde{v}_j,$$

where a_{ij} $(1 \le i < j \le n)$ are complex constants, and $\tilde{v}_i$ $(1 \le i \le n)$ are the restriction of the vector fields v_i $(1 \le i \le n)$ on T_N. Thus $\tilde{\pi}$ can be written as

$$\tilde{\pi} = \sum_{1 \le i < j \le n} a_{ij} \tilde{v}_i \wedge \tilde{v}_j.$$

Since T_N is a dense open set of X_Δ, we have

$$\pi = \sum_{1 \le i < j \le n} a_{ij} v_i \wedge v_j.$$

$\square$

Example 2.15. Let $X = \mathbb{C}\mathbf{P}^n$ and let $[z_0, z_1, \ldots, z_n]$ be homogenous coordinates on it. As we have shown in Example 2.6 and in Example 2.11, $X = \mathbb{C}\mathbf{P}^n$ is a toric variety. Let $P = \mathbb{C}^{n+1} \backslash \{0\} = \{(z_0, z_1, \ldots, z_n) \mid z_0, z_1, \ldots, z_n$ are not all zeros$\}$, and let $p : P = \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}\mathbf{P}^n$ be the canonical projection. Then $v_i = p_*(z_i \frac{\partial}{\partial z_i})$ $(i = 0, 1, \ldots, n)$ are holomorphic toric-invariant vector fields on X, and $\sum_{i=0}^n v_i = 0$. Moreover, by Equation (2.4) and Equation (2.5), we have

$$v_i = \rho(e_i) \quad \text{for} \quad i = 0, 1, ..., n.$$

Thus any holomorphic toric Poisson structures on X can be written as

$$\pi = \sum_{1 \le i < j \le n} a_{ij} v_i \wedge v_j,$$

where a_{ij} $(1 \le i < j \le n$ are complex constants.

2.3. **The standard Poisson structure on $\mathbb{C}\mathbf{P}^n$.** In [?B-G-Y06, ?G-Y09], Brown, Goodear and Yakimov studied the geometry of the standard Poisson structures on affine spaces and flag varieties. Let us review the definition of the standard Poisson structure on flag varieties.

Let G be a connected complex reductive algebraic group with maximal torus H. Denote the corresponding Lie algebra by $\mathfrak{g}$ and $\mathfrak{h}$. Denote Δ_+ (Δ_-) the set of all positive (negative) roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

The standard r-matrix of $\mathfrak{g}$ is given by

$$r_{\mathfrak{g}} = \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha}, \tag{2.6}$$

where e_α and $e_{-\alpha}$ are root vectors of α and $-\alpha$, normalized by $\langle e_\alpha, f_\alpha \rangle = 1$. The standard Poisson structure on G is given by

$$\pi_G = L(r_{\mathfrak{g}}) - R(r_{\mathfrak{g}}),$$

where $L(r_{\mathfrak{g}})$ and $R(r_{\mathfrak{g}})$ refer to the left and right invariant bi-vector fields on G associated to $r_{\mathfrak{g}} \in \wedge^2 \mathfrak{g} \cong \wedge^2 T_e G$.

For a parabolic group $\mathcal{P}$ containing H, $X = G/\mathcal{P}$ is a flag variey. The action of G on $X = G/\mathcal{P}$ induces a map $\mu : \mathfrak{g} \to \mathfrak{X}(X)$. By abuse of notations, the induced maps $\wedge^k \mathfrak{g} \to \mathfrak{X}^k(X)$ are also denoted by μ. The natural projection

$$\phi : G \to X = G/\mathcal{P}$$

induces the following Poisson structure on the flag variety $X = G/\mathcal{P}$:

$$\pi_{st} = \phi_*(\pi_G) = \mu(r_{\mathfrak{g}}), \tag{2.7}$$

called the standard Poisson structure on the flag varieties. The standard Poisson structure π_{st} is a holomorphic Poisson structure on the flag variety $G/\mathcal{P}$.

Next we will focus on the standard Poisson structure on $\mathbb{C}\mathbf{P}^n$.

Set $G = GL(n+1, \mathbb{C})$, H consisting of the diagonal matrices in $GL(n+1, \mathbb{C})$, $\mathcal{P}$ consisting of matrices of the following form $\begin{pmatrix} \lambda & b \\ 0 & D \end{pmatrix}$, where $\lambda \in \mathbb{C}^*, b \in \mathbb{C}^n, D \in GL(n, \mathbb{C})$. Then $X = G/\mathcal{P}$ becomes the projective space $\mathbb{C}\mathbf{P}^n$.

The left action of $GL(n+1, \mathbb{C})$ on $X = \mathbb{C}\mathbf{P}^n$ can be written as:

$$A \cdot [z_0, z_1, ..., z_n] \mapsto p((z_0, z_1, ..., z_n)A^t),$$

where $A \in GL(n+1, \mathbb{C})$, $[z_0, z_1, ..., z_n] \in \mathbb{C}\mathbf{P}^n$, $(z_0, z_1, ..., z_n) \in \mathbb{C}^{n+1}$, and p is the canonical projection

$$\mathbb{C}^{n+1} \backslash \{0\} \xrightarrow{p} \mathbb{C}\mathbf{P}^n : (z_0, z_1, ..., z_n) \to [z_0, z_1, ..., z_n].$$

The standard r-matrix of $\mathfrak{g} = gl(n+1, \mathbb{C})$ can be written as

$$r_{\mathfrak{g}} = \sum_{0 \le i < j \le n} e_{ij} \wedge e_{ji}, \tag{2.8}$$

where e_{ij} denotes the matrix having 1 in the $(i+1, j+1)$ position and 0 elsewhere. Now we are ready to compute the standard Poisson structure on $\mathbb{C}\mathbf{P}^n$.

Lemma 2.16. *Let $X = \mathbb{C}\mathbf{P}^n = GL(n+1, \mathbb{C})/\mathcal{P}$. Let $v_i = p_*(z_i \frac{\partial}{\partial z_i})$ $(i = 0, 1, ..., n)$. Then the standard Poisson structure on $X = \mathbb{C}\mathbf{P}^n$ can be written as*

$$\pi_{st} = \sum_{1 \le i < j \le n} v_i \wedge v_j. \tag{2.9}$$

Proof. By computation, we have

$$\mu(e_{ij}) = p_*(z_i \frac{\partial}{\partial z_j}).$$

Therefore the standard Poisson structure on $X = \mathbb{C}\mathbf{P}^n$ can be written as

$$\begin{aligned}
\pi_{st} = \mu(r_{\mathfrak{g}}) &= \sum_{0\le i<j\le n} \mu(e_{ij}) \wedge \mu(e_{ji}) \\
&= \sum_{0\le i<j\le n} p_*(z_i \frac{\partial}{\partial z_j}) \wedge p_*(z_j \frac{\partial}{\partial z_i}) \\
&= \sum_{0\le i<j\le n} p_*(z_i \frac{\partial}{\partial z_i}) \wedge p_*(z_j \frac{\partial}{\partial z_j}) \\
&= \sum_{0\le i<j\le n} v_i \wedge v_j \\
&= \sum_{1\le i<j\le n} v_i \wedge v_j.
\end{aligned}$$

The last step holds as $\sum_{i=0}^n v_i = 0$. □

2.4. Some exact sequences related to $\mathbb{C}\mathbf{P}^n$.

Theorem 2.17. [1] *Let P be a principle bundle over X with group G. Then there exists an exact sequence of vector bundles over X:*

$$0 \to P \times_G \mathfrak{g} \to TP/G \to TX \to 0, \tag{2.10}$$

where $P \times_G \mathfrak{g}$ is the bundle associated to P by the adjoint representation of G on $\mathfrak{g} = Lie(G)$, and TP/G is the bundle of invariant vector fields on P.

Recall that for a principle G-bundle P over X, and a representation of G on a vector space V, the associated vector bundle over X is defined to be $P \times_G V = (P \times V)/\sim$, and $(x.g, v) \sim ((x, g.v)$ $\forall x \in P$, $g \in G, v \in V$.

Let $P = \mathbb{C}^{n+1}\backslash\{0\}$, $X = \mathbb{C}\mathbf{P}^n$, and $p : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathbf{P}^n$ being the canonical projection. $G = \mathbb{C}^*$ operates by right multiplication on $P = \mathbb{C}^{n+1}\backslash\{0\}$:

$$\lambda : v \to v\lambda, \quad v \in \mathbb{C}^{n+1}\backslash\{0\}, \quad \lambda \in \mathbb{C}^*.$$

Then P is a principle $\mathbb{C}^*$-bundle over X. Let $L = P \times_{\mathbb{C}^*} \mathbb{C}$ be the associated line bundle with the $\mathbb{C}^*$ action on $\mathbb{C}$ by multiplication. Then $L = O(-1)$ is isomorphic to the tautological line bundle of $\mathbb{C}\mathbf{P}^n$, and $L^* = O(1)$, where $O(1)$ denotes the line bundle corresponding to a hyperplane section.

