PhD-FSTC-2016-41
The Faculty of Sciences, Technology and Communication

Faculté des Sciences et Technologies
Institut Elie Cartan de Lorraine
Ecole Doctorale IAEM Lorraine

DISSERTATION

Defense held on 22/09/2016 in Luxembourg
to obtain the degree of

DOCTEUR DE L'UNIVERSITÉ DU LUXEMBOURG
EN MATHÉMATIQUES

AND

DOCTEUR DE L'UNIVERSITÉ DE LORRAINE
EN MATHÉMATIQUES

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THE C^*-ALGEBRAS OF CERTAIN LIE GROUPS

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Abstract

In this doctoral thesis, the C^*-algebras of the connected real two-step nilpotent Lie groups and the Lie group $SL(2,\mathbb{R})$ are characterized. Furthermore, as a preparation for an analysis of its C^*-algebra, the topology of the spectrum of the semidirect product $U(n) \ltimes \mathbb{H}_n$ is described, where $\mathbb{H}_n$ denotes the Heisenberg Lie group and $U(n)$ the unitary group acting by automorphisms on $\mathbb{H}_n$. For the determination of the group C^*-algebras, the operator valued Fourier transform is used in order to map the respective C^*-algebra into the algebra of all bounded operator fields over its spectrum. One has to find the conditions that are satisfied by the image of this C^*-algebra under the Fourier transform and the aim is to characterize it through these conditions. In the present thesis, it is proved that both the C^*-algebras of the connected real two-step nilpotent Lie groups and the C^*-algebra of $SL(2,\mathbb{R})$ fulfill the same conditions, namely the "norm controlled dual limit" conditions. Thereby, these C^*-algebras are described in this work and the "norm controlled dual limit" conditions are explicitly computed in both cases. The methods used for the two-step nilpotent Lie groups and the group $SL(2,\mathbb{R})$ are completely different from each other. For the two-step nilpotent Lie groups, one regards their coadjoint orbits and uses the Kirillov theory, while for the group $SL(2,\mathbb{R})$ one can accomplish the calculations more directly.

Résumé

Dans la présente thèse de doctorat, les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux et du groupe de Lie $SL(2,\mathbb{R})$ sont caractérisées. En outre, comme préparation à une analyse de sa C^*-algèbre, la topologie du spectre du produit semi-direct $U(n) \ltimes \mathbb{H}_n$ est décrite, où $\mathbb{H}_n$ dénote le groupe de Lie de Heisenberg et $U(n)$ le groupe unitaire qui agit sur $\mathbb{H}_n$ par automorphismes. Pour la détermination des C^*-algèbres de groupes, la transformation de Fourier à valeurs opérationnelles est utilisée pour appliquer chaque C^*-algèbre dans l'algèbre de tous les champs d'opérateurs bornés sur son spectre. On doit trouver les conditions que satisfait l'image de cette C^*-algèbre sous la transformation de Fourier et l'objectif est de la caractériser par ces conditions. Dans cette thèse, il est démontré que les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux et la C^*-algèbre de $SL(2,\mathbb{R})$ satisfont les mêmes conditions, des conditions appelées "limites duales sous contrôle normique". De cette manière, ces C^*-algèbres sont décrites dans ce travail et les conditions "limites duales sous contrôle normique" sont explicitement calculées dans les deux cas. Les méthodes utilisées pour les groupes de Lie nilpotents de pas deux et pour le groupe $SL(2,\mathbb{R})$ sont très différentes l'une de l'autre. Pour les groupes de Lie nilpotents de pas deux, on regarde leurs orbites coadjointes et on utilise la théorie de Kirillov, alors que pour le groupe $SL(2,\mathbb{R})$, on peut mener les calculs plus directement.

Zusammenfassung

In dieser Doktorarbeit werden die C^*-Algebren der zusammenhängenden reellen zweistufig nilpotenten Lie-Gruppen und der Lie-Gruppe $SL(2,\mathbb{R})$ charakterisiert. Außerdem wird die Topologie des Spektrums des semidirekten Produktes $U(n) \ltimes \mathbb{H}_n$ in Vorbereitung auf eine Analyse der zugehörigen C^*-Algebra beschrieben, wobei $\mathbb{H}_n$ die Heisenberg-Lie-Gruppe und $U(n)$ die unitäre Gruppe, die durch Automorphismen auf $\mathbb{H}_n$ wirkt, bezeichnen. Für die Bestimmung der Gruppen-C^*-Algebren wird die operatorwertige Fouriertransformation benutzt, um die jeweilige C^*-Algebra in die Algebra der beschränkten Operatorenfelder über ihrem Spektrum abzubilden. Das Ziel ist, die Bedingungen zu finden, die das Bild dieser C^*-Algebra unter der Fouriertransformation erfüllt und sie durch eben diese zu charakterisieren. In der vorliegenden Arbeit wird bewiesen, dass sowohl die C^*-Algebren der zusammenhängenden reellen zweistufig nilpotenten Lie-Gruppen, als auch die C^*-Algebra von $SL(2,\mathbb{R})$ dieselben Bedingungen erfüllen, nämlich die "normkontrollierter dualer Limes"-Bedingungen. Dadurch werden diese C^*-Algebren beschrieben und die "normkontrollierter dualer Limes"-Bedingungen werden in beiden Fällen explizit nachgewiesen. Die Methoden, die für die zweistufig nilpotenten Lie-Gruppen und die Gruppe $SL(2,\mathbb{R})$ verwendet werden, sind dabei komplett unterschiedlich. Für die zweistufig nilpotenten Lie-Gruppen betrachtet man deren koadjungierte Bahnen und benutzt die Kirillov-Theorie, während man für die Gruppe $SL(2,\mathbb{R})$ die Berechnungen direkter durchführen kann.

1 Introduction

In this doctoral thesis, the structure of the C^*-algebras of connected real two-step nilpotent Lie groups and the structure of the C^*-algebra of the Lie group $SL(2,\mathbb{R})$ will be analyzed. Moreover, as a preparative work for an examination of its C^*-algebra, the topology of the spectrum of the semidirect product $U(n) \ltimes \mathbb{H}_n$ will be described, where $\mathbb{H}_n$ denotes the Heisenberg Lie group and $U(n)$ the unitary group acting by automorphisms on $\mathbb{H}_n$.

The research of C^*-algebras – an abstraction of algebras of bounded linear operators on Hilbert spaces – began in the 1930s due to a need in quantum mechanics and more precisely in order to serve as mathematical models for algebras of physical observables. Quantum systems are described with the help of self-adjoint operators on Hilbert spaces. Hence, algebras of bounded operators on these spaces were regarded. The term "C^*-algebra" was first introduced in the 1940s by I.Segal for describing norm-closed subalgebras of the algebra of bounded linear operators on a Hilbert space. By now, C^*-algebras represent an important tool in the theory of unitary representations of locally compact groups and in the mathematical description of quantum mechanics.
Lie groups were introduced in the 1870s by S.Lie in the framework of the Lie theory in order to examine continuous symmetries in differential equations and their theory has been developed further in the 20th century. Today Lie groups are used in many mathematical fields and in theoretical physics, as for example in particle physics.
As Lie groups and other symmetry groups hold an important position in physics, the examination of their group C^*-algebras – C^*-algebras that are constructed by building a completion of the space of all L^1-functions on the respective groups – has been continued and advanced. This task represents the mission of this thesis.
The method of analyzing a group C^*-algebra employed in this work has been initiated by G.M.Fell in the 1960s. It uses the non-abelian Fourier transform in order to write the C^*-algebras as algebras of operator fields and to study their elements in that way. A requirement to make use of this method and thus to understand these C^*-algebras is the knowledge of their spectrum and its topology which are unknown for many groups. Hence, this represents a major problem of Fell's procedure. The advancement of this method that is adopted in this thesis has been elaborated in the last years (see [24] and [26]) and makes the study of a large class of group C^*-algebras possible.

In order to be able to understand C^*-algebras, the Fourier transform is an important tool. Denoting by the unitary dual or spectrum $\widehat{A}$ of a C^*-algebra A the set of all equivalence classes of the irreducible unitary representations of A, the Fourier transform $\mathcal{F}(a) = \hat{a}$ of an element $a \in A$ is defined in the following way: One chooses in every equivalence class $\gamma \in \widehat{A}$ a representation $(\pi_\gamma, \mathcal{H}_\gamma)$ and defines

$$\mathcal{F}(a)(\gamma) := \pi_\gamma(a) \in \mathcal{B}(\mathcal{H}_\gamma),$$

where $\mathcal{B}(\mathcal{H}_\gamma)$ denotes the C^*-algebra of bounded linear operators on $\mathcal{H}_\gamma$. Then, $\mathcal{F}(a)$ is contained in the algebra of all bounded operator fields over $\widehat{A}$

$$l^\infty(\widehat{A}) = \Big\{ \phi = \big(\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma)\big)_{\gamma\in\widehat{A}} \;\big|\; \|\phi\|_\infty := \sup_{\gamma\in\widehat{A}} \|\phi(\pi_\gamma)\|_{op} < \infty \Big\}$$

and the mapping

$$\mathcal{F} : A \to l^{\infty}(\widehat{A}),\ a \mapsto \hat{a}$$

is an isometric $*$-homomorphism.
Denoting by dx the Haar measure of the locally compact group G, one can define on $L^1(G) := L^1(G, \mathbb{C})$ the convolution product $*$ in the following way

$$f * f'(g) := \int\limits_G f(x) f'\big(x^{-1}g\big) dx \quad \forall f, f' \in L^1(G)\ \forall g \in G$$

and obtains a Banach algebra $\big(L^1(G), *\big)$. Moreover, one gets an isometric involution on this algebra by letting

$$f^*(g) := \Delta_G(g)^{-1} \overline{f\big(g^{-1}\big)} \quad \forall g \in G,$$

Δ_G being the modular function which is defined to be the positive function fulfilling

$$\int\limits_G f\big(tg^{-1}\big) dt = \Delta_G(g) \int\limits_G f(t) dt \quad \forall f \in C_0(G)\ \forall g \in G,$$

where $C_0(G)$ is the space of all compactly supported continuous functions.
For every irreducible unitary representation $(\pi, \mathcal{H})$ of G, a representation $(\tilde{\pi}, \mathcal{H})$ of $L^1(G)$ can be obtained as follows

$$\tilde{\pi}(f) := \int\limits_G f(g) \pi(g) dg \quad \forall f \in L^1(G).$$

This representation turns out to be irreducible and unitary.
Now, the C^*-algebra of G is defined as the completion of the convolution algebra $L^1(G)$ with respect to the C^*-norm of $L^1(G)$, i.e.

$$C^*(G) := \overline{L^1(G, dx)}^{\|\cdot\|_{C^*(G)}} \quad \text{with} \quad \|f\|_{C^*(G)} := \sup_{[\pi] \in \widehat{G}} \|\pi(f)\|_{op},$$

where $\widehat{G}$ is the spectrum of G, defined as above as the set of all equivalence classes of the irreducible unitary representations of G.
Furthermore, every irreducible unitary representation $(\tilde{\pi}, \mathcal{H})$ of $L^1(G)$ can be uniquely written as an integral in the above shown way and hence, one obtains an irreducible unitary representation $(\pi, \mathcal{H})$ of G. One therefore gets the following well-known result, that can be found in [9], Chapter 13.9, and states that the spectrum of $C^*(G)$ coincides with the spectrum of G:

$$\widehat{C^*(G)} = \widehat{G}.$$

The structure of the group C^*-algebras is already known for certain classes of Lie groups, such as the Heisenberg and the thread-like Lie groups (see [26]) and the $ax + b$-like groups (see [24]). Furthermore, the C^*-algebras of the 5-dimensional nilpotent Lie groups have been determined in [31], while those of all 6-dimensional nilpotent Lie groups have been characterized by H.Regeiba in [30].
In this thesis, a result shown by H.Regeiba and J.Ludwig (see [31], Theorem 3.5) shall be utilized for the characterization of the C^*-algebras mentioned at the beginning.

For the C^*-algebras of the two-step nilpotent Lie groups, the approach will partly be similar but more complex, to the one used for the characterization of the C^*-algebra of the Heisenberg Lie group (see [26]), which is also two-step nilpotent and thus serves as an example.
In the case of the Heisenberg Lie group, the situation is a lot easier than for general two-step nilpotent Lie groups, as in this special case the appearing coadjoint orbits can only have two different dimensions, while in the general case, there are many different dimensions that can appear. Consequently, much more complicated cases emerge.
The 5- and 6-dimensional nilpotent Lie groups, in turn, have a more complicated structure than the two-step nilpotent Lie groups. Here, non-flat coadjoint orbits appear, whereas in the two-step nilpotent case, one only has to deal with flat coadjoint orbits. In these cases of 5- and 6-dimensional nilpotent Lie groups, one therefore examines every group by itself, while for the two-step nilpotent Lie groups, one covers this whole class of groups without knowing their exact structure.
As mentioned above, the approach for the group $SL(2,\mathbb{R})$ will be completely different. Since one treats one single group which is explicitly given, in this case, one can carry out more concrete computations.
For semisimple Lie groups in general, there is no explicit description of their group C^*-algebras given in literature. However, for those semisimple Lie groups whose spectrum is classified, the procedure of the determination of the group C^*-algebra used in this work might be successfully applied in a similar way. A description of the C^*-algebra of the Lie group $SL(2,\mathbb{C})$ is given in [13], Theorem 5.3 and Theorem 5.4, and a characterization of reduced group C^*-algebras of semisimple Lie groups can be found in [34].
In the present doctoral thesis, the group C^*-algebras of connected real two-step nilpotent Lie groups and of $SL(2,\mathbb{R})$ shall be described very explicitly. It will be shown that they are characterized by specific conditions that are called "norm controlled dual limit" conditions which will be given below. The main objective is to concretely compute these conditions and to construct the required "norm controls". In an abstract existence result (see [2], Theorem 4.6), these conditions are shown to hold true for all simply connected connected nilpotent Lie groups. They are explicitly checked for all 5- and 6-dimensional nilpotent Lie groups (see [31]), for the Heisenberg Lie groups and the thread-like Lie groups (see [26]).
The results about the group C^*-algebras of the connected real two-step nilpotent Lie groups have been published in the "Revista Matemática Complutense" as a joint article with J.Ludwig (see [17]). A further article about the group C^*-algebra of $SL(2,\mathbb{R})$ has been submitted to the "Journal of Lie Theory" and can be found on arXiv (see [16]).
Since for the determination of a group C^*-algebra the spectrum of the group and its topology are necessary, in this doctoral thesis, the topology of the spectrum of the groups $G_n = U(n) \ltimes \mathbb{H}_n$ for $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ shall be analyzed as well. This work was started by M.Elloumi and J.Ludwig (see [11], Chapter 3) and their preprint will be elaborated in this thesis.
Like for several other classes of Lie groups, for example the exponential solvable Lie groups and the Euclidean motion groups, the spectrum $\widehat{G_n}$ can be parametrized by the space of all admissible coadjoint orbits in the dual $\mathfrak{g}_n^*$ of the Lie algebra $\mathfrak{g}_n$ of G_n.
The aim is to show that the bijection between the space of all admissible coadjoint orbits equipped with the quotient topology and the spectrum $\widehat{G_n}$ of G_n is a homeomorphism. This has already been shown for the classical motion groups $SO(n) \ltimes \mathbb{R}^n$ for $n \geq 2$ in [12].

For the groups $G_n = U(n) \ltimes \mathbb{H}_n$, this claim has been proved for $n = 1$. For general $n \in \mathbb{N}^*$, it could be shown that the mapping between the spectrum $\widehat{G_n}$ and the space of all admissible coadjoint orbits of G_n is continuous. The remaining parts of the proof of the claim are work in progress.

In this section, the definition of a C^*-algebra with "norm controlled dual limits" will be given. The conditions required by this definition characterize group C^*-algebras. Thereafter, in Section 3, they will be shown to hold true for the two-step nilpotent Lie groups. After having completed this proof, an example of a two-step nilpotent Lie group will be examined. In Section 4, these conditions will then be proved in the case of the Lie group $SL(2,\mathbb{R})$, in a very different approach than the one used for the two-step nilpotent Lie groups.

Definition 1.1.
A C^*-algebra A is called C^*-algebra with ***norm controlled dual limits*** *if it fulfills the following conditions:*

- **Condition 1**: *Stratification of the spectrum:*

  (a) *There is a finite increasing family $S_0 \subset S_1 \subset \ldots \subset S_r = \widehat{A}$ of closed subsets of the spectrum $\widehat{A}$ of A in such a way that for $i \in \{1, \cdots, r\}$ the subsets $\Gamma_0 := S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff in their relative topologies and such that S_0 consists of all the characters of A.*

  (b) *For every $i \in \{0, \cdots, r\}$, there is a Hilbert space $\mathcal{H}_i$ and for every $\gamma \in \Gamma_i$, there is a concrete realization $(\pi_\gamma, \mathcal{H}_i)$ of γ on the Hilbert space $\mathcal{H}_i$.*

- **Condition 2**: *CCR C^*-algebra:*
  A is a separable CCR (or liminal) C^*-algebra, i.e. a separable C^*-algebra such that the image of every irreducible representation $(\pi, \mathcal{H})$ of A is contained in the algebra of compact operators $\mathcal{K}(\mathcal{H})$ (which implies that the image equals $\mathcal{K}(\mathcal{H})$).

- **Condition 3**: *Changing of layers:*
  Let $a \in A$.

  (a) *The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on the different sets Γ_i.*

  (b) *For any $i \in \{0, \cdots, r\}$ and for any converging sequence contained in Γ_i with limit set outside Γ_i (thus in S_{i-1}), there is a properly converging subsequence $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}}$, as well as a constant $C > 0$ and for every $k \in \mathbb{N}$ an involutive linear mapping $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma},k} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$, which is bounded by $C\|\cdot\|_{S_{i-1}}$ (uniformly in k), such that*

  $$\lim_{k\to\infty} \left\| \mathcal{F}(a)(\gamma_k) - \tilde{\nu}_k\big(\mathcal{F}(a)_{|S_{i-1}}\big) \right\|_{op} = 0.$$

  Here, $CB(S_{i-1})$ is the $*$-algebra of all the uniformly bounded fields of operators $\big(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_l)\big)_{\gamma\in\Gamma_l, l=0,\cdots,i-1}$, which are operator norm continuous on the subsets Γ_l for every $l \in \{0, \cdots, i-1\}$, provided with the infinity-norm

  $$\|\psi\|_{S_{i-1}} := \sup_{\gamma \in S_{i-1}} \|\psi(\gamma)\|_{\mathrm{op}}.$$

Theorem 1.2 (Main result)**.**
The C^*-algebras of the connected real two-step nilpotent Lie groups G and of the Lie group $G = SL(2,\mathbb{R})$ have norm controlled dual limits.

In order to prove Theorem 1.2, concrete subsets Γ_i and S_i of $\widehat{C^*(G)} \cong \widehat{G}$ will be defined and the mappings $\left(\tilde{\nu}_k\right)_{k\in\mathbb{N}}$ will be constructed.

The norm controlled dual limit conditions completely characterize the structure of a C^*-algebra in the following sense: Taking the number r, the Hilbert spaces $\mathcal{H}_i$, the sets Γ_i and S_i for $i \in \{0,...,r\}$ and the mappings $\left(\tilde{\nu}_k\right)_{k\in\mathbb{N}}$ required in the above definition, by [31], Theorem 3.5, one gets the result below for the C^*-algebras of the two-step nilpotent Lie groups G and of $G = SL(2,\mathbb{R})$.

Theorem.
The C^*-algebra $C^*(G)$ of a connected real two-step nilpotent Lie group G and the C^*-algebra of the Lie group $G = SL(2,\mathbb{R})$, respectively, are isomorphic (under the Fourier transform) to the set of all operator fields φ defined over the spectrum $\widehat{G}$ of the respective group such that:

1. *$\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $i \in \{1,...,r\}$ and every $\gamma \in \Gamma_i$.*
2. *$\varphi \in l^\infty(\widehat{G})$.*
3. *The mappings $\gamma \mapsto \varphi(\gamma)$ are norm continuous on the different sets Γ_i.*
4. *For any sequence $(\gamma_k)_{k\in\mathbb{N}} \subset \widehat{G}$ going to infinity, $\lim_{k\to\infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.*
5. *For $i \in \{1,...,r\}$ and any properly converging sequence $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}} \subset \Gamma_i$ whose limit set is contained in S_{i-1} (taking a subsequence if necessary) and for the mappings $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma},k} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$, one has*

$$\lim_{k\to\infty} \left\|\varphi(\gamma_k) - \tilde{\nu}_k\left(\varphi_{|S_{i-1}}\right)\right\|_{op} = 0.$$

In the case of the Lie group $G = SL(2,\mathbb{R})$, this result can still be simplified and more concretely described in Theorem 4.21.

2 General definitions and results

2.1 Nilpotent Lie groups

2.1.1 Preliminaries

Let $\mathfrak{g}$ be a real nilpotent Lie algebra.
Fix a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ and take on $\mathfrak{g}$ the Campbell-Baker-Hausdorff multiplication

$$u \cdot v = u + v + \frac{1}{2}[u,v] + \frac{1}{12}\big[u,[u,v]\big] + \frac{1}{12}\big[v,[v,u]\big] + ... \quad \forall u,v \in \mathfrak{g},$$

which is a finite sum, as $\mathfrak{g}$ is nilpotent.
This gives the simply connected connected Lie group $G = (\mathfrak{g},\cdot)$ with Lie algebra $\mathfrak{g}$. The exponential mapping $\exp : \mathfrak{g} \to G = (\mathfrak{g},\cdot)$ is in this case the identity mapping.
The Haar measure of this group is a Lebesgue measure which is denoted by dx.

Convention 2.1.
Throughout this work, all function spaces are spaces of complex valued functions.

For a vector space V, denote by the Schwartz space $\mathcal{S}(V)$ the space of all rapidly decreasing smooth functions on V. Then, as $G = (\mathfrak{g},\cdot)$, one can define $\mathcal{S}(G) := \mathcal{S}(\mathfrak{g})$.
Define on $\mathfrak{g}$ and thus on G a norm by $\|\cdot\| := |\langle\cdot,\cdot\rangle|^{\frac{1}{2}}$. If $\{X_1, ..., X_n\}$ represents an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, define $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Moreover, denote by f'_α the derivative of $f \in \mathcal{S}(G)$ in the direction of X^α for every $\alpha \in \mathbb{N}^n$. Then, the semi-norms on $\mathcal{S}(G)$ can be given by

$$\|f\|_{(N)} = \sum_{|\alpha| \leq N} \int_G \big(1 + \|g\|^2\big)^N \big|f'_\alpha(g)\big| dg \quad \forall f \in \mathcal{S}(G)\ \forall N \in \mathbb{N}.$$

The space $\mathcal{S}(G)$ is dense in $L^1(G)$ and in turn, for a vector space V, a well-known result states that the space $\big\{f \in \mathcal{S}(V)|\ \hat{f} \text{ is compactly supported}\big\}$ is dense in $\mathcal{S}(V)$. Hence, $\big\{f \in \mathcal{S}(G)|\ \hat{f} \text{ is compactly supported}\big\}$ is dense in $\mathcal{S}(G)$ as well.

Now, for a linear functional ℓ of $\mathfrak{g}$, consider the skew-bilinear form

$$B_\ell(X,Y) := \langle \ell, [X,Y] \rangle$$

on $\mathfrak{g}$. Moreover, let

$$\mathfrak{g}(\ell) := \big\{X \in \mathfrak{g}|\ \langle \ell, [X,\mathfrak{g}] \rangle = \{0\}\big\}$$

be the radical of B_ℓ and the stabilizer of the linear functional ℓ.

Definition 2.2 (Polarization)**.**
A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, that is subordinated to ℓ (i.e. that fulfills $\langle \ell, [\mathfrak{p},\mathfrak{p}] \rangle = \{0\}$) and that has the dimension

$$dim(\mathfrak{p}) = \frac{1}{2}\Big(dim(\mathfrak{g}) + dim(\mathfrak{g}(\ell))\Big),$$

which means that $\mathfrak{p}$ is maximal isotropic for B_ℓ, is called a polarization of ℓ.

If $\mathfrak{p} \subset \mathfrak{g}$ is any subalgebra of $\mathfrak{g}$ which is subordinated to ℓ, the linear functional ℓ defines a unitary character χ_ℓ of $P := \exp(\mathfrak{p})$:

$$\chi_\ell(x) := e^{-2\pi i\langle \ell, \log(x)\rangle} = e^{-2\pi i\langle \ell, x\rangle} \quad \forall x \in P.$$

Definition 2.3 (The coadjoint action).
Define for all $g \in G$ the mappings

$$\alpha_g : G \to G,\ x \mapsto gxg^{-1} \quad \text{and} \quad \mathrm{Ad}\,(g) := d(\alpha_g)_e : \mathfrak{g} \to \mathfrak{g}.$$

Then, $\mathrm{Ad} : G \to GL(\mathfrak{g})$ *is a group representation. The coadjoint action* Ad^* *can be defined as follows:*

$$\mathrm{Ad}^*(g) : \mathfrak{g}^* \to \mathfrak{g}^* \quad \forall g \in G, \quad \mathrm{Ad}^*(g)\ell(X) = \ell\Big(\mathrm{Ad}\,\big(g^{-1}\big)X\Big) \quad \forall \ell \in \mathfrak{g}^*\ \forall X \in \mathfrak{g}.$$

$\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ *is another group representation.*
The space of coadjoint orbits $\{\,\mathrm{Ad}^*(G)\ell|\ \ell \in \mathfrak{g}^*\}$ *is denoted by* $\mathfrak{g}^*/G$*.*
Furthermore, define the mapping

$$\mathrm{ad}^*(X) : \mathfrak{g}^* \to \mathfrak{g}^* \quad \forall X \in \mathfrak{g}, \quad \mathrm{ad}^*(X)\ell(Y) = \ell([Y, X]).$$

$\mathrm{ad}^* : \mathfrak{g} \to GL(\mathfrak{g}^*)$ *is a representation of the Lie algebra* $\mathfrak{g}$*.*

Definition 2.4 (Induced representation).
Let H be a closed subgroup of G and define

$$L^2(G/H, \chi_\ell) \ := \ \Big\{\xi : G \to \mathbb{C} \ \Big|\ \xi \text{ measurable},\ \xi(gh) = \overline{\chi_\ell(h)}\xi(g)\ \forall g \in G\ \forall h \in H,$$
$$\|\xi\|_2^2 := \int\limits_{G/H} |\xi(g)|^2 d\dot{g} < \infty\Big\},$$

where $d\dot{g}$ is an invariant measure on G/H which always exists for nilpotent G.
$L^2(G/H, \chi_\ell)$ *is a Hilbert space and one can define the induced representation*

$$\mathrm{ind}_H^G \chi_\ell(g)\xi(u) := \xi\big(g^{-1}u\big) \quad \forall g, u \in G\ \forall \xi \in L^2(G/H, \chi_\ell).$$

This is a unitary representation. If the associated Lie algebra $\mathfrak{h}$ of H is a polarization of ℓ, this representation is also irreducible.

