ON THE GENERALIZED ASSOCIATIVITY EQUATION

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ABSTRACT. The so-called generalized associativity functional equation

$$G(J(x,y),z) = H(x,K(y,z))$$

has been investigated under various assumptions, for instance when the unknown functions G, H, J, and K are real, continuous, and strictly monotonic in each variable. In this note we investigate the following related problem: given the functions J and K, find every function F that can be written in the form

$$F(x,y,z) = G(J(x,y),z) = H(x,K(y,z))$$

for some functions G and H. We show how this problem can be solved when any of the inner functions J and K has the same range as one of its sections.

1. Introduction

Let X, Y, Z, U_J, U_K, and U be nonempty sets and consider the functional equation

$$(1) \qquad G(J(x,y),z) = H(x,K(y,z)), \qquad x \in X,\ y \in Y,\ z \in Z,$$

where $J\colon X \times Y \to U_J$, $K\colon Y \times Z \to U_K$, $G\colon U_J \times Z \to U$, and $H\colon X \times U_K \to U$ are unknown functions. This functional equation, called the *generalized associativity equation*, has been investigated under various solvability conditions, in particular when the unknown functions are real, continuous, and strictly monotonic in each variable (see, e.g, [1, 2] and the references therein).

In this paper we are interested in the following problem, which is closely related to that of solving the generalized associativity equation (1). Throughout this paper we denote the domain and range of any function f by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively.

Problem 1. *Given two functions $J\colon X \times Y \to U_J$ and $K\colon Y \times Z \to U_K$, determine the class $\mathcal{F}_{J,K}$ of functions $F\colon X \times Y \times Z \to \mathrm{ran}(F)$ for which there exist $G\colon U_J \times Z \to \mathrm{ran}(F)$ and $H\colon X \times U_K \to \mathrm{ran}(F)$ such that*

$$(2) \qquad F(x,y,z) = G(J(x,y),z) = H(x,K(y,z)), \qquad x \in X,\ y \in Y,\ z \in Z.$$

Contrary to the problem of solving the generalized associativity equation, here we assume that the inner functions J and K are given beforehand and we search for all functions F which have the form given in (2). For instance, searching for the real functions $F\colon \mathbb{R}^3 \to \mathbb{R}$ that can be expressed in the form

$$F(x,y,z) = G(x-y,z) = H(x,y-z), \qquad x,y,z \in \mathbb{R},$$

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for some functions $G,H\colon\mathbb{R}^2\to\mathbb{R}$ is a special case of Problem 1. As we will see in Example 4, these functions are all of the form

$$F(x,y,z) = f(x-y+z),$$

where $f\colon\mathbb{R}\to\mathbb{R}$ is an arbitrary function.

The following problem gives an equivalent but simpler reformulation of Problem 1, where the functions G and H are not explicitly involved.

Problem 2. *Given two functions $J\colon X\times Y\to U_J$ and $K\colon Y\times Z\to U_K$, determine the class $\mathcal{F}_{J,K}$ of functions $F\colon X\times Y\times Z\to\operatorname{ran}(F)$ satisfying the conditions*

$$\begin{aligned} J(x,y) &= J(x',y') &\Rightarrow\quad F(x,y,z) &= F(x',y',z)\\ K(y,z) &= K(y',z') &\Rightarrow\quad F(x,y,z) &= F(x,y',z') \end{aligned}$$

for all $x,x'\in X$, all $y,y'\in Y$, and $z,z'\in Z$.

It is easy to see that Problem 1 and Problem 2 are equivalent in the sense that they define the same class $\mathcal{F}_{J,K}$ of functions. We also observe that $\mathcal{F}_{J,K}$ is never empty since it contains all the constant functions. More generally, we have the following fact.

Fact 1. *If $F\in\mathcal{F}_{J,K}$, then $f\circ F\in\mathcal{F}_{J,K}$ for every function f defined on* $\operatorname{ran}(F)$.

Solving Problem 1, or equivalently Problem 2, seems not easy in general. However, solutions can be found as soon as certain assumptions are made on the functions J and K. In Section 2 we show how this problem can be solved whenever any of the functions J and K has the same range as one of its sections. In Section 3 we focus on the special case where $X=Z=A$ and $Y=A^{n-2}$ for some nonempty set A and some integer $n\geqslant 3$ (in which case any function in $\mathcal{F}_{J,K}$ is defined on the Cartesian power A^n) and we provide conditions on J and K for the functions in $\mathcal{F}_{J,K}$ to be expressible in terms of their diagonal sections (i.e., every $F\in\mathcal{F}_{J,K}$ is of the form $F=\delta_F\circ M$ for some function $M\colon A^n\to A$, where $\delta_F\colon A\to A$ is defined by $\delta_F(x)=F(x,\ldots,x)$).

We use the following notation. The identity function on any nonempty set E is denoted by id_E. We denote the set of quasi-inverses of a function f by $Q(f)$, where a quasi-inverse g of a function f is defined by the conditions (see, e.g., [4, Sect. 2.1])

$$f\circ g|_{\operatorname{ran}(f)}=\mathrm{id}_{\operatorname{ran}(f)}\qquad\text{and}\qquad \operatorname{ran}(g|_{\operatorname{ran}(f)})=\operatorname{ran}(g).$$

Throughout this paper we assume that every function has at least one quasi-inverse. It is well known that this assumption is equivalent to the Axiom of Choice. Recall also that the relation of being quasi-inverse is symmetric: if $g\in Q(f)$ then $f\in Q(g)$; moreover, we have $\operatorname{ran}(g)\subseteq\operatorname{dom}(f)$ and $\operatorname{ran}(f)\subseteq\operatorname{dom}(g)$ and the functions $f|_{\operatorname{ran}(g)}$ and $g|_{\operatorname{ran}(f)}$ are one-to-one (in particular if $\operatorname{ran}(g)=\operatorname{dom}(f)$ and $\operatorname{ran}(f)=\operatorname{dom}(g)$, then f and g are inverses of each other).