Let us show the Atiyah's exact sequence (2.10) in this case.

Since $G = \mathbb{C}^*$ is abelian, the adjoint representation is trivial, we have $P \times_G \mathfrak{g} \cong X \times \mathbb{C}$. The $G = \mathbb{C}^*$ action on $TP \cong \mathbb{C}^{n+1}\backslash\{0\} \times \mathbb{C}^{n+1}$ is given by:

$$(x \times v)\lambda = x\lambda \times v\lambda, \qquad x \in \mathbb{C}^{n+1}\backslash\{0\}, v \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}^*.$$

Hence $TP/G \cong P \times_{\mathbb{C}^*} \mathbb{C}^{n+1}$, where $P \times_{\mathbb{C}^*} \mathbb{C}^{n+1}$ is the associated bundle of P by the $\mathbb{C}^*$ representation ρ on $\mathbb{C}^{n+1}$ given by:

$$\rho(\lambda)v = \lambda^{-1}v, \qquad v \in \mathbb{C}^{n+1}, \lambda \in \mathbb{C}^*.$$

Thus $TP/G \cong L^* \otimes \mathbb{C}^{n+1} \cong O(1)^{\oplus(n+1)}$, where $\mathbb{C}^{n+1}$ denotes the trivial bundle $X \times \mathbb{C}^{n+1}$. So in this case, the Atiyah exact sequence (2.10) becomes

$$0 \to \mathbb{C} \to O(1)^{\oplus(n+1)} \to T\mathbb{C}\mathbf{P}^n \to 0, \tag{2.11}$$

which is exactly the Euler exact sequence for $\mathbb{C}\mathbf{P}^n$.

By a similar way, we can prove the vector bundles isomorphisms

$$(\wedge^j TP)/G \cong \wedge^j (TP/G) \cong O(j)^{\oplus \binom{n+1}{j}}, \tag{2.12}$$

where $(\wedge^j TP)/G$ denotes the vector bundle of $\mathbb{C}^*$-invariant j-vector fields on P, $1 \leqslant j \leqslant n+1$.

Let us choose $(z_0, z_1, \ldots, z_n)$ as the coordinates on $\mathbb{C}^{n+1} \supset P = \mathbb{C}^{n+1}\backslash\{0\}$. Then the canonical map $p : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathbf{P}^n$ becomes $(z_0, z_1, \ldots, z_n) \to [z_0, z_1, \ldots, z_n]$, where $[z_0, z_1, \ldots, z_n]$ are the homogenous coordinates on $\mathbb{C}\mathbf{P}^n$. Under the isomorphism $TP \cong \mathbb{C}^{n+1}\backslash\{0\} \times \mathbb{C}^{n+1}$, we choose $\frac{\partial}{\partial z_0}, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ as a basis for $\mathbb{C}^{n+1}$ (the tangent part of TP).

Since $TP/G \cong O(1)^{\oplus(n+1)}$, any $\mathbb{C}^*$-invariant holomorphic vector field on P can be written as

$$\sum_{0 \leqslant i,j \leqslant n} a_i^j z_j \frac{\partial}{\partial z_i},$$

where a_i^j are complex constants. And since $(\wedge^j TP)/G \cong O(j)^{\oplus \binom{n+1}{j}}$, any $\mathbb{C}^*$-invariant holomorphic k-vector field on P can be written as

$$\sum_{0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} f_{i_1, i_2, \ldots, i_k} \frac{\partial}{\partial z_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial z_{i_k}}, \tag{2.13}$$

where $f_{i_1, i_2, \ldots, i_k}$ are homogenous polynomials of degree k with variables $z_0, z_1, \ldots, z_n$.

The map $p : P \to \mathbb{C}\mathbf{P}^n$ induces a map $TP/G \to T\mathbb{C}\mathbf{P}^n$, which can be identified with the map $O(1)^{\oplus(n+1)} \to T\mathbb{C}\mathbf{P}^n$ in the Euler exact sequence (2.11). By abuse of notation, the map $O(1)^{\oplus(n+1)} \to T\mathbb{C}\mathbf{P}^n$ will be denoted by p_*, with $\ker p_*$ being a trivial line bundle generated by the Euler vector fields $\overrightarrow{e} = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$. Then the Euler exact sequence (2.11) can be written as

$$0 \to \mathbb{C} \hookrightarrow O(1)^{\oplus(n+1)} \xrightarrow{p_*} T\mathbb{C}\mathbf{P}^n \to 0, \tag{2.14}$$

where $\mathbb{C} \hookrightarrow O(1)^{\oplus(n+1)}$ is considered as the embedding map $\ker p_* \hookrightarrow O(1)^{\oplus(n+1)}$.

Lemma 2.18. *Let us denote $L = \mathbb{C}\overrightarrow{e}$ as the trivial line bundle $\mathbb{C}$ in Euler exact sequence* (2.14) *and let $E = O(1)^{\oplus(n+1)}$. Then we have exact sequences*

$$0 \to L \wedge (\wedge^{j-1} E) \hookrightarrow \wedge^j E \xrightarrow{\overrightarrow{e}\wedge\cdot} L \wedge (\wedge^j E) \to 0 \tag{2.15}$$

and

$$0 \to L \wedge (\wedge^{j-1} E) \hookrightarrow \wedge^j E \xrightarrow{p_*} \wedge^j TX \to 0 \tag{2.16}$$

for all $j \geq 1$, where

(a) $L \wedge (\wedge^j E) = \mathbb{C}\overrightarrow{e} \wedge (\wedge^j E)$ *is a subbundle of* $\wedge^{j+1} E$,
(b) $L \wedge (\wedge^{j-1} E) \hookrightarrow \wedge^j E$ *is the embedding of* $L \wedge (\wedge^{j-1} E)$ *as a subbundle of* $\wedge^j E$,
(c) $\wedge^j E \xrightarrow{\overrightarrow{e}\wedge\cdot} L \wedge (\wedge^j E)$ *is defined by the wedge of* $\overrightarrow{e}$ *with elements in* $\wedge^j E$,
(d) $\wedge^j E \xrightarrow{p_*} \wedge^j TX$ *is induced by the map* $E \xrightarrow{p_*} T\mathbb{C}\mathbf{P}^n$ *in* (2.14).

Proof. At each point $x \in X = \mathbb{C}\mathbf{P}^n$, for any $\alpha_x \in \wedge^j E\,|_x$, we have that

$$\overrightarrow{e}_x \wedge \alpha_x = 0$$

if and only if there exist $\beta_x \in \wedge^{j-1} E\mid_x$, such that

$$\alpha_x = \overrightarrow{e}_x \wedge \beta_x.$$

It implies that (2.15) is an exact sequence for all $j \geq 1$.

By (2.14), the kernel of $E \xrightarrow{p_*} T\mathbb{C}\mathbf{P}^n$ is the trivial bundle $L = \mathbb{C}\overrightarrow{e}$. At each point $x \in X$, the kernel of the map

$$E\mid_x \xrightarrow{p_*} TX\mid_x$$

is $\mathbb{C}\overrightarrow{e}_x$. As a consequence, the kernel of

$$\wedge^j E\mid_x \xrightarrow{p_*} \wedge^j TX\mid_x$$

is

$$\overrightarrow{e}_x \wedge (\wedge^{j-1} E\mid_x).$$

Thus (2.15) is an exact sequence for all $j \geq 1$. $\square$

3. The cohomology group $H^i(\mathbb{C}\mathbf{P}^n, \wedge^j \mathcal{T}_{\mathbb{C}\mathbf{P}^n})$

3.1. The vanishing of the cohomology group $H^i(\mathbb{C}\mathbf{P}^n, \wedge^j \mathcal{T}_{\mathbb{C}\mathbf{P}^n})$ for $i > 0$ and $0 \leqslant j \leqslant n$.

Lemma 3.1. *Let us denote L as the trivial line bundle $\mathbb{C}$ in Euler exact sequence* (2.14) *and let $E = O(1)^{\oplus(n+1)}$. Then we have*

$$H^i(X, \wedge^j TX) \cong H^i(X, L \wedge (\wedge^j E)) \cong H^{i+1}(X, L \wedge (\wedge^{j-1} E)) \tag{3.1}$$

for all $i > 0$ and $j \geq 1$.