2.1.2 Orbit method

By the Kirillov theory (see [8], Section 2.2), for every representation class $\gamma \in \widehat{G}$, there exists an element $\ell \in \mathfrak{g}^*$ and a polarization $\mathfrak{p}$ of ℓ in $\mathfrak{g}$ such that $\gamma = \big[\mathrm{ind}_P^G \chi_\ell\big]$, where $P := \exp(\mathfrak{p})$. Moreover, if $\ell, \ell' \in \mathfrak{g}^*$ are located in the same coadjoint orbit $\mathcal{O} \in \mathfrak{g}^*/G$ and $\mathfrak{p}$ and $\mathfrak{p}'$ are polarizations in ℓ and ℓ', respectively, the induced representations $\mathrm{ind}_P^G \chi_\ell$ and $\mathrm{ind}_{P'}^G \chi_{\ell'}$ are equivalent and the Kirillov map which goes from the coadjoint orbit space $\mathfrak{g}^*/G$ to the spectrum $\widehat{G}$ of G

$$K : \mathfrak{g}^*/G \to \widehat{G},\ \mathrm{Ad}^*(G)\ell \mapsto \big[\mathrm{ind}_P^G \chi_\ell\big]$$

is a homeomorphism (see [6] or [23], Chapter 3). Therefore,

$$\mathfrak{g}^*/G \cong \widehat{G}$$

as topological spaces.

Definition 2.5 (Properly converging).
Let T be a second countable topological space and suppose that T is not Hausdorff, which means that converging sequences can have many limit points. Denote by $L((t_k)_{k\in\mathbb{N}})$ the collection of all the limit points of a sequence $(t_k)_{k\in\mathbb{N}}$ in T. A sequence $(t_k)_{k\in\mathbb{N}}$ is called properly converging if $(t_k)_{k\in\mathbb{N}}$ has limit points and if every subsequence of $(t_k)_{k\in\mathbb{N}}$ has the same limit set as $(t_k)_{k\in\mathbb{N}}$.

It is well-known that every converging sequence in T admits a properly converging subsequence.

2.1.3 The C^*-algebra $C^*(G/U, \chi_\ell)$

Let $\mathfrak{u} \subset \mathfrak{g}$ be an ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$, $U := \exp(\mathfrak{u})$ and let $\ell \in \mathfrak{g}^*$ such that $\langle \ell, [\mathfrak{g}, \mathfrak{u}]\rangle = \{0\}$ and such that $\mathfrak{u} \subset \mathfrak{g}(\ell)$. Then, the character χ_ℓ of the group $U = \exp(\mathfrak{u})$ is G-invariant. One can thus define the involutive Banach algebra $L^1(G/U, \chi_\ell)$ as

$$L^1(G/U,\chi_\ell) := \Big\{ f : G \to \mathbb{C} \;\Big|\; f \text{ measurable}, \ f(gu) = \chi_\ell(u^{-1}) f(g) \ \forall g \in G \ \forall u \in U, \quad \|f\|_1 := \int_{G/U} |f(g)| d\dot{g} < \infty \Big\}.$$

The convolution

$$f * f'(g) := \int_{G/U} f(x) f'(x^{-1}g) d\dot{x} \quad \forall g \in G$$

and the involution

$$f^*(g) := \overline{f(g^{-1})} \quad \forall g \in G$$

are well-defined for $f, f' \in L^1(G/U, \chi_\ell)$ and

$$\|f * f'\|_1 \le \|f\|_1 \ \|f'\|_1.$$

Moreover, the linear mapping

$$p_{G/U} : L^1(G) \to L^1(G/U, \chi_\ell), \ p_{G/U}(F)(g) := \int_U F(gu)\chi_\ell(u) du \quad \forall F \in L^1(G) \ \forall g \in G$$

is a surjective $*$-homomorphism between the algebras $L^1(G)$ and $L^1(G/U, \chi_\ell)$.
Let

$$\widehat{G}_{\mathfrak{u},\ell} := \Big\{ [\pi] \in \widehat{G} \big| \ \pi_{|U} = \chi_{\ell|U} \mathbb{I}_{\mathcal{H}_\pi} \Big\}.$$

Then, $\widehat{G}_{\mathfrak{u},\ell}$ is a closed subset of $\widehat{G}$, which can be identified with the spectrum of the algebra $L^1(G/U, \chi_\ell)$. Indeed, it is easy to see that every irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ whose equivalence class is located in $\widehat{G}_{\mathfrak{u},\ell}$ defines an irreducible representation $(\tilde{\pi}, \mathcal{H}_\pi)$ of the algebra $L^1(G/U, \chi_\ell)$ as follows:

$$\tilde{\pi}\big(p_{G/U}(F)\big) := \pi(F) \quad \forall F \in L^1(G).$$

Similarly, if $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ is an irreducible unitary representation of $L^1(G/U, \chi_\ell)$, then $[\pi]$ given by

$$\pi := \tilde{\pi} \circ p_{G/U}$$

defines an element of $\widehat{G}_{\mathfrak{u},\ell}$.
Let $\mathfrak{s} \subset \mathfrak{g}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}(\ell) \oplus \mathfrak{s}$. Since $\mathfrak{u}$ contains $[\mathfrak{g}, \mathfrak{g}]$, one has

$$\widehat{G}_{\mathfrak{u},\ell} = \big\{ \big[\chi_q \otimes \pi_\ell\big] \big|\ q \in (\mathfrak{u} + \mathfrak{s})^\perp \big\},$$

letting $\pi_\ell := \mathrm{ind}_P^G \chi_\ell$ for a polarization $\mathfrak{p}$ of ℓ and $P := \exp(\mathfrak{p})$.
Denote by $C^*(G/U, \chi_\ell)$ the C^*-algebra of $L^1(G/U, \chi_\ell)$, whose spectrum can also be identified with $\widehat{G}_{\mathfrak{u},\ell}$.
For $\pi_{\ell+q} := \mathrm{ind}_P^G \chi_{\ell+q}$, the Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}(a)(q) := \pi_{\ell+q}(a) \quad \forall q \in (\mathfrak{u} + \mathfrak{s})^\perp$$

then maps the C^*-algebra $C^*(G/U, \chi_\ell)$ onto the algebra $C_\infty\big((\mathfrak{u} + \mathfrak{s})^\perp, \mathcal{K}(\mathcal{H}_{\pi_\ell})\big)$ of the continuous mappings $\varphi : (\mathfrak{u} + \mathfrak{s})^\perp \to \mathcal{K}(\mathcal{H}_{\pi_\ell})$ vanishing at infinity with values in the algebra of compact operators on the Hilbert space $\mathcal{H}_{\pi_\ell}$ of the representation π_ℓ.

If one restricts $p_{G/U}$ to the Fréchet algebra $\mathcal{S}(G) \subset L^1(G)$, its image is the Fréchet algebra

$$\begin{aligned}\mathcal{S}(G/U, \chi_\ell) \quad = \quad & \big\{ f \in L^1(G/U, \chi_\ell) \big|\ f \text{ smooth and for every subspace } \mathfrak{s}' \subset \mathfrak{g} \text{ with } \mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{u} \\ & \text{and for } S' = \exp(\mathfrak{s}'),\ f_{|S'} \in \mathcal{S}(S') \big\}.\end{aligned}$$

2.2 General results

In this section, some general results shall be listed which are later needed in several proofs.

Theorem 2.6 (Schur's lemma)**.**
A unitary representation π of a topological group G on a Hilbert space $\mathcal{H}$ is irreducible if and only if the only bounded linear operators on $\mathcal{H}$ commuting with $\pi(g)$ for all $g \in G$ are the scalar operators.

For the proof of this theorem see [20], Chapter I.3.

Lemma 2.7.
Let G be a Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Define the canonical projection p_G going from $\mathfrak{g}^*$ to the space of coadjoint orbits $\mathfrak{g}^*/G$ and equip the space $\mathfrak{g}^*/G$ with the quotient topology, i.e. a subset U in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(U)$ is open in $\mathfrak{g}^*$. Then, a sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O} \in \mathfrak{g}^*/G$ if and only if for any $\ell \in \mathcal{O}$ there exists $\ell_k \in \mathcal{O}_k$ for every $k \in \mathbb{N}$ such that $\ell = \lim\limits_{k\to\infty} \ell_k$.

For the proof of this Lemma, see [23], Chapter 3.1.

Now, let G be a second countable locally compact group and $(\pi, \mathcal{H}_\pi)$ an irreducible unitary representation on the Hilbert space $\mathcal{H}_\pi$. A function of positive type of π is defined to be a linear functional

$$C_\xi^\pi : G \longrightarrow \mathbb{C},\ g \longmapsto \langle \pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi},$$

where ξ is a cyclic vector in $\mathcal{H}_\pi$.

The spectrum $\widehat{G}$ has a natural topology which can be characterized in the following way:

Theorem 2.8 (Topology of the spectrum)**.**
Let $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ be a family of irreducible unitary representations of G. Then, $(\pi_k)_{k\in\mathbb{N}}$ converges to $(\pi, \mathcal{H}_\pi)$ in $\widehat{G}$ if and only if for some non-zero (respectively for every) vector ξ in $\mathcal{H}_\pi$, for every $k \in \mathbb{N}$ there exists $\xi_k \in \mathcal{H}_{\pi_k}$ such that the sequence $\big(C^{\pi_k}_{\xi_k}\big)_{k\in\mathbb{N}} = \big(\big\langle \pi_k(\cdot)\xi_k, \xi_k\big\rangle_{\mathcal{H}_{\pi_k}}\big)_{k\in\mathbb{N}}$ converges uniformly on compacta to $C^\pi_\xi = \big\langle \pi(\cdot)\xi, \xi\big\rangle_{\mathcal{H}_\pi}$.

The topology of $\widehat{G}$ can also be expressed by the weak convergence of the coefficient functions:

Theorem 2.9.
Let $\big(\pi_k, \mathcal{H}_{\pi_k}\big)_{k\in\mathbb{N}}$ be a sequence of irreducible unitary representations of G. Then, $\big(\pi_k, \mathcal{H}_{\pi_k}\big)_{k\in\mathbb{N}}$ converges to $(\pi, \mathcal{H}_\pi)$ in $\widehat{G}$ if and only if for some (respectively for every) unit vector ξ in $\mathcal{H}_\pi$, there is for every $k \in \mathbb{N}$ a vector $\xi_k \in \mathcal{H}_{\pi_k}$ with $\|\xi_k\|_{\mathcal{H}_{\pi_k}} \leq 1$ such that the sequence of linear functionals $\big(C^{\pi_k}_{\xi_k}\big)_{k\in\mathbb{N}}$ converges weakly on some dense subspace of $C^*(G)$ to C^π_ξ.

The proofs of the Theorems 2.8 and 2.9 can be found in [9], Theorem 13.5.2.

For the following corollary, let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Furthermore, for a unitary representation $(\pi, \mathcal{H}_\pi)$ of G, let $\mathcal{H}^\infty_\pi := \big\{ f \in \mathcal{H}_\pi |\ G \ni g \mapsto \pi(g)f \in \mathcal{H}_\pi \text{ is smooth}\big\}$, the subspace of $\mathcal{H}_\pi$ of smooth vectors.

Corollary 2.10.
Let $\big(\pi_k, \mathcal{H}_{\pi_k}\big)_{k\in\mathbb{N}}$ be a sequence of irreducible unitary representations of the Lie group G. If $\big(\pi_k, \mathcal{H}_{\pi_k}\big)_{k\in\mathbb{N}}$ converges to $(\pi, \mathcal{H}_\pi)$ in $\widehat{G}$, then for each unit vector ξ in $\mathcal{H}^\infty_\pi$, there exists for every $k \in \mathbb{N}$ a vector $\xi_k \in \mathcal{H}^\infty_{\pi_k}$ such that the sequence $\big(\langle d\pi_k(D)\xi_k, \xi_k\rangle_{\mathcal{H}_{\pi_k}}\big)_{k\in\mathbb{N}}$ converges to $\langle d\pi(D)\xi, \xi\rangle_{\mathcal{H}_\pi}$ for each $D \in \mathcal{U}(\mathfrak{g})$.

The proof below of this corollary originates from [11].

Proof:
Let $\xi \in \mathcal{H}^\infty_\pi$ a unit vector. It follows from [10], Theorem 3.3, that there are functions $f_1, ..., f_s$ in $C^\infty_0(G)$, the space of all compactly supported $C^\infty(G)$-functions, and linearly independent vectors $\xi_1, ..., \xi_s \in \mathcal{H}_\pi$, such that $\xi = \pi(f_1)\xi_1 + ... + \pi(f_s)\xi_s$. Since π is irreducible, for any non-zero vector $\eta \in \mathcal{H}_\pi$ and every $j \in \{1..., s\}$, there exist elements q_j in the C^*-algebra of G fulfilling $\xi_j = \pi(q_j)\eta$. Hence, $\xi = \sum\limits_{j=1}^{s} \pi(f_j)\pi(q_j)\eta$.

Now, choose for every $k \in \mathbb{N}$ vectors $\eta_k \in \mathcal{H}_{\pi_k}$ such that $\big(C^{\pi_k}_{\eta_k}\big)_{k\in\mathbb{N}}$ converges weakly to C^π_η. Furthermore, for $k \in \mathbb{N}$ let $\xi_k := \sum\limits_{j=1}^{s} \pi_k(f_j)\pi_k(q_j)\eta_k$. Then, for $D \in \mathcal{U}(\mathfrak{g})$ it follows that

$$\begin{aligned}
\lim_{k\to\infty} \langle d\pi_k(D)\xi_k, \xi_k\rangle_{\mathcal{H}_{\pi_k}} &= \lim_{k\to\infty} \Big\langle \sum_{j=1}^{s} \pi_k(D * f_j)\pi_k(q_j)\eta_k, \sum_{i=1}^{s} \pi_k(f_i)\pi_k(q_i)\eta_k \Big\rangle_{\mathcal{H}_{\pi_k}} \\
&= \sum_{i=1}^{s}\sum_{j=1}^{s} \lim_{k\to\infty} \Big\langle \pi_k\big(q_i^* * f_i^* * D * f_j * q_j\big)\eta_k, \eta_k \Big\rangle_{\mathcal{H}_{\pi_k}} \\
&= \sum_{i=1}^{s}\sum_{j=1}^{s} \Big\langle \pi\big(q_i^* * f_i^* * D * f_j * q_j\big)\eta, \eta \Big\rangle_{\mathcal{H}_\pi} \;=\; \langle d\pi(D)\xi, \xi\rangle_{\mathcal{H}_\pi}.
\end{aligned}$$

□

3 The C^*-algebras of connected real two-step nilpotent Lie groups

In this section, the connected real two-step nilpotent Lie groups will be examined. In its first part, some preliminaries about two-step nilpotent Lie groups will be given which are needed in order to understand the setting and to prepare the proof of the norm controlled dual limit conditions listed in Definition 1.1. In Section 3.2, the Conditions 1, 2 and 3(a) will be verified and in Section 3.3, Condition 3(b) will be examined. Its proof is divided into three cases which are treated in the three following subsections. Subsequently, a result describing the C^*-algebras of connected real two-step nilpotent Lie groups will be given. After having finished the characterization of general connected real two-step nilpotent Lie groups, the example of the free two-step nilpotent Lie groups will be regarded.

The proof of Condition 1 of Definition 1.1 consists in the construction of the different layers of the spectrum of G and is based on a method of construction of a polarization developed in [27]. It is rather technical but not very long. Condition 2 follows directly from a result in [8] and Condition 3(a) is due to the construction of the above-mentioned polarization. The main work of the examination of the C^*-algebras of two-step nilpotent Lie groups consists in the proof of Property 3(b) of Definition 1.1 and in particular in the construction of the mappings $(\tilde{\nu}_k)_{k\in\mathbb{N}}$.

3.1 Preliminaries

3.1.1 Two-step nilpotent Lie groups

Let $\mathfrak{g}$ be a real Lie algebra which is nilpotent of step two. This means that

$$[\mathfrak{g},\mathfrak{g}] := \operatorname{span}\big\{[X,Y] \,|\, X,Y\in\mathfrak{g}\big\}$$

is contained in the center of $\mathfrak{g}$.
Fix a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ like for general nilpotent Lie algebras. The Campbell-Baker-Hausdorff multiplication is then of the form

$$u\cdot v = u+v+\frac{1}{2}[u,v] \quad \forall u,v\in\mathfrak{g}$$

and one gets again the simply connected connected Lie group $G=(\mathfrak{g},\cdot)$ with Lie algebra $\mathfrak{g}$.

As $\mathfrak{g}$ is two-step nilpotent, $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}(\ell)$ and thus $\mathfrak{g}(\ell)$ is an ideal of $\mathfrak{g}$.
Furthermore, for $\ell\in\mathfrak{g}^*$, every maximal isotropic subspace $\mathfrak{p}$ of $\mathfrak{g}$ for B_ℓ containing $[\mathfrak{g},\mathfrak{g}]$ is a polarization of ℓ.

3.1.2 Induced representations

The induced representation $\sigma_{\ell,\mathfrak{p}}=\operatorname{ind}_P^G\chi_\ell$ for a polarization $\mathfrak{p}$ of ℓ and $P:=\exp(\mathfrak{p})$ can be described in the following way:
Since $\mathfrak{p}$ contains $[\mathfrak{g},\mathfrak{g}]$ and even the center $\mathfrak{z}$ of $\mathfrak{g}$, one can write $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{p}$ and $\mathfrak{p}=\mathfrak{t}\oplus\mathfrak{z}$ for two subspaces $\mathfrak{t}$ and $\mathfrak{s}$ of $\mathfrak{g}$. The quotient space G/P is then homeomorphic to $\mathfrak{s}$ and the Lebesgue measure ds on $\mathfrak{s}$ defines an invariant Borel measure $d\dot{g}$ on G/P. The group G acts by the left translation $\sigma_{\ell,\mathfrak{p}}$ on the Hilbert space $L^2(G/P,\chi_\ell)$.

Now, if one uses the coordinates $G = \mathfrak{s} \cdot \mathfrak{p}$, one can identify the Hilbert spaces $L^2(G/P, \chi_\ell)$ and $L^2(\mathfrak{s}) = L^2(\mathfrak{s}, ds)$:
Let $U_\ell : L^2(\mathfrak{s}, ds) \to L^2(G/P, \chi_\ell)$ be defined by

$$U_\ell(\varphi)(S \cdot Y) := \chi_\ell(-Y)\varphi(S) \quad \forall Y \in \mathfrak{p} \; \forall S \in \mathfrak{s} \; \forall \varphi \in L^2(\mathfrak{s}).$$

Then, U_ℓ is a unitary operator and one can transform the representation $\sigma_{\ell,\mathfrak{p}}$ into a representation $\pi_{\ell,\mathfrak{p}}$ on the space $L^2(\mathfrak{s})$:

$$\pi_{\ell,\mathfrak{p}} := U_\ell^* \circ \sigma_{\ell,\mathfrak{p}} \circ U_\ell. \tag{1}$$

Furthermore, one can express the representation $\sigma_{\ell,\mathfrak{p}}$ in the following way:

$$\begin{aligned}
\sigma_{\ell,\mathfrak{p}}(S \cdot Y)\xi(R) &= \xi\big(Y^{-1}S^{-1}R\big) \\
&= \xi\Big((R-S) \cdot \Big(-Y + \frac{1}{2}[R,S] - \frac{1}{2}[R-S,Y]\Big)\Big) \\
&= e^{2\pi i \langle \ell, -Y + \frac{1}{2}[R,S] - \frac{1}{2}[R-S,Y]\rangle}\xi(R-S) \quad \forall R,S \in \mathfrak{s} \; \forall Y \in \mathfrak{p} \; \forall \xi \in L^2(G/P, \chi_\ell)
\end{aligned}$$

and hence,

$$\pi_{\ell,\mathfrak{p}}(S \cdot Y)\varphi(R) = e^{2\pi i \langle \ell, -Y + \frac{1}{2}[R,S] - \frac{1}{2}[R-S,Y]\rangle}\varphi(R-S) \quad \forall R,S \in \mathfrak{s} \; \forall Y \in \mathfrak{p} \; \forall \varphi \in L^2(\mathfrak{s}). \tag{2}$$

3.1.3 Orbit method

In the case of two-step nilpotent Lie groups, for every $\ell \in \mathfrak{g}^*$ and for every $g \in G = (\mathfrak{g}, \cdot)$, the element $\operatorname{Ad}^*(g)\ell$ can be computed to be

$$\operatorname{Ad}^*(g)\ell = \big(\mathbb{I}_{\mathfrak{g}^*} + \operatorname{ad}^*(g)\big)\ell \in \ell + \mathfrak{g}(\ell)^\perp.$$

Therefore, as $\operatorname{ad}^*(\mathfrak{g})\ell = \mathfrak{g}(\ell)^\perp$,

$$\mathcal{O}_\ell := \operatorname{Ad}^*(G)\ell = \{\operatorname{Ad}^*(g)\ell \mid g \in G\} = \ell + \mathfrak{g}(\ell)^\perp \quad \forall \ell \in \mathfrak{g}^*, \tag{3}$$

i.e. only flat orbits appear. Hence,

$$\mathfrak{g}^*/G = \{\operatorname{Ad}^*(G)\ell \mid \ell \in \mathfrak{g}^*\} = \big\{\ell + \mathfrak{g}(\ell)^\perp \mid \ell \in \mathfrak{g}^*\big\}.$$

Thus, by the Kirillov theory, one gets

$$\widehat{G} \cong \mathfrak{g}^*/G = \big\{\ell + \mathfrak{g}(\ell)^\perp \mid \ell \in \mathfrak{g}^*\big\}$$

as topological spaces.

Now, let $\big([\pi_k]\big)_{k\in\mathbb{N}} \subset \widehat{G}$ be a properly converging sequence in $\widehat{G}$ with limit set $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$. Let $\mathcal{O} \in \mathfrak{g}^*/G$ be the coadjoint orbit of some $[\pi] \in L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$, $\mathcal{O}_k$ the coadjoint orbit of $[\pi_k]$ for every $k \in \mathbb{N}$ and let $\ell \in \mathcal{O}$. Then, by Lemma 2.7, there exists for every $k \in \mathbb{N}$ an element $\ell_k \in \mathcal{O}_k$, such that $\lim\limits_{k\to\infty} \ell_k = \ell$ in $\mathfrak{g}^*$. One can assume that, passing to a subsequence if necessary, the sequence $(\mathfrak{g}(\ell_k))_{k\in\mathbb{N}}$ converges in the topology of the Grassmannian to a subalgebra $\mathfrak{u}$ of $\mathfrak{g}(\ell)$ and that there exists a number $d \in \mathbb{N}$, such that $\dim(\mathcal{O}_k) = 2d$ for every $k \in \mathbb{N}$. Then, it follows from (3) that

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} \ell_k + \mathfrak{g}(\ell_k)^{\perp} = \ell + \mathfrak{u}^{\perp} = \mathcal{O}_\ell + \mathfrak{u}^{\perp} \subset \mathfrak{g}^*/G. \tag{4}$$

Since in the two-step nilpotent case $\mathfrak{g}(\ell_k)$ contains $[\mathfrak{g},\mathfrak{g}]$ for every $k \in \mathbb{N}$, the subspace $\mathfrak{u}$ also contains $[\mathfrak{g},\mathfrak{g}]$. Hence, as the Kirillov mapping is a homeomorphism and $\mathfrak{u}^{\perp}$ consists only of characters of $\mathfrak{g}$, the limit set $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$ of the sequence $\big([\pi_k]\big)_{k\in\mathbb{N}}$ in $\widehat{G}$ is the "affine" subset

$$L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big) = \big\{\big[\chi_q \otimes \mathrm{ind}_P^G \chi_\ell\big] \mid q \in \mathfrak{u}^{\perp}\big\}$$

for a polarization $\mathfrak{p}$ of ℓ and $P := \exp(\mathfrak{p})$.

The observations above lead to the following proposition.

Proposition 3.1.
There are three different types of possible limit sets of the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ of coadjoint orbits:

1. *dim$(\mathcal{O}_\ell) = 2d$: In this case, the limit set $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ is the singleton $\mathcal{O}_\ell = \ell + \mathfrak{g}(\ell)^{\perp}$, i.e. $\mathfrak{u} = \mathfrak{g}(\ell)$.*
2. *dim$(\mathcal{O}_\ell) = 0$: Here, $L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \ell + \mathfrak{u}^{\perp}$ and the limit set $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$ consists only of characters, i.e. $q\big([\mathfrak{g},\mathfrak{g}]\big) = \{0\}$ for all $\mathcal{O}_q \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$.*
3. *The dimension of the orbit $\mathcal{O}_\ell$ is strictly larger than 0 and strictly smaller than $2d$. In this case, $0 <$ dim$(\mathcal{O}) < 2d$ for every $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$ and*

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \bigcup_{\mathfrak{q}\in\mathfrak{u}^{\perp}} q + \mathcal{O}_\ell, \quad \text{i.e.} \quad L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big) = \bigcup_{q\in\mathfrak{u}^{\perp}} \big[\chi_q \otimes \mathrm{ind}_P^G \chi_\ell\big]$$

for a polarization $\mathfrak{p}$ of ℓ and $P :=$ exp$(\mathfrak{p})$.