Fact 2. *If $g\in Q(f)$ and $\operatorname{ran}(h)\subseteq\operatorname{ran}(f)$, then $f\circ g\circ h=h$.*

Remark 1. Consider the class $\mathcal{F}_{J,K}$ as defined in Problem 1 and let $F\in\mathcal{F}_{J,K}$. Then we have $G(a,b)=F(\phi(a),b)$ for every $\phi\in Q(J)$ and every $(a,b)\in\operatorname{ran}(J)\times Z$. Therefore, G is completely determined from F. Similarly, we have $H(a,b)=F(a,\psi(b))$ for every $\psi\in Q(K)$ and every $(a,b)\in X\times\operatorname{ran}(K)$, and hence H is completely determined from F. Thus, when such quasi-inverses ϕ and ψ can be given explicitly, Problem 1 amounts to solving the generalized associativity equation (1) for given functions J and K and an arbitrary set U.

2. Main results

For any $a \in Z$ we define the section $K_2^a\colon Y \to U_K$ of K as the function $K_2^a(y) = K(y,a)$. The following theorem provides a first step in the resolution of Problem 1 whenever $\mathrm{ran}(K) = \mathrm{ran}(K_2^a)$ for some $a \in Z$.

Theorem 3. *Assume that* $\mathrm{ran}(K) = \mathrm{ran}(K_2^a)$ *for some* $a \in Z$ *and let* $F \in \mathcal{F}_{J,K}$. *Then there exists* $f\colon U_J \to \mathrm{ran}(F)$ *such that* $F = f \circ R_k$ *for every* $k \in Q(K_2^a)$, *where* $R_k\colon X \times Y \times Z \to U_J$ *is defined by*

$$R_k(x,y,z) \; = \; J(x, k \circ K(y,z)).$$

Proof. Let $F \in \mathcal{F}_{J,K}$. Then, there exist $G\colon U_J \times Z \to \mathrm{ran}(F)$ and $H\colon X \times U_K \to \mathrm{ran}(F)$ such that (2) holds. Let $f = G_2^a$ and $k \in Q(K_2^a)$. For any $(y,z) \in Y \times Z$, by Fact 2 we have

$$K(y,z) \; = \; K_2^a \circ k \circ K(y,z) \; = \; K(k \circ K(y,z), a).$$

For every $(x,y,z) \in X \times Y \times Z$, we then have

$$\begin{aligned} F(x,y,z) \; &= \; H(x, K(y,z)) \; = \; H(x, K(k \circ K(y,z), a)) \\ &= \; G(J(x, k \circ K(y,z)), a) \; = \; f \circ R_k(x,y,z), \end{aligned}$$

which completes the proof. □

Remark 2. We observe that, although the quasi-inverse k of K_2^a need not be unique, the identity $F = f \circ R_k$ in Theorem 3 does not depend on the choice of this quasi-inverse.

Example 4. Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be a function for which there exist $G, H\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$F(x,y,z) \; = \; G(x-y, z) \; = \; H(x, y-z), \qquad x,y,z \in \mathbb{R}. \tag{3}$$

Searching for all possible such functions F reduces to describing the functions in $\mathcal{F}_{J,K}$ that range in $\mathbb{R}$, where J and K are defined by $J(x,y) = x - y$ and $K(y,z) = y - z$. Since $K_2^0 = \mathrm{id}_{\mathbb{R}}$, we have $\mathrm{ran}(K_2^0) = \mathbb{R} = \mathrm{ran}(K)$ and hence we can apply Theorem 3 with $a = 0$. We then have $k = (K_2^0)^{-1} = \mathrm{id}_{\mathbb{R}}$ and $R_k(x,y,z) = x - y + z$. Therefore any function $F \in \mathcal{F}_{J,K}$ ranging in $\mathbb{R}$ is of the form

$$F(x,y,z) \; = \; f(x - y + z), \qquad x,y,z \in \mathbb{R}, \tag{4}$$

for some $f\colon \mathbb{R} \to \mathbb{R}$. Conversely any such function clearly lies in $\mathcal{F}_{J,K}$. Therefore we necessarily have

$$\{F \in \mathcal{F}_{J,K} \mid \mathrm{ran}(F) \subseteq \mathbb{R}\} \; = \; \{(x,y,z) \mapsto f(x-y+z) \mid f\colon \mathbb{R} \to \mathbb{R}\}.$$

Finally, setting $y = 0$ in (3) and (4) we obtain $G(x,z) = H(x,-z) = f(x+z)$ for every $x, z \in \mathbb{R}$. □

Example 5. Assume that $(A, \vee)$ is a bounded join-semilattice, with 0 as the least element and let $F\colon A^n \to \mathrm{ran}(F)$ be a function for which there exist $G, H\colon A^2 \to \mathrm{ran}(F)$ such that

$$F(x_1, \ldots, x_n) \; = \; G(x_1 \vee \cdots \vee x_{n-1}, x_n) \; = \; H(x_1, x_2 \vee \cdots \vee x_n).$$

The class of all possible functions F satisfying this condition is nothing other than the set $\mathcal{F}_{J,K}$, where the functions $J,K\colon A^{n-1}\to A$ are defined by $J(x_1,\ldots,x_{n-1}) = K(x_1,\ldots,x_{n-1}) = x_1\vee\cdots\vee x_{n-1}$. Using Theorem 3 with $a=0$, we can easily see that

$$\mathcal{F}_{J,K} = \{(x_1,\ldots,x_n)\mapsto f(x_1\vee\cdots\vee x_n) \mid \mathrm{dom}(f)=A\}. \quad \square$$

The following theorem is the dual version of Theorem 3. The proof is similar to that of Theorem 3 and hence is omitted. For any $b\in X$ we define the section $J_1^b\colon Y\to U_J$ of J as the function $J_1^b(y)=J(b,y)$.