Proof. The exact sequence (2.15) in Lemma 2.18 induces a long exact sequence

$$\ldots \to H^i(X, \wedge^j E) \to H^i(X, L \wedge (\wedge^j E)) \to H^{i+1}(X, L \wedge (\wedge^{j-1} E)) \to H^{i+1}(X, \wedge^j E) \to \ldots. \tag{3.2}$$

As

$$\wedge^j E = \wedge^j (O(1)^{\oplus(n+1)}) = O(j)^{\oplus\binom{n+1}{j}},$$

we have that

$$\begin{aligned} H^i(X, \wedge^j E) &= H^i(X, O(j)^{\oplus\binom{n+1}{j}}) \\ &= H^i(X, O(j))^{\oplus\binom{n+1}{j}} \\ &= H^i(X, K_X \otimes O(n+1+j))^{\oplus\binom{n+1}{j}}, \end{aligned}$$

where $K_X \cong O(-n-1)$ is the canonical bundle of $X = \mathbb{C}\mathbf{P}^n$. By Kodaira vanishing theorem, we have

$$H^i(X, K_X \otimes O(n+1+j)) = 0$$

for $i > 0$. Thus

$$H^i(X, \wedge^j E) = 0 \quad (i > 0).$$

As a consequence, by the exact sequence (3.2), we have

$$H^i(X, L \wedge (\wedge^j E)) \cong H^{i+1}(X, L \wedge (\wedge^{j-1} E)) \tag{3.3}$$

for all $i > 0$ and $j \geq 1$.

Similarly, by the exact sequence (2.16) in Lemma 2.18, we can prove that

$$H^i(X, \wedge^j TX) \cong H^{i+1}(X, L \wedge (\wedge^{j-1} E)) \tag{3.4}$$

for all $i > 0$ and $j \geq 1$.

Combine (3.3) and (3.4), we proved the lemma. □

Theorem 3.2. *For $X = \mathbb{CP}^n$, we have*

$$H^i(X, \wedge^j \mathcal{T}_X) = 0 \tag{3.5}$$

for all $i > 0$ and $0 \leqslant j \leqslant n$.

Proof.

(1) In the case of $j = 0$, $H^i(X, \mathcal{O}_X) = 0$ $(i > 0)$ is a well known result. It comes directly from $H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{K}_X \otimes \mathcal{O}(n+1))$ and Kodaira vanishing theorem, where $\mathcal{K}_X \cong \mathcal{O}(-n-1)$ is the sheaf of canonical bundle.

(2) In the case of $j \geq 1$, by Lemma 3.1, we have

$$H^i(X, \wedge^j TX) \cong H^i(X, L \wedge (\wedge^j E)) \cong H^{i+1}(X, L \wedge (\wedge^{j-1} E)) \cong \cdots \cong H^{i+j}(X, L)$$

As L is a trivial line bundle, we have $H^{i+j}(X, L) = H^{i+j}(X, O_X) = 0$ for $i > 0$ and $j \geq 1$.

Hence

$$H^i(X, \wedge^j \mathcal{T}_X) = 0$$

for all $i > 0$ and $j \geq 1$.

□

Remark 3.3. For $X = \mathbb{CP}^n$, in the case of $j = 1$, the conclusion $H^i(X, \mathcal{T}_X) = 0$ $(i > 0)$ is a special case of Theorem VII in [3].

3.2. **Holomorphic vector fields and multi-vector fields on $\mathbb{CP}^n$.** In this section, we will give a description of the holomorphic vector fields and multi-vector fields on $\mathbb{CP}^n$.

Lemma 3.4. *Let us denote $L = \mathbb{C}\overrightarrow{e}$ as the trivial line bundle $\mathbb{C}$ in Euler exact sequence* (2.14) *and let $E = O(1)^{\oplus(n+1)}$. Then we have exact sequences*

$$0 \to H^0(X, L \wedge (\wedge^{j-1} E)) \to H^0(X, \wedge^j E) \xrightarrow{p_*} H^0(X, \wedge^j TX) \to 0 \tag{3.6}$$

for all $j \geq 1$.

Proof. By the exact sequence (2.16), we have

$$0 \to H^0(X, L \wedge (\wedge^{j-1} E)) \to H^0(X, \wedge^j E) \xrightarrow{p_*} H^0(X, \wedge^j TX) \to H^1(X, L \wedge (\wedge^{j-1} E)) \to \cdots$$

for all $j \geq 1$. By Lemma 3.1, we have

$$H^1(X, L \wedge (\wedge^{j-1} E)) \cong H^2(X, L \wedge (\wedge^{j-2} E)) \cong \cdots \cong H^j(X, L) = H^j(X, O_X) = 0$$

for all $j \geq 1$. Thus we have

$$0 \to H^0(X, L \wedge (\wedge^{j-1} E)) \to H^0(X, \wedge^j E) \xrightarrow{p_*} H^0(X, \wedge^j TX) \to 0$$

for all $j \geq 1$. □

Remark 3.5. In the case $j = 1$, the exact sequence (3.6) becomes

$$0 \to \mathbb{C} \to H^0(X, \mathcal{O}(1))^{\oplus(n+1)} \to H^0(X, \mathcal{T}_X) \to 0. \tag{3.7}$$

Notice that $\wedge^j E = O(j)^{\oplus\binom{n+1}{j}}$. Next we will give a description of the space $H^0(X, \wedge^j E) = H^0(X, O(j))^{\oplus\binom{n+1}{j}}$.

Let V_k $(1 \leqslant k \leqslant n+1)$ be the complex vector space of the k-vector fields

$$\sum_{0\leqslant i_1<i_2<\ldots<i_k\leqslant n} f_{i_1,i_2,\ldots,i_k}\frac{\partial}{\partial z_{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z_{i_k}}$$

on $\mathbb{C}^{n+1}$, where $f_{i_1,i_2,\ldots,i_k}$ are homogenous polynomials of $z_0, z_1, \ldots, z_n$ with degree k. The restriction of the k-vector fields in V_k on $P = \mathbb{C}^{n+1}\backslash\{0\}$ forms a vector space, which will be denoted by $\tilde{V}_k$. As we have shown in Equation (2.13), $\tilde{V}_k$ coincide with the space of the $\mathbb{C}^*$-invariant holomorphic k-vector fields on P.

Lemma 3.6. *For $1 \leqslant j \leqslant n+1$, the complex vector spaces below are isomorphic:*

(a) V_j,
(b) $H^0(X, O(j))^{\oplus\binom{n+1}{j}}$,
(c) *the space of $\mathbb{C}^*$-invariant holomorphic j-vector fields on P.*

Proof. $(a) \cong (b)$: The vector space V_j and $H^0(X, O(j))^{\oplus\binom{n+1}{j}}$ are isomorphic by identifying the $\binom{n+1}{j}$ polynomials $f_{i_1,i_2,\ldots,i_k}$ with the different components of $H^0(X, O(j))^{\oplus\binom{n+1}{j}}$.

$(a) \cong (c)$: Since $\tilde{V}_j$ coincide with the space of $\mathbb{C}^*$-invariant holomorphic j-vector fields on P, we have that V_j and the space of $\mathbb{C}^*$-invariant holomorphic j-vector fields on P are isomorphic.

$(b) \cong (c)$: As $(\wedge^j TP)/G \cong O(j)^{\oplus\binom{n+1}{j}}$, $H^0(X, O(j))^{\oplus\binom{n+1}{j}}$ and the space of $\mathbb{C}^*$-invariant holomorphic j-vector fields on P are isomorphic. $\square$

By Lemma 3.4 and Lemma 3.6, we have

Lemma 3.7. [2] *Let $p : P = \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}\mathbf{P}^n : (z_0, z_1, \ldots, z_n) \to [z_0, z_1, \ldots, z_n]$ being the canonical projection. Then we have*

(1) *The holomorphic k-vector fields on $\mathbb{C}\mathbf{P}^n$ can be written as*

$$p_*(\sum_{0\leqslant i_1<i_2<\ldots<i_k\leqslant n} g_{i_1,i_2,\ldots,i_k}\frac{\partial}{\partial z_{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z_{i_k}}),$$

where $g_{i_1,i_2,\ldots,i_k}$ are homogenous polynomials with variables $z_0, z_1, ..., z_n$ of degree k. Or in other words,

$$H^0(X, \wedge^k \mathcal{T}_X) = p_*(\tilde{V}_k).$$

(2) *For the map $p_* : \tilde{V}_k \to H^0(X, \wedge^k \mathcal{T}_X)$,*

$$\ker p_* = (\sum_{i=0}^n z_i \frac{\partial}{\partial z_i}) \wedge \tilde{V}_{k-1}.$$

Next we will introduce some notations, which are important for the paper.