3.2 Conditions 1, 2 and 3(a)

Now, to start with the proof of the conditions listed in Definition 1.1, the families of sets $(S_i)_{i\in\{0,\dots,r\}}$ and $(\Gamma_i)_{i\in\{0,\dots,r\}}$ are going to be defined and the Properties 1, 2 and 3(a) of Definition 1.1 are going to be verified.
In order to be able to define the required families $(\Gamma_i)_{i\in\{0,\dots,r\}}$ and $(S_i)_{i\in\{0,\dots,r\}}$, one needs to construct a polarization $\mathfrak{p}_\ell^V$ of $\ell \in \mathfrak{g}^*$ in the following way:

Let $\{H_1, ..., H_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$, in such a way that $\mathfrak{g}_i := \mathrm{span}\{H_i, ..., H_n\}$ for $i \in \{0, ..., n\}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is two-step nilpotent, one can first choose a basis $\{H_{\tilde{n}}, \cdots, H_n\}$ of $[\mathfrak{g},\mathfrak{g}]$ and then add the vectors $H_1, \cdots, H_{\tilde{n}-1}$ to obtain a basis of $\mathfrak{g}$. Let

$$I_\ell^{Puk} := \{i \le n \mid \mathfrak{g}(\ell) \cap \mathfrak{g}_i = \mathfrak{g}(\ell) \cap \mathfrak{g}_{i+1}\}$$

be the Pukanszky index set for $\ell \in \mathfrak{g}^*$. The number of elements $\big|I_\ell^{Puk}\big|$ of I_ℓ^{Puk} is the dimension of the orbit $\mathcal{O}_\ell$ of ℓ.
Moreover, if one denotes by $\mathfrak{g}_i(\ell_{|\mathfrak{g}_i})$ the stabilizer of $\ell_{|\mathfrak{g}_i}$ in $\mathfrak{g}_i$,

$$\mathfrak{p}_\ell^V := \sum_{i=1}^{n} \mathfrak{g}_i(\ell_{|\mathfrak{g}_i})$$

is the Vergne polarization of ℓ in $\mathfrak{g}$. Its construction will now be analyzed by a method developed in [27], Section 1.

So, let $\ell \in \mathfrak{g}^*$. Then, choose the largest index $j_1(\ell) \in \{1, ..., n\}$ such that $H_{j_1(\ell)} \notin \mathfrak{g}(\ell)$ and let $Y_1^{V,\ell} := H_{j_1(\ell)}$. Moreover, choose the index $k_1(\ell) \in \{1, ..., n\}$ such that $\langle \ell, [H_{k_1(\ell)}, H_{j_1(\ell)}]\rangle \neq 0$ and $\langle \ell, [H_i, H_{j_1(\ell)}]\rangle = 0$ for all $i > k_1(\ell)$ and let $X_1^{V,\ell} := H_{k_1(\ell)}$.

Next, let $\mathfrak{g}^{1,\ell} := \{U \in \mathfrak{g} | \ \langle \ell, [U, Y_1^{V,\ell}]\rangle = 0\}$. Then, $\mathfrak{g}^{1,\ell}$ is an ideal in $\mathfrak{g}$ which does not contain $X_1^{V,\ell}$, and $\mathfrak{g} = \mathbb{R}X_1^{V,\ell} \oplus \mathfrak{g}^{1,\ell}$. Now, the Jordan-Hölder basis will be changed, taking out $H_{k_1(\ell)}$. Consider the Jordan-Hölder basis $\{H_1^{1,\ell}, ..., H_{k_1(\ell)-1}^{1,\ell}, H_{k_1(\ell)+1}^{1,\ell}, ..., H_n^{1,\ell}\}$ of $\mathfrak{g}^{1,\ell}$, where

$$H_i^{1,\ell} := H_i \quad \forall i > k_1(\ell) \quad \text{and} \quad H_i^{1,\ell} := H_i - \frac{\langle \ell, [H_i, Y_1^{V,\ell}]\rangle X_1^{V,\ell}}{\langle \ell, [X_1^{V,\ell}, Y_1^{V,\ell}]\rangle} \quad \forall i < k_1(\ell).$$

Then, choose the largest index $j_2(\ell) \in \{1, ..., k_1(\ell) - 1, k_1(\ell) + 1, ..., n\}$ in such a way that $H_{j_2(\ell)}^{1,\ell} \notin \mathfrak{g}^{1,\ell}(\ell_{|\mathfrak{g}^{1,\ell}})$ and let $Y_2^{V,\ell} := H_{j_2(\ell)}^{1,\ell}$. Again, choose $k_2(\ell) \in \{1, ..., k_1(\ell)-1, k_1(\ell)+1, ..., n\}$ in such a way that $\langle \ell, [H_{k_2(\ell)}^{1,\ell}, H_{j_2(\ell)}^{1,\ell}]\rangle \neq 0$ and $\langle \ell, [H_i^{1,\ell}, H_{j_2(\ell)}^{1,\ell}]\rangle = 0$ for all $i > k_2(\ell)$ and set $X_2^{V,\ell} := H_{k_2(\ell)}^{1,\ell}$.

Iterating this procedure, one gets sets $\{Y_1^{V,\ell}, ..., Y_d^{V,\ell}\}$ and $\{X_1^{V,\ell}, ..., X_d^{V,\ell}\}$ for $d \in \{0, ..., [\frac{n}{2}]\}$ with the properties

$$\mathfrak{p}_\ell^V = \text{span}\{Y_1^{V,\ell}, ..., Y_d^{V,\ell}\} \oplus \mathfrak{g}(\ell)$$

and

$$\langle \ell, [X_i^{V,\ell}, Y_i^{V,\ell}]\rangle \neq 0, \quad \langle \ell, [X_i^{V,\ell}, Y_j^{V,\ell}]\rangle = 0 \quad \forall i \neq j \in \{1, ..., d\} \quad \text{and}$$
$$\langle \ell, [Y_i^{V,\ell}, Y_j^{V,\ell}]\rangle = 0 \quad \forall i, j \in \{1, ..., d\}.$$

Now, let

$$J(\ell) := \{j_1(\ell), \cdots, j_d(\ell)\} \quad \text{and} \quad K(\ell) := \{k_1(\ell), \cdots, k_d(\ell)\}.$$

Then,

$$I_\ell^{Puk} = J(\ell) \ \dot{\cup} \ K(\ell) \quad \text{and} \quad j_1(\ell) > ... > j_d(\ell).$$

The index sets I_ℓ^{Puk}, $J(\ell)$ and $K(\ell)$ are the same on every coadjoint orbit (see [27]) and can therefore also be denoted by $I_\mathcal{O}^{Puk}$, $J(\mathcal{O})$ and $K(\mathcal{O})$ if ℓ is located in the coadjoint orbit $\mathcal{O}$.

Now, for the parametrization of $\mathfrak{g}^*/G$ and thus of $\widehat{G}$ and for the choice of the concrete realization of a representation required in Property 1(b) of Definition 1.1, let $\mathcal{O} \in \mathfrak{g}^*/G$. A theorem of L.Pukanszky (see [29], Part II, Chapter I.3 or [27], Corollary 1.2.5) states that there exists one unique $\ell_\mathcal{O} \in \mathcal{O}$ such that $\ell_\mathcal{O}(H_i) = 0$ for all $i \in I_\mathcal{O}^{Puk}$. Choose this $\ell_\mathcal{O}$, let $P_{\ell_\mathcal{O}}^V := \exp(\mathfrak{p}_{\ell_\mathcal{O}}^V)$ and define the irreducible unitary representation

$$\pi_{\ell_\mathcal{O}}^V := \text{ind}_{P_{\ell_\mathcal{O}}^V}^G \chi_{\ell_\mathcal{O}}$$

associated to the orbit $\mathcal{O}$ and acting on $L^2(G/P_{\ell_\mathcal{O}}^V, \chi_{\ell_\mathcal{O}}) \cong L^2(\mathbb{R}^d)$.

Next, one has to construct the demanded sets Γ_i for $i \in \{0, ..., r\}$.
For this, define for a pair of sets (J, K) such that $J, K \subset \{1, \cdots, n\}$, $|J| = |K|$ and $J \cap K = \emptyset$ the subset $(\mathfrak{g}^*/G)_{(J,K)}$ of $\mathfrak{g}^*/G$ by

$$(\mathfrak{g}^*/G)_{(J,K)} := \{\mathcal{O} \in \mathfrak{g}^*/G |\ (J, K) = (J(\mathcal{O}), K(\mathcal{O}))\}.$$

Moreover, let

$$\mathcal{M} := \{(J, K)|\ J, K \subset \{1, ..., n\},\ J \cap K = \emptyset,\ |J| = |K|,\ (\mathfrak{g}^*/G)_{(J,K)} \neq \emptyset\}$$

and

$$(\mathfrak{g}^*/G)_{2d} := \Big\{\mathcal{O} \in \mathfrak{g}^*/G \big|\ \big|I^{Puk}_{\mathcal{O}}\big| = 2d\Big\}.$$

Then,

$$(\mathfrak{g}^*/G)_{2d} = \bigcup_{\substack{(J,K):\ J,K\subset\{1,...,n\},\\ |J|=|K|=d,\ J\cap K=\emptyset}} (\mathfrak{g}^*/G)_{(J,K)}$$

and

$$\mathfrak{g}^*/G = \bigcup_{d\in\{0,...,[\frac{n}{2}]\}} (\mathfrak{g}^*/G)_{2d} = \bigcup_{(J,K)\in\mathcal{M}} (\mathfrak{g}^*/G)_{(J,K)}.$$

Now, an order on the set $\mathcal{M}$ shall be introduced.
First, if $|J| = |K| = d$, $|J'| = |K'| = d'$ and $d < d'$, then the pair (J, K) is defined to be smaller than the pair (J', K'): $(J, K) < (J', K')$.
If $|J| = |K| = |J'| = |K'| = d$, $J = \{j_1, ..., j_d\}$, $J' = \{j'_1, ..., j'_d\}$ and $j_1 < j'_1$, the pair (J, K) is again defined to be smaller than (J', K').
Otherwise, if $j_1 = j'_1$, one has to consider the sets $K = \{k_1, ..., k_d\}$ and $K' = \{k'_1, ..., k'_d\}$ and here again, compare the first elements k_1 and k'_1. So, if $j_1 = j'_1$ and $k_1 < k'_1$, again $(J, K) < (J', K')$. But if $k_1 = k'_1$, one compares j_2 and j'_2 and continues in that way.
If $r \in \mathbb{N}$ with $r + 1 = |\mathcal{M}|$, one can identify the ordered set $\mathcal{M}$ with the set $\{0, \cdots, r\}$ and assign to each pair $(J, K) \in \mathcal{M}$ a number $i_{JK} \in \{0, ..., r\}$.
Finally, one can therefore define the sets $\Gamma_{i_{JK}}$ and $S_{i_{JK}}$ as

$$\Gamma_{i_{JK}} := \Big\{\big[\pi^V_{\ell_{\mathcal{O}}}\big] \big|\ \mathcal{O} \in (\mathfrak{g}^*/G)_{(J,K)}\Big\} \quad \text{and}$$

$$S_{i_{JK}} := \bigcup_{i\in\{0,...,i_{JK}\}} \Gamma_i.$$

Then, obviously, the family $(S_i)_{i\in\{0,...,r\}}$ is an increasing family in $\widehat{G}$.
Furthermore, the set S_i is closed for every $i \in \{0, ..., r\}$. This can easily be deduced from the definition of the index sets $J(\ell)$ and $K(\ell)$. The indices $j_m(\ell)$ and $k_m(\ell)$ for $m \in \{1, ..., d\}$ are chosen such that they are the largest to fulfill a condition of the type $\big\langle \ell, [H^{m-1,\ell}_{j_m(\ell)}, \cdot]\big\rangle \neq 0$ or $\big\langle \ell, [H^{m-1,\ell}_{k_m(\ell)}, \cdot]\big\rangle \neq 0$, respectively. For a more detailed proof of the closure of the sets $(S_i)_{i\in\{0,...,r\}}$, see Lemma 7.1.
In addition, the sets Γ_i are Hausdorff. For this, let $i = i_{JK}$ for $(J, K) \in \mathcal{M}$ and $(\mathcal{O}_k)_{k\in\mathbb{N}}$ in $(\mathfrak{g}^*/G)_{(J,K)}$ be a sequence of orbits such that the sequence $\big([\pi^V_{\ell_{\mathcal{O}_k}}]\big)_{k\in\mathbb{N}}$ converges in Γ_i, i.e. $(\mathcal{O}_k)_{k\in\mathbb{N}}$ converges in $(\mathfrak{g}^*/G)_{(J,K)}$ and thus has a limit point $\mathcal{O}$ in $(\mathfrak{g}^*/G)_{(J,K)}$. If now $\mathcal{O}_k \ni \ell_k \overset{k\to\infty}{\longrightarrow} \ell \in \mathcal{O}$, then by (4), it follows that the limit $\mathfrak{u}$ of the sequence $(\mathfrak{g}(\ell_k))_{k\in\mathbb{N}}$ is equal

to $\mathfrak{g}(\ell)$. Therefore, the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ and thus also the sequence $\big([\pi^V_{\ell_{\mathcal{O}_k}}]\big)_{k\in\mathbb{N}}$ have unique limits and hence, Γ_i is Hausdorff.
Moreover, one can still observe that for $d = 0$ the choice $J = K = \emptyset$ represents the only possibility to get $|J| = |K| = d$. So, the pair $(\emptyset, \emptyset)$ is the first element in the above defined order and therefore corresponds to 0. Thus,

$$\Gamma_0 = \Big\{ [\pi^V_{\ell_{\mathcal{O}}}] \,\big|\, I^{Puk}_{\mathcal{O}} = \emptyset \Big\},$$

which is equivalent to the fact that $\mathfrak{g}(\ell_{\mathcal{O}}) = \mathfrak{g}$ which again is equivalent to the fact that every $\pi^V_{\ell_{\mathcal{O}}} \in \Gamma_0$ is a character. Hence, $S_0 = \Gamma_0$ is the set of all characters of G, as demanded.

Since one can identify $G/P^V_{\ell_{\mathcal{O}}}$ with $\mathbb{R}^d$ by means of the subspace $\mathfrak{s}_{\ell_{\mathcal{O}}} = \text{span}\{X^{V,\ell_{\mathcal{O}}}_1, ..., X^{V,\ell_{\mathcal{O}}}_d\}$, one can thus identify the Hilbert space $L^2\big(G/P^V_{\ell_{\mathcal{O}}}, \chi_{\ell_{\mathcal{O}}}\big)$, that the representation $\pi^V_{\ell_{\mathcal{O}}}$ acts on, with the Hilbert space $L^2(\mathbb{R}^d)$ for every $\mathcal{O} \in (\mathfrak{g}^*/G)_{2d}$ as in (1). Therefore, one can take for all $J, K \in \{1, ..., n\}$ with $|J| = |K| = d$ the Hilbert space $L^2(\mathbb{R}^d)$ as the desired Hilbert space $\mathcal{H}_{i_{JK}}$ and $\big(\pi^V_{\ell_{\mathcal{O}}}, L^2(\mathbb{R}^d)\big)$ thus represents the demanded concrete realization.

Hence, the first condition of Definition 1.1 is fulfilled. For the proof of the Properties 2 and 3(a), a proposition will be shown.

Proposition 3.2.
For every $a \in C^*(G)$ and every $(J, K) \in \mathcal{M}$ with $|J| = |K| = d \in \{0, ..., [\frac{n}{2}]\}$, the mapping

$$\Gamma_{i_{JK}} \to L^2(\mathbb{R}^d),\ \gamma \mapsto \mathcal{F}(a)(\gamma)$$

is norm continuous and the operator $\mathcal{F}(a)(\gamma)$ is compact for all $\gamma \in \Gamma_{i_{JK}}$.

Proof:
The compactness follows directly from a general theorem which can be found in [8], Chapter 4.2 or [29], Part II, Chapter II.5 and states that the C^*-algebra $C^*(G)$ of every connected nilpotent Lie group G fulfills the CCR condition, i.e. the image of every irreducible representation of $C^*(G)$ is a compact operator.

Next, let $d \in \{0, ..., [\frac{n}{2}]\}$ and $(J, K) \in \mathcal{M}$ such that $|J| = |K| = d$.
First, one has to observe that the polarization $\mathfrak{p}^V_\ell$ is continuous in ℓ on the set $\{\ell_{\mathcal{O}'} |\ \mathcal{O}' \in (\mathfrak{g}^*/G)_{(J,K)}\}$ with respect to the topology of the Grassmannian. This can be seen by the construction of the vectors $\{Y^{V,\ell}_1, ..., Y^{V,\ell}_d\}$.
Now, let $(\mathcal{O}_k)_{k\in\mathbb{N}}$ be a sequence of orbits in $(\mathfrak{g}^*/G)_{(J,K)}$ and $\mathcal{O} \in (\mathfrak{g}^*/G)_{(J,K)}$ such that $[\pi^V_{\ell_{\mathcal{O}_k}}] \overset{k\to\infty}{\longrightarrow} [\pi^V_{\ell_{\mathcal{O}}}]$ and let $a \in C^*(G)$. Then, $\ell_{\mathcal{O}_k} \overset{k\to\infty}{\longrightarrow} \ell_{\mathcal{O}}$ and by the observation above, the associated sequence of polarizations $\big(\mathfrak{p}^V_{\ell_{\mathcal{O}_k}}\big)_{k\in\mathbb{N}}$ converges to the polarization $\mathfrak{p}^V_{\ell_{\mathcal{O}}}$. By the proof of Theorem 2.3 in [31], thus, $\pi^V_{\ell_{\mathcal{O}_k}}(a) \overset{k\to\infty}{\longrightarrow} \pi^V_{\ell_{\mathcal{O}}}(a)$ in the operator norm.

□

Since $C^*(G)$ is obviously separable, this proposition proves the desired Properties 2 and 3(a) of Definition 1.1 and hence, it remains to show Property 3(b).

3.3 Condition 3(b)

3.3.1 Introduction to the setting

For simplicity, in the following, the representations will be identified with their equivalence classes.

Let $d \in \{0, ..., [\frac{n}{2}]\}$ and $(J, K) \in \mathcal{M}$ with $|J| = |K| = d$. Furthermore, fix $i = i_{JK} \in \{0, ..., r\}$.

Let $\left(\pi_k^V\right)_{k\in\mathbb{N}} = \left(\pi_{\ell_{\mathcal{O}_k}}^V\right)_{k\in\mathbb{N}}$ be a sequence in Γ_i whose limit set is located outside Γ_i. Since every converging sequence has a properly converging subsequence, it will be assumed that $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ is properly converging and the transition to a subsequence will be omitted. Hence, the sequence $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ takes on the role of the subsequence $(\gamma_k)_{k\in\mathbb{N}}$ in Condition 3(b) of Definition 1.1. Therefore, for every $k \in \mathbb{N}$, one has to construct an involutive linear mapping $\tilde{\nu}_k : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$, bounded by $C\|\cdot\|_{S_{i-1}}$ and fulfilling

$$\lim_{k\to\infty} \left\|\mathcal{F}(a)\left(\pi_k^V\right) - \tilde{\nu}_k\left(\mathcal{F}(a)_{|S_{i-1}}\right)\right\|_{op} = 0,$$

with a constant $C > 0$ independent of k.
In order to do so, regard the sequence of coadjoint orbits $(\mathcal{O}_k)_{k\in\mathbb{N}}$ corresponding to the sequence $\left(\pi_k^V\right)_{k\in\mathbb{N}}$. It is contained in $(\mathfrak{g}^*/G)_{(J,K)}$ and in particular, every $\mathcal{O}_k$ has the same dimension $2d$. Moreover, it converges properly to a set of orbits $L((\mathcal{O}_k)_{k\in\mathbb{N}})$.
In addition, since S_i is closed, the limit set $L\left(\left(\pi_k^V\right)_{k\in\mathbb{N}}\right)$ of the sequence $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ is contained in $S_{i-1} = \bigcup_{j\in\{0,\ldots,i-1\}} \Gamma_j$ and therefore, for every element $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$ there exists a pair $(J_{\mathcal{O}}, K_{\mathcal{O}}) < (J, K)$ such that $\pi_{\ell_{\mathcal{O}}}^V \in \Gamma_{i_{J_{\mathcal{O}}K_{\mathcal{O}}}}$ or equivalently, $\mathcal{O} \in (\mathfrak{g}^*/G)_{(J_{\mathcal{O}},K_{\mathcal{O}})}$.

Now, the Lie algebra $\mathfrak{g}$ will be examined and divided into different parts. With their help, a new sequence of representations $(\pi_k)_{k\in\mathbb{N}}$ which are equivalent to the representations $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ will be defined. Then, in the second and third case mentioned in Proposition 3.1, $(\pi_k)_{k\in\mathbb{N}}$ will be analyzed and for this new sequence, mappings $(\nu_k)_{k\in\mathbb{N}}$ with certain properties similar to the ones required in Condition 3(b) of Definition 1.1 will be constructed. At the end, the requested convergence will be deduced from the convergence of $(\pi_k)_{k\in\mathbb{N}}$ together with the equivalence of the representations π_k and π_k^V.

3.3.2 Changing the Jordan-Hölder basis

Let $\tilde{\ell} \in \tilde{\mathcal{O}} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$. Then, there exists a sequence $\left(\tilde{\ell}_k\right)_{k\in\mathbb{N}}$ in $(\mathcal{O}_k)_{k\in\mathbb{N}}$ such that $\tilde{\ell} = \lim_{k\to\infty} \tilde{\ell}_k$.
Since one is interested in the orbits $\mathcal{O}_k = \tilde{\ell}_k + \mathfrak{g}\left(\tilde{\ell}_k\right)^{\perp}$, one can change the sequence $\left(\tilde{\ell}_k\right)_{k\in\mathbb{N}}$ to a sequence $(\ell_k)_{k\in\mathbb{N}}$ by letting $\ell_k(A) = 0$ for every $A \in \mathfrak{g}\left(\tilde{\ell}_k\right)^{\perp} = \mathfrak{g}(\ell_k)^{\perp}$.
Thus, one obtains another converging sequence $(\ell_k)_{k\in\mathbb{N}}$ in $(\mathcal{O}_k)_{k\in\mathbb{N}}$ whose limit ℓ is located in an orbit $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$.
In the following, this sequence $(\ell_k)_{k\in\mathbb{N}}$ will be investigated and with its help, the above-mentioned splitting of $\mathfrak{g}$ will be accomplished.

As above, one can suppose that the subalgebras $(\mathfrak{g}(\ell_k))_{k\in\mathbb{N}}$ converge to a subalgebra $\mathfrak{u}$, whose corresponding Lie group $\exp(\mathfrak{u})$ is denoted by U. These subalgebras $\mathfrak{g}(\ell_k)$ for $k \in \mathbb{N}$ can be written as

$$\mathfrak{g}(\ell_k) = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{s}_k,$$

where $\mathfrak{s}_k \subset [\mathfrak{g}, \mathfrak{g}]^\perp$. In addition, let $\mathfrak{n}_{k,0}$ be the kernel of $\ell_{k|[\mathfrak{g},\mathfrak{g}]}$ and $\mathfrak{s}_{k,0}$ the kernel of $\ell_{k|\mathfrak{s}_k}$ for all $k \in \mathbb{N}$. One can assume that $\mathfrak{s}_{k,0} \neq \mathfrak{s}_k$ and choose $T_k \in \mathfrak{s}_k$ of length 1 orthogonal to $\mathfrak{s}_{k,0}$. The case $\mathfrak{s}_{k,0} = \mathfrak{s}_k$ for $k \in \mathbb{N}$, being easier, will be omitted.
Similarly, choose $Z_k \in [\mathfrak{g}, \mathfrak{g}]$ of length 1 orthogonal to $\mathfrak{n}_{k,0}$. One can easily see that such a Z_k exists: If $\ell_{k|[\mathfrak{g},\mathfrak{g}]} = 0$, then $\pi^V_{\ell_{\mathcal{O}_k}}$ is a character and thus contained in $S_0 = \Gamma_0$. This happens only in the case $i = 0$ and then the whole sequence $\big(\pi^V_{\ell_{\mathcal{O}_k}}\big)_{k\in\mathbb{N}}$ is contained in S_0. But $S_0 = \Gamma_0$ is closed and thus, $\big(\pi^V_{\ell_{\mathcal{O}_k}}\big)_{k\in\mathbb{N}}$ cannot have a limit set outside Γ_0, as regarded in the setting of Condition 3(b) of Definition 1.1.

Furthermore, let $\mathfrak{r}_k = \mathfrak{g}(\ell_k)^\perp \subset \mathfrak{g}$.
One can assume that, passing to a subsequence if necessary, $\lim\limits_{k\to\infty} Z_k =: Z$, $\lim\limits_{k\to\infty} T_k =: T$ and $\lim\limits_{k\to\infty} \mathfrak{r}_k =: \mathfrak{r}$ exist.