Theorem 6. *Assume that* $\mathrm{ran}(J)=\mathrm{ran}(J_1^b)$ *for some* $b\in X$ *and let* $F\in\mathcal{F}_{J,K}$. *Then there exists* $g\colon U_K\to\mathrm{ran}(F)$ *such that* $F=g\circ S_j$ *for every* $j\in Q(J_1^b)$, *where* $S_j\colon X\times Y\times Z\to U_K$ *is defined by*

$$S_j(x,y,z) = K(j\circ J(x,y),z).$$

We observe that each of Theorems 3 and 6 provides only necessary conditions for a function to be in $\mathcal{F}_{J,K}$. Examples 4 and 5 show that the use of only one of these theorems may sometimes be sufficient to derive a complete characterization of the class $\mathcal{F}_{J,K}$. The following example shows that using both theorems may somewhat simplify the quest for such a characterization.

Example 7. Let $F\colon\mathbb{R}^3\to\mathrm{ran}(F)$ be a function for which there exist $G,H\colon\mathbb{R}^2\to\mathrm{ran}(F)$ such that

$$F(x,y,z) = G(xy,z) = H(x,y+z), \qquad x,y,z\in\mathbb{R}.$$

Using both Theorems 3 and 6 with $J(x,y)=xy$, $K(y,z)=y+z$, $a=0$, and $b=1$, we obtain the expressions

$$F(x,y,z) = f(xy+xz) = g(xy+z)$$

for some functions $f,g\colon\mathbb{R}\to\mathrm{ran}(F)$. Setting $y=0$ and $z=1$ in these equations shows that $f=g$ must be a constant function. Therefore $\mathcal{F}_{J,K}$ consists of the class of constant functions. Note that using Theorem 3 only would have been sufficient here. Indeed, taking $a=0$ and then $a=1$ would lead to the identity $f(xy+xz)=f'(xy+xz-x)$ for some functions $f,f'\colon\mathbb{R}\to\mathrm{ran}(F)$, from which we would reach the same conclusion by setting $y=1$ and $z=0$. $\square$

Example 7 may suggest that the set $\mathcal{F}_{J,K}$ reduces to the class of constant functions whenever the functions J and K do not coincide. To see that this is not true, just replace J and K in Example 7 with the functions $J(x,y)=y$ and $K(y,z)=y$, respectively. Any $F\in\mathcal{F}_{J,K}$ is then of the form $F(x,y,z)=f(y)$ for some function $f\colon\mathbb{R}\to\mathrm{ran}(F)$.

Interestingly, $\mathcal{F}_{J,K}$ may reduce to the class of constant functions even if J and K coincide. The following example illustrates this fact.

Example 8. Let $A=[0,+\infty[$ and let $F\colon A^3\to\mathrm{ran}(F)$ be a function for which there exist $G,H\colon A^2\to\mathrm{ran}(F)$ such that

$$F(x,y,z) = G(\max(1,x+y),z) = H(x,\max(1,y+z)), \qquad x,y,z\in A.$$

Using both Theorems 3 and 6 with $J(x,y)=K(x,y)=\max(1,x+y)$ and $a=b=0$ and choosing $j=k=\mathrm{id}_{[1,+\infty[}$, we obtain the expressions

$$F(x,y,z) = f(\max(1+x,x+y+z)) = g(\max(1+z,x+y+z))$$

for some functions $f, g\colon [1, +\infty[\to \mathrm{ran}(F)$. Setting first $x \in [0,1]$ and $y = z = 0$ and then $x = y = 0$ and $z \in [0,1]$ in these identities, we obtain that $f = g$ is constant on $[1,2]$. Then, setting $x \geqslant 1$ and $y = z = 0$ and then $x = y = 0$ and $z \geqslant 1$, we obtain that $f = g$ is constant on $[1, +\infty[$. Therefore $\mathcal{F}_{J,K}$ consists of the class of constant functions. $\square$

The following two propositions give sufficient conditions on the functions R_k and S_j (as defined in Theorems 3 and 6) to obtain a characterization of the class $\mathcal{F}_{J,K}$.

Proposition 9. *Assume that* $\mathrm{ran}(K) = \mathrm{ran}(K_2^a)$ *for some* $a \in Z$. *Let* $R_k\colon X \times Y \times Z \to U_J$ *be defined as in Theorem 3. If* $R_k \in \mathcal{F}_{J,K}$, *then* $\mathcal{F}_{J,K} = \{f \circ R_k \mid \mathrm{dom}(f) = U_J\}$.