- Let $v_i = p_*(z_i\frac{\partial}{\partial z_i})$ $(0 \le i \le n)$. As we have shown in Example 2.15, $v_i = \rho(e_i)$ $(0 \le i \le n)$, and $v_0 = -\sum_{i=1}^n v_i = 0$. Let W^1 be the n-dimensional $\mathbb{C}$-vector space generated by $v_1, ..., v_n$. Then we have $W = \rho(N_{\mathbb{C}})$. Set $W^k = \wedge^k W$ $(1 \le k \le n)$, $W^0 = \mathbb{C}$. Then W^k can be considered as a subspace of $H^0(X, \wedge^k \mathcal{T}_X)$.
- For monominals $z_0^{m_0}...z_n^{m_n}$ satisfying $\sum_{i=0}^n m_i = 0$, the derivatives of $z_0^{m_0}...z_n^{m_n}$ satisfy
$$v_i(z_0^{m_0}...z_n^{m_n}) = \langle e_i, m\rangle z_0^{m_0}...z_n^{m_n},$$
where $m = (m_1, ..., m_n) \in M$. Let
$$\tilde{M} = \{(m_0, m_1, ..., m_n) \in \mathbb{Z}^{n+1} \mid \sum_{i=0}^n m_i = 0\}.$$
Then $\tilde{M} \cong M$.
- For $I = (m_0, m_1, ..., m_n) \in \tilde{M}$ satisfying $m_i \ge -1$ $(i = 0, 1, ...n)$, suppose $\{m_{i_1}, ..., m_{i_l} \mid 0 \le i_1 < ... < i_l \le n\}$ are all the elements equal to -1 in the set $\{m_0, m_1, ..., m_n\}$. Set
$$|I| = l, \quad Z^I = z_0^{m_0}...z_n^{m_n}, \quad V_I = v_{i_1} \wedge ... \wedge v_{i_l} \in W^l,$$
$$e_I = e_{i_1} \wedge ... \wedge e_{i_l} \in \wedge^l N, \quad m(I) = (m_1, ..., m_n) \in M.$$

Theorem 3.8. *Let $X = \mathbb{C}\mathbf{P}^n$. We have*
$$H^0(X, \wedge^k \mathcal{T}_X) = \oplus_{I \in S_k} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|} \tag{3.8}$$
for $0 \le k \le n$, where S_k is the subset of $\tilde{M}$ consisting of all $I \in \tilde{M}$ satisfying the conditions
$$\langle m(I), e_i\rangle = m_i \ge -1 \quad (0 \le i \le n) \tag{3.9}$$
and
$$|I| \le k. \tag{3.10}$$

Proof. (1) First, we will prove that
$$H^0(X, \wedge^k \mathcal{T}_X) = \sum_{I \in S_k} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}. \quad (0 \le k \le n) \tag{3.11}$$

By Lemma 3.7, any holomorphic k-vector field Ξ on $\mathbb{C}\mathbf{P}^n$ can be written as
$$\begin{aligned}\Xi &= p_*(\sum_{0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} g_{i_1, i_2, \ldots, i_k} \frac{\partial}{\partial z_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial z_{i_k}}) \\ &= \sum_{0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} f_{i_1, i_2, \ldots, i_k} v_{i_1} \wedge \ldots \wedge v_{i_k},\end{aligned}$$
where $f_{i_1, i_2, \ldots, i_k} = \dfrac{g_{i_1, i_2, \ldots, i_k}}{\prod_{s=1}^k z_{i_s}} = \sum c^{i_1, \ldots, i_k}_{m_0, \ldots, m_n} z_0^{m_0} ... z_n^{m_n}$, with $c^{i_1, \ldots, i_k}_{m_0, \ldots, m_n}$ being complex constants, and
(a) $m_{i_s} \ge -1$ for $1 \le s \le k$,
(b) $m_j \ge 0$ for $j \notin \{i_s \mid 1 \le s \le k\}$,
(c) $\sum_{i=0}^n m_i = 0$.

$$\Xi = \sum c^{i_1,\ldots,i_k}_{m_0,\ldots,m_n} z_0^{m_0}\ldots z_n^{m_n} v_{i_1}\wedge\ldots\wedge v_{i_k}, \tag{3.12}$$

with m_i $(0\le i\le n)$ satisfying the above conditions.

Without loss of generality, suppose $\{m_{i_1}, m_{i_2}, ..., m_{i_l}\}$ are all the elements equal to -1 in the set $\{m_0, m_1, ..., m_n\}$. Then

$$z_0^{m_0}\ldots z_n^{m_n} v_{i_1}\wedge\ldots\wedge v_{i_k} = (z_0^{m_0}\ldots z_n^{m_n} v_{i_1}\wedge\ldots\wedge v_{i_l})\wedge(v_{i_{l+1}}\wedge\ldots\wedge v_{i_k}).$$

For $I=(m_0, m_1, ..., m_n)$, we have

$$z_0^{m_0}\ldots z_n^{m_n} v_{i_1}\wedge\ldots\wedge v_{i_l} = Z^I\cdot V_I,$$

and

$$v_{i_{l+1}}\wedge\ldots\wedge v_{i_k}\in W^{k-|I|}.$$

By Equation (3.12), $\Xi\in\sum_{I\in S_k}\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}\quad (0\le k\le n)$.

Thus we have

$$H^0(X,\wedge^k\mathcal{T}_X)\subseteq\sum_{I\in S_k}\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}\quad (0\le k\le n).$$

On the other hand, for $I=(m_0, m_1, ..., m_n)\in S\subset\tilde{M}$, suppose $m_{i_1}, m_{i_2}, ..., m_{i_l}$ $(0\le i_1<i_2<\ldots<i_l\le n)$ are all the elements equal to -1 in the set $\{m_0, m_1, ..., m_n\}$. Then we have

$$\begin{aligned}
Z^I\cdot V_I &= z_0^{m_0}\ldots z_n^{m_n} v_{i_1}\wedge\ldots\wedge v_{i_l}\\
&= z_0^{m_0}\ldots z_n^{m_n} p_*(z_{i_1}\frac{\partial}{\partial z_{i_1}})\wedge\ldots\wedge p_*(z_{i_1}\frac{\partial}{\partial z_{i_1}})\\
&= p_*((z_{i_1}\ldots z_{i_l}).(z_0^{m_0}\ldots z_n^{m_n})\frac{\partial}{\partial z_{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z_{i_l}})\\
&= p_*((\prod_{i\notin\{i_1,\ldots i_l\}} z_i^{m_i})\frac{\partial}{\partial z_{i_1}}\wedge\ldots\wedge\frac{\partial}{\partial z_{i_l}}).
\end{aligned}$$

Since $m_i\ge 0$ for $i\notin\{i_1, ...i_l\}$, we know that $\prod_{i\notin\{i_1,\ldots i_l\}} z_i^{m_i}$ is a polynominal with variables $z_0, z_1, ...z_n$ of degree l. By Lemma 3.7, $Z^I\cdot V_I$ is a holomorphic l-vector field on $X=\mathbb{C}\mathbf{P}^n$.

As $W^{k-|I|}=W^{k-l}$ is a subspace of $H^0(X,\wedge^{k-l}\mathcal{T}_X)$, we have that $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$ is a subspace of $H^0(X,\wedge^k\mathcal{T}_X)$. Thus we have

$$\sum_{I\in S_k}\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}\subseteq H^0(X,\wedge^k\mathcal{T}_X).$$

By the argument above, the Equation (3.12) holds.

(2) Next we will prove that for different I and J in $S\subset\tilde{M}$ satisfying the conditions 3.9 and 3.10,

$$\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}\cap\mathbb{C}(Z^J\cdot V_J)\wedge W^{k-|J|}=0, \tag{3.13}$$

where $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$ and $\mathbb{C}(Z^J\cdot V_J)\wedge W^{k-|J|}$ are considered as the subspaces of $H^0(X,\wedge^k\mathcal{T}_X)$.

Let $R: H^0(X,\wedge^k\mathcal{T}_X)\to H^0(X,\wedge^k\mathcal{T}_{T_N})$ be the restriction of the holomorphic k-vector fields on the algebraic torus $T_N\subset X$. Then we have

$$R(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|})\subseteq\mathbb{C}Z^I\cdot R(W^k)$$

and

$$R(\mathbb{C}(Z^J \cdot V_J) \wedge W^{k-|J|}) \subseteq \mathbb{C}Z^J \cdot R(W^k),$$

where $R(W^k)$ denotes the restriction of W^k on $T_N \subset X$. Since $\wedge^k T(T_N)$ is a triival bundle, with $\{R(v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_k}) \mid 1 \leq i_1 < i_2 < ... < i_k \leq n\}$ as a basis, and since $\{Z^I \mid I \in M\}$ are $\mathbb{C}$-linear independent functions on T_N, it is easy to verify that for different I and J in $\tilde{M}$,

$$\mathbb{C}Z^I \cdot R(W^k) \cap \mathbb{C}Z^J \cdot R(W^k) = 0.$$

As a consequence, we have

$$R(\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}) \cap R(\mathbb{C}(Z^J \cdot V_J) \wedge W^{k-|J|}) = 0.$$

As T_N is an open dense subset in $X = \mathbb{C}\mathbf{P}^n$, we proved Equation (3.13).