Now, new polarizations $\mathfrak{p}_k$ of ℓ_k are going to be constructed in order to define the represenations $(\pi_k)_{k\in\mathbb{N}}$ with their help.
The restriction to $\mathfrak{r}_k$ of the skew-form B_k defined by

$$B_k(V, W) := \langle \ell_k, [V, W]\rangle, \quad \forall V, W \in \mathfrak{g}$$

is non-degenerate on $\mathfrak{r}_k$ and there exists an invertible endomorphism S_k of $\mathfrak{r}_k$ such that

$$\langle x, S_k(x')\rangle = B_k(x, x') \quad \forall x, x' \in \mathfrak{r}_k.$$

S_k is skew-symmetric, i.e. $S_k^t = -S_k$, and with the help of Lemma 7.2, one can decompose $\mathfrak{r}_k$ into an orthogonal direct sum

$$\mathfrak{r}_k = \sum_{j=1}^{d} V_k^j$$

of two-dimensional S_k-invariant subspaces. Choose an orthonormal basis $\{X_j^k, Y_j^k\}$ of V_k^j. Then,

$$\begin{aligned}
&[X_i^k, X_j^k] \in \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, ..., d\}, \\
&[Y_i^k, Y_j^k] \in \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, ..., d\} \quad \text{and} \\
&[X_i^k, Y_j^k] = \delta_{i,j} c_j^k Z_k \bmod \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, ..., d\},
\end{aligned}$$

where $0 \neq c_j^k \in \mathbb{R}$ and $\sup\limits_{k\in\mathbb{N}} c_j^k < \infty$ for every $j \in \{1, ..., d\}$.
Again, by passing to a subsequence if necessary, the sequence $(c_j^k)_{k\in\mathbb{N}}$ converges for every $j \in \{1, ..., d\}$ to some $c_j \in \mathbb{R}$.
Since $X_j^k, Y_j^k \in \mathfrak{r}_k$ and $\ell_k(A) = 0$ for every $A \in \mathfrak{r}_k$, $\ell_k(X_j^k) = \ell_k(Y_j^k) = 0$ for all $j \in \{1, ..., d\}$.
Furthermore, one can suppose that the sequences $(X_j^k)_{k\in\mathbb{N}}, (Y_j^k)_{k\in\mathbb{N}}$ converge in $\mathfrak{g}$ to vectors X_j, Y_j which form a basis modulo $\mathfrak{u}$ in $\mathfrak{g}$.

It follows that

$$\begin{aligned} \langle \ell_k, [X_j^k, Y_j^k] \rangle &= c_j^k \lambda_k, \text{ where} \\ \lambda_k &= \langle \ell_k, Z_k \rangle \overset{k\to\infty}{\longrightarrow} \langle \ell, Z \rangle \ =: \lambda. \end{aligned}$$

As Z_k was chosen orthogonal to $\mathfrak{n}_{k,0}$, $\lambda_k \neq 0$ for every $k \in \mathbb{N}$.

Now, let

$$\mathfrak{p}_k := \text{span}\{Y_1^k, \cdots, Y_d^k, \mathfrak{g}(\ell_k)\}$$

and $P_k := \exp(\mathfrak{p}_k)$. Then, $\mathfrak{p}_k$ is a polarization of ℓ_k. Furthermore, define the representation π_k as

$$\pi_k := \text{ind}_{P_k}^G \chi_{\ell_k}.$$

Since π_k as well as π_k^V are induced representations of polarizations and of the characters χ_{ℓ_k} and $\chi_{\ell_{\mathcal{O}_k}}$, where ℓ_k and $\ell_{\mathcal{O}_k}$ lie in the same coadjoint orbit $\mathcal{O}_k$, the two representations are equivalent, as observed in Section 2.1.2.

Let $\mathfrak{a}_k := \mathfrak{n}_{k,0} + \mathfrak{s}_{k,0}$. Then, $\mathfrak{a}_k$ is an ideal of $\mathfrak{g}$ on which ℓ_k is 0. Therefore, the normal subgroup $\exp(\mathfrak{a}_k)$ is contained in the kernel of the representation π_k. Moreover, let $\mathfrak{a} := \lim\limits_{k\to\infty} \mathfrak{a}_k$.
In addition, let $p \in \mathbb{N}$, $\tilde{p} \in \{1, ..., p\}$ and let $\{A_1^k, \cdots, A_{\tilde{p}}^k\}$ denote an orthonormal basis of $\mathfrak{n}_{k,0}$, the part of $\mathfrak{a}_k$ which is located inside $[\mathfrak{g}, \mathfrak{g}]$, and $\{A_{\tilde{p}+1}^k, \cdots, A_p^k\}$ an orthonormal basis of $\mathfrak{s}_{k,0}$, the part of $\mathfrak{a}_k$ outside $[\mathfrak{g}, \mathfrak{g}]$. Then, $\{A_1^k, \cdots, A_p^k\}$ is an orthonormal basis of $\mathfrak{a}_k$ and as above, one can assume that $\lim\limits_{k\to\infty} A_j^k = A_j$ exists for all $j \in \{1, ..., p\}$.
Now, for every $k \in \mathbb{N}$, one can take as an orthonormal basis for $\mathfrak{g}$ the set of vectors

$$\{X_1^k, \cdots, X_d^k, Y_1^k, \cdots, Y_d^k, T_k, Z_k, A_1^k, ..., A_p^k\}$$

as well as the set

$$\{X_1, \cdots, X_d, Y_1, \cdots, Y_d, T, Z, A_1, ..., A_p\}.$$

This gives the following Lie brackets:

$$\begin{aligned} [X_i^k, Y_j^k] &= \delta_{i,j} c_j^k Z_k \text{ mod } \mathfrak{a}_k, \\ [X_i^k, X_j^k] &= 0 \ \text{ mod } \mathfrak{a}_k \quad \text{and} \\ [Y_i^k, Y_j^k] &= 0 \ \text{ mod } \mathfrak{a}_k. \end{aligned}$$

The vectors Z_k and T_k are central modulo $\mathfrak{a}_k$.

3.3.3 Definitions

Before starting the analysis of $(\pi_k)_{k\in\mathbb{N}}$, some notations have to be introduced.

Choose for $j \in \{1, ..., d\}$ the Schwartz functions $\eta_j \in \mathcal{S}(\mathbb{R})$ such that $\|\eta_j\|_{L^2(\mathbb{R})} = 1$ and $\|\eta_j\|_{L^\infty(\mathbb{R})} \leq 1$.
Furthermore, for $x_1, ..., x_d, y_1, ..., y_d, t, z, a_1, \cdots, a_p \in \mathbb{R}$, write

$$(x)_k := (x_1, ..., x_d)_k := \sum_{j=1}^d x_j X_j^k, \quad (y)_k \ := (y_1, ..., y_d)_k := \sum_{j=1}^d y_j Y_j^k, \quad (t)_k := tT_k,$$

$$(z)_k := zZ_k, \quad (\dot{a})_k := (a_1, ..., a_{\tilde{p}})_k := \sum_{j=1}^{\tilde{p}} a_j A_j^k, \quad (\ddot{a})_k := (a_{\tilde{p}+1}, ..., a_p)_k := \sum_{j=\tilde{p}+1}^{p} a_j A_j^k$$

$$\text{and} \quad (a)_k := (\dot{a}, \ddot{a})_k = (a_1, ..., a_p)_k = \sum_{j=1}^{p} a_j A_j^k,$$

where $(\cdot, ..., \cdot)_k$ is defined to be the d-, $\tilde{p}$-, $(p - \tilde{p})$- or the p-tuple with respect to the bases $\{X_1^k, ..., X_d^k\}$, $\{Y_1^k, ..., Y_d^k\}$, $\{A_1^k, ..., A_{\tilde{p}}^k\}$, $\{A_{\tilde{p}+1}^k, ..., A_p^k\}$ and $\{A_1^k, ..., A_p^k\}$, respectively, and let

$$\begin{aligned}
(g)_k &:= (x_1, ..., x_d, y_1, ..., y_d, t, z, a_1, ..., a_p)_k := ((x)_k, (y)_k, (t)_k, (z)_k, (\dot{a})_k, (\ddot{a})_k) \\
&= ((x)_k, (h)_k) \\
&= \sum_{j=1}^{d} x_j X_j^k \ + \ \sum_{j=1}^{d} y_j Y_j^k \ + \ tT_k \ + \ zZ_k \ + \ \sum_{j=1}^{p} a_j A_j^k,
\end{aligned}$$

where $(h)_k$ is in the polarization $\mathfrak{p}_k$ and the $(2d+2+p)$-tuple $(\cdot, ..., \cdot)_k$ is regarded with respect to the basis $\{X_1^k, ..., X_d^k, Y_1^k, ..., Y_d^k, T_k, Z_k, A_1^k, ..., A_p^k\}$.
Moreover, define the limits

$$(x)_\infty := (x_1, ..., x_d)_\infty := \lim_{k\to\infty} (x)_k = \sum_{j=1}^{d} x_j X_j, \quad (y)_\infty := (y_1, ..., y_d)_\infty := \lim_{k\to\infty} (y)_k = \sum_{j=1}^{d} y_j Y_j,$$

$$(t)_\infty := \lim_{k\to\infty} (t)_k = tT, \ (z)_\infty := \lim_{k\to\infty} (z)_k = zZ, \ (\dot{a})_\infty := (a_1, ..., a_{\tilde{p}})_\infty := \lim_{k\to\infty} (\dot{a})_k = \sum_{j=1}^{\tilde{p}} a_j A_j,$$

$$(\ddot{a})_\infty := (a_{\tilde{p}+1}, ..., a_p)_\infty := \lim_{k\to\infty} (\ddot{a})_k = \sum_{j=\tilde{p}+1}^{p} a_j A_j,$$

$$(a)_\infty := (\dot{a}, \ddot{a})_\infty = (a_1, ..., a_p)_\infty := \lim_{k\to\infty} (a)_k = \sum_{j=1}^{p} a_j A_j \quad \text{and}$$

$$(g)_\infty := (x, y, t, z, \dot{a}, \ddot{a})_\infty := \lim_{k\to\infty} (g)_k \ = \ \sum_{j=1}^{d} x_j X_j \ + \ \sum_{j=1}^{d} y_j Y_j \ + \ tT \ + \ zZ \ + \ \sum_{j=1}^{p} a_j A_j.$$

3.3.4 Formula for π_k

Now, the representations $(\pi_k)_{k\in\mathbb{N}}$ can be computed.
Let $f \in L^1(G)$.
With $\rho_k := \langle \ell_k, T_k \rangle$, $c^k := (c_1^k, ..., c_d^k)$ and for $s_1, ..., s_d \in \mathbb{R}$ and $(s)_k := (s_1, ..., s_d)_k = \sum_{j=1}^{d} s_j X_j^k$, where again $(\cdot, ..., \cdot)_k$ is the d-tuple with respect to the basis $\{X_1^k, ..., X_d^k\}$, as in (2), the representation π_k acts on $L^2(G/P_k, \chi_{\ell_k})$ in the following way:

$$\begin{aligned}
\pi_k\big((g)_k\big)\xi\big((s)_k\big) &= \xi\big((g)_k^{-1}\cdot(s)_k\big)\\
&= \xi\Big((s)_k-(x)_k-(h)_k+\frac{1}{2}[-(x)_k-(h)_k,(s)_k]\Big)\\
&= \xi\Big(\big((s)_k-(x)_k\big)\cdot\Big(-(h)_k+[(s)_k,(h)_k]-\frac{1}{2}[(x)_k,(h)_k+(s)_k]\Big)\Big)\\
&= e^{2\pi i\langle \ell_k,-(h)_k+[(s)_k,(h)_k]-\frac{1}{2}[(x)_k,(h)_k+(s)_k]\rangle}\xi\big((s-x)_k\big)\\
&= e^{2\pi i\langle \ell_k,-(y)_k-(t)_k-(z)_k-(\dot a)_k-(\ddot a)_k+[(s)_k-\frac{1}{2}(x)_k,(y)_k]-\frac{1}{2}[(x)_k,(s)_k]\rangle}\xi\big((s-x)_k\big)\\
&= e^{2\pi i\left(-t\rho_k-z\lambda_k+\sum\limits_{j=1}^{d}\lambda_k c_j^k(s_j-\frac{1}{2}x_j)y_j\right)}\xi\big((s-x)_k\big) \qquad (5)\\
&= e^{2\pi i(-t\rho_k-z\lambda_k+\lambda_k c^k((s)_k-\frac{1}{2}(x)_k)(y)_k)}\xi\big((s-x)_k\big),
\end{aligned}$$

since $\ell_k(Y_j^k)=0$ for all $j\in\{1,...,d\}$.

Now, identify G with $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde p}\times\mathbb{R}^{p-\tilde p}\cong\mathbb{R}^{2d+2+p}$, let $\xi\in L^2(\mathbb{R}^d)$ and $s\in\mathbb{R}^d$. Moreover, identify π_k with a representation acting on $L^2(\mathbb{R}^d)$ which will also be called π_k. To stress the dependence on k of the function $f\in L^1(G)$ fixed above, denote by $f_k\in L^1(\mathbb{R}^{2d+2+p})$ the function f applied to an element in the k-basis:

$$f_k(g):=f((g)_k)=f\left(\sum_{j=1}^{d}x_jX_j^k\;+\;\sum_{j=1}^{d}y_jY_j^k\;+\;tT_k\;+\;zZ_k\;+\;\sum_{j=1}^{p}a_jA_j^k\right).$$

Denote by $f'_{\alpha,k}$ for $f\in\mathcal{S}(G)$ and $\alpha\in\mathbb{N}^{2d+2+p}$ the derivative of f in the direction of $(X_1^k)^{\alpha_1}\cdots(X_d^k)^{\alpha_d}(Y_1^k)^{\alpha_{d+1}}\cdots(Y_d^k)^{\alpha_{2d}}(T_k)^{\alpha_{2d+1}}(Z_k)^{\alpha_{2d+2}}(A_1^k)^{\alpha_{2d+3}}\cdots(A_p^k)^{\alpha_{2d+2+p}}$. Then,

$$\|f_k\|_{(N)}=\sum_{|\alpha|\le N}\int_{\mathbb{R}^{2d+2+p}}\big(1+\|g\|^2\big)^N\big|f'_{\alpha,k}(g)\big|dg.$$

Since the k-basis of G converges to the ∞-basis, one can estimate the expressions $\big|f'_{\alpha,k}(g)\big|$ for $g\in\mathbb{R}^{2d+2+p}$ and $|\alpha|\le N$ by constants independent of k and therefore gets an estimate for $\|f_k\|_{(N)}$ that does not depend on k.
Now, denoting by $\hat f_k^{2,3,4,5,6}$ the Fourier transform in the 2nd, 3rd, 4th, 5th and 6th variable,

$$\begin{aligned}
&\pi_k(f)\xi(s)\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde p}\times\mathbb{R}^{p-\tilde p}} f_k(g)\pi_k(g)\xi(s)dg\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde p}\times\mathbb{R}^{p-\tilde p}} f_k(x,y,t,z,\dot a,\ddot a)e^{2\pi i\left(-t\rho_k-z\lambda_k+\lambda_k c^k(s-\frac{1}{2}x)y\right)}\xi(s-x)d(x,y,t,z,\dot a,\ddot a)\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde p}\times\mathbb{R}^{p-\tilde p}} f_k(s-x,y,t,z,\dot a,\ddot a)e^{2\pi i\left(-t\rho_k-z\lambda_k+\frac{1}{2}\lambda_k c^k(s+x)y\right)}\xi(x)d(x,y,t,z,\dot a,\ddot a)\\
&= \int_{\mathbb{R}^d}\hat f_k^{2,3,4,5,6}\Big(s-x,-\frac{\lambda_k c^k}{2}(s+x),\rho_k,\lambda_k,0,0\Big)\xi(x)dx. \qquad (6)
\end{aligned}$$

3.4 Condition 3(b) – First case

First, consider the case that $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ consists of one single limit point $\mathcal{O}$, the first case described in Proposition 3.1.
In this case, for every $k \in \mathbb{N}$,

$$2d = \dim(\mathcal{O}_k) = \dim(\mathcal{O}).$$

Thus, the regarded situation occurs if and only if $\lambda \neq 0$ and $c_j \neq 0$ for every $j \in \{1, ..., d\}$.
The first case is the easiest case. After a few observations, here the operators $(\tilde{\nu}_k)_{k\in\mathbb{N}}$ can be defined immediately. Moreover, the claims of Condition 3(b) of Definition 1.1 can be shown easily. In this first case, the transition to the representations $(\pi_k)_{k\in\mathbb{N}}$ and the definitions and computations accomplished in the Sections 3.3.3 and 3.3.4 are not needed.
Consider again the sequence $(\ell_k)_{k\in\mathbb{N}}$ chosen above which converges to $\ell \in \mathcal{O}$. As the dimensions of the orbits $\mathcal{O}_k$ and $\mathcal{O}$ are the same, there exists a subsequence of $(\ell_k)_{k\in\mathbb{N}}$ (which will also be denoted by $(\ell_k)_{k\in\mathbb{N}}$ for simplicity) such that $\mathfrak{p} := \lim\limits_{k\to\infty} \mathfrak{p}^V_{\ell_k}$ is a polarization of ℓ, but not necessarily the Vergne polarization. Moreover, define $P := \exp(\mathfrak{p}) = \lim\limits_{k\to\infty} P^V_{\ell_k}$ and let

$$\pi := \operatorname{ind}_P^G \chi_\ell.$$

Now, if one identifies the Hilbert spaces $\mathcal{H}_{\pi^V_{\ell_k}}$ and $\mathcal{H}_\pi$ of the representations $\pi^V_{\ell_k} = \operatorname{ind}^G_{P^V_{\ell_k}} \chi_{\ell_k}$ and π with $L^2(\mathbb{R}^d)$, from [31], Theorem 2.3, one can conclude that

$$\left\|\pi^V_{\ell_k}(a) - \pi(a)\right\|_{op} = \left\|\operatorname{ind}^G_{P^V_{\ell_k}} \chi_{\ell_k}(a) - \operatorname{ind}^G_P \chi_\ell(a)\right\|_{op} \overset{k\to\infty}{\longrightarrow} 0$$

for every $a \in C^*(G)$.
Since π and $\pi^V_\ell = \operatorname{ind}^G_{P^V_\ell} \chi_\ell$ are both induced representations of polarizations and of the same character χ_ℓ, they are equivalent and hence, there exists a unitary intertwining operator

$$F : \mathcal{H}_{\pi^V_\ell} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_\pi \cong L^2(\mathbb{R}^d) \quad \text{such that} \quad F \circ \pi^V_\ell(a) = \pi(a) \circ F \quad \forall a \in C^*(G).$$

Furthermore, the two representations $\pi^V_k = \pi^V_{\ell_{\mathcal{O}_k}} = \operatorname{ind}^G_{P^V_{\ell_{\mathcal{O}_k}}} \chi_{\ell_{\mathcal{O}_k}}$ and $\pi^V_{\ell_k} = \operatorname{ind}^G_{P^V_{\ell_k}} \chi_{\ell_k}$ are equivalent for every $k \in \mathbb{N}$ because $\ell_{\mathcal{O}_k}$ and ℓ_k are located in the same coadjoint orbit $\mathcal{O}_k$ and $\mathfrak{p}^V_{\ell_{\mathcal{O}_k}}$ and $\mathfrak{p}^V_{\ell_k}$ are polarizations. Thus, there exist further unitary intertwining operators

$$F_k : \mathcal{H}_{\pi^V_k} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi^V_{\ell_k}} \cong L^2(\mathbb{R}^d) \quad \text{such that} \quad F_k \circ \pi^V_k(a) = \pi^V_{\ell_k}(a) \circ F_k \quad \forall a \in C^*(G).$$

Now, define the required operators $\tilde{\nu}_k$ for every $k \in \mathbb{N}$ as

$$\tilde{\nu}_k(\varphi) := F_k^* \circ F \circ \varphi\big(\pi^V_\ell\big) \circ F^* \circ F_k \quad \forall \varphi \in CB(S_{i-1}),$$

which is reasonable since π^V_ℓ is a limit point of the sequence $\big(\pi^V_k\big)_{k\in\mathbb{N}}$ and hence contained in S_{i-1}, as seen in Section 3.3.1.
As $\varphi\big(\pi^V_\ell\big) \in \mathcal{B}(L^2(\mathbb{R}^d))$ and F and F_k are intertwining operators and thus bounded, the image of $\tilde{\nu}_k$ is contained in $\mathcal{B}(L^2(\mathbb{R}^d))$, as requested.

Next, it needs to be shown that $\tilde{\nu}_k$ is bounded. By the definition of $\|\cdot\|_{S_{i-1}}$, one has for every $\varphi \in CB(S_{i-1})$,

$$\begin{aligned}
\|\tilde{\nu}_k(\varphi)\|_{op} &= \left\|F_k^* \circ F \circ \varphi\left(\pi_\ell^V\right) \circ F^* \circ F_k\right\|_{op} \\
&\leq \|F_k^*\|_{op}\ \|F\|_{op}\ \left\|\varphi\left(\pi_\ell^V\right)\right\|_{op}\ \|F^*\|_{op}\ \|F_k\|_{op} \\
&= \left\|\varphi\left(\pi_\ell^V\right)\right\|_{op} \\
&\leq \|\varphi\|_{S_{i-1}}.
\end{aligned}$$

In addition, one can easily observe that $\tilde{\nu}_k$ is involutive. For every $\varphi \in CB(S_{i-1})$,

$$\tilde{\nu}_k(\varphi)^* = \left(F_k^* \circ F \circ \varphi\left(\pi_\ell^V\right) \circ F^* \circ F_k\right)^* = F_k^* \circ F \circ \varphi^*\left(\pi_\ell^V\right) \circ F^* \circ F_k = \tilde{\nu}_k(\varphi^*).$$

Now, the last thing to check is the required convergence of Condition 3(b). For all $a \in C^*(G)$,

$$\begin{aligned}
\left\|\pi_k^V(a) - \tilde{\nu}_k\left(\mathcal{F}(a)_{|S_{i-1}}\right)\right\|_{op} &= \left\|\pi_k^V(a) - F_k^* \circ F \circ \mathcal{F}(a)_{|S_{i-1}}\left(\pi_\ell^V\right) \circ F^* \circ F_k\right\|_{op} \\
&= \left\|\pi_k^V(a) - F_k^* \circ F \circ \pi_\ell^V(a) \circ F^* \circ F_k\right\|_{op} \\
&= \left\|F_k^* \circ \pi_{\ell_k}^V(a) \circ F_k - F_k^* \circ \pi(a) \circ F_k\right\|_{op} \\
&= \left\|F_k^* \circ \left(\pi_{\ell_k}^V - \pi\right)(a) \circ F_k\right\|_{op} \\
&\leq \left\|\pi_{\ell_k}^V(a) - \pi(a)\right\|_{op} \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, the representations $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ and the constructed $(\tilde{\nu}_k)_{k\in\mathbb{N}}$ fulfill Condition 3(b) of Definition 1.1 and thus, in this case, the claim is shown.

3.5 Condition 3(b) – Second case

In the second case of Condition 3(b) of Definition 1.1 described in Proposition 3.1, the situation that $\lambda = 0$ or $c_j = 0$ for every $j \in \{1, ..., d\}$ is going to be considered.
In this case,

$$\left\langle \ell_k, [X_j^k, Y_j^k]\right\rangle = c_j^k \lambda_k \overset{k\to\infty}{\longrightarrow} c_j\lambda = 0 \quad \forall j \in \{1, ..., d\},$$

while $c_j^k \lambda_k \neq 0$ for every k and every $j \in \{1, ..., d\}$.
Then, $\ell_{|[\mathfrak{g},\mathfrak{g}]} = 0$ and since $\tilde{\ell}_{|[\mathfrak{g},\mathfrak{g}]}$ obviously also vanishes for all $\tilde{\ell} \in \mathfrak{u}^\perp$, every $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$, which means that the associated representation is a character. Therefore, every limit orbit $\mathcal{O}$ in the set $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ has the dimension 0.

3.5.1 Convergence of $(\pi_k)_{k\in\mathbb{N}}$ in $\widehat{G}$

In order to better understand the sequence of representations $(\pi_k)_{k\in\mathbb{N}}$ and to get an idea of how to construct the required linear mappings $(\nu_k)_{k\in\mathbb{N}}$, in this section, the convergence in $\widehat{G}$ of $(\pi_k)_{k\in\mathbb{N}}$ is going to be analyzed. This shall serve as a motivation for the construction of the mappings $(\nu_k)_{k\in\mathbb{N}}$ which will be carried out subsequently.

As in Calculation (6) in Section 3.3.4, identify G again with $\mathbb{R}^{2d+2+p}$. From now on, this identification will be used most of the time. Only in some cases where one applies ℓ_k or ℓ and thus it is important to know whether one is using the basis depending on k or the limit basis, the calculation will be done in the above defined bases $(\cdot)_k$ or $(\cdot)_\infty$.

Now, adapt the methods developed in [26] to this given situation.
Let $s = (s_1, ..., s_d)$, $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$ and define

$$\eta_{k,\alpha,\beta}(s) = \eta_{k,\alpha,\beta}(s_1, \cdots, s_d) := e^{2\pi i \alpha s} \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}}\left(s_j + \frac{\beta_j}{\lambda_k c_j^k}\right)\right)$$

for the Schwartz functions η_j for $j \in \{1, ..., d\}$ chosen at the beginning of Section 3.3.3. This definition is reasonable because $\lambda_k c_j^k \neq 0$ for every k and every $j \in \{1, ..., d\}$.