Proof. Inclusion '$\subseteq$' follows from Theorem 3. Inclusion '$\supseteq$' follows from both the hypothesis and Fact 1. $\square$

Proposition 10. *Assume that* $\mathrm{ran}(J) = \mathrm{ran}(J_1^b)$ *for some* $b \in X$. *Let* $S_j\colon X \times Y \times Z \to U_K$ *be defined as in Theorem 6. If* $S_j \in \mathcal{F}_{J,K}$, *then* $\mathcal{F}_{J,K} = \{g \circ S_j \mid \mathrm{dom}(g) = U_K\}$.

Remark 3. Finding necessary and sufficient conditions on functions J and K for R_k (or S_j) to be in $\mathcal{F}_{J,K}$ remains an interesting problem.

The following proposition states that if the functions $f \circ R_k$ and $g \circ S_j$ defined in Theorems 3 and 6 are equal, then they belong to the class $\mathcal{F}_{J,K}$.

Proposition 11. *Assume that* $\mathrm{ran}(K) = \mathrm{ran}(K_2^a)$ *and* $\mathrm{ran}(J) = \mathrm{ran}(J_1^b)$ *for some* $a \in Z$ *and* $b \in X$. *Let* $R_k\colon X \times Y \times Z \to U_J$ *and* $S_j\colon X \times Y \times Z \to U_K$ *be defined as in Theorems 3 and 6. If* $f \circ R_k = g \circ S_j$ *for some functions* f *and* g *such that* $\mathrm{dom}(f) = U_J$ *and* $\mathrm{dom}(g) = U_K$, *then* $f \circ R_k \in \mathcal{F}_{J,K}$.

Proof. Since the identity $f \circ R_k = g \circ S_j$ can be rewritten as condition (1) for some function G defined on $U_J \times Z$ and some function H defined on $X \times U_K$, the function $f \circ R_k$ is necessarily in $\mathcal{F}_{J,K}$. $\square$

Remark 4. Proposition 11 is particularly useful, when combined with any of the Propositions 9 and 10, if for instance $U_J = U_K$ and $f = g$ is the identity function (or a one-to-one function by Fact 1). Example 12 illustrates this observation.

Example 12. Consider the class $\mathcal{F}_{J,K}$, where $J, K\colon [0,1]^2 \to [0,1]$ are defined by $J(x,y) = K(x,y) = \frac{1}{2}\max(1, x+y)$. Consider also the functions R_k and S_j defined in Theorems 3 and 6 by choosing the values $a = b = 1$ and the functions $j, k\colon [\frac{1}{2}, 1] \to [0,1]$ defined by $j(x) = k(x) = 2x - 1$. Then we have

$$R_k(x,y,z) \;=\; S_j(x,y,z) \;=\; \tfrac{1}{2}\max(1, x+y+z-1).$$

By Proposition 11 (and in view of Remark 4), we can immediately see that

$$\mathcal{F}_{J,K} \;=\; \{(x,y,z) \mapsto f(\tfrac{1}{2}\max(1, x+y+z-1)) \mid \mathrm{dom}(f) = [\tfrac{1}{2}, 1]\}. \quad \square$$

We observe that Problem 1 can also be generalized to functions J and K that are defined on subsets of $X \times Y$ and $Y \times Z$, respectively. Such a generalization can be useful for instance when the assumption of Theorem 3 is not satisfied (i.e., when $\mathrm{ran}(K) \neq \mathrm{ran}(K_2^a)$ for all $a \in Z$). For the interested reader we elaborate on this generalization in the Appendix.

3. When the domain of F is a Cartesian power

We now particularize Problem 1 to the case where $X = Z = A$ and $Y = A^{n-2}$ for some nonempty set A and some integer $n \geqslant 3$. We then have $X \times Y \times Z = A^n$ and both functions J and K have $n-1$ arguments (like in Example 5).

Recall that the *diagonal section* of a function $F: A^n \to \mathrm{ran}(F)$ is the function $\delta_F: A \to \mathrm{ran}(F)$ defined by $\delta_F(x) = F(x, \ldots, x)$. Also, a function $F: A^n \to A$ is said to be *range-idempotent* if $\delta_F \circ F = F$. It is said to be *idempotent* if $\delta_F = \mathrm{id}_A$.

In this section we provide conditions on J and K for each function F in $\mathcal{F}_{J,K}$ to be expressible as $F = \delta_F \circ M$ for some function $M: A^n \to A$. Under idempotence and nondecreasing monotonicity (assuming A is an ordered set), such a function M is then called a *Chisini mean associated with* F (see [3]). This observation could be useful in applications where aggregation functions, and especially mean functions, are considered.

Let us first consider an important but simple lemma.

Lemma 13. *Let $R: A^n \to U$ be a function such that $\mathrm{ran}(R) = \mathrm{ran}(\delta_R)$ and consider the functions $f: U \to V$ and $F = f \circ R$. The following assertions hold.*

(a) *We have $F = \delta_F \circ r \circ R$ for every $r \in Q(\delta_R)$.*
(b) *If $F' = f' \circ R$ satisfies $\delta_F = \delta_{F'}$ for some $f': U \to V$, then we have $F = F'$.*
(c) *If F is idempotent, then $F = r \circ R$ for every $r \in Q(\delta_R)$.*
(d) *If $r \circ R$ is not idempotent for some $r \in Q(\delta_R)$, then F is not idempotent.*
(e) *For every $r \in Q(\delta_R)$, the function $r \circ R$ is range-idempotent (i.e., $r \circ \delta_R \circ r \circ R = r \circ R$). It is idempotent if and only if δ_R is one-to-one.*

Proof. By Fact 2 we have $\delta_R \circ r \circ R = R$, which proves assertion (e). We also derive the identities $F = f \circ R = f \circ \delta_R \circ r \circ R = \delta_F \circ r \circ R$, which prove assertion (a). Assertions (b) and (c) immediately follow from (a). Assertion (d) follows from (c). □

Whenever its assumptions are satisfied, Lemma 13 provides interesting properties of function F. Assertions (a) and (c) give an explicit expression of F in terms of its diagonal section. Assertion (b) shows that F depends only on δ_F and R. Assertion (d) is nothing other than the contrapositive of assertion (c). Finally, assertion (e) reveals a surprising property of $r \circ R$.