(3) By Equation (3.11) and Equation (3.13), we have

$$H^0(X, \wedge^k \mathcal{T}_X) = \oplus_{I \in S_k} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|} \quad (0 \leq k \leq n).$$

□

In Theorem 3.8, S_k is the set of all $I \in \tilde{M}$ satisfying conditions 3.9 and 3.10. Let us denote $S(i)$ $(0 \leq i \leq n)$ as the set of all all $I \in \tilde{M}$ satisfying the condition 3.9 and $|I| = i$. Then $S_k = \uplus_{0 \leq i \leq k} S(i)$. And we have

$$S_0 \subseteq S_1 \subseteq S_2 \ldots \subseteq S_n.$$

Proposition 3.9. *Let $X = \mathbb{C}\mathbf{P}^n$. We have*

$$H^0(X, \wedge^k \mathcal{T}_X) = (H^0(X, \wedge^{k-1} \mathcal{T}_X) \wedge W) \oplus (\oplus_{I \in S(k)} \mathbb{C}Z^I \cdot W^k) \quad (1 \leq k \leq n), \tag{3.14}$$

where $S(k)$ is the set of all all $I \in \tilde{M}$ satisfying the condition 3.9 and $|I| = k$.

Proof. By Theorem 3.8, we have

$$\begin{aligned} H^0(X, \wedge^k \mathcal{T}_X) &= \oplus_{I \in S_k} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|} \\ &= \oplus_{i=0}^{k} (\oplus_{I \in S(i)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}) \\ &= \oplus_{i=0}^{k-1} (\oplus_{I \in S(i)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}) \oplus (\oplus_{I \in S(k)} \mathbb{C}Z^I \cdot W^k) \\ &= (\oplus_{i=0}^{k-1} (\oplus_{I \in S(i)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-1-|I|}) \wedge W) \oplus (\oplus_{I \in S(k)} \mathbb{C}Z^I \cdot W^k) \\ &= (H^0(X, \wedge^{k-1} \mathcal{T}_X) \wedge W) \oplus (\oplus_{I \in S(k)} \mathbb{C}Z^I \cdot W^k), \end{aligned}$$

where the last step holds since

$$H^0(X, \wedge^{k-1} \mathcal{T}_X) = \oplus_{i=0}^{k-1} (\oplus_{I \in S(i)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-1-|I|}).$$

□

4. Poisson cohomology of $\mathbb{C}\mathbf{P}^n$

4.1. **Poisson cohomology of toric Poisson structures on $\mathbb{C}\mathbf{P}^n$.** Let $X = \mathbb{C}\mathbf{P}^n$. To start the main theorem, we need some preparations.

- Let $\pi = \sum_{1\le i<j\le n} a_{ij} v_i \wedge v_j$ be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$, where $v_i = p_*(z_i \frac{\partial}{\partial z_i}) = \rho(e_i)$ $(0 \le i \le n)$ as we have shown in Example 2.15. Set
$$\Pi = \sum_{1\le i<j\le n} a_{ij} e_i \wedge e_j.$$
Then we have $\Pi \in \wedge^2 N_{\mathbb{C}}$.
- For $I = (m_0, ..., m_n) \in \tilde{M}$, $m(I) = (m_1, ..., m_n) \in M$, we have
$$\imath_{m(I)}\Pi \in N_{\mathbb{C}},$$
where $\imath_{m(I)}\Pi$ denotes the contraction of $m(I) \in M$ with $\Pi \in \wedge^2 N_{\mathbb{C}}$ by the $\mathbb{C}$-linear extension of the pairing $\langle , \rangle : M \times N \to \mathbb{Z}$. And $\rho(\imath_{m(I)}\Pi)$ is a holomorphic vector field on X, where $\rho : N_{\mathbb{C}} \to \mathfrak{X}(X)$ is the map we have defined in Equation (2.3).

Lemma 4.1. *Let $\pi = \sum_{1\le i<j\le n} a_{ij} v_i \wedge v_j$ be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$. For any $I \in \tilde{M}$, we have*

$$[\pi, Z^I] = Z^I \cdot \rho(\imath_{m(I)}\Pi). \tag{4.1}$$

and

$$[\pi, Z^I \cdot V_I] = \rho((\imath_{m(I)}\Pi) \wedge e_I). \tag{4.2}$$

Proof. (1) For $I \in \tilde{M}$ and $0 \le i \le n$, we have

$$v_i(Z^I) = \langle e_i, m(I)\rangle Z^I, \tag{4.3}$$

where Z^I is considered as a rational function on $X = \mathbb{C}\mathbf{P}^n$, $v_i(Z^I)$ denotes the derivative of Z^I along the vector field v_i.

Since that $T_N \cong (\mathbb{C}^*)^n$ is commutative, we have

$$[v_i, v_j] = 0 \tag{4.4}$$

for $0 \le i, j \le n$.

The Lemma can be proved by a simple computation using Equation (4.3) and Equation (4.4).

(2) Since $\rho(e_I) = V_I$, by Equation 4.1 we have
$$\begin{aligned}[\pi, Z^I \cdot V_I] &= \rho(\imath_{m(I)}\Pi) \wedge V_I \\ &= \rho(\imath_{m(I)}\Pi) \wedge \rho(e_I) \\ &= \rho((\imath_{m(I)}\Pi) \wedge e_I).\end{aligned}$$

∎

By Lemma 2.4 and Theorem 3.2, the Poisson cohomology group $H^\bullet_\pi(X)$ is isomorphic to the cohomology of the complex

$$H^0(X,\mathcal{O}_X) \xrightarrow{d_\pi} H^0(X,T_X) \xrightarrow{d_\pi} H^0(X,\wedge^2 T_X) \xrightarrow{d_\pi} \dots \xrightarrow{d_\pi} H^0(X,\wedge^n T_X) \tag{4.5}$$

where $d_\pi = [\pi,\cdot]$.

Lemma 4.2. *Let π be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$.*

(1) *For any $I \in S_k$ $(0 \le k \le n)$, where S_k is the set of all $I \in \tilde{M}$ satisfying conditions 3.9 and 3.10, we have*

$$d_\pi(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}) \subseteq \mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|+1},$$

where $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$ is considered as a subspace of $H^0(X,\wedge^k T_X)$, $d_\pi(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|})$ denotes the image of $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$ under the map $H^0(X,\wedge^k T_X) \xrightarrow{d_\pi} H^0(X,\wedge^{k+1}T_X)$, $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|+1}$ is considered as a subspace of $H^0(X,\wedge^{k+1}T_X)$.

(2) *For any $I \in S_k$ $(0 \le k \le n)$ satisfying the equation*

$$(\imath_{m(I)}\Pi)\wedge e_I = 0,$$

we have

$$d_\pi(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}) = 0.$$

Proof. (1) For any element $\Psi = Z^I\cdot V_I\wedge w$ in $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$, where $I \in S_k$ and $w \in W^{k-|I|}$, by Lemma 4.1, we have

$$\begin{aligned} d_\pi(\Psi) &= [\pi, Z^I\cdot V_I\wedge w] \\ &= \rho(\imath_{m(I)}\Pi)\wedge(Z^I\cdot V_I\wedge w) \\ &= (-1)^{|I|}Z^I\cdot V_I\wedge(\rho(\imath_{m(I)}\Pi)\wedge w. \end{aligned}$$

Since $\rho(\imath_{m(I)}\Pi)\wedge w \in W^{k-|I|+1}$, $d_\pi(\Psi)$ is an element in $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|+1}$.