Next, the convergence of $(\pi_k)_{k\in\mathbb{N}}$ to $\chi_{\ell+\ell_{\alpha,\beta}}$ in $\widehat{G}$ will be shown. For this, let $c^k_{\alpha,\beta}$ be the coefficient function defined by

$$c^k_{\alpha,\beta}(g) := \langle \pi_k(g)\eta_{k,\alpha,\beta}, \eta_{k,\alpha,\beta}\rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}$$

and $\ell_{\alpha,\beta}$ the linear functional

$$\ell_{\alpha,\beta}(g) = \ell_{\alpha,\beta}(x, y, t, z, a) := \alpha x + \beta y \quad \forall g = (x, y, t, z, a) \in G \cong \mathbb{R}^{2d+2+p}.$$

Since the Hilbert space of the character $\chi_{\ell+\ell_{\alpha,\beta}}$ is one-dimensional, one can choose $\{1\}$ as a basis and gets the following computation:
Using Relation (5), for $\rho := \langle \ell, T\rangle$ and $g = (x, y, t, z, \dot{a}, \ddot{a}) \in \mathbb{R}^{2d+2+p}$,

$$\begin{aligned}
& c^k_{\alpha,\beta}(g) \\
= & \int_{\mathbb{R}^d} \pi_k(g)\eta_{k,\alpha,\beta}(s)\overline{\eta_{k,\alpha,\beta}(s)}ds \\
= & \int_{\mathbb{R}^d} e^{2\pi i\left(-t\rho_k - z\lambda_k + \sum_{j=1}^{d} \lambda_k c_j^k (s_j - \frac{1}{2}x_j)y_j\right)} \eta_{k,\alpha,\beta}(s_1 - x_1, ..., s_d - x_d) \\
& \quad \overline{\eta_{k,\alpha,\beta}(s_1, ..., s_d)} d(s_1, ..., s_d) \\
= & \int_{\mathbb{R}^d} e^{2\pi i\left(-t\rho_k - z\lambda_k + \sum_{j=1}^{d} \lambda_k c_j^k (s_j - \frac{1}{2}x_j)y_j\right)} e^{-2\pi i \alpha x} \\
& \quad \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}}\left(s_j - x_j + \frac{\beta_j}{\lambda_k c_j^k}\right)\right) \\
& \quad \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{4}} \overline{\eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}}\left(s_j + \frac{\beta_j}{\lambda_k c_j^k}\right)\right)} d(s_1, ..., s_d)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-2\pi i(t\rho_k+z\lambda_k)}e^{-2\pi i\alpha x}\prod_{j=1}^{d}\Bigg(e^{-2\pi i\frac{1}{2}x_jy_jc_j^k\lambda_k}\left|\lambda_k c_j^k\right|^{\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}} e^{2\pi i y_j s_j c_j^k\lambda_k}\eta_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\Big(s_j-x_j+\frac{\beta_j}{\lambda_k c_j^k}\Big)\Big)\overline{\eta_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\Big(s_j+\frac{\beta_j}{\lambda_k c_j^k}\Big)\Big)}ds_j\Bigg) \\
&= e^{-2\pi i(t\rho_k+z\lambda_k)}e^{-2\pi i\alpha x}\prod_{j=1}^{d}\Bigg(e^{-2\pi i\frac{1}{2}x_jy_jc_j^k\lambda_k}\left|\lambda_k c_j^k\right|^{\frac{1}{2}} \\
&\qquad \int_{\mathbb{R}} e^{2\pi i y_j\left(s_j-\frac{\beta_j}{\lambda_k c_j^k}\right)c_j^k\lambda_k}\eta_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}(s_j-x_j)\Big)\overline{\eta_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}(s_j)\Big)}ds_j\Bigg) \\
&= e^{-2\pi i(t\rho_k+z\lambda_k)}e^{-2\pi i\alpha x}\prod_{j=1}^{d}\Bigg(e^{-2\pi i\frac{1}{2}x_jy_jc_j^k\lambda_k} \\
&\qquad \int_{\mathbb{R}} e^{2\pi i y_j\left(s_j sgn(\lambda_k c_j^k)|\lambda_k c_j^k|^{\frac{1}{2}}-\beta_j\right)}\eta_j\Big(s_j-\left|\lambda_k c_j^k\right|^{\frac{1}{2}}x_j\Big)\overline{\eta_j(s_j)}ds_j\Bigg) \\
&\overset{k\to\infty}{\longrightarrow} e^{-2\pi i t\rho}e^{-2\pi i z\lambda}e^{-2\pi i\alpha x}\prod_{j=1}^{d}\Bigg(\int_{\mathbb{R}} e^{-2\pi i y_j\beta_j}\eta_j(s_j)\overline{\eta_j(s_j)}ds_j\Bigg) \\
&\overset{\|\eta_j\|_2=1}{=} e^{-2\pi i t\rho}e^{-2\pi i z\lambda}e^{-2\pi i\alpha x}e^{-2\pi i y\beta} = \chi_{\ell+\ell_{\alpha,\beta}}(g) = \big\langle\chi_{\ell+\ell_{\alpha,\beta}}(g)1,1\big\rangle,
\end{aligned}
$$

where this convergence holds uniformly on compacta.

3.5.2 Definition of ν_k

Finally, the linear mappings $(\nu_k)_{k\in\mathbb{N}}$ which are needed for the sequence $(\pi_k)_{k\in\mathbb{N}}$ to fulfill conditions similar to those required in 3(b) are going to be defined and analyzed.

For $0 \neq \tilde{\eta} \in L^2(G/P_k,\chi_{\ell_k}) \cong L^2(\mathbb{R}^d)$, let

$$P_{\tilde{\eta}} : L^2(\mathbb{R}^d) \to \mathbb{C}\tilde{\eta},\ \xi \mapsto \tilde{\eta}\langle\xi,\tilde{\eta}\rangle.$$

$P_{\tilde{\eta}}$ is the orthogonal projection onto the space $\mathbb{C}\tilde{\eta}$.
Let $h \in C^*(G/U,\chi_\ell)$. Again, identify G/U with $\mathbb{R}^d\times\mathbb{R}^d \cong \mathbb{R}^{2d}$ and as already introduced in Section 3.3.4 in order to show the dependence on k, here, the utilization of the limit basis will be expressed by an index ∞ if necessary:

$$h_\infty(x,y) := h((x,y)_\infty).$$

Now, $\hat{h}_\infty$ can be regarded as a function in $C_\infty(\ell+\mathfrak{u}^\perp) \cong C_\infty(\mathbb{R}^{2d})$ and, using this identification, define the linear operator

$$\nu_k(h) := \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x},\tilde{y})P_{\eta_{k,\tilde{x},\tilde{y}}}\ \frac{d(\tilde{x},\tilde{y})}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|}.$$

Then, the following proposition holds:

Proposition 3.3.

1. *For all $k \in \mathbb{N}$ and $h \in \mathcal{S}(G/U, \chi_\ell)$, the integral defining $\nu_k(h)$ converges in the operator norm.*
2. *The operator $\nu_k(h)$ is compact and $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U,\chi_\ell)}$.*
3. *ν_k is involutive, i.e. $\nu_k(h)^* = \nu_k(h^*)$ for every $h \in C^*(G/U, \chi_\ell)$.*

Proof:
1) Let $h \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$. Since

$$\begin{aligned}
\|P_{\eta_{k,\alpha,\beta}}\|_{op} &= \|\eta_{k,\alpha,\beta}\|_2^2 \\
&= \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{2}} \int_{\mathbb{R}} \left| e^{2\pi i \alpha_j s_j}\ \eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}} \left(s_j + \frac{\beta_j}{\lambda_k c_j^k} \right)\right)\right|^2 ds_j \\
&= \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{2}} \int_{\mathbb{R}} |\eta_j(s_j)|^2 |\lambda_k c_j^k|^{-\frac{1}{2}} ds_j \\
&= \prod_{j=1}^{d} \|\eta_j\|_2^2 \ = \ 1,
\end{aligned}$$

one can estimate the operator norm of $\nu_k(h)$ as follows:

$$\begin{aligned}
\|\nu_k(h)\|_{op} &= \left\| \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y})\ P_{\eta_{k,\tilde{x},\tilde{y}}}\ \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^{d} |\lambda_k c_j^k|} \right\|_{op} \\
&\leq \int_{\mathbb{R}^{2d}} \left|\hat{h}_\infty(\tilde{x}, \tilde{y})\right| \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^{d} |\lambda_k c_j^k|} \ = \ \frac{\|\hat{h}\|_{L^1(\mathbb{R}^{2d})}}{\prod_{j=1}^{d} |\lambda_k c_j^k|}.
\end{aligned}$$

Therefore, the convergence of the integral $\nu_k(h)$ in the operator norm is shown for $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_\ell)$.

2) First, let $h \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$.
Define for $s = (s_1, ..., s_d) \in \mathbb{R}^d$,

$$\eta_{k,\beta}(s) := \prod_{j=1}^{d} |\lambda_k c_j^k|^{\frac{1}{4}}\ \eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}} \left(s_j + \frac{\beta_j}{\lambda_k c_j^k} \right)\right).$$

Then,

$$\eta_{k,\alpha,\beta}(s) = e^{2\pi i \alpha s} \eta_{k,\beta}(s)$$

and thus, one has for $\xi \in \mathcal{S}(\mathbb{R}^d)$ and $s \in \mathbb{R}^d$,

$$\begin{aligned}
\nu_k(h)\xi(s) &= \int\limits_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x},\tilde{y})\langle \xi, \eta_{k,\tilde{x},\tilde{y}}\rangle \eta_{k,\tilde{x},\tilde{y}}(s)\ \frac{d(\tilde{x},\tilde{y})}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|} \\
&= \int\limits_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x},\tilde{y})\Big(\int\limits_{\mathbb{R}^{d}} \xi(r)\overline{\eta}_{k,\tilde{x},\tilde{y}}(r)dr\Big)\eta_{k,\tilde{x},\tilde{y}}(s)\ \frac{d(\tilde{x},\tilde{y})}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|} \\
&= \int\limits_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x},\tilde{y})\Big(\int\limits_{\mathbb{R}^{d}} \xi(r)e^{-2\pi i\tilde{x}r}\overline{\eta}_{k,\tilde{y}}(r)dr\Big)e^{2\pi i\tilde{x}s}\eta_{k,\tilde{y}}(s)\ \frac{d(\tilde{x},\tilde{y})}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|} \\
&= \int\limits_{\mathbb{R}^{d}}\int\limits_{\mathbb{R}^{d}}\int\limits_{\mathbb{R}^{d}} \hat{h}_\infty(\tilde{x},\tilde{y})e^{2\pi i\tilde{x}(s-r)}\xi(r)\overline{\eta}_{k,\tilde{y}}(r)dr\ \eta_{k,\tilde{y}}(s)\ \frac{d\tilde{x}d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|} \\
&= \int\limits_{\mathbb{R}^{d}}\int\limits_{\mathbb{R}^{d}} \hat{h}^2_\infty(s-r,\tilde{y})\xi(r)\overline{\eta}_{k,\tilde{y}}(r)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|}\ dr, \qquad (7)
\end{aligned}$$

where, similar as above, $\hat{h}^2_\infty$ denotes the Fourier transform of $\hat{h}_\infty$ in the second variable. Hence, as the kernel function

$$h_K(s,r) := \int\limits_{\mathbb{R}^{d}} \hat{h}^2_\infty(s-r,\tilde{y})\overline{\eta}_{k,\tilde{y}}(r)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|}$$

of $\nu_k(h)$ is contained in $\mathcal{S}(\mathbb{R}^{2d})$, the operator $\nu_k(h)$ is compact.
Now, it will be shown that

$$\|\nu_k(h)\|_{op} \leq \left\|\hat{h}\right\|_\infty.$$

Again, for $\xi \in \mathcal{S}(\mathbb{R}^d)$ one has

$$\begin{aligned}
&\|\nu_k(h)\xi\|_2^2 \\
= &\int\limits_{\mathbb{R}^{d}} \left|\int\limits_{\mathbb{R}^{d}}\int\limits_{\mathbb{R}^{d}} \hat{h}^2_\infty(s-r,\tilde{y})\xi(r)\overline{\eta}_{k,\tilde{y}}(r)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|}\ dr\right|^2 ds \\
= &\int\limits_{\mathbb{R}^{d}} \left|\int\limits_{\mathbb{R}^{d}} \hat{h}^2_\infty(\cdot,\tilde{y}) * (\xi\overline{\eta}_{k,\tilde{y}})(s)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|}\right|^2 ds \\
\underset{\|\eta_j\|_\infty \leq 1}{\overset{\substack{Cauchy- \\ Schwarz,}}{\leq}} &\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|^{\frac{1}{2}} \int\limits_{\mathbb{R}^{d}}\int\limits_{\mathbb{R}^{d}} \left|\hat{h}^2_\infty(\cdot,\tilde{y}) * (\xi\overline{\eta}_{k,\tilde{y}})(s)\right|^2\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\left|\lambda_k c_j^k\right|^2}\ ds
\end{aligned}$$

$$\overset{Plancherel}{\leq} \quad \left\|\hat{h}\right\|_\infty^2 \prod_{j=1}^d \frac{1}{\left|\lambda_k c_j^k\right|^{\frac{3}{2}}} \int\limits_{\mathbb{R}^d} \|\xi\overline{\eta}_{k,\tilde{y}}\|_2^2 d\tilde{y}$$

$$= \quad \left\|\hat{h}\right\|_\infty^2 \prod_{j=1}^d \frac{1}{\left|\lambda_k c_j^k\right|^{\frac{3}{2}}} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} |\xi(s)|^2|\overline{\eta}_{k,\tilde{y}}(s)|^2 ds d\tilde{y}$$

$$= \quad \left\|\hat{h}\right\|_\infty^2 \prod_{j=1}^d \frac{1}{\left|\lambda_k c_j^k\right|} \int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} \left|\eta_j\left(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\left(s_j + \frac{\tilde{y}_j}{\lambda_k c_j^k}\right)\right)\right|^2 d\tilde{y}_1 \cdots d\tilde{y}_d$$

$$|\xi(s_1, ..., s_d)|^2 ds_1 \cdots ds_d$$

$$= \quad \left\|\hat{h}\right\|_\infty^2 \prod_{j=1}^d \int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \cdots \int\limits_{\mathbb{R}} |\eta_j(\tilde{y}_j)|^2 d\tilde{y}_1 \cdots d\tilde{y}_d \; |\xi(s_1, ..., s_d)|^2 ds_1 \cdots ds_d$$

$$\overset{\|\eta_j\|_2=1}{=} \quad \left\|\hat{h}\right\|_\infty^2 \|\xi\|_2^2.$$

Thus, since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$,

$$\|\nu_k(h)\|_{op} = \sup_{\substack{\xi \in L^2(\mathbb{R}^d), \\ \|\xi\|_2=1}} \|\nu_k(h)(\xi)\|_2 \leq \left\|\hat{h}\right\|_\infty$$

for $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_\ell)$. Therefore, with the density of $\mathcal{S}(G/U, \chi_\ell)$ in $C^*(G/U, \chi_\ell)$, one gets the compactness of the operator $\nu_k(h)$ for $h \in C^*(G/U, \chi_\ell)$, as well as the desired inequality

$$\|\nu_k(h)\|_{op} \leq \left\|\hat{h}\right\|_\infty = \|h\|_{C^*(G/U,\chi_\ell)}.$$

3) For $h \in C^*(G/U, \chi_\ell) \cong C^*(\mathbb{R}^{2d})$, one has

$$\nu_k(h)^* \quad = \quad \left(\int\limits_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) \; P_{\eta_{k,\tilde{x},\tilde{y}}} \; \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d \left|\lambda_k c_j^k\right|} \right)^*$$

$$\overset{P=P^*}{=} \quad \int\limits_{\mathbb{R}^{2d}} \overline{\hat{h}_\infty(\tilde{x}, \tilde{y})} \; P_{\eta_{k,\tilde{x},\tilde{y}}} \; \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d \left|\lambda_k c_j^k\right|}$$

$$= \quad \int\limits_{\mathbb{R}^{2d}} \widehat{h^*_\infty}(\tilde{x}, \tilde{y}) \; P_{\eta_{k,\tilde{x},\tilde{y}}} \; \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d \left|\lambda_k c_j^k\right|} \quad = \quad \nu_k(h^*).$$

□

This proposition firstly shows that the image of the operator ν_k is located in $\mathcal{B}(L^2(\mathbb{R}^d)) = \mathcal{B}(\mathcal{H}_i)$ as required in Condition 3(b) of Definition 1.1. Secondly, the proposition gives the boundedness and the involutivity of the linear mappings ν_k for every $k \in \mathbb{N}$. For the analysis of the sequence $(\pi_k)_{k\in\mathbb{N}}$, it remains to show a convergence, from which the convergence for the original sequence $(\pi_k^V)_{k\in\mathbb{N}}$ demanded in Condition 3(b) can be deduced.

3.5.3 Theorem – Second Case

Theorem 3.4.
Define as in Section 2.1.3

$$p_{G/U} : L^1(G) \to L^1(G/U, \chi_\ell),\ p_{G/U}(f)(\tilde{g}) := \int\limits_U f(\tilde{g}u)\chi_\ell(u)du \quad \forall \tilde{g} \in G\ \forall f \in L^1(G)$$

and canonically extend $p_{G/U}$ to a mapping going from $C^*(G)$ to $C^*(G/U, \chi_\ell)$.
Furthermore, let $a \in C^*(G)$. Then,

$$\lim_{k\to\infty} \left\| \pi_k(a) - \nu_k\big(p_{G/U}(a)\big) \right\|_{op} = 0.$$

Proof:
First, it is well-known that $p_{G/U}$ is a surjective homomorphism from $C^*(G)$ to $C^*(G/U, \chi_\ell)$ and from $L^1(G)$ to $L^1(G/U, \chi_\ell)$ as well as from $\mathcal{S}(G)$ to $\mathcal{S}(G/U, \chi_\ell)$.
Furthermore, for $u = (t, z, \dot{a}, \ddot{a})_\infty \in U = \text{span}\{T, Z, A_1, ..., A_{\tilde{p}}, A_{\tilde{p}+1}, ..., A_p\}$,

$$\chi_\ell(u) = e^{-2\pi i\langle \ell,(t,z,\dot{a},\ddot{a})_\infty\rangle} = e^{-2\pi i(t\rho+z\lambda)}$$

and therefore, identifying U again with $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde{p}}\times\mathbb{R}^{p-\tilde{p}}$ and $L^1(G/U, \chi_\ell)$ with $L^1(\mathbb{R}^{2d})$, for $f \in L^1(G) \cong L^1(\mathbb{R}^{2d+2+p})$ and $\tilde{g} = (\tilde{x}, \tilde{y}, 0, 0, 0, 0) \in \mathbb{R}^{2d}$, one has

$$\begin{aligned} \big(p_{G/U}(f)\big)_\infty(\tilde{g}) &= \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde{p}}\times\mathbb{R}^{p-\tilde{p}}} f_\infty(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{z}, \tilde{\dot{a}}, \tilde{\ddot{a}})e^{-2\pi i(\tilde{t}\rho+\tilde{z}\lambda)}d(0, 0, \tilde{t}, \tilde{z}, \tilde{\dot{a}}, \tilde{\ddot{a}}) \\ &= \hat{f}^{3,4,5,6}_\infty(\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0), \end{aligned} \tag{8}$$

where $f_\infty(\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0) = f((\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0)_\infty)$.

Now, let $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$ such that its Fourier transform in $[\mathfrak{g}, \mathfrak{g}]$ has a compact support in $G \cong \mathbb{R}^{2d+2+p}$. If one writes the elements g of G as $g = (x, y, t, z, \dot{a}, \ddot{a})$ like above, where

$$x \in \text{span}\{X_1\} \times ... \times \text{span}\{X_d\}, \quad y \in \text{span}\{Y_1\} \times ... \times \text{span}\{Y_d\}, \quad t \in \text{span}\{T\},$$
$$z \in \text{span}\{Z\}, \quad \dot{a} \in \text{span}\{A_1\} \times ... \times \text{span}\{A_{\tilde{p}}\}, \quad \ddot{a} \in \text{span}\{A_{\tilde{p}+1}\} \times ... \times \text{span}\{A_p\}$$

or

$$x \in \text{span}\{X^k_1\} \times ... \times \text{span}\{X^k_d\}, \quad y \in \text{span}\{Y^k_1\} \times ... \times \text{span}\{Y^k_d\}, \quad t \in \text{span}\{T_k\},$$
$$z \in \text{span}\{Z_k\}, \quad \dot{a} \in \text{span}\{A^k_1\} \times ... \times \text{span}\{A^k_{\tilde{p}}\}, \quad \ddot{a} \in \text{span}\{A^k_{\tilde{p}+1}\} \times ... \times \text{span}\{A^k_p\},$$

respectively, this means that the partial Fourier transform $\hat{f}^{4,5}$ has a compact support in G, since $[\mathfrak{g}, \mathfrak{g}] = \text{span}\{Z_k, A^k_1, ..., A^k_{\tilde{p}}\} = \text{span}\{Z, A_1, ..., A_{\tilde{p}}\}$.

Moreover, let $\xi \in L^2(\mathbb{R}^d)$ and $s \in \mathbb{R}^d$ and define

$$\eta_{k,0}(s) := \prod_{j=1}^d \left|\lambda_k c^k_j\right|^{\frac{1}{4}} \eta_j\Big(\left|\lambda_k c^k_j\right|^{\frac{1}{2}}(s_j)\Big).$$

(Compare the definition of $\eta_{k,\beta}$ in the last proof.)

To prove the theorem, the expression $\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)\xi$ is going to be regarded, using Results (6) from Section 3.3.4 and (7) from the proof of the proposition above. The obtained integrals are going to be divided into five parts and the norm of each part is going to be estimated separately. For this, G will again be identified with $\mathbb{R}^{2d+2+p}$ and therefore, f will be replaced by the functions f_k and f_∞, respectively, which are defined as above.

$$
\begin{aligned}
&\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)\xi(s) \\
\overset{(6),(7)}{=}\ & \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho_k, \lambda_k, 0, 0\Big)\xi(r)dr \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{p_{G/U}(f)}^{2}_{\infty}(s-r,\tilde{y})\xi(r)\overline{\eta}_{k,\tilde{y}}(r)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\big|\lambda_k c_j^k\big|}\ dr \\
\underset{(8)}{\overset{\|\eta_{k,0}\|_2=1}{=}}\ & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho_k, \lambda_k, 0, 0\Big)\xi(r)\overline{\eta}_{k,0}(\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y}dr \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{f}_\infty^{2,3,4,5,6}(s-r,\tilde{y},\rho,\lambda,0,0)\xi(r)\overline{\eta}_{k,\tilde{y}}(r)\eta_{k,\tilde{y}}(s)\ \frac{d\tilde{y}}{\prod_{j=1}^{d}\big|\lambda_k c_j^k\big|}\ dr \\
=\ & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho_k, \lambda_k, 0, 0\Big)\xi(r)\overline{\eta}_{k,0}(\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y}dr \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{f}_\infty^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big)\xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\eta_{k,0}(\tilde{y})d\tilde{y}dr.
\end{aligned}
$$

As described above, the integrals just obtained are now going to be divided into five parts. To do so, the functions q_k, u_k, v_k, n_k and w_k will be defined.

$$
\begin{aligned}
q_k(s,\tilde{y}) \ :=\ & \int_{\mathbb{R}^d} \xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\Big(\hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho_k,\lambda_k,0,0\Big) \\
&\qquad - \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda_k,0,0\Big)\Big)dr, \\
u_k(s,\tilde{y}) \ :=\ & \int_{\mathbb{R}^d} \xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\Big(\hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda_k,0,0\Big) \\
&\qquad - \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda,0,0\Big)\Big)dr, \\
v_k(s,\tilde{y}) \ :=\ & \int_{\mathbb{R}^d} \xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\Big(\hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda,0,0\Big) \\
&\qquad - \hat{f}_k^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big)\Big)dr,
\end{aligned}
$$

$$n_k(s,\tilde{y}) := \int\limits_{\mathbb{R}^d} \xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\Big(\hat{f}_k^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big) - \hat{f}_\infty^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big)\Big)dr$$

and

$$w_k(s) := \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \xi(r)\eta_{k,0}(\tilde{y})\big(\overline{\eta}_{k,0}(\tilde{y}) - \overline{\eta}_{k,0}(\tilde{y}+r-s)\big) \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho_k,\lambda_k,0,0\Big)dr d\tilde{y}.$$

Then,

$$\big(\pi_k(f)-\nu_k\big(p_{G/U}(f)\big)\big)\xi(s) = \int\limits_{\mathbb{R}^d} q_k(s,\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y} + \int\limits_{\mathbb{R}^d} u_k(s,\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y} + \int\limits_{\mathbb{R}^d} v_k(s,\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y} + \int\limits_{\mathbb{R}^d} n_k(s,\tilde{y})\eta_{k,0}(\tilde{y})d\tilde{y} + w_k(s).$$

In order to show that

$$\big\|\pi_k(f)-\nu_k\big(p_{G/U}(f)\big)\big\|_{op} \overset{k\to\infty}{\longrightarrow} 0,$$

it suffices to prove that there are κ_k, γ_k, δ_k, ω_k and ϵ_k which are tending to 0 for $k\to\infty$, such that

$$\|q_k\|_2 \le \kappa_k\|\xi\|_2, \quad \|u_k\|_2 \le \gamma_k\|\xi\|_2, \quad \|v_k\|_2 \le \delta_k\|\xi\|_2, \quad \|n_k\|_2 \le \omega_k\|\xi\|_2 \quad \text{and} \quad \|w_k\|_2 \le \epsilon_k\|\xi\|_2.$$

First, regard the last factor of the function q_k:

$$\begin{aligned} & \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho_k,\lambda_k,0,0\Big) - \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda_k,0,0\Big) \\ = & (\rho_k-\rho)\int\limits_0^1 \partial_3\hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho+t(\rho_k-\rho),\lambda_k,0,0\Big)dt. \end{aligned}$$

Since f is a Schwartz function, one can find a constant $C_1>0$ (depending on f) such that

$$\begin{aligned} & \Big|\ \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho_k,\lambda_k,0,0\Big) - \hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho,\lambda_k,0,0\Big)\ \Big| \\ \le & |\rho_k-\rho|\ \frac{C_1}{(1+\|s-r\|)^{2d}}. \end{aligned}$$

Hence, one gets the following estimate for q_k:

$$\begin{aligned}
\|q_k\|_2^2 &= \int\limits_{\mathbb{R}^{2d}} |q_k(s,\tilde{y})|^2 d(s,\tilde{y}) \\
&\leq \int\limits_{\mathbb{R}^{2d}} \Bigg(\int\limits_{\mathbb{R}^d} \big|\xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\big| |\rho_k-\rho| \, \frac{C_1}{(1+\|s-r\|)^{2d}} \, dr \Bigg)^2 d(s,\tilde{y}) \\
&\overset{Cauchy-}{\underset{Schwarz}{\leq}} C_1^2 |\rho_k-\rho|^2 \int\limits_{\mathbb{R}^{2d}} \Bigg(\int\limits_{\mathbb{R}^d} \left| \frac{\xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)}{(1+\|s-r\|)^d} \right|^2 dr \Bigg) \\
&\qquad \Bigg(\int\limits_{\mathbb{R}^d} \left(\frac{1}{(1+\|s-r\|)^d} \right)^2 dr \Bigg) d(s,\tilde{y}) \\
&= C_1' |\rho_k-\rho|^2 \int\limits_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 d(r,s,\tilde{y}) \\
&\overset{\|\eta_{k,0}\|_2=1}{\leq} C_1'' |\rho_k-\rho|^2 \|\xi\|_2^2
\end{aligned}$$

for matching constants $C_1' > 0$ and $C_1'' > 0$ depending on f. Thus, for $\kappa_k := \sqrt{C_1''}\,|\rho_k - \rho|$, $\kappa_k \overset{k\to\infty}{\longrightarrow} 0$ since $\rho_k \overset{k\to\infty}{\longrightarrow} \rho$, and

$$\|q_k\|_2 \leq \kappa_k \|\xi\|_2.$$

As $\lambda_k \overset{k\to\infty}{\longrightarrow} \lambda$, the estimation for the function u_k can be done analogously.