Example 14. Let $R: \mathbb{R}^n \to \mathbb{R}$ be the sum function $R(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$. Assertion (a) of Lemma 13 shows that, for any function $f: \mathbb{R} \to \mathbb{R}$, the function $F = f \circ R$ can be written as

$$F(x_1, \ldots, x_n) = \delta_F\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big), \qquad x_1, \ldots, x_n \in \mathbb{R}. \quad \square$$

For any function $F: A^n \to U$, any $k \in \{1, \ldots, n\}$, and any $a \in A$, we define the section $F_k^a: A^{n-1} \to U$ of F by

$$F_k^a(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) = F(x_1, \ldots, x_{k-1}, a, x_{k+1}, \ldots, x_n).$$

Combining Theorem 3 with Lemma 13, we obtain the following result. First define $\mathcal{F}_{J,K}^A = \{F \in \mathcal{F}_{J,K} \mid \mathrm{ran}(F) \subseteq A\}$.

Theorem 15. *Assume $\mathrm{ran}(K) = \mathrm{ran}(K_{n-1}^a)$ for some $a \in A$ and let $F \in \mathcal{F}_{J,K}$. Assume also that $\mathrm{ran}(R) = \mathrm{ran}(\delta_R)$, where R is one of the functions R_k ($k \in Q(K_{n-1}^a)$) defined in Theorem 3. Then the assertions (a)–(e) of Lemma 13 hold* *(where U and V are to be replaced with U_J and* $\operatorname{ran}(F)$*, respectively). If $F \in \mathcal{F}^A_{J,K}$, then for every $r \in Q(\delta_R)$ we have $r \circ R \in \mathcal{F}^A_{J,K}$ if and only if*

$$\mathcal{F}^A_{J,K} \;=\; \{f \circ r \circ R \mid f\colon A \to A\}.$$

Proof. By Theorem 3 there exists $f\colon U_J \to \operatorname{ran}(F)$ such that $F = f \circ R$ and hence Lemma 13 applies. Let us now establish the last part of the theorem. To see that the condition is sufficient, just take $f = \mathrm{id}_A$. Let us show that it is necessary. The inclusion '$\supseteq$' follows from Fact 1. To see that the inclusion '$\subseteq$' holds, let $F' \in \mathcal{F}^A_{J,K}$. Then $F' = \delta_{F'} \circ r \circ R$ (by assertion (a)) and hence we can take $f = \delta_{F'}$. □

The dual statement of Theorem 15 can be derived immediately. We then have the following theorem.

Theorem 16. *Assume that* $\operatorname{ran}(J) = \operatorname{ran}(J_1^b)$ *for some $b \in A$ and let $F \in \mathcal{F}_{J,K}$. Assume also that* $\operatorname{ran}(S) = \operatorname{ran}(\delta_S)$*, where S is one of the functions S_j $(j \in Q(J_1^b))$ defined in Theorem 6. Then the assertions (a)–(e) of Lemma 13 hold (where R, r, U, and V are to be replaced with S, s, U_K and* $\operatorname{ran}(F)$*, respectively). If $F \in \mathcal{F}^A_{J,K}$, then for every $s \in Q(\delta_S)$ we have $s \circ S \in \mathcal{F}^A_{J,K}$ if and only if*

$$\mathcal{F}^A_{J,K} \;=\; \{f \circ s \circ S \mid f\colon A \to A\}.$$

Example 17. Considering again Example 12, where

$$R(x, y, x) \;=\; \tfrac{1}{2}\max(1, x + y + z - 1)$$

on $[0, 1]^3$, we clearly see that $\operatorname{ran}(R) = \operatorname{ran}(\delta_R) = [\frac{1}{2}, 1]$. Applying Theorem 15, with $r(x) = \frac{2x+1}{3}$ on $[\frac{1}{2}, 1]$ for instance, we obtain

$$r \circ R(x, y, z) \;=\; \tfrac{1}{3}\max(2, x + y + z)$$

and for any $F \in \mathcal{F}_{J,K}$ we have $F(x, y, z) = \delta_F(\frac{1}{3}\max(2, x + y + z))$. By identifying the variables in the latter identity we then obtain $\delta_F(x) = \delta_F(\frac{1}{3}\max(2, 3x))$, which shows that δ_F is constant on $[0, \frac{2}{3}]$. □

Remark 5. Let A be a nonempty real interval possibly unbounded. Recall that if a function $F\colon A^n \to \mathbb{R}$ is nondecreasing in each variable and satisfies $\operatorname{ran}(F) = \operatorname{ran}(\delta_F)$, then there always exists a function $M\colon A^n \to A$ (called a *Chisini mean*) that is idempotent and nondecreasing in each variable such that $F = \delta_F \circ M$ (see [3] for a constructive proof). For instance, considering again the functions in Example 17, we can write $F(x, y, z) = \delta_F(\frac{x+y+z}{3})$, where δ_F is constant on $[0, \frac{2}{3}]$.