Thus we have

$$d_\pi(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}) \subseteq \mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|+1} \quad \text{for all} \quad I \in S_k.$$

(2) If $I \in S_k$ satisfies the equation

$$(\imath_{m(I)}\Pi)\wedge e_I = 0,$$

then

$$\rho((\imath_{m(I)}\Pi)\wedge e_I) = \rho(\imath_{m(I)}\Pi)\wedge V_I = 0.$$

By Lemma 4.1, with the similar argument as above, we have

$$d_\pi(\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}) = 0.$$

□

Lemma 4.3. *Let π be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$. For any holomorphic k-vector field Ψ in $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|}$ with $I \in S_{k-1} \subseteq S_k$ $(1 \le k \le n)$, if*

$$d_\pi(\Psi) = 0 \quad \textit{and} \quad (\imath_{m(I)}\Pi)\wedge e_I \neq 0,$$

then there exists a holomorphic $(k-1)$-vector field Φ in $\mathbb{C}(Z^I\cdot V_I)\wedge W^{k-|I|-1}$, such that

$$\Psi = d_\pi(\Phi).$$

Proof. For any holomorphic k-vector field $\Psi = Z^I \cdot V_I \wedge w \in \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$, where $I \in S_{k-1} \subseteq S_k$ and $w \in W^{k-|I|}$, by Lemma 4.1, we have

$$\begin{aligned} d_\pi(\Psi) &= [\pi, \Psi] = [\pi, Z^I \cdot V_I \wedge w] \\ &= \rho(\imath_{m(I)}\Pi) \wedge (Z^I \cdot V_I \wedge w) \\ &= Z^I \cdot \rho(\imath_{m(I)}\Pi) \wedge v_{i_1} \wedge \ldots \wedge v_{i_l} \wedge w \end{aligned}$$

If $(\imath_{m(I)}\Pi) \wedge e_I \neq 0$, then $\imath_{m(I)}\Pi$ and $e_{i_1}, e_{i_2} \ldots e_{i_l}$ are $\mathbb{C}$-linear independent vectors in $N_{\mathbb{C}}$, $\rho(\imath_{m(I)}\Pi)$ and $v_{i_1}, v_{i_2} \ldots v_{i_l}$ are $\mathbb{C}$-linear independent vectors in $W = \rho(N_{\mathbb{C}})$.

If $d_\pi(\Psi) = Z^I \cdot \rho(\imath_{m(I)}\Pi) \wedge v_{i_1} \wedge \ldots \wedge v_{i_l} \wedge w = 0$, we have

$$\rho(\imath_{m(I)}\Pi) \wedge v_{i_1} \wedge \ldots \wedge v_{i_l} \wedge w = 0.$$

By simple linear algebra we know that $w \in W^{k-|I|}$ can be written as

$$w = \rho(\imath_{m(I)}\Pi) \wedge w_0 + \sum_{s=1}^{l} v_{i_s} \wedge w_i,$$

where $w_0, w_1, \ldots, w_l$ are elements in $W^{k-|I|-1}$. Moreover, we have

$$\begin{aligned} \Psi &= Z^I \cdot V_I \wedge w \\ &= Z^I (v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_l}) \wedge (\rho(\imath_{m(I)}\Pi) \wedge w_0 + \sum_{s=1}^{l} v_{i_s} \wedge w_s) \\ &= Z^I (v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_l}) \wedge (\rho(\imath_{m(I)}\Pi) \wedge w_0) \\ &= (-1)^{|I|} \rho(\imath_{m(I)}\Pi) \wedge (Z^I \cdot V_I \wedge w_0) \end{aligned}$$

Let $\Phi = (-1)^{|I|} Z^I \cdot V_I \wedge w_0$. Then Φ is a holomorphic $(k-1)$-vector fields in the space $\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|-1}$.

A simple computation using Lemma 4.1 shows that

$$\Psi = d_\pi(\Phi).$$

$\square$

Theorem 4.1. *Let π be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$. We have*

(1) *for $0 \leq k \leq n$,*

$$H^k_\pi(X) = \oplus_{I \in S_k(\pi)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}, \tag{4.6}$$

where $S_k(\pi)$ is the set consisting of all $I \in \tilde{M}$ satisfying

$$\langle m(I), e_i \rangle = m_i \geq -1 \quad (0 \leq i \leq n), \tag{4.7}$$

$$|I| \leq k, \tag{4.8}$$

and the equation

$$(\imath_{m(I)}\Pi) \wedge e_I = 0. \tag{4.9}$$

(2) *$H^k_\pi(X) = 0$ for $k > n$.*

Remark 4.4. (1) $S_k(\pi)$ is a subset of S_k consisting of all $I \in S_k$ satisfying Equation 4.9. By Theorem 3.8, $\oplus_{I\in S_k(\pi)}\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$ is a subspace of $H^0(X, \wedge^k T_X)$. it is identified with the quotient space

$$\frac{\ker : H^0(X,\wedge^k T_X) \xrightarrow{d_\pi} H^0(X,\wedge^{k+1} T_X)}{\mathrm{Im} : H^0(X,\wedge^{k-1} T_X) \xrightarrow{d_\pi} H^0(X,\wedge^k T_X)}$$

in the Theorem 4.1.

(2) For $k = 0$, $H^0_\pi(X) = \mathbb{C}$, consisting of the complex constants on X.

(3) For $k = n$, Theorem 4.1 can be state in the following equivalent way:

$$H^n_\pi(X) = \oplus_I \mathbb{C}Z^I \cdot v_1 \wedge ... \wedge v_n \tag{4.10}$$

for all $I \in S_n \subset \tilde{M}$ satisfying one of the following conditions

$$\begin{cases} |I| = n, \\ (\imath_{m(I)}\Pi) \wedge e_I = 0. \end{cases} \tag{4.11}$$

(4) For each $I \in \tilde{M}$, Equation (4.9) can be written as

$$\sum_{i=1}^{n} a^i(I,\pi)m_i, \tag{4.12}$$

where $a^i(I, \pi)$ are complex constants depending on I and π.

Proof of Theorem 4.1:

Proof. (1) For $k = 0$, S_0 consists of only $(0, ...0) \in \tilde{M}$, and $W^{k-|i|} = W^0 = \mathbb{C}$. Thus we have

$$H^0_\pi(X) = \mathbb{C}.$$

(2) For $1 \le k \le n$, by Theorem 3.8, any holomorphic k-vector field $\Psi \in H^0(X, \wedge^k T_X)$ can be written as

$$\Psi = \sum_{I\in S_k} \Psi_I, \qquad \Psi_I \in \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}.$$

As

$$d_\pi(\Psi) = \sum_{I\in S_k} d_\pi(\Psi_I),$$

by Lemma 4.2, we have that

$$d_\pi(\Psi) = 0 \Longleftrightarrow d_\pi(\Psi_I) = 0 \quad \text{for all} \quad I \in S_k.$$

(a) For any $I \in S_{k-1} \subseteq S_k$ satisfying $(\imath_{m(I)}\Pi) \wedge e_I \neq 0$, by Lemma 4.3, $d_\pi(\Psi_I) = 0$ imples that there exist $\Phi_I \in \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|-1}$ such that $\Psi_I = d_\pi(\Phi_I)$. Thus there exists only zero Poisson cohomology class in $\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$ for $I \in S_{k-1} \subseteq S_k$ satisfying $(\imath_{m(I)}\Pi) \wedge e_I \neq 0$.

(b) For any $I \in S_{k-1} \subseteq S_k$ satisfying $(\imath_{m(I)}\Pi) \wedge e_I = 0$, i.e., $I \in S_{k-1}(\pi)$, by Lemma 4.2, we have

$$d_\pi(\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|-1}) = 0 \quad \text{and} \quad d_\pi(\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}) = 0.$$

Thus for each nonzero element $\Psi_I \in \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$ with $I \in S_{k-1}(\pi)$, it represents a nonzero cohomology class in the Poisson cohomology group. And

$$\oplus_{I \in S_{k-1}(\pi)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$$

can be seen as subspace of the Poisson cohomology group $H^k_\pi(X)$.

(c) For any $I \in S_k(\pi) \backslash S_{k-1}(\pi)$, i.e, $I \in S(k)$ satisfying $(\imath_{m(I)} \Pi) \wedge e_I = 0$, by Lemma 4.3, we have

$$d_\pi(\mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}) = d_\pi(\mathbb{C}(Z^I \cdot V_I)) = 0.$$

By Theorem 3.8, we have

$$H^0(X, \wedge^{k-1} T_X) = \oplus_{I \in S_{k-1}} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|-1}.$$

And by Lemma 4.2, we get that

$$d_\pi(H^0(X, \wedge^{k-1} T_X)) \subseteq \oplus_{I \in S_{k-1}} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}.$$

Thus for each nonzero element $\Psi_I \in \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$ with $I \in S_k(\pi) \backslash S_{k-1}(\pi)$, it represents a nonzero cohomology class in the Poisson cohomology group. And

$$\oplus_{I \in S_k(\pi) \backslash S_{k-1}(\pi)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$$

can be seen as a subspace of the Poisson cohomology group $H^k_\pi(X)$.

By the argument above, we have

$$H^k_\pi(X) = \oplus_{I \in S_k(\pi)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}$$

for $1 \leq k \leq n$.

(3) For $k > n$, $H^k_\pi(X) = 0$ comes directly from Lemma 2.4 and Theorem 3.2.