Now, regard v_k. Like for q_k and u_k, one has

$$\begin{aligned}
& \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho, \lambda, 0, 0\Big) - \hat{f}_k^{2,3,4,5,6}\big(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0\big) \\
= \;& \lambda_k c^k \Big(\frac{1}{2}(r-s) - (r-s+\tilde{y}) \Big) \\
& \cdot \int\limits_0^1 \partial_2 \hat{f}_k^{2,3,4,5,6}\Big(s-r, \lambda_k c^k(\tilde{y}-s) + t\lambda_k c^k \Big(\frac{1}{2}(r-s) - (r-s+\tilde{y})\Big), \rho, \lambda, 0, 0 \Big) dt,
\end{aligned}$$

where $\cdot$ is the scalar product, and hence there exists again a constant C_3 depending on f such that

$$\begin{aligned}
& \left| \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho, \lambda, 0, 0\Big) - \hat{f}_k^{2,3,4,5,6}\big(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0\big) \right| \\
\leq \;& |\lambda_k| \, \Big(\big\|c^k(r-s)\big\| + \big\|c^k(r-s+\tilde{y})\big\| \Big) \, \frac{C_3}{(1+\|s-r\|)^{2d+1}}.
\end{aligned}$$

Therefore, defining $\tilde{\eta}_j(t) := \|t\|\eta_j(t)$, one gets a similar estimation for v_k:

$$
\begin{aligned}
&\|v_k\|_2^2 \\
=\;& \int_{\mathbb{R}^{2d}} |v_k(s,\tilde{y})|^2 d(s,\tilde{y}) \\
\leq\;& \int_{\mathbb{R}^{2d}} \Bigg(\int_{\mathbb{R}^d} \big|\xi(r)\overline{\eta}_{k,0}(\tilde{y}+r-s)\big||\lambda_k| \, \Big(\big\|c^k(r-s)\big\| + \big\|c^k(r-s+\tilde{y})\big\|\Big) \\
& \qquad \frac{C_3}{(1+\|s-r\|)^{2d+1}} \, dr\Bigg)^2 d(s,\tilde{y}) \\
\overset{Cauchy-}{\underset{Schwarz}{\leq}}\;& C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d+2}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 |\lambda_k|^2 \\
& \qquad \Big(\big\|c^k(r-s)\big\| + \big\|c^k(r-s+\tilde{y})\big\|\Big)^2 d(r,s,\tilde{y}) \\
\leq\;& 2C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d+2}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 |\lambda_k|^2 \big\|c^k(r-s+\tilde{y})\big\|^2 d(r,s,\tilde{y}) \\
& + 2C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d+2}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 |\lambda_k|^2 \big\|c^k(r-s)\big\|^2 d(r,s,\tilde{y}) \\
\leq\;& 2C_3'|\lambda_k|^2 \Bigg(\int_{\mathbb{R}^{2d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d+2}} \, d(r,s) \Bigg) \Bigg(\int_{\mathbb{R}^d} \big\|c^k\tilde{y}\big\|^2 \big|\eta_{k,0}(\tilde{y})\big|^2 d\tilde{y} \Bigg) \\
& + 2C_3' \big\|\lambda_k c^k\big\|^2 \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 \, d(r,s,\tilde{y}) \\
\leq\;& 2C_3' \big\|\lambda_k c^k\big\| \Bigg(\int_{\mathbb{R}^{2d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d+2}} \, d(r,s) \Bigg) \\
& \Bigg(\prod_{j=1}^{d} \big|\lambda_k c_j^k\big|^{\frac{1}{2}} \int_{\mathbb{R}} \Big\| |\lambda_k c_j^k|^{\frac{1}{2}} \tilde{y}_j \Big\|^2 \, \Big|\eta_j\Big(|\lambda_k c_j^k|^{\frac{1}{2}}(\tilde{y}_j)\Big)\Big|^2 d\tilde{y}_j \Bigg) \\
& + 2C_3' \big\|\lambda_k c^k\big\|^2 \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1+\|s-r\|)^{2d}} \, \big|\eta_{k,0}(\tilde{y}+r-s)\big|^2 d(r,s,\tilde{y}) \\
\overset{\|\eta_{k,0}\|_2=1}{\leq}\;& 2C_3' \big\|\lambda_k c^k\big\| \; \|\xi\|_2^2 \Bigg(\prod_{j=1}^{d} \int_{\mathbb{R}} |\tilde{\eta}_j(\tilde{y}_j)|^2 d\tilde{y}_j \Bigg) \; + \; 2C_3' \big\|\lambda_k c^k\big\|^2 \|\xi\|_2^2 \\
=\;& C_3'' \Bigg(\prod_{j=1}^{d} \|\tilde{\eta}_j\|_2^2 + \big\|\lambda_k c^k\big\| \Bigg) \big\|\lambda_k c^k\big\| \; \|\xi\|_2^2
\end{aligned}
$$

with constants $C_3' > 0$ and $C_3'' > 0$, again depending on f.

Now, since $\lambda_k c^k \overset{k\to\infty}{\longrightarrow} 0$, $\delta_k := \left(C_3''\Big(\prod_{j=1}^{d} \|\tilde{\eta}_j\|_2^2 + \big\|\lambda_k c^k\big\|\Big)\big\|\lambda_k c^k\big\|\right)^{\frac{1}{2}}$ fulfills $\delta_k \overset{k\to\infty}{\longrightarrow} 0$ and

$$\|v_k\|_2 \leq \delta_k \|\xi\|_2.$$

For the estimation of n_k, the fact that the Fourier transform in $[\mathfrak{g},\mathfrak{g}]$, i.e. in the 4th and 5th variable, $\hat{f}^{4,5} =: \tilde{f}$ has a compact support will be needed. Therefore, let the support of f be located in the compact set $K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\tilde{p}} \times \mathbb{R}^{p-\tilde{p}}$ and let $K := K_2 \times K_3 \times K_6 \subset \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{p-\tilde{p}}$.
Furthermore, since a Fourier transform is independent of the choice of the basis, the value of $\tilde{f}$ on $[\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{Z_k, A_1^k, ..., A_{\tilde{p}}^k\} = \mathrm{span}\{Z, A_1, ..., A_{\tilde{p}}\}$ expressed in the k-basis and its value expressed in the limit basis are the same:

$$f(\cdot,\cdot,\cdot,(z,\dot{a})_k,\cdot) = f(\cdot,\cdot,\cdot,(z,\dot{a})_\infty,\cdot).$$

So, in the course of this proof, the limit basis will be chosen for the representation of the 4th and 5th position of an element g.
Then,

$$\begin{aligned}
&\hat{f}_k^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big) - \hat{f}_\infty^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde{y}-s),\rho,\lambda,0,0\big)\\
= &\int\limits_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{p-\tilde{p}}} \Big(\tilde{f}_k(s-r,y,t,\lambda,0,\ddot{a}) - \tilde{f}_\infty(s-r,y,t,\lambda,0,\ddot{a})\Big) e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y+\rho t)} d(y,t,\ddot{a})\\
= &\int\limits_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{p-\tilde{p}}} \Big(\tilde{f}((s-r,y,t)_k,(\lambda,0)_\infty,(\ddot{a})_k) - \tilde{f}((s-r,y,t,\lambda,0,\ddot{a})_\infty)\Big)\\
&\qquad e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y+\rho t)} d(y,t,\ddot{a})\\
= &\int\limits_K \Big(\tilde{f}((s-r,y,t)_k,(\lambda,0)_\infty,(\ddot{a})_k) - \tilde{f}((s-r,y,t,\lambda,0,\ddot{a})_\infty)\Big) e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y+\rho t)} d(y,t,\ddot{a}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\tilde{f}((s-r,y,t)_k,(\lambda,0)_\infty,(\ddot{a})_k) - \tilde{f}((s-r,y,t,\lambda,0,\ddot{a})_\infty)\\
= &\ \tilde{f}\Big(\sum_{i=1}^{d}(s_i-r_i)X_i^k + \sum_{i=1}^{d} y_i Y_i^k + tT_k + \lambda Z + \sum_{i=\tilde{p}+1}^{p} a_i A_i^k\Big)\\
&- \tilde{f}\Big(\sum_{i=1}^{d}(s_i-r_i)X_i + \sum_{i=1}^{d} y_i Y_i + tT + \lambda Z + \sum_{i=\tilde{p}+1}^{p} a_i A_i\Big)\\
= &\Big(\sum_{i=1}^{d}(s_i-r_i)(X_i^k - X_i) + \sum_{i=1}^{d} y_i(Y_i^k - Y_i) + t(T_k - T) + \sum_{i=\tilde{p}+1}^{p} a_i(A_i^k - A_i)\Big)\\
&\cdot \int_0^1 \partial\tilde{f}\Big(\sum_{i=1}^{d}(s_i-r_i)X_i + \sum_{i=1}^{d} y_i Y_i + tT + \lambda Z + \sum_{i=\tilde{p}+1}^{p} a_i A_i\\
&\quad + \tilde{t}\Big(\sum_{i=1}^{d}(s_i-r_i)(X_i^k - X_i) + \sum_{i=1}^{d} y_i(Y_i^k - Y_i) + t(T_k - T) + \sum_{i=\tilde{p}+1}^{p} a_i(A_i^k - A_i)\Big)\Big) d\tilde{t}.
\end{aligned}$$

Since $(X_1^k,...,X_d^k,Y_1^k,...,Y_d^k,T_k,Z_k,A_1^k,...,A_p^k) \overset{k\to\infty}{\longrightarrow} (X_1,...,X_d,Y_1,...,Y_d,T,Z,A_1,...,A_p)$, there exist $\omega_k^i \overset{k\to\infty}{\longrightarrow} 0$ for $i\in\{1,...,4\}$ and a constant $C_4>0$ depending on f such that

$$\begin{aligned}&\Big|\ \tilde f((s-r,y,t)_k,(\lambda,0)_\infty,(\ddot a)_k)-\tilde f((s-r,y,t,\lambda,0,\ddot a)_\infty)\Big|\\ \leq\ &\Big(\|s-r\|\omega_k^1+\|y\|\omega_k^2+|t|\omega_k^3+\|\ddot a\|\omega_k^4\Big)\ \frac{C_4}{(1+\|s-r\|)^{2d+1}}.\end{aligned}$$

Now, with the help of the two computations above, $\|n_k\|_2^2$ can be estimated:

$$\begin{aligned}
&\|n_k\|_2^2\\
=\ &\int_{\mathbb{R}^{2d}} |n_k(s,\tilde y)|^2 d(s,\tilde y)\\
=\ &\int_{\mathbb{R}^{2d}}\Big|\int_{\mathbb{R}^d}\xi(r)\overline{\eta}_{k,0}(\tilde y+r-s)\Big(\hat f_k^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde y-s),\rho,\lambda,0,0\big)\\
&\qquad -\hat f_\infty^{2,3,4,5,6}\big(s-r,\lambda_k c^k(\tilde y-s),\rho,\lambda,0,0\big)\Big)dr\Big|^2 d(s,\tilde y)\\
\leq\ &\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}\big|\xi(r)\overline{\eta}_{k,0}(\tilde y+r-s)\big|\int_K\Big(\|s-r\|\omega_k^1+\|y\|\omega_k^2+|t|\omega_k^3+\|\ddot a\|\omega_k^4\Big)\\
&\qquad \frac{C_4}{(1+\|s-r\|)^{2d+1}}\ \big|e^{-2\pi i(\lambda_k c^k(\tilde y-s)y+\rho t)}\big|d(y,t,\ddot a)dr\Big)^2 d(s,\tilde y)\\
=\ &C_4^2\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}\frac{1}{(1+\|s-r\|)^{2d+1}}\ |\xi(r)|\big|\overline{\eta}_{k,0}(\tilde y+r-s)\big|\Big(\|s-r\|\omega_k^5+\omega_k^6\Big)dr\Big)^2 d(s,\tilde y)\\
\overset{Cauchy-Schwarz}{\leq}\ &C_4^2\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d}\frac{1}{(1+\|s-r\|)^{2d+2}}\ |\xi(r)|^2\big|\overline{\eta}_{k,0}(\tilde y+r-s)\big|^2\Big(\|s-r\|\omega_k^5+\omega_k^6\Big)^2dr\Big)\\
&\qquad\Big(\int_{\mathbb{R}^d}\frac{1}{(1+\|s-r\|)^{2d}}\ dr\Big)d(s,\tilde y)\\
\leq\ &C_4'\omega_k^7\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^d}\frac{1}{(1+\|s-r\|)^{2d}}\ |\xi(r)|^2\big|\overline{\eta}_{k,0}(\tilde y+r-s)\big|^2 dr d(s,\tilde y)\\
\overset{\|\eta_{k,0}\|_2=1}{=}\ &C_4'\omega_k^7\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+\|s-r\|)^{2d}}\ |\xi(r)|^2 dr ds\\
=\ &C_4''\omega_k^7\|\xi\|_2^2
\end{aligned}$$

with constants $C_4'>0$ and $C_4''>0$ depending on f and $\omega_k^i \overset{k\to\infty}{\longrightarrow} 0$ for $i\in\{5,...,7\}$. Thus, $\omega_k := \sqrt{C_4''\omega_k^7}$ fulfills $\omega_k \overset{k\to\infty}{\longrightarrow} 0$ and

$$\|n_k\|_2 \leq \omega_k\|\xi\|_2.$$

Last, it still remains to examine w_k:

$$\begin{aligned} & \overline{\eta}_{k,0}(\tilde{y}) - \overline{\eta}_{k,0}(\tilde{y}+r-s) \\ = & \sum_{j=1}^{d}(r_j-s_j)\int\limits_0^1 \partial_j\overline{\eta}_{k,0}(\tilde{y}+t(r-s))dt \\ = & \sum_{j=1}^{d}(r_j-s_j)\int\limits_0^1 \Bigg(\prod_{\substack{i=1\\ i\neq j}}^{d} \left|\lambda_k c_i^k\right|^{\frac{1}{4}}\overline{\eta}_i\Big(\left|\lambda_k c_i^k\right|^{\frac{1}{2}}(\tilde{y}_i+t(r_i-s_i))\Big) \\ & \qquad \left|\lambda_k c_j^k\right|^{\frac{3}{4}}\partial\overline{\eta}_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}(\tilde{y}_j+t(r_j-s_j))\Big)\Bigg)dt. \end{aligned}$$

Since f and the functions $(\eta_j)_{j\in\{1,\dots,d\}}$ are Schwartz functions, one can find a constant C_5 depending on $(\eta_j)_{j\in\{1,\dots,d\}}$ such that

$$\begin{aligned} & \left|\Big(\overline{\eta}_{k,0}(\tilde{y}) - \overline{\eta}_{k,0}(\tilde{y}+r-s)\Big)\hat{f}_k^{2,3,4,5,6}\Big(s-r,-\frac{\lambda_k c^k}{2}(s+r),\rho_k,\lambda_k,0,0\Big)\right| \\ \leq & \|r-s\|\Bigg(\sum_{j=1}^{d}\prod_{i=1}^{d}\left|\lambda_k c_i^k\right|^{\frac{1}{4}}\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\Bigg)\frac{C_5}{(1+\|r-s\|)^{2d+1}}. \end{aligned}$$

Now, one has the following estimation for $\|w_k\|_2$, which is again similar to the above ones:

$$\begin{aligned} \|w_k\|_2^2 \quad = & \int\limits_{\mathbb{R}^d}|w_k(s)|^2 ds \\ \leq & \int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}|\xi(r)|\big|\eta_{k,0}(\tilde{y})\big|\ \|r-s\|\Bigg(\sum_{j=1}^{d}\prod_{i=1}^{d}\left|\lambda_k c_i^k\right|^{\frac{1}{4}}\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\Bigg) \\ & \qquad \frac{C_5}{(1+\|r-s\|)^{2d+1}}\ dr d\tilde{y}\Bigg)^2 ds \\ \overset{Cauchy-Schwarz}{\leq} & C_5'\Bigg(\sum_{j=1}^{d}\prod_{i=1}^{d}\left|\lambda_k c_i^k\right|^{\frac{1}{2}}\left|\lambda_k c_j^k\right|\Bigg)\int\limits_{\mathbb{R}^{3d}}\frac{|\xi(r)|^2}{(1+\|r-s\|)^{2d+2}}\ \big|\eta_{k,0}(\tilde{y})\big|^2\|r-s\|^2 d(r,\tilde{y},s) \\ \overset{\|\eta_{k,0}\|_2=1}{\leq} & C_5''\Bigg(\sum_{j=1}^{d}\prod_{i=1}^{d}\left|\lambda_k c_i^k\right|^{\frac{1}{2}}\left|\lambda_k c_j^k\right|\Bigg)\|\xi\|_2^2, \end{aligned}$$

where the constants $C_5' > 0$ and $C_5'' > 0$ depend on the functions $(\eta_j)_{j\in\{1,\dots,d\}}$. Therefore, for $\epsilon_k := \Bigg(C_5''\Bigg(\sum\limits_{j=1}^{d}\prod\limits_{i=1}^{d}\left|\lambda_k c_i^k\right|^{\frac{1}{2}}\left|\lambda_k c_j^k\right|\Bigg)\Bigg)^{\frac{1}{2}}$, the desired properties $\epsilon_k \overset{k\to\infty}{\longrightarrow} 0$ and

$$\|w_k\|_2 \leq \epsilon_k\|\xi\|_2$$

are fulfilled.

Thus, for those $f \in \mathcal{S}(\mathbb{R}^{2d+2+p}) \cong \mathcal{S}(G)$ whose Fourier transform in $[\mathfrak{g},\mathfrak{g}]$ has a compact support,

$$\left\|\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\right\|_{op} = \sup_{\substack{\xi\in L^2(\mathbb{R}^d)\\ \|\xi\|_2=1}} \left\|\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)(\xi)\right\|_2 \overset{k\to\infty}{\longrightarrow} 0.$$

Because of the density in $L^1(G)$ and thus in $C^*(G)$ of the set of Schwartz functions $f \in \mathcal{S}(G)$ whose partial Fourier transform has a compact support, the claim is true for general elements $a \in C^*(G)$ by Corollary 7.6 in the appendix.

□

3.5.4 Transition to $\big(\pi_k^V\big)_{k\in\mathbb{N}}$

Finally, from the results shown above for the sequence $(\pi_k)_{k\in\mathbb{N}}$, the assertion for the original sequence $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ shall be deduced.

Since for every $k \in \mathbb{N}$ the two representations π_k and π_k^V are equivalent, there exist unitary intertwining operators

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{such that} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall a \in C^*(G).$$

Furthermore, since the limit set $L\big(\big(\pi_k^V\big)_{k\in\mathbb{N}}\big)$ of the sequence $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ is contained in S_{i-1}, as discussed in Section 3.3.1, by identifying $\widehat{G}$ with the set of coadjoint orbits $\mathfrak{g}^*/G$, one can restrict an operator field $\varphi \in CB(S_{i-1})$ to $L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \ell + \mathfrak{u}^\perp$ and obtains an element in $CB(\ell + \mathfrak{u}^\perp)$. Thus, as $\big\{\mathcal{F}(a)_{|L((\mathcal{O}_k)_{k\in\mathbb{N}})}|\ a \in C^*(G)\big\} = C_\infty(L((\mathcal{O}_k)_{k\in\mathbb{N}})) = C_\infty(\ell + \mathfrak{u}^\perp)$, one can define the $*$-isomorphism

$$\tau : C_\infty(\mathbb{R}^{2d}) \cong C_\infty(\ell + \mathfrak{u}^\perp) \to C^*(G/U, \chi_\ell) \cong C^*(\mathbb{R}^{2d}),\ \mathcal{F}(a)_{|L((\mathcal{O}_k)_{k\in\mathbb{N}})} \mapsto p_{G/U}(a).$$

Now, for $k \in \mathbb{N}$, define $\tilde{\nu}_k$ as

$$\tilde{\nu}_k(\varphi) := F_k^* \circ (\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k \quad \forall \varphi \in CB(S_{i-1}).$$

Since the image of ν_k is in $\mathcal{B}(L^2(\mathbb{R}^d))$ and F_k is an intertwining operator and thus bounded, the image of $\tilde{\nu}_k$ is contained in $\mathcal{B}(L^2(\mathbb{R}^d))$ as well.
Moreover, the operator $\tilde{\nu}_k$ is bounded: From the boundedness of ν_k (see Proposition 3.3) and using that τ is an isomorphism, one gets for every $\varphi \in CB(S_{i-1})$,

$$\begin{aligned} \|\tilde{\nu}_k(\varphi)\|_{op} &= \left\|F_k^* \circ (\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k\right\|_{op} \\ &\leq \left\|(\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big)\right\|_{op} \\ &\leq \left\|\tau\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big)\right\|_{C^*(\mathbb{R}^{2d})} \\ &\leq \left\|\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big)\right\|_\infty \ \leq\ \|\varphi\|_{S_{i-1}}. \end{aligned}$$

Next, the involutivity of $\tilde{\nu}_k$ will be shown. With the involutivity of ν_k (see Proposition 3.3) and τ, for every $\varphi \in CB(S_{i-1})$,

$$\begin{aligned}
\tilde{\nu}_k(\varphi)^* &= \Big(F_k^* \circ (\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k\Big)^* \\
&= F_k^* \circ (\nu_k \circ \tau)^*\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k \\
&= F_k^* \circ \nu_k\Big(\tau^*\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big)\Big) \circ F_k \\
&= F_k^* \circ (\nu_k \circ \tau)\big(\varphi^*_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k \;=\; \tilde{\nu}_k(\varphi^*).
\end{aligned}$$

Finally, the demanded convergence of Condition 3(b) of Definition 1.1 can also be shown: With the equivalence of the representations π_k and π_k^V stated above, one gets

$$\begin{aligned}
\big\|\pi_k^V(a) - \tilde{\nu}_k\big(\mathcal{F}(a)_{|S_{i-1}}\big)\big\|_{op} &= \Big\|F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ (\nu_k \circ \tau)\big(\mathcal{F}(a)_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k\Big\|_{op} \\
&= \big\|F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ \nu_k\big(p_{G/U}(a)\big) \circ F_k\big\|_{op} \\
&= \big\|F_k^* \circ \big(\pi_k(a) - \nu_k\big(p_{G/U}(a)\big)\big) \circ F_k\big\|_{op} \\
&\leq \big\|\nu_k\big(p_{G/U}(a)\big) - \pi_k(a)\big\|_{op} \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, the representations $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ fulfill Property 3(b) and thus, the conditions of Definition 1.1 are proved.

3.6 Condition 3(b) – Third case

Now, the third case mentioned in Proposition 3.1 is going to be considered. The approach will be similar to the one in the second case.

In the third and last case of Condition 3(b) of Definition 1.1, $\lambda \neq 0$ and there exists $1 \leq m < d$ such that $c_j \neq 0$ for every $j \in \{1, ..., m\}$ and $c_j = 0$ for every $j \in \{m+1, ..., d\}$.
This means that

$$\big\langle \ell_k, [X_j^k, Y_j^k]\big\rangle = c_j^k \lambda_k \overset{k\to\infty}{\longrightarrow} c_j\lambda = 0 \quad \Longleftrightarrow \quad j \in \{m+1, ..., d\}.$$

In this case, $\mathfrak{p} := \operatorname{span}\{X_{m+1}, ..., X_d, Y_1, ..., Y_d, T, Z, A_1, ..., A_p\}$ is a polarization of ℓ.
Moreover, for $\tilde{\mathfrak{p}}_k := \operatorname{span}\{X_{m+1}^k, ..., X_d^k, Y_1^k, ..., Y_d^k, T_k, Z_k, A_1^k, ..., A_p^k\}$, one has $\tilde{\mathfrak{p}}_k \overset{k\to\infty}{\longrightarrow} \mathfrak{p}$.
Let $P := \exp(\mathfrak{p})$ and $\tilde{P}_k := \exp(\tilde{\mathfrak{p}}_k)$.

3.6.1 Convergence of $(\pi_k)_{k\in\mathbb{N}}$ in $\widehat{G}$

In this section, the convergence in $\widehat{G}$ of the sequence $(\pi_k)_{k\in\mathbb{N}}$ is going to be examined in order to motivate the choice of the linear mappings $(\nu_k)_{k\in\mathbb{N}}$.