We observe that the function $r \circ R$ is idempotent in Example 14 while it is not in Example 17. Actually, under the assumptions of Theorem 15, the function $r \circ R$ is idempotent whenever there exists $F \in \mathcal{F}_{J,K}$ such that δ_F is one-to-one. Indeed, we then have $r \circ R = \delta_F^{-1} \circ F$ and hence $\delta_{r \circ R} = \delta_F^{-1} \circ \delta_F = \mathrm{id}_A$, which shows that $r \circ R$ is idempotent. Clearly, the dual version of this fact can be derived by considering the assumptions of Theorem 16.

We also have the following result.

Proposition 18. *Under the assumptions of both Theorems 15 and 16, the following assertions are equivalent.*

(i) *There exists $F \in \mathcal{F}_{J,K}$ such that δ_F is one-to-one.*
(ii) *$r \circ R = s \circ S$ is idempotent.*

(iii) *$r \circ R$ is idempotent and lies in $\mathcal{F}_{J,K}$.*
(iv) *$s \circ S$ is idempotent and lies in $\mathcal{F}_{J,K}$.*

Proof. Clearly (iii) or (iv) implies (i). Let us prove that (i) implies (ii). As observed above, we have $r \circ R = \delta_F^{-1} \circ F = s \circ S$ and hence the function $r \circ R = s \circ S$ is idempotent. Finally, (ii) implies (iii) and (iv) by Proposition 11. □

Remark 6. We observe that the proof of Lemma 13 does not rely on the very concept of diagonal section. Actually, Lemma 13 can be easily generalized as follows. Consider the functions $R: X \to U$, $f: U \to V$, $F = f \circ R$, and $\Pi: X \to X$ and assume that $\mathrm{ran}(R) = \mathrm{ran}(R \circ \Pi)$. Denote by $\ker(f)$ the *kernel* of any function f, that is, the relation $\{(a,b) \in \mathrm{dom}(f)^2 \mid f(a) = f(b)\}$. Then the following assertions hold.

(a) We have $F = F \circ \Pi \circ r \circ R$ for every $r \in Q(R \circ \Pi)$.
(b) If $F' = f' \circ R$ satisfies $F \circ \Pi = F' \circ \Pi$ for some $f': U \to V$, then we have $F = F'$.
(c) If $F \circ \Pi = \Pi$, then $F = \Pi \circ r \circ R$ for every $r \in Q(R \circ \Pi)$.
(d) If $\Pi \circ r \circ R \circ \Pi \neq \Pi$ for some $r \in Q(R \circ \Pi)$, then we have $F \circ \Pi \neq \Pi$.
(e) If $\Pi \circ \Pi = \Pi$, then $T_r \circ \Pi \circ T_r = T_r$ for every $r \in Q(R \circ \Pi)$, where $T_r = \Pi \circ r \circ R$. In this case we have $T_r \circ \Pi = \Pi$ if and only if $\ker(T_r \circ \Pi) = \ker(\Pi)$.

Appendix

We consider a generalization of Problem 1 in which the functions J and K are defined on subsets of $X \times Y$ and $Y \times Z$, respectively.

Problem 3. *Given two functions $J: D_J \to U_J$ and $K: D_K \to U_K$, where $D_J \subseteq X \times Y$ and $D_K \subseteq Y \times Z$, determine the class $\mathcal{F}_{J,K}$ of functions $F: D_{J,K} \to \mathrm{ran}(F)$, where $D_{J,K} = \{(x,y,z) \mid (x,y) \in D_J$ and $(y,z) \in D_K\}$, for which there exist $G: U_J \times Z \to \mathrm{ran}(F)$ and $H: X \times U_K \to \mathrm{ran}(F)$ such that*

$$F(x,y,z) \; = \; G(J(x,y),z) \; = \; H(x,K(y,z)), \qquad (x,y,z) \in D_{J,K}.$$

This generalization of Problem 1 can be useful for instance when the assumption of Theorem 3 is not satisfied (i.e., when $\mathrm{ran}(K) \neq \mathrm{ran}(K_2^a)$ for all $a \in Z$). Indeed, it then may be possible to restrict the domain of K to a subset $D_K \subseteq Y \times Z$ on which the assumption is satisfied. These situations are illustrated in the following results (whose proofs can be obtained by a simple adaptation of Theorems 3 and 6) and examples.

If a function f is defined on $D \subseteq X \times Y$ and if $a \in Y$, then we denote by f_2^a the function defined on $\{x \in X \mid (x,a) \in D\}$ by $f_2^a(x) = f(x,a)$. Similarly, if $b \in X$, then we denote by f_1^b the function defined on $\{y \in Y \mid (b,y) \in D\}$ by $f_1^b(y) = f(b,y)$.

Theorem 19. *Under the notation of Problem 3, assume that $\mathrm{ran}(K) = \mathrm{ran}(K_2^a)$ for some $a \in Z$ and let $F \in \mathcal{F}_{J,K}$. Then there exists $f: U_J \to \mathrm{ran}(F)$ such that for every $k \in Q(K_2^a)$ we have $F(x,y,z) = f \circ R_k(x,y,z)$ for every $(x,y,z) \in D_{J,K}$ such that $(x, k \circ K(y,z)) \in D_J$, where $R_k: X \times Y \times Z \to U_J$ is defined by $R_k(x,y,z) = J(x, k \circ K(y,z))$.*