□

Let us denoted $S(i, \pi)$ as the set of all $I \in \tilde{M}$ satisfying $|I| = i$ and the conditions (4.7), (4.9). Then we have $S_k(\pi) = \uplus_{0 \leq i \leq k} S(i, \pi)$. By a similar way as in Proposition 3.9, we can prove that

Proposition 4.5. *Let π be a holomorphic toric Poisson structure on $X = \mathbb{C}\mathbf{P}^n$. For $1 \leq k \leq n$, we have*

$$H^k_\pi(X) = (H^{k-1}_\pi(X) \wedge W) \oplus (\oplus_{I \in S(k,\pi)} \mathbb{C}(Z^I \cdot V_I) \wedge W^{k-|I|}), \tag{4.13}$$

where $S(k, \pi)$ is the set of all $I \in \tilde{M}$ satisfying $|I| = k$ and the conditions (4.7)*,* (4.9).

4.2. **Poisson cohomology of the standard Poisson structure on $\mathbb{C}\mathbf{P}^n$.** As we have shown in Lemma 2.16, the standard Poisson structure on $X = \mathbb{C}\mathbf{P}^n$ can be written as

$$\pi_{st} = \sum_{1 \leq i < j \leq n} v_i \wedge v_j,$$

where $v_i = p_*(z_i \frac{\partial}{\partial z_i}) = \rho(e_i)$ $(0 \leq i \leq n)$ as we shown in Example 2.15. And $\Pi_{st} = \sum_{1 \leq i < j \leq n} e_i \wedge e_j \in \wedge^2 N_{\mathbb{C}}$.

We can apply Theorem 4.1 to compute the Poisson cohomology of the standard Poisson structure on $\mathbb{C}\mathbf{P}^n$. Here we only list the Poisson cohomology groups in the case $n = 2$ and $n = 3$. For other cases it could be done similarly, but more complicated.

Proposition 4.6. *Let $X = \mathbb{C}\mathbf{P}^2$ and let $[z_0, z_1, z_2]$ be the homogenous coordinates on it. Let $v_i = p_*(z_i \frac{\partial}{\partial z_i})$ $(0 \leq i \leq 2)$. The standard Poisson structure on $X = \mathbb{C}\mathbf{P}^2$ can be written as*

$$\pi_{st} = v_1 \wedge v_2.$$

The Poisson cohomology group of (X, π_{st}) can be written as

(1) $H^0_{\pi_{st}}(X) = \mathbb{C}$, *and* $\dim H^0_{\pi_{st}}(X) = 1$.
(2) $H^1_{\pi_{st}}(X)$ *has a basis* $\{v_1, v_2\}$, *and* $\dim H^1_{\pi_{st}}(X) = 2$.
(3) $H^2_{\pi_{st}}(X)$ *has a basis* $\{(z_0^{m_0} z_1^{m_1} z_2^{m_2}) v_1 \wedge v_2\}$ *with* (m_0, m_1, m_2) *in the set*

$$\left\{ \begin{array}{ccc} (0,0,0), & & \\ (-1,-1,2), & (-1,2,-1), & (2,-1,-1) \end{array} \right\}.$$

Thence $\dim H^2_{\pi_{st}}(X) = 4$.
(4) $H^k_{\pi_{st}}(X) = 0$ *for* $k > 2$.

The Proposition 4.6 verified the results about Poisson cohomology of $\mathbb{C}\mathbf{P}^2$ in [`?Hong-Xu11`].

Proposition 4.7. *Let $X = \mathbb{C}\mathbf{P}^3$ and let $[z_0, z_1, z_2, z_3]$ be the homogenous coordinates on it. Let $v_i = p_*(z_i \frac{\partial}{\partial z_i})$ $(0 \leq i \leq 3)$. The standard Poisson structure on $X = \mathbb{C}\mathbf{P}^3$ can be written as*

$$\pi_{st} = v_1 \wedge v_2 + v_1 \wedge v_3 + v_2 \wedge v_3.$$

The Poisson cohomology group of (X, π_{st}) can be written as

(1) $H^0_{\pi_{st}}(X) = \mathbb{C}$, *and* $\dim H^0_{\pi_{st}}(X) = 1$.
(2) $H^1_{\pi_{st}}(X)$ *has a basis* $\{v_1, v_2, v_3\}$, *and* $\dim H^1_{\pi_{st}}(X) = 3$.
(3) $H^2_{\pi_{st}}(X)$ *has a basis as the union of three parts:*
   (a) $\{v_1 \wedge v_2, \quad v_1 \wedge v_3, \quad v_2 \wedge v_3\}$,
   (b) $\{(z_0^{m_0} z_1^{m_1} z_2^{m_2} z_3^{m_3}) v_0 \wedge v_2\}$ *with* (m_0, m_1, m_2, m_3) *in the set*

$$\{\ (-1,1,,-1,1), \quad (-1,2,-1,0), \quad (-1,0,-1,2)\ \}$$

   where $v_0 = -\sum_{i=1}^3 v_i$,
   (c) $\{(z_0^{m_0} z_1^{m_1} z_2^{m_2} z_3^{m_3}) v_1 \wedge v_3\}$ *with* (m_0, m_1, m_2, m_3) *in the set*

$$\{\ (1,-1,,1,-1), \quad (2,-1,0,-1), \quad (0,-1,2,-1)\ \}.$$

Thence $\dim H^2_{\pi_{st}}(X) = 9$.
(4) $H^3_{\pi_{st}}(X)$ *has a basis* $\{(z_0^{m_0} z_1^{m_1} z_2^{m_2} z_3^{m_3}) v_1 \wedge v_2 \wedge v_3\}$ *with* (m_0, m_1, m_2, m_3) *in the set*

$$\left\{ \begin{array}{cccc} (0,0,0,0), & & & \\ (-1,1,,-1,1), & (-1,2,-1,0), & (-1,0,-1,2), & \\ (1,-1,1,-1), & (2,-1,0,-1), & (0,-1,2,-1), & \\ (-1,-1,-1,3), & (-1,-1,3,-1), & (-1,3,-1,-1), & (3,-1,-1,-1) \end{array} \right\}.$$

Thence $\dim H^3_{\pi_{st}}(X) = 11$.
(5) $H^k_{\pi_{st}}(X) = 0$ *for* $k > 3$.

For general $\mathbb{C}\mathbf{P}^n$ equipped with the standard Poisson structure, it is interesting to explore the meaning of the Poisson cohomology groups. Here we will give an explicit description of the first Poisson cohomology group of $(\mathbb{C}\mathbf{P}^n, \pi_{st})$.

Theorem 4.8. *For $X = \mathbb{C}\mathbf{P}^n$ equipped with the standard Poisson structure*

$$\pi_{st} = \sum_{1 \le i < j \le n} v_i \wedge v_j,$$

we have

$$H^1_{\pi_{st}}(X) = W \quad \textit{and} \quad \dim H^1_{\pi_{st}}(X) = n.$$

To prove Theorem 4.8, we need the following lemma.

Lemma 4.9. *For $X = \mathbb{C}\mathbf{P}^n$ equipped with the standard Poisson structure*

$$\pi_{st} = \sum_{1 \le i < j \le n} v_i \wedge v_j,$$

we have that

$$S(1, \pi_{st}) = \varnothing,$$

where $S(1, \pi_{st})$ is the set of all $I \in S(1)$ satisfying the Equation

$$(\imath_{m(I)} \Pi_{st}) \wedge e_I = 0, \tag{4.14}$$

and $\Pi_{st} = \sum_{1 \le i < j \le n} e_i \wedge e_j$.

Proof. For $I \in S(1)$, Equation (4.14) is equivalent to

$$\imath_{m(I)} \Pi_{st} = \lambda e_I, \qquad \lambda \in \mathbb{C}. \tag{4.15}$$

Let us denote $\alpha_{i,j}$ $(0 \le i \ne j \le n)$ as the element in $\tilde{M}$, with $m_i = -1$, $m_j = 1$ and 0 elsewhere. Then

$$S(1) = \{\alpha_{i,j} \mid 0 \le i \ne j \le n\}.$$

For $I = \alpha_{i,j}$, we have that $e_I = e_i$. The Equation (4.15) becomes

$$\imath_{m(I)} \Pi_{st} = \lambda e_i, \qquad \lambda \in \mathbb{C}. \tag{4.16}$$

Let $\{e_1^*, e_2^*, ..., e_n^*\} \subset M$ be the dual basis of $\{e_1, e_2, ..., e_n\}$. Then we have

$$m(I) = \begin{cases} e_j^*, & \text{for} \quad i = 0,\ j \ne 0. \\ -e_i^*, & \text{for} \quad i \ne 0,\ j = 0. \\ -e_i^* + e_j^*, & \text{for} \quad i \ne 0,\ j \ne 0. \end{cases}$$

(a) In the case $i = 0$ and $j \ne 0$, Equation (4.16) becomes

$$\imath_{e_j^*} \Pi = \lambda e_0 = -\lambda (\sum_{s=1}^n e_s),$$

which implies

$$\Pi(e_j^*, e_s^*) = \langle \imath_{e_j^*} \Pi, e_s^* \rangle = -\lambda \tag{4.17}$$

for all $1 \le s \le n$.