Let

$$(x)_\infty = (\dot{x}, \ddot{x})_\infty \;\text{ with }\; (\dot{x})_\infty := (x_1, ..., x_m)_\infty \;\text{ and }\; (\ddot{x})_\infty := (x_{m+1}, ..., x_d)_\infty$$

and analogously

$$(y)_\infty = (\dot{y}, \ddot{y})_\infty \;\text{ with }\; (\dot{y})_\infty := (y_1, ..., y_m)_\infty \;\text{ and }\; (\ddot{y})_\infty := (y_{m+1}, ..., y_d)_\infty.$$

Moreover, as in Section 3.3.3 above, let

$$(a)_\infty = (\dot a, \ddot a)_\infty \;\text{ with }\; (\dot a)_\infty = (a_1, ..., a_{\tilde p})_\infty \;\text{ and }\; (\ddot a)_\infty = (a_{\tilde p+1}, ..., a_p)_\infty$$

and let

$$(g)_\infty = (\dot x, \ddot x, \dot y, \ddot y, t, z, \dot a, \ddot a)_\infty = (x, y, t, z, a)_\infty = (x, h)_\infty.$$

Now, let $\ddot\alpha := (\alpha_{m+1}, ..., \alpha_d) \in \mathbb{R}^{d-m}$ and $\ddot\beta := (\beta_{m+1}, ..., \beta_d) \in \mathbb{R}^{d-m}$, consider $\ddot\alpha$ and $\ddot\beta$ as elements of $\mathbb{R}^d$ identifying them with $(0, ..., 0, \alpha_{m+1}, ..., \alpha_d)$ and $(0, ..., 0, \beta_{m+1}, ..., \beta_d)$, respectively, and let

$$\tilde\pi := \tilde\pi_{\ddot\alpha,\ddot\beta} := \mathrm{ind}_P^G \chi_{\ell+\ell_{\ddot\alpha,\ddot\beta}}.$$

Then, for a function $\dot\xi$ in the corresponding Hilbert space $\mathcal{H}_{\tilde\pi} = L^2\big(G/P, \chi_{\ell+\ell_{\ddot\alpha,\ddot\beta}}\big)$ of $\tilde\pi$ and for $s_1, ..., s_m \in \mathbb{R}$, $(\dot s)_\infty = (s_1, ..., s_m)_\infty \in \mathrm{span}\{X_1\} \times ... \times \mathrm{span}\{X_m\}$ and $\dot c = (c_1, ..., c_m)$, letting $\rho := \langle \ell, T\rangle$, one has similarly as in (5):

$$\begin{aligned}
& \tilde\pi\big((g)_\infty\big)\dot\xi\big((\dot s)_\infty\big) \\
= \; & \dot\xi\big((g)_\infty^{-1} \cdot (\dot s)_\infty\big) \\
= \; & \dot\xi\Big((\dot s)_\infty - (x)_\infty - (h)_\infty + \frac{1}{2}[-(x)_\infty - (h)_\infty, (\dot s)_\infty]\Big) \\
= \; & \dot\xi\Big(\big((\dot s)_\infty - (\dot x)_\infty\big) \cdot \Big(- (h)_\infty - (\ddot x)_\infty + [(\dot s)_\infty, (h)_\infty + (\ddot x)_\infty] \\
& \qquad - \frac{1}{2}[(\dot x)_\infty, (h)_\infty + (\dot s)_\infty + (\ddot x)_\infty]\Big)\Big) \\
= \; & e^{2\pi i\langle \ell+\ell_{\ddot\alpha,\ddot\beta}, -(h)_\infty - (\ddot x)_\infty + [(\dot s)_\infty, (h)_\infty + (\ddot x)_\infty] - \frac{1}{2}[(\dot x)_\infty, (h)_\infty + (\dot s)_\infty + (\ddot x)_\infty]\rangle} \dot\xi\big((\dot s - \dot x)_\infty\big) \\
= \; & e^{2\pi i\langle \ell+\ell_{\ddot\alpha,\ddot\beta}, -(\dot y)_\infty - (\ddot y)_\infty - (t)_\infty - (z)_\infty - (a)_\infty - (\ddot x)_\infty + [(\dot s)_\infty, (\dot y)_\infty + (\ddot y)_\infty + (t)_\infty + (z)_\infty + (a)_\infty + (\ddot x)_\infty]\rangle} \\
& e^{2\pi i\langle \ell+\ell_{\ddot\alpha,\ddot\beta}, -\frac{1}{2}[(\dot x)_\infty, (\dot y)_\infty + (\ddot y)_\infty + (t)_\infty + (z)_\infty + (a)_\infty + (\dot s)_\infty + (\ddot x)_\infty]\rangle} \dot\xi\big((\dot s - \dot x)_\infty\big) \qquad (9) \\
= \; & e^{2\pi i\langle \ell, -(t)_\infty - (z)_\infty + [(\dot s)_\infty, (\dot y)_\infty] - \frac{1}{2}[(\dot x)_\infty, (\dot y)_\infty]\rangle} e^{2\pi i\langle \ell_{\ddot\alpha,\ddot\beta}, -(\ddot x)_\infty - (\ddot y)_\infty\rangle} \dot\xi\big((\dot s - \dot x)_\infty\big) \\
= \; & e^{2\pi i\big(-t\rho - z\lambda + \sum_{j=1}^{m} \lambda c_j (s_j - \frac{1}{2}x_j) y_j\big)} e^{-2\pi i(\ddot\alpha(\ddot x)_\infty + \ddot\beta(\ddot y)_\infty)} \dot\xi\big((\dot s - \dot x)_\infty\big) \\
= \; & e^{2\pi i(-t\rho - z\lambda + \lambda\dot c(((\dot s)_\infty - \frac{1}{2}(\dot x)_\infty)(\dot y)_\infty))} e^{-2\pi i(\ddot\alpha(\ddot x)_\infty + \ddot\beta(\ddot y)_\infty)} \dot\xi\big((\dot s - \dot x)_\infty\big),
\end{aligned}$$

since $\ell(Y_j) = \ell(X_j) = 0$ for all $j \in \{1, ..., d\}$.

From now on again, most of the time, G will be identified with $\mathbb{R}^{2d+2+p}$. $C^*(G)$ does not coincide with $C^*(\mathbb{R}^{2d+2+p})$ though, since this last algebra is commutative while $C^*(G)$ is non-commutative. To avoid this inconvenience, most of the calculations will be accomplished in $L^1(G)$ or $\mathcal{S}(G)$ which in turn are isomorphic to $L^1(\mathbb{R}^{2d+2+p})$ and $\mathcal{S}(\mathbb{R}^{2d+2+p})$, respectively, as Fréchet spaces. Because of the density of $L^1(G)$ and $\mathcal{S}(G)$ in $C^*(G)$, the results will then follow.
So, in the sequel, the isomorphisms will be considered as Fréchet space-isomorphisms and not as algebra-isomorphisms.

Define for $\ddot{s} = (s_{m+1}, ..., s_d) \in \text{span}\{X_{m+1}\} \times ... \times \text{span}\{X_d\} \cong \mathbb{R}^{d-m}$,

$$\ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(\ddot{s}) := e^{2\pi i \ddot{\alpha}\ddot{s}} \prod_{j=m+1}^{d} \left|\lambda_k c_j^k\right|^{\frac{1}{4}} \eta_j\left(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}\left(s_j + \frac{\beta_j}{\lambda_k c_j^k}\right)\right)$$

and furthermore, for a function $\dot{\xi} \in \mathcal{H}_{\tilde{\pi}} = L^2\big(G/P, \chi_{\ell+\ell_{\ddot{\alpha},\ddot{\beta}}}\big) \cong L^2(\mathbb{R}^m)$ and an element $s = (\dot{s}, \ddot{s}) \in \big(\text{span}\{X_1\} \times ... \times \text{span}\{X_m\}\big) \times \big(\text{span}\{X_{m+1}\} \times ... \times \text{span}\{X_d\}\big) \cong \mathbb{R}^m \times \mathbb{R}^{d-m}$,

$$\xi_k(s) := \dot{\xi}(\dot{s})\ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(\ddot{s}).$$

As above in the second case, using Relation (5), one therefore gets for an element $g = (x, y, t, z, a) = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a}) \in \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\tilde{p}} \times \mathbb{R}^{p-\tilde{p}}$,

$$\begin{aligned}
& \langle \pi_k(g)\xi_k, \xi_k\rangle \\
= \; & \int_{\mathbb{R}^d} \pi_k(g)\xi_k(s)\overline{\xi_k(s)}ds \\
\overset{(5)}{=} \; & \int_{\mathbb{R}^d} e^{2\pi i\left(-t\rho_k - z\lambda_k + \sum_{j=1}^{d} \lambda_k c_j^k (s_j - \frac{1}{2}x_j)y_j\right)} \dot{\xi}(s_1 - x_1, ..., s_m - x_m) \\
& \quad \ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(s_{m+1} - x_{m+1}, ..., s_d - x_d)\overline{\dot{\xi}(s_1, ..., s_m)\ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(s_{m+1}, ..., s_d)}d(s_1, ..., s_d) \\
\underset{2nd\ case}{\overset{as\ in}{=}} \; & \int_{\mathbb{R}^m} \dot{\xi}(s_1 - x_1, ..., s_m - x_m)\overline{\dot{\xi}(s_1, ..., s_m)}\, e^{2\pi i \sum_{j=1}^{m} \lambda_k c_j^k (s_j - \frac{1}{2}x_j)y_j} d(s_1, ..., s_m) \\
& e^{-2\pi i(t\rho_k + z\lambda_k)} e^{-2\pi i \ddot{\alpha}\ddot{x}} \prod_{j=m+1}^{d} \Bigg(e^{-2\pi i \frac{1}{2} x_j y_j c_j^k \lambda_k} \int_{\mathbb{R}} e^{2\pi i y_j (s_j sgn(\lambda_k c_j^k)|\lambda_k c_j^k|^{\frac{1}{2}} - \beta_j)} \\
& \qquad \eta_j\left(s_j - \left|\lambda_k c_j^k\right|^{\frac{1}{2}} x_j\right)\overline{\eta_j(s_j)}ds_j \Bigg) \\
\overset{k\to\infty}{\longrightarrow} \; & \int_{\mathbb{R}^m} \dot{\xi}(s_1 - x_1, ..., s_m - x_m)\overline{\dot{\xi}(s_1, ..., s_m)}\, e^{2\pi i \sum_{j=1}^{m} \lambda c_j (s_j - \frac{1}{2}x_j)y_j} d(s_1, ..., s_m) \\
& e^{-2\pi i t\rho} e^{-2\pi i z\lambda} e^{-2\pi i \ddot{\alpha}\ddot{x}} \prod_{j=m+1}^{d} \left(\int_{\mathbb{R}} e^{-2\pi i y_j \beta_j} \eta_j(s_j)\overline{\eta_j(s_j)}ds_j \right) \\
\overset{\|\eta_j\|_2 = 1}{=} \; & e^{-2\pi i t\rho} e^{-2\pi i z\lambda} e^{-2\pi i \ddot{\alpha}\ddot{x}} e^{-2\pi i \ddot{\beta}\ddot{y}} \left(\int_{\mathbb{R}^m} e^{2\pi i \lambda \dot{c}(\dot{s} - \frac{1}{2}\dot{x})\dot{y}} \dot{\xi}(\dot{s} - \dot{x})\overline{\dot{\xi}(\dot{s})}d\dot{s} \right) \\
= \; & \left\langle \tilde{\pi}(g)\dot{\xi}, \dot{\xi}\right\rangle
\end{aligned}$$

and the coefficient functions $\left(c^k_{\ddot{\alpha},\ddot{\beta}}\right)_{k\in\mathbb{N}}$ defined by

$$c^k_{\ddot{\alpha},\ddot{\beta}}(g) := \langle \pi_k(g)\xi_k, \xi_k\rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}$$

converge uniformly on compacta to $c_{\ddot{\alpha},\ddot{\beta}}$ which in turn is defined by

$$c_{\ddot{\alpha},\ddot{\beta}}(g) := \left\langle \tilde{\pi}_{\ddot{\alpha},\ddot{\beta}}(g)\dot{\xi}, \dot{\xi}\right\rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}.$$

3.6.2 Definition of ν_k

Again, the linear mappings $(\nu_k)_{k\in\mathbb{N}}$ are going to be defined and analyzed.

For $0 \neq \eta' \in L^2(\tilde{P}_k/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^{d-m})$, let

$$P_{\eta'} : L^2(\mathbb{R}^{d-m}) \to \mathbb{C}\eta',\ \xi \mapsto \eta'\langle \xi, \eta'\rangle.$$

Then, $P_{\eta'}$ is the orthogonal projection onto the space $\mathbb{C}\eta'$.
Define now for $k \in \mathbb{N}$ and $h \in C^*(G/U, \chi_\ell)$ the linear operator

$$\nu_k(h) := \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x},\tilde{y})}(h) \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}} \frac{d(\tilde{x},\tilde{y})}{\prod_{j=m+1}^{d} \left|\lambda_k c_j^k\right|},$$

where $\pi_{\ell+(\tilde{x},\tilde{y})}$ is defined as $\operatorname{ind}_P^G \chi_{\ell+(\tilde{x},\tilde{y})}$ for an element $\ell + (\tilde{x},\tilde{y})$ which is located in $\ell + \Big(\big(\operatorname{span}\{X_{m+1}\} \times ... \times \operatorname{span}\{X_d\}\big) \times \big(\operatorname{span}\{Y_{m+1}\} \times ... \times \operatorname{span}\{Y_d\}\big)\Big)^* \cong \ell + \mathbb{R}^{2(d-m)}$.
Thus, for $L^2(\mathbb{R}^d) \ni \xi = \sum_{i=1}^{\infty} \dot{\xi}_i \otimes \ddot{\xi}_i$ with $\dot{\xi}_i \in L^2(\mathbb{R}^m)$ and $\ddot{\xi}_i \in L^2(\mathbb{R}^{d-m})$ for all $i \in \mathbb{N}$, one has

$$\nu_k(h)(\xi) := \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x},\tilde{y})}(h)(\dot{\xi}_i) \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{\xi}_i) \frac{d(\tilde{x},\tilde{y})}{\prod_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}$$

and for $s = (s_1, ..., s_d) \in \mathbb{R}^d$,

$$\nu_k(h)\xi(s) := \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x},\tilde{y})}(h)(\dot{\xi}_i)(s_1, ..., s_m) \cdot P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{\xi}_i)(s_{m+1}, ..., s_d) \frac{d(\tilde{x},\tilde{y})}{\prod_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}.$$

Since the operators

$$\pi_{\ell+(\tilde{x},\tilde{y})}(h) : L^2(G/P, \chi_\ell) \cong L^2(\mathbb{R}^m) \to L^2(G/P, \chi_\ell) \cong L^2(\mathbb{R}^m) \quad \text{and}$$

$$P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}} : L^2(\tilde{P}_k/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^{d-m}) \to L^2(\tilde{P}_k/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^{d-m})$$

are bounded and the tensor product of two bounded operators on Hilbert spaces is bounded by the product of the two operator norms, the operator

$$\pi_{\ell+(\tilde{x},\tilde{y})}(h) \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is bounded on $L^2(\mathbb{R}^d)$ by $\|\pi_{\ell+(\tilde{x},\tilde{y})}(h)\|_{op}\|P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}\|_{op}$.

Proposition 3.5.

1. *For all $k \in \mathbb{N}$ and $h \in \mathcal{S}(G/U, \chi_\ell)$, the integral defining $\nu_k(h)$ converges in the operator norm.*
2. *The operator $\nu_k(h)$ is compact and $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U,\chi_\ell)}$.*
3. *ν_k is involutive.*

Proof:
Let $\mathcal{K} = \mathcal{K}(L^2(\mathbb{R}^m))$ be the C^*-algebra of the compact operators on the Hilbert space $L^2(\mathbb{R}^m)$ and $C_\infty(\mathbb{R}^{2(d-m)}, \mathcal{K})$ the C^*-algebra of all continuous mappings from $\mathbb{R}^{2(d-m)}$ into $\mathcal{K}$ vanishing at infinity.
Define for $\varphi \in C_\infty(\mathbb{R}^{2(d-m)}, \mathcal{K})$ and $k \in \mathbb{N}$ the linear operator

$$\mu_k(\varphi) := \int\limits_{\mathbb{R}^{2(d-m)}} \varphi(\tilde{x}, \tilde{y}) \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}} \frac{d(\tilde{x}, \tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}$$

on $L^2(\mathbb{R}^d)$. Then, as $\mathcal{F}(h) \in C_\infty(\mathbb{R}^{2(d-m)}, \mathcal{K})$ for every $h \in C^*(G/U, \chi_\ell)$,

$$\nu_k(h) = \mu_k(\mathcal{F}(h)).$$

1) Since $\mathcal{F}(h) \in \mathcal{S}(\mathbb{R}^{2(d-m)}, \mathcal{K})$ for $h \in \mathcal{S}(G/U, \chi_\ell)$ and since

$$\|\mu_k(\varphi)\|_{\text{op}} \leq \int\limits_{\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x}, \tilde{y})\|_{\text{op}} \frac{d(\tilde{x}, \tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^{2(d-m)}, \mathcal{K})$ and $k \in \mathbb{N}$, the first assertion follows immediately.

2) As $p_{G/U}$ is surjective from $\mathcal{S}(G)$ to $\mathcal{S}(G/U, \chi_\ell)$, for every $h \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$, there exists a function $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$ such that $h = p_{G/U}(f)$ and, as shown in the second case, for $\tilde{g} = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, 0, 0, 0, 0) \in G/U \cong \mathbb{R}^{2d}$, one has

$$h_\infty(\tilde{g}) = \hat{f}_\infty^{5,6,7,8}(\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, \rho, \lambda, 0, 0), \tag{10}$$

where again $h_\infty = h((\cdot)_\infty)$ and $f_\infty = f((\cdot)_\infty)$.

Now, let $s_1, ..., s_m \in \mathbb{R}$ and $\dot{s} = (s_1, ..., s_m)_\infty = \sum\limits_{j=1}^{m} s_j X_j$ and moreover, with $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, \dot{a}, \ddot{a}$ and $\dot{c}$ as above, let $(g)_\infty = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a})_\infty = (x, y, t, z, a)_\infty = (x, h)_\infty$. By Calculation (9) for $\ell + (\tilde{x}, \tilde{y})$ instead of $\ell + \ell_{\ddot{\alpha},\ddot{\beta}}$, one gets for $\dot{\xi}_i \in L^2(\mathbb{R}^m)$,

$$\begin{aligned}
& \pi_{\ell+(\tilde{x},\tilde{y})}\big((g)_\infty\big)\dot{\xi}_i\big((\dot{s})_\infty\big) \\
= \; & e^{2\pi i\langle \ell+(\tilde{x},\tilde{y}), -(\dot{y})_\infty-(\ddot{y})_\infty-(t)_\infty-(z)_\infty-(a)_\infty-(\ddot{x})_\infty+[(\dot{s})_\infty,(\dot{y})_\infty+(\ddot{y})_\infty+(t)_\infty+(z)_\infty+(a)_\infty+(\ddot{x})_\infty]\rangle} \\
& e^{2\pi i\langle \ell+(\tilde{x},\tilde{y}), -\frac{1}{2}[(\dot{x})_\infty,(\dot{y})_\infty+(\ddot{y})_\infty+(t)_\infty+(z)_\infty+(a)_\infty+(\dot{s})_\infty+(\ddot{x})_\infty]\rangle}\dot{\xi}_i\big((\dot{s}-\dot{x})_\infty\big) \\
= \; & e^{2\pi i\langle \ell, -(t)_\infty-(z)_\infty+[(\dot{s})_\infty,(\dot{y})_\infty]-\frac{1}{2}[(\dot{x})_\infty,(\dot{y})_\infty]\rangle+\langle(\tilde{x},\tilde{y}),-(\ddot{y})_\infty-(\ddot{x})_\infty\rangle}\dot{\xi}_i\big((\dot{s}-\dot{x})_\infty\big) \\
= \; & e^{2\pi i(-t\rho-z\lambda+\lambda\dot{c}(((\dot{s})_\infty-\frac{1}{2}(\dot{x})_\infty)(\dot{y})_\infty)-\langle\tilde{x},(\ddot{x})_\infty\rangle-\langle\tilde{y},(\ddot{y})_\infty\rangle)}\dot{\xi}_i\big((\dot{s}-\dot{x})_\infty\big).
\end{aligned}$$

Identifying with $\mathbb{R}^{2d+2+p}$ again, one gets with (10) for $h = p_{G/U}(f) \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$ for a function $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$,

$$
\begin{aligned}
&\pi_{\ell+(\tilde{x},\tilde{y})}(h)\dot{\xi}_i(\dot{s}) \\
= &\int\limits_{\mathbb{R}^{2d}} \big(p_{G/U}(f)\big)_\infty(\tilde{g})\pi_{\ell+(\tilde{x},\tilde{y})}(\tilde{g})\dot{\xi}_i(\dot{s})d\tilde{g} \\
= &\int\limits_{\mathbb{R}^{2d}} \int\limits_{\mathbb{R}^{2+\tilde{p}+(p-\tilde{p})}} f_\infty(\tilde{g}u)\chi_\ell(u)du\ \pi_{\ell+(\tilde{x},\tilde{y})}(\tilde{g})\dot{\xi}_i(\dot{s})d\tilde{g} \\
= &\int\limits_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{\tilde{p}}\times\mathbb{R}^{p-\tilde{p}}} f_\infty(\dot{x},\ddot{x},\dot{y},\ddot{y},t,z,\dot{a},\ddot{a})e^{2\pi i(\lambda\dot{c}((\dot{s}-\frac{1}{2}\dot{x})\dot{y})-\langle\tilde{x},\ddot{x}\rangle-\langle\tilde{y},\ddot{y}\rangle)} \\
&\qquad e^{-2\pi i(t\rho+z\lambda)}\dot{\xi}_i(\dot{s}-\dot{x})d(\dot{x},\ddot{x},\dot{y},\ddot{y},t,z,\dot{a},\ddot{a}) \\
= &\int\limits_{\mathbb{R}^{2m}} \hat{f}_\infty^{2,4,5,6,7,8}(\dot{x},\tilde{x},\dot{y},\tilde{y},\rho,\lambda,0,0)e^{2\pi i\lambda\dot{c}((\dot{s}-\frac{1}{2}\dot{x})\dot{y})}\dot{\xi}_i(\dot{s}-\dot{x})d(\dot{x},\dot{y}) \\
\overset{(10)}{=} &\int\limits_{\mathbb{R}^{2m}} \hat{h}_\infty^{2,4}(\dot{x},\tilde{x},\dot{y},\tilde{y})e^{2\pi i\lambda\dot{c}((\dot{s}-\frac{1}{2}\dot{x})\dot{y})}\dot{\xi}_i(\dot{s}-\dot{x})d(\dot{x},\dot{y}) \\
= &\int\limits_{\mathbb{R}^m} \hat{h}_\infty^{2,3,4}\Big(\dot{x},\tilde{x},\lambda\dot{c}\Big(\frac{1}{2}\dot{x}-\dot{s}\Big),\tilde{y}\Big)\dot{\xi}_i(\dot{s}-\dot{x})d\dot{x}. \qquad (11)
\end{aligned}
$$

Regard now the second factor $P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}$ of the tensor product.
As in the second case above, define

$$\ddot{\eta}_{k,\ddot{\beta}}(\ddot{s}) := \prod_{j=m+1}^{d} \big|\lambda_k c_j^k\big|^{\frac{1}{4}}\ \eta_j\bigg(\big|\lambda_k c_j^k\big|^{\frac{1}{2}}\bigg(s_j + \frac{\beta_j}{\lambda_k c_j^k}\bigg)\bigg).$$

Then,

$$\ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(\ddot{s}) = e^{2\pi i\ddot{\alpha}\ddot{s}}\ddot{\eta}_{k,\ddot{\beta}}(\ddot{s})$$

and therefore, with $\ddot{\xi}_i \in L^2(\mathbb{R}^{d-m})$,

$$
\begin{aligned}
P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{\xi}_i)(\ddot{s}) &= \langle\ddot{\xi}_i,\ddot{\eta}_{k,\tilde{x},\tilde{y}}\rangle\ddot{\eta}_{k,\tilde{x},\tilde{y}}(\ddot{s}) \\
&= \bigg(\int\limits_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r})\overline{\ddot{\eta}}_{k,\tilde{x},\tilde{y}}(\ddot{r})d\ddot{r}\bigg)\ddot{\eta}_{k,\tilde{x},\tilde{y}}(\ddot{s}) \\
&= \bigg(\int\limits_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r})e^{-2\pi i\tilde{x}\ddot{r}}\ \overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\ddot{r}\bigg)e^{2\pi i\tilde{x}\ddot{s}}\ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \\
&= \int\limits_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r})e^{2\pi i\tilde{x}(\ddot{s}-\ddot{r})}\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\ddot{r}\ \ddot{\eta}_{k,\tilde{y}}(\ddot{s}).
\end{aligned}
$$

Joining together the calculation above and the one for the first factor of the tensor product (11), one gets for $\mathcal{S}(\mathbb{R}^d) \ni \xi = \sum\limits_{i=1}^{\infty} \dot{\xi}_i \otimes \ddot{\xi}_i$ with $\dot{\xi}_i \in \mathcal{S}(\mathbb{R}^m)$ and $\ddot{\xi}_i \in \mathcal{S}(\mathbb{R}^{d-m})$ for all $i \in \mathbb{N}$, $h \in \mathcal{S}(\mathbb{R}^{2d})$ and $s = (\dot{s}, \ddot{s}) \in \mathbb{R}^d$,

$$
\begin{aligned}
&\nu_k(h)(\xi)(s)\\
&= \sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x},\tilde{y})}(h)(\dot{\xi}_i)(\dot{s}) \cdot P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{\xi}_i)(\ddot{s}) \; \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&= \sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^{2(d-m)}} \Bigg(\int\limits_{\mathbb{R}^m} \hat{h}_{\infty}^{2,3,4}\Big(\dot{x}, \tilde{x}, \lambda\dot{c}\Big(\frac{1}{2}\dot{x} - \dot{s}\Big), \tilde{y}\Big)\dot{\xi}_i(\dot{s} - \dot{x})d\dot{x}\Bigg)\\
&\quad \cdot\Bigg(\int\limits_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r})e^{2\pi i\tilde{x}(\ddot{s}-\ddot{r})}\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\ddot{r} \; \ddot{\eta}_{k,\tilde{y}}(\ddot{s})\Bigg) \; \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&= \sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^{2(d-m)}} \int\limits_{\mathbb{R}^{d-m}} \int\limits_{\mathbb{R}^m} \hat{h}_{\infty}^{2,3,4}\Big(\dot{x}, \tilde{x}, \lambda\dot{c}\Big(\frac{1}{2}\dot{x} - \dot{s}\Big), \tilde{y}\Big)\dot{\xi}_i(\dot{s} - \dot{x})\\
&\qquad\qquad \ddot{\xi}_i(\ddot{r})e^{2\pi i\tilde{x}(\ddot{s}-\ddot{r})}\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\dot{x}d\ddot{r} \; \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \; \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&= \sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^{d-m}} \int\limits_{\mathbb{R}^{d-m}} \int\limits_{\mathbb{R}^m} \hat{h}_{\infty}^{3,4}\Big(\dot{x}, \ddot{s} - \ddot{r}, \lambda\dot{c}\Big(\frac{1}{2}\dot{x} - \dot{s}\Big), \tilde{y}\Big)\dot{\xi}_i(\dot{s} - \dot{x})\\
&\qquad\qquad \ddot{\xi}_i(\ddot{r})\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\dot{x}d\ddot{r} \; \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \; \frac{d\tilde{y}}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&= \sum_{i=1}^{\infty} \int\limits_{\mathbb{R}^{d-m}} \int\limits_{\mathbb{R}^{d-m}} \int\limits_{\mathbb{R}^m} \hat{h}_{\infty}^{3,4}\Big(\dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2}\dot{c}(-\dot{x} - \dot{s}), \tilde{y}\Big)\dot{\xi}_i(\dot{x})\\
&\qquad\qquad \ddot{\xi}_i(\ddot{r})\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})d\dot{x}d\ddot{r} \; \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \; \frac{d\tilde{y}}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&= \int\limits_{\mathbb{R}^{d}} \int\limits_{\mathbb{R}^{d-m}} \hat{h}_{\infty}^{3,4}\Big(\dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2}\dot{c}(-\dot{x} - \dot{s}), \tilde{y}\Big)\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})\ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \; \frac{d\tilde{y}}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\\
&\qquad \xi(\dot{x},\ddot{r})d(\dot{x},\ddot{r}).
\end{aligned} \tag{12}
$$

Therefore, the kernel function

$$
h_K\big((\dot{s},\ddot{s}),(\dot{x},\ddot{r})\big) := \int\limits_{\mathbb{R}^{d-m}} \hat{h}_{\infty}^{3,4}\Big(\dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2}\dot{c}(-\dot{x} - \dot{s}), \tilde{y}\Big)\overline{\ddot{\eta}}_{k,\tilde{y}}(\ddot{r})\ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \; \frac{d\tilde{y}}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}
$$

of $\nu_k(h)$ is in $\mathcal{S}(\mathbb{R}^{2d})$ and thus, $\nu_k(h)$ is a compact operator for $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_\ell)$. With the density of $\mathcal{S}(G/U, \chi_\ell)$ in $C^*(G/U, \chi_\ell)$, it is compact for every $h \in C^*(G/U, \chi_\ell)$.