Theorem 20. *Under the notation of Problem 3, assume that $\mathrm{ran}(J) = \mathrm{ran}(J_1^b)$ for some $b \in X$ and let $F \in \mathcal{F}_{J,K}$. Then there exists $g: U_K \to \mathrm{ran}(F)$ such that for every $j \in Q(J_1^b)$ we have $F(x,y,z) = g \circ S_j(x,y,z)$ for every $(x,y,z) \in D_{J,K}$ such that $(j \circ J(x,y), z) \in D_K$, where $S_j: X \times Y \times Z \to U_K$ is defined by $S_j(x,y,z) = K(j \circ J(x,y), z)$.*

Example 21. Let A be a real interval, let $J,K\colon A^2\to A$ be defined by $J(x,y)=K(x,y)=\frac{x+y}{2}$, and let $F\colon A^3\to\mathbb{R}$ be a function for which there exist $G,H\colon A^2\to\mathbb{R}$ such that

$$F(x,y,z) \;=\; G(J(x,y),z) \;=\; H(x,K(y,z)),\qquad x,y,z\in A.$$

If $A=\mathbb{R}$, then $\operatorname{ran}(K)=\operatorname{ran}(K_2^0)$ and by Theorem 3 the function F is of the form $F(x,y,z)=f(x+y+z)$ for some $f\colon\mathbb{R}\to\mathbb{R}$. If $A=[0,1]$, then we have $\operatorname{ran}(K)\neq\operatorname{ran}(K_2^a)$ for every $a\in[0,1]$. Let us then use Theorem 19 by considering the sets

$$D_J \;=\; [0,1]^2 \quad\text{and}\quad D_K \;=\; \{(y,z)\in[0,1]^2\mid y+z\leqslant 1\}.$$

We then see that $\operatorname{ran}(K|_{D_K})=\operatorname{ran}((K|_{D_K})_2^0)$ and $D_{J,K}=[0,1]\times D_K$. Also, we may define $k\colon[0,\frac{1}{2}]\to[0,1]$ by $k(x)=2x$. By Theorem 19, for every $F\in\mathcal{F}_{J,K}$ there exists $f\colon[0,2]\to\mathbb{R}$ such that

$$F(x,y,z) \;=\; f(x+y+z)\qquad\text{when } y+z\leqslant 1.$$

Similarly, considering the set

$$D'_K \;=\; \{(y,z)\in[0,1]^2\mid y+z\geqslant 1\},$$

we see that $\operatorname{ran}(K|_{D'_K})=\operatorname{ran}((K|_{D'_K})_2^1)$ and then there exists $f'\colon[1,3]\to\mathbb{R}$ such that

$$F(x,y,z) \;=\; f'(x+y+z)\qquad\text{when } y+z\geqslant 1.$$

It follows that on the whole domain $[0,1]^3$ we have $F(x,y,z)=f(x+y+z)$ for some $f\colon[0,3]\to\mathbb{R}$. □

Note that the assumption $(x,k\circ K(y,z))\in D_J$ of Theorem 19 need not be satisfied for every $(x,y,z)\in D_{J,K}$. The following example illustrates this case.

Example 22. Let J and K be the real functions defined on

$$D \;=\; \{(x,y)\in\mathbb{R}^2\mid y\neq x+1\}$$

by

$$J(x,y) \;=\; K(x,y) \;=\; \frac{x+y}{2(x-y+1)}$$

and let $F\colon C\to\mathbb{R}$ be a function for which there exist $G,H\colon\mathbb{R}^2\to\mathbb{R}$ such that

$$F(x,y,z) \;=\; G(J(x,y),z) \;=\; H(x,K(y,z)),\qquad (x,y,z)\in C,\tag{5}$$

where $C=\{(x,y,z)\in\mathbb{R}^3\mid y\neq x+1,\ z\neq y+1\}$.

Since $K(x,x)=x$, we have $\operatorname{ran}(K)=\mathbb{R}$. We also have $\operatorname{ran}(K_2^a)=\mathbb{R}\setminus\{\frac{1}{2}\}$ if $a\in\mathbb{R}\setminus\{\frac{1}{2}\}$ and $\operatorname{ran}(K_2^{1/2})=\{\frac{1}{2}\}$, which shows that $\operatorname{ran}(K)\neq\operatorname{ran}(K_2^a)$ for every $a\in\mathbb{R}$. We then can apply Theorem 19 if we choose an appropriate restriction of function K. Let us use the notation of Problem 3 and Theorem 19 with the sets

$$D_J \;=\; D \quad\text{and}\quad D_K \;=\; D\setminus\{(y,z)\in\mathbb{R}^2\mid z=\tfrac{1}{2}\}.$$

We then have $K_2^0(y)=\frac{y}{2(1+y)}$ for every $y\neq -1$ and $\operatorname{ran}(K)=\operatorname{ran}(K_2^0)=\mathbb{R}\setminus\{\frac{1}{2}\}$. The unique quasi-inverse of K_2^0 is the function $k\colon\mathbb{R}\setminus\{\frac{1}{2}\}\to\mathbb{R}\setminus\{-1\}$ defined by $k(t)=\frac{2t}{1-2t}$. Moreover we have

$$D_{J,K} \;=\; \{(x,y,z)\in\mathbb{R}^3\mid y\neq x+1,\ z\neq y+1,\ z\neq\tfrac{1}{2}\}$$

and the condition $(x, k \circ K(y,z)) \in D_J$ is then equivalent to $\frac{y+z}{1-2z} \neq x+1$. It follows from Theorem 19 that for every function $F{:}\, D_{J,K} \to \mathrm{ran}(F)$ in $\mathcal{F}_{J,K}$ there is a function $f{:}\,\mathbb{R} \to \mathrm{ran}(F)$ such that

$$F(x,y,z) \;=\; f\Big(\frac{x+y+z-2xz}{2+2x-2y-6z-4xz}\Big), \tag{6}$$

for every $(x,y,z) \in \mathbb{R}^3$ such that $y \neq x+1$, $z \neq y+1$, $z \neq \frac{1}{2}$, and $\frac{y+z}{1-2z} \neq x+1$.