As

$$\Pi_{st} = \sum_{1 \le i < j \le n} e_i \wedge e_j,$$

Equation (4.17) can not be true since that

$$\Pi(e_j^*, e_j^*) = 0$$

and

$$\Pi(e_j^*, e_s^*) = \pm 1$$

for $1 \leq s \neq j \leq n$.

Thus in this case, Equation (4.14) has no solution.

(b) In the case $i \neq 0$ and $j = 0$, Equation (4.16) becomes

$$\imath_{(-e_i^*)}\Pi = \lambda e_i,$$

which implies

$$\Pi(-e_i^*, e_s^*) = 0$$

for all $1 \leq s \neq i \leq n$.

As

$$\Pi_{st} = \sum_{1 \leq i < j \leq n} e_i \wedge e_j,$$

we have

$$\Pi(-e_i^*, e_s^*) = \pm 1$$

for all $1 \leq s \neq i \leq n$. Thus Equation (4.16) has no solution in this case.

(c) In the case $i \neq 0$ and $j \neq 0$, Equation (4.16) becomes

$$\imath_{(-e_i^* + e_j^*)}\Pi = \lambda e_i.$$

It can not be true since that

$$\Pi(-e_i^* + e_j^*, e_j^*) = \pm 1,$$

but

$$\langle \lambda e_i, e_j^* \rangle = 0.$$

By the argument above, we have $S(1, \pi_{st}) = \varnothing$. □

Proof of Theorem 4.8:

Proof. By Theorem 4.1, we have

$$H^1_{\pi_{st}}(X) = \oplus_{I \in S_1(\pi_{st})} \mathbb{C}(Z^I \cdot V_I) \wedge W^{1-|I|}.$$

As $S_1(\pi_{st}) = S(0, \pi_{st}) \uplus S(1, \pi_{st})$, we have

$$\begin{aligned} H^1_{\pi_{st}}(X) &= \oplus_{I \in S_1(\pi_{st})} \mathbb{C}(Z^I \cdot V_I) \wedge W^{1-|I|} \\ &= (\oplus_{I \in S(0,\pi_{st})} \mathbb{C}(Z^I \cdot V_I) \wedge W^{1-|I|}) \oplus (\oplus_{I \in S(1,\pi_{st})} \mathbb{C}(Z^I \cdot V_I) \wedge W^{1-|I|}). \end{aligned}$$

Since $S(0, \pi_{st})$ consists only one element $I = (0, \ldots, 0)$, we have

$$\oplus_{I \in S(0,\pi_{st})} \mathbb{C}(Z^I \cdot V_I) \wedge W^{1-|I|} = W.$$

On the other hand, by Lemma 4.9, we have $S(1, \pi_{st}) = \varnothing$. Thus we have

$$H^1_{\pi_{st}}(X) = W \quad \text{and} \quad \dim H^1_{\pi_{st}}(X) = n.$$

□

For $X = \mathbb{C}\mathbf{P}^n$, there is a cyclic group $\mathbb{Z}_{n+1}$ action on $X = \mathbb{C}\mathbf{P}^n$, generated by

$$[z_0, z_1, \ldots, z_{n-1}, z_n] \xrightarrow{\sigma} [z_1, z_2, \ldots, z_n, z_0], \tag{4.18}$$

where σ is a generator of $\mathbb{Z}_{n+1}$. By $\sum_{i=0}^{n} v_i = 0$, we have that

$$\begin{aligned}\pi_{st} = \sum_{1\le i<j\le n} v_i \wedge v_j &= \sum_{0\le i<j\le n} v_i \wedge v_j \\ &= \sum_{0\le i<j\le n-1} v_i \wedge v_j \\ &= \sigma_*^{-1}(\pi_{st}).\end{aligned}$$

Thus the standard Poisson structure π_{st} on $X = \mathbb{C}\mathbf{P}^n$ is invariant under the $\mathbb{Z}_{n+1}$-action defined in (4.18).

Proposition 4.10. *The standard Poisson structure*

$$\pi_{st} = \sum_{1\le i<j\le n} v_i \wedge v_j$$

on $X = \mathbb{C}\mathbf{P}^n$ is invariant under the $\mathbb{Z}_{n+1}$-action defined in Equation (4.18). As a consequence, the $\mathbb{Z}_{n+1}$-action on X induces a $\mathbb{Z}_{n+1}$-action on the Poisson cohomology group $H^k_{\pi_{st}}(X)$ for $0 \le k \le n$.

In the cases of $X = \mathbb{C}\mathbf{P}^2$ (Proposition 4.6) and $X = \mathbb{C}\mathbf{P}^3$ (Proposition 4.7), it is easy to find the $\mathbb{Z}_3$-action and the $\mathbb{Z}_4$-action on the Poisson cohomology group.

There should be more interesting thing about the Poisson cohomology groups of the standard Poisson structure to be explored. However, those will be future works.

5. General Conjectures

Let X_Δ be a nonsingular toric veriety associated with a fan Δ in $N_\mathbb{R}$. Let α_i $(1 \le i \le r)$ be all the one dimensional cones in $\Delta(1)$, and let$n(\alpha_i) \in N$ be the corresponding primitive elements. Set $W = \rho(N_\mathbb{C})$, $W^k = \wedge^k W$ and $W^0 = \mathbb{C}$. Then W^k can be considered as a subspace of $H^0(X_\Delta, \wedge^k \mathcal{T}_{X_\Delta})$.

For any $I \in M$, let

$$m_i(I) = \langle I, n(\alpha_i)\rangle \ (1 \le i \le r).$$

Suppose $m_{i_1}(I), ..., m_{i_l}(I)$ $(1 \le i_1 < ... < i_l \le r)$ are all the elements equal to -1 in the set $\{m_1(I), ..., m_r(I)\}$. Let us introduce some notations:

$$|I| = l, \quad V_I = \rho(n(\alpha_{i_1})) \wedge \ldots \wedge \rho(n(\alpha_{i_l})) \in W^l, \quad n_I = n(\alpha_{i_1}) \wedge \ldots \wedge n(\alpha_{i_l}) \in \wedge^l N.$$

Conjecture 5.1. Let X_Δ be a nonsingular toric veriety associated with a fan Δ in $N_\mathbb{R}$. Then

$$H^0(X, \wedge^k \mathcal{T}_X) = \oplus_{I\in S_k} \mathbb{C}(\chi^I \cdot V_I) \wedge W^{k-|I|}$$

for $0 \le k \le n$, where S_k is the subset of M consisting of all $I \in M$ satisfying the conditions

$$m_i(I) \ge -1 \quad (1 \le i \le r)$$

and

$$|I| \le k.$$

Remark 5.1. (1) As we have shown in Theorem 3.8, Conjecture 5.1 is true for $X = \mathbb{C}\mathbf{P}^n$.

(2) For the general toric variety X_Δ, in the case of $k = 1$, by [8] (Proposition 7, p. 571) (one may consult [19]), Conjecture 5.1 retains true.

If Conjecture 5.1 retains true, then we can prove the following conjecture by similar way as we have done for $X = \mathbb{C}\mathbf{P}^n$.

Conjecture 5.2. Let X_Δ be a nonsingular toric veriety satisfying $H^i(X, \wedge^j \mathcal{T}_{X_\Delta}) = 0$ for all $i > 0$ and $0 \leq j \leq n$. Let π be a holomorphic toric structure on X_Δ, and let Π be the element in $\wedge^2 N_\mathbb{C}$ determined by $\rho(\Pi) = \pi$. Then we have

(1) for $0 \leq k \leq n$,
$$H^k_\pi(X) = \oplus_{I \in S_k(\pi)} \mathbb{C}(\chi^I \cdot V_I) \wedge W^{k-|I|},$$
where $S_k(\pi)$ is the set consisting of all $I \in M$ satisfying
$$\begin{aligned} &m_i(I) \geq -1 \quad (1 \leq i \leq r), \\ &|I| \leq k, \end{aligned}$$
and the equation
$$(\imath_I \Pi) \wedge n_I = 0.$$
(2) $H^k_\pi(X) = 0$ for $k > n$.

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