We can apply Theorem 20 similarly. Let us use the corresponding notation with the sets

$$D_K \;=\; D \quad \text{and} \quad D_J \;=\; D \smallsetminus \{(x,y) \in \mathbb{R}^2 \mid x = -\tfrac{1}{2}\}.$$

We then have $J_1^0(y) = \frac{y}{2(1-y)}$ for every $y \neq 1$ and $\mathrm{ran}(J) = \mathrm{ran}(J_1^0) = \mathbb{R} \smallsetminus \{-\frac{1}{2}\}$. The unique quasi-inverse of J_1^0 is the function $j{:}\,\mathbb{R} \smallsetminus \{-\frac{1}{2}\} \to \mathbb{R} \smallsetminus \{1\}$ defined by $j(t) = \frac{2t}{1+2t}$. Moreover we have

$$D_{J,K} \;=\; \{(x,y,z) \in \mathbb{R}^3 \mid y \neq x+1,\; z \neq y+1,\; x \neq -\tfrac{1}{2}\}$$

and the condition $(j \circ J(x,y), z) \in D_K$ is then equivalent to $\frac{x+y}{1+2x} \neq z-1$. It follows from Theorem 20 that for every function $F{:}\, D_{J,K} \to \mathrm{ran}(F)$ in $\mathcal{F}_{J,K}$ there is a function $g{:}\,\mathbb{R} \to \mathrm{ran}(F)$ such that

$$F(x,y,z) \;=\; g\Big(\frac{x+y+z+2xz}{2+6x+2y-2z-4xz}\Big), \tag{7}$$

for every $(x,y,z) \in \mathbb{R}^3$ such that $y \neq x+1$, $z \neq y+1$, $x \neq -\frac{1}{2}$, and $\frac{x+y}{1+2x} \neq z-1$.

Now, let us consider the function F given in (6) and (7) on the domain

$$E \;=\; \{(x,y,z) \in \mathbb{R}^3 \mid y \neq x+1,\; z \neq y+1,\; x \neq -\tfrac{1}{2},\; z \neq \tfrac{1}{2},\; \tfrac{y+z}{1-2z} \neq x+1,\; \tfrac{x+y}{1+2x} \neq z-1\}.$$

Substituting $x = -\frac{1}{3}(t+1)$, $y = t$, and $z = 1$ in (6) and (7), we obtain

$$f\Big(-\frac{2t+2}{2t+5}\Big) \;=\; g(0), \qquad t \in \mathbb{R} \smallsetminus \{-\tfrac{5}{2}, 0, \tfrac{1}{2}\}. \tag{8}$$

Similarly, substituting $x = -\frac{1}{5}(t+2)$, $y = t$, and $z = 2$, we obtain

$$f\Big(-\frac{4t+8}{2t+19}\Big) \;=\; g(0), \qquad t \in \mathbb{R} \smallsetminus \{-\tfrac{19}{2}, \tfrac{1}{2}, 1\}. \tag{9}$$

It follows from conditions (8) and (9) that $f{:}\,\mathbb{R} \to \mathrm{ran}(F)$ is a constant map on $\mathbb{R} \smallsetminus \{-\frac{1}{2}\}$. Using (6) it is then easy to see that F is constant on E.

Let us now consider equation (5) when $x = -\frac{1}{2}$. We have

$$F(-\tfrac{1}{2}, y, z) \;=\; G(-\tfrac{1}{2}, z) \;=\; H(-\tfrac{1}{2}, K(y,z))$$

for every $(y,z) \in \mathbb{R}^2$ such that $y \neq \frac{1}{2}$ and $z \neq y+1$. It follows that the identity

$$H(-\tfrac{1}{2}, K(y,z)) \;=\; H(-\tfrac{1}{2}, K(y',z))$$

holds for any $y, y' \in \mathbb{R} \smallsetminus \{\frac{1}{2}\}$ and any $z \in \mathbb{R} \smallsetminus \{y+1, y'+1\}$. Since $\mathrm{ran}(K_2^z) = \mathbb{R} \smallsetminus \{\frac{1}{2}\}$ for every $z \neq \frac{1}{2}$, we see that the function $t \mapsto H(-\frac{1}{2}, t)$ is constant on $\mathbb{R} \smallsetminus \{\frac{1}{2}\}$. Therefore F is constant on $\{(-\frac{1}{2}, y, z) \in \mathbb{R}^3 \mid z \neq \frac{1}{2},\; y \neq \frac{1}{2},\; z \neq y+1\}$. We can show similarly that F is constant on $\{(x, y, \frac{1}{2}) \in \mathbb{R}^3 \mid x \neq -\frac{1}{2},\; y \neq -\frac{1}{2},\; y \neq x+1\}$. Finally, F is clearly constant on $\{(-\frac{1}{2}, y, \frac{1}{2}) \in \mathbb{R}^3 \mid y \neq \pm\frac{1}{2}\}$. Now, in order to complete the resolution of this exercise, it would remain to know whether or not these constant values are related and to search for the behavior of F when $\frac{y+z}{1-2z} = x+1$ or $\frac{x+y}{1+2x} = z-1$. $\square$

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