Now, it will be shown that for every $\varphi \in C_\infty(\mathbb{R}^{2(d-m)}, \mathcal{K})$,

$$\|\mu_k(\varphi)\|_{\text{op}} \leq \|\varphi\|_\infty := \sup_{(\tilde{x},\tilde{y})\in\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x},\tilde{y})\|_{\text{op}}.$$

For this, for any $\psi \in L^2(\mathbb{R}^d)$, define

$$f_{\psi,k}(\tilde{x},\tilde{y})(\dot{s}) := \int\limits_{\mathbb{R}^{d-m}} \psi(\dot{s},\ddot{s})\overline{\ddot{\eta}}_{k,\tilde{x},\tilde{y}}(\ddot{s})d\ddot{s} \quad \forall(\tilde{x},\tilde{y}) \in \mathbb{R}^{2(d-m)}\ \forall \dot{s} \in \mathbb{R}^m.$$

Then, as

$$\mathbb{I}_{L^2(\mathbb{R}^{d-m})} = \int\limits_{\mathbb{R}^{2(d-m)}} P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|},$$

one gets the identity

$$\|\psi\|^2_{L^2(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(\tilde{x},\tilde{y})\|^2_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}. \tag{13}$$

Now, for $\xi, \psi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}
&\left|\langle\mu_k(\varphi)\xi,\psi\rangle_{L^2(\mathbb{R}^d)}\right| \\
=\ &\left|\int\limits_{\mathbb{R}^{2(d-m)}} \big\langle(\varphi(\tilde{x},\tilde{y}) \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}})\xi,\psi\big\rangle_{L^2(\mathbb{R}^d)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\right| \\
=\ &\Bigg|\int\limits_{\mathbb{R}^{2(d-m)}} \big\langle(\varphi(\tilde{x},\tilde{y}) \otimes \mathbb{I}_{L^2(\mathbb{R}^{d-m})}) \circ (\mathbb{I}_{L^2(\mathbb{R}^m)} \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}})\xi, \\
&\qquad (\mathbb{I}_{L^2(\mathbb{R}^m)} \otimes P_{\ddot{\eta}_{k,\tilde{x},\tilde{y}}})\psi\big\rangle_{L^2(\mathbb{R}^d)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\Bigg| \\
=\ &\left|\int\limits_{\mathbb{R}^{2(d-m)}} \langle\varphi(\tilde{x},\tilde{y})f_{\xi,k}(\tilde{x},\tilde{y}), f_{\psi,k}(\tilde{x},\tilde{y})\rangle_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\right| \\
\overset{\substack{\text{Cauchy–}\\ \text{Schwarz}}}{\leq}\ &\left(\int\limits_{\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x},\tilde{y})f_{\xi,k}(\tilde{x},\tilde{y})\|^2_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\right)^{\frac{1}{2}} \\
&\left(\int\limits_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(\tilde{x},\tilde{y})\|^2_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|}\right)^{\frac{1}{2}}
\end{aligned}$$

$$\overset{(13)}{\leq} \sup_{(\tilde{x},\tilde{y})\in\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x},\tilde{y})\|_{\text{op}}$$

$$\left(\int\limits_{\mathbb{R}^{2(d-m)}} \|f_{\xi,k}(\tilde{x},\tilde{y})\|^2_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x},\tilde{y})}{\prod\limits_{j=m+1}^{d} \left|\lambda_k c_j^k\right|} \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^d)}$$

$$\overset{(13)}{\leq} \|\varphi\|_\infty \|\xi\|_{L^2(\mathbb{R}^d)} \|\psi\|_{L^2(\mathbb{R}^d)}.$$

Hence, for every $h \in C^*(G/U,\chi_\ell)$,

$$\|\nu_k(h)\|_{\text{op}} = \|\mu_k(\mathcal{F}(h))\|_{\text{op}} \leq \|\mathcal{F}(h)\|_\infty = \|h\|_{C^*(G/U,\chi_\ell)}.$$

3) To show that ν_k is involutive is straightforward as in the second case.

□

Like in the second case mentioned in Proposition 3.1, this proves that the image of ν_k is contained in $\mathcal{B}(\mathcal{H}_i)$, as well as the required boundedness and involutiveness of the mappings ν_k for every $k \in \mathbb{N}$. Hence, only a modification of the demanded convergence of Condition 3(b) of Definition 1.1 remains to be shown.

3.6.3 Theorem – Third Case

Theorem 3.6.
For $a \in C^*(G)$

$$\lim_{k\to\infty} \left\|\pi_k(a) - \nu_k\big(p_{G/U}(a)\big)\right\|_{op} = 0.$$

Proof:
Let $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$ such that its Fourier transform in $[\mathfrak{g},\mathfrak{g}]$ has a compact support in $G \cong \mathbb{R}^{2d+2+p}$. In the setting of this third case, this means that $\hat{f}^{6,7}$ has a compact support in G (compare the proof of Theorem 3.4).

Identify G with $\mathbb{R}^{2d+2+p}$ again, let $\xi \in L^2(\mathbb{R}^d)$ and $s = (s_1,...,s_d) = (\dot{s},\ddot{s}) \in \mathbb{R}^m \times \mathbb{R}^{d-m} \cong \mathbb{R}^d$ and define

$$\ddot{\eta}_{k,0}(\ddot{s}) := \prod_{j=m+1}^{d} \left|\lambda_k c_j^k\right|^{\frac{1}{4}} \eta_j\Big(\left|\lambda_k c_j^k\right|^{\frac{1}{2}}(s_j)\Big).$$

Let $\dot{c} = (c_1,...,c_m)$, $\ddot{c} = (c_{m+1},...,c_d) = (0,...,0)$, $\dot{c}^k = (c_1^k,...,c_m^k)$ and $\ddot{c}^k = (c_{m+1}^k,...,c_d^k)$.
As in the second case, the expression $\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)$ will now be regarded, divided into several parts and then estimated. For this, Equation (6) from Section 3.3.4 will be used again but its notation needs to be adapted.

$$\begin{aligned}
\pi_k(f)(s) &\overset{(6)}{=} \int\limits_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6}\Big(s-r, -\frac{\lambda_k c^k}{2}(s+r), \rho_k, \lambda_k, 0, 0\Big)\xi(r)dr \\
&= \int\limits_{\mathbb{R}^d} \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}), \rho_k, \lambda_k, 0, 0\Big) \\
&\qquad \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}).
\end{aligned}$$

Using the above equation, Result (12) and the fact that $p_{G/U}(f) = \hat{f}^{5,6,7,8}(\cdot,\cdot,\cdot,\cdot,\rho,\lambda,0,0)$, one gets

$$
\begin{aligned}
&\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)\xi(s) \\
\overset{(12)}{=}\ &\int_{\mathbb{R}^d} \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho_k,\lambda_k,0,0\Big)\xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}) \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-m}} \widehat{p_{G/U}(f)}_\infty^{3,4}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},\frac{\lambda}{2}\dot{c}(-\dot{r}-\dot{s}),\tilde{y}\Big)\overline{\ddot{\eta}_{k,\tilde{y}}}(\ddot{r})\ddot{\eta}_{k,\tilde{y}}(\ddot{s})\ \frac{d\tilde{y}}{\prod_{j=m+1}^{d}\big|\lambda_k c_j^k\big|} \\
&\qquad \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}) \\
\overset{\|\ddot{\eta}_{k,0}\|_2=1}{=}\ &\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-m}} \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho_k,\lambda_k,0,0\Big) \\
&\qquad \overline{\ddot{\eta}_{k,0}}(\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y}\ \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}) \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-m}} \hat{f}_\infty^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},\frac{\lambda}{2}\dot{c}(-\dot{r}-\dot{s}),\tilde{y},\rho,\lambda,0,0\Big)\overline{\ddot{\eta}_{k,\tilde{y}}}(\ddot{r}) \\
&\qquad \ddot{\eta}_{k,\tilde{y}}(\ddot{s})\ \frac{d\tilde{y}}{\prod_{j=m+1}^{d}\big|\lambda_k c_j^k\big|}\ \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}) \\
=\ &\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-m}} \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho_k,\lambda_k,0,0\Big) \\
&\qquad \overline{\ddot{\eta}_{k,0}}(\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y}\ \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}) \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-m}} \hat{f}_\infty^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big) \\
&\qquad \overline{\ddot{\eta}_{k,0}}(\tilde{y}+\ddot{r}-\ddot{s})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y}\ \xi(\dot{r},\ddot{r})d(\dot{r},\ddot{r}).
\end{aligned}
$$

Similarly as in the second case, functions q_k, u_k, v_k, o_k, n_k and w_k are going to be defined in order to divide the above integrals into six parts.

$$
\begin{aligned}
q_k(s,\tilde{y}) \ := \ &\int_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\overline{\ddot{\eta}_{k,0}}(\tilde{y}+\ddot{r}-\ddot{s}) \\
&\Big(\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho_k,\lambda_k,0,0\Big) \\
&\quad - \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho,\lambda_k,0,0\Big)\Big)d(\dot{r},\ddot{r}),
\end{aligned}
$$

$$u_k(s,\tilde{y}) \;:=\; \int\limits_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})$$
$$\Big(\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k\ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho,\lambda_k,0,0\Big)$$
$$-\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k\ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho,\lambda,0,0\Big)\Big)d(\dot{r},\ddot{r}),$$

$$v_k(s,\tilde{y}) \;:=\; \int\limits_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})$$
$$\Big(\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k\ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho,\lambda,0,0\Big)$$
$$-\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big)\Big)d(\dot{r},\ddot{r}),$$

$$o_k(s,\tilde{y}) \;:=\; \int\limits_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})$$
$$\Big(\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big)$$
$$-\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big)\Big)d(\dot{r},\ddot{r}),$$

$$n_k(s,\tilde{y}) \;:=\; \int\limits_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})$$
$$\Big(\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big)$$
$$-\hat{f}_\infty^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}),\lambda_k\ddot{c}^k(\tilde{y}-\ddot{s}),\rho,\lambda,0,0\Big)\Big)d(\dot{r},\ddot{r})$$

and

$$w_k(s) \;:=\; \int\limits_{\mathbb{R}^{d-m}}\int\limits_{\mathbb{R}^d} \xi(\dot{r},\ddot{r})\ddot{\eta}_{k,0}(\tilde{y})\Big(\,\overline{\ddot{\eta}}_{k,0}(\tilde{y})-\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})\Big)$$
$$\hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r},\ddot{s}-\ddot{r},-\frac{\lambda_k\dot{c}^k}{2}(\dot{s}+\dot{r}),-\frac{\lambda_k\ddot{c}^k}{2}(\ddot{s}+\ddot{r}),\rho_k,\lambda_k,0,0\Big)d(\dot{r},\ddot{r})d\tilde{y}.$$

Then,

$$\big(\pi_k(f)-\nu_k\big(p_{G/U}(f)\big)\big)\xi(s) \;=\; \int\limits_{\mathbb{R}^{d-m}} q_k(s,\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y} \;+\; \int\limits_{\mathbb{R}^{d-m}} u_k(s,\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y}$$
$$+\int\limits_{\mathbb{R}^{d-m}} v_k(s,\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y} \;+\; \int\limits_{\mathbb{R}^{d-m}} o_k(s,\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y}$$
$$+\int\limits_{\mathbb{R}^{d-m}} n_k(s,\tilde{y})\ddot{\eta}_{k,0}(\tilde{y})d\tilde{y} \;+\; w_k(s).$$

As in the second case, to show that

$$\left\|\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\right\|_{op} \overset{k\to\infty}{\longrightarrow} 0,$$

one has to prove that there are κ_k, γ_k, δ_k, τ_k, ω_k and ϵ_k which are tending to 0 for $k \to \infty$ such that

$$\|q_k\|_2 \leq \kappa_k\|\xi\|_2, \quad \|u_k\|_2 \leq \gamma_k\|\xi\|_2, \quad \|v_k\|_2 \leq \delta_k\|\xi\|_2, \quad \|o_k\|_2 \leq \tau_k\|\xi\|_2,$$

$$\|n_k\|_2 \leq \omega_k\|\xi\|_2 \quad \text{and} \quad \|w_k\|_2 \leq \epsilon_k\|\xi\|_2.$$

The estimation of the functions q_k, u_k, v_k, n_k and w_k is similar to their estimation in the second case and will thus be skipped. So, only the estimation of o_k remains.

For this, first regard the last factor of the function o_k:

$$\begin{aligned}
& \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}), \lambda_k \ddot{c}^k(\tilde{y}-\ddot{s}), \rho, \lambda, 0, 0\Big) \\
& - \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, -\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}), \lambda_k \ddot{c}^k(\tilde{y}-\ddot{s}), \rho, \lambda, 0, 0\Big) \\
= \; & \Big(\frac{1}{2}(\lambda\dot{c} - \lambda_k\dot{c}^k)(\dot{s}+\dot{r})\Big) \cdot \int_0^1 \partial_3 \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, \\
& \qquad -\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}) + t\Big(\frac{1}{2}(\lambda\dot{c} - \lambda_k\dot{c}^k)(\dot{s}+\dot{r})\Big), \lambda_k \ddot{c}^k(\tilde{y}-\ddot{s}), \rho, \lambda, 0, 0\Big) dt.
\end{aligned}$$

Thus, there exists a constant $C_1 > 0$ depending on f such that

$$\begin{aligned}
& \Big| \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s}+\dot{r}), \lambda_k \ddot{c}^k(\tilde{y}-\ddot{s}), \rho, \lambda, 0, 0\Big) \\
& \quad - \hat{f}_k^{3,4,5,6,7,8}\Big(\dot{s}-\dot{r}, \ddot{s}-\ddot{r}, -\frac{\lambda}{2}\dot{c}(\dot{s}+\dot{r}), \lambda_k \ddot{c}^k(\tilde{y}-\ddot{s}), \rho, \lambda, 0, 0\Big)\Big| \\
\leq \; & \big\|\lambda\dot{c} - \lambda_k\dot{c}^k\big\| \|\dot{s}+\dot{r}\| \; \frac{C_1}{(1+\|\dot{s}+\dot{r}\|)^{2d+1}\,(1+\|\ddot{s}-\ddot{r}\|)^{2d}}.
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}
& \|o_k\|_2^2 \\
= \; & \int_{\mathbb{R}^{d+(d-m)}} |o_k(s,\tilde{y})|^2 d(s,\tilde{y}) \\
\leq \; & \int_{\mathbb{R}^{d+(d-m)}} \Big(\int_{\mathbb{R}^d} |\xi(\dot{r},\ddot{r})| \big|\overline{\eta}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})\big| \big\|\lambda\dot{c} - \lambda_k\dot{c}^k\big\| \|\dot{s}+\dot{r}\| \\
& \qquad \frac{C_1}{(1+\|\dot{s}+\dot{r}\|)^{2d+1}\,(1+\|\ddot{s}-\ddot{r}\|)^{2d}} \, d(\dot{r},\ddot{r})\Big)^2 d(\dot{s},\ddot{s},\tilde{y})
\end{aligned}$$

$$\overset{Cauchy-}{\underset{Schwarz}{\leq}} C_1^2 \big\|\lambda\dot{c} - \lambda_k \dot{c}^k\big\|^2 \int\limits_{\mathbb{R}^{d+(d-m)}} \Bigg(\int\limits_{\mathbb{R}^d} \frac{1}{(1+\|\dot{s}+\dot{r}\|)^{2d}\,(1+\|\ddot{s}-\ddot{r}\|)^{2d}}\ d(\dot{r},\ddot{r}) \Bigg)$$

$$\Bigg(\int\limits_{\mathbb{R}^d} |\xi(\dot{r},\ddot{r})|^2 \big|\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})\big|^2 \frac{\|\dot{s}+\dot{r}\|^2}{(1+\|\dot{s}+\dot{r}\|)^{2d+2}(1+\|\ddot{s}-\ddot{r}\|)^{2d}}\ d(\dot{r},\ddot{r}) \Bigg) d(\dot{s},\ddot{s},\tilde{y})$$

$$\leq C_1' \big\|\lambda\dot{c} - \lambda_k \dot{c}^k\big\|^2 \int\limits_{\mathbb{R}^{d+(d-m)}} \int\limits_{\mathbb{R}^d} |\xi(\dot{r},\ddot{r})|^2 \big|\overline{\ddot{\eta}}_{k,0}(\tilde{y}+\ddot{r}-\ddot{s})\big|^2$$

$$\frac{1}{(1+\|\dot{s}+\dot{r}\|)^{2d}\,(1+\|\ddot{s}-\ddot{r}\|)^{2d}}\ d(\dot{r},\ddot{r}) d(\dot{s},\ddot{s},\tilde{y})$$

$$\overset{\|\eta_{k,0}\|_2=1}{=} C_1'' \big\|\lambda\dot{c} - \lambda_k \dot{c}^k\big\|^2 \|\xi\|_2^2,$$

with matching constants $C_1' > 0$ and $C_1'' > 0$ depending on f. Hence, $\tau_k := \sqrt{C_1''}\ \big\|\lambda\dot{c} - \lambda_k\dot{c}^k\big\|$ fulfills $\tau_k \overset{k\to\infty}{\longrightarrow} 0$ and

$$\|o_k\|_2 \leq \tau_k \|\xi\|_2.$$

Thus, for those $f \in \mathcal{S}(\mathbb{R}^{2d+2+p}) \cong \mathcal{S}(G)$ whose Fourier transform in $[\mathfrak{g},\mathfrak{g}]$ has a compact support,

$$\big\|\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big\|_{op} = \sup_{\substack{\xi\in L^2(\mathbb{R}^d)\\ \|\xi\|_2=1}} \big\|\big(\pi_k(f) - \nu_k\big(p_{G/U}(f)\big)\big)(\xi)\big\|_2 \overset{k\to\infty}{\longrightarrow} 0.$$

As in the second case, because of the density in $C^*(G)$ of the set of Schwartz functions whose partial Fourier transform has a compact support, the claim follows for all $a \in C^*(G)$ by Corollary 7.6.

□

3.6.4 Transition to $\big(\pi_k^V\big)_{k\in\mathbb{N}}$

Now, the assertions for the sequence $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ can be deduced.

Again, because of the equivalence of the representations π_k and π_k^V for every $k \in \mathbb{N}$, there exist unitary intertwining operators

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{such that} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall a \in C^*(G).$$

With the injective $*$-homomorphism

$$\tau : C_\infty\big(\mathbb{R}^{2(d-m)},\mathcal{K}\big) \to C^*(G/U,\chi_\ell),\ \mathcal{F}(a)_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})} \mapsto p_{G/U}(a),$$

define

$$\tilde{\nu}_k(\varphi) := F_k^* \circ (\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k \quad \forall \varphi \in CB(S_{i-1}).$$

Then, like in the second case, $\tilde{\nu}_k$ complies with the demanded requirements and thus, the original representations $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ fulfill Property 3(b).

This completes the proof of the three conditions of Definition 1.1 which are needed in order to be able to use the above-mentioned theorem of H.Regeiba and J.Ludwig (see [31], Theorem 3.5) for the determination of $C^*(G)$.

3.7 Result for the connected real two-step nilpotent Lie groups

Finally, with the families of subsets $(S_i)_{i\in\{0,...,r\}}$ and $(\Gamma_i)_{i\in\{0,...,r\}}$ of $\widehat{G}$ that were constructed in Section 3.2 and the Hilbert spaces $(\mathcal{H}_i)_{i\in\{0,...,r\}}$ which were defined in the same section fulfilling the conditions listed in Definition 1.1, one obtains the following result for the connected real two-step nilpotent Lie groups which was already stated in Section 1:

Theorem 3.7.
The C^*-algebra $C^*(G)$ of a connected real two-step nilpotent Lie group G is isomorphic (under the Fourier transform) to the set of all operator fields φ defined over the spectrum $\widehat{G}$ of the respective group such that:

1. $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ *for every* $i \in \{1, ..., r\}$ *and every* $\gamma \in \Gamma_i$.
2. $\varphi \in l^\infty(\widehat{G})$.
3. *The mappings* $\gamma \mapsto \varphi(\gamma)$ *are norm continuous on the different sets* Γ_i.
4. *For any sequence* $(\gamma_k)_{k\in\mathbb{N}} \subset \widehat{G}$ *going to infinity,* $\lim_{k\to\infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.
5. *For* $i \in \{1, ..., r\}$ *and any properly converging sequence* $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}} \subset \Gamma_i$ *whose limit set is contained in* S_{i-1} *(taking a subsequence if necessary) and for the mappings* $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma},k} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ *constructed in the preceding sections, one has*
$$\lim_{k\to\infty} \left\|\varphi(\gamma_k) - \tilde{\nu}_k\left(\varphi_{|S_{i-1}}\right)\right\|_{op} = 0.$$

3.8 Example: The free two-step nilpotent Lie groups of 3 and 4 generators

Now, as an example of two-step nilpotent Lie groups, the free two-step nilpotent Lie groups of n generators shall be considered in order to closely regard the cases $n = 3$ and $n = 4$. More detailed calculations can be found in the doctoral thesis of R.Lahiani (see [21]).
Let $\mathfrak{f}(2, n)$ be the free two-step nilpotent Lie algebra with n generators $H_1, ..., H_n$ and $F(2, n)$ the associated connected simply connected Lie group. Furthermore, let $0 \neq U_{ij} := [H_i, H_j]$ for all $1 \leq i < j \leq n$. Then,

$$\mathfrak{f}(2, n) = \text{span}\{H_k, U_{ij} |\ 1 \leq k \leq n, 1 \leq i < j \leq n\}$$

and since $\mathfrak{f}(2, n)$ is two-step nilpotent,

$$[U_{ij}, \mathfrak{f}(2, n)] = 0 \quad \forall 1 \leq i < j \leq n.$$

Moreover, if $X, Y \in \mathfrak{f}(2, n)$, $X = \sum_{i=1}^{n} x_i H_i + \tilde{X}$ and $Y = \sum_{i=1}^{n} y_i H_i + \tilde{Y}$ for $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{R}$ and $\tilde{X}, \tilde{Y} \in \text{span}\{U_{ij} |\ 1 \leq i < j \leq n\}$, one gets

$$[X, Y] = \sum_{i,j=1}^{n} (x_i y_j - x_j y_i) U_{ij}.$$

One can consider U_{ij} as a skew-symmetric matrix, identifying it with the matrix $E_{ij} - E_{ji}$, where E_{ij} is the $n \times n$ matrix whose only non-zero entry is a 1 in the i-th entry of the j-th column. Moreover, one can identify H_i with the i-th canonical basis vector of $\mathbb{R}^n$ and thus,

$$F(2, n) \cong \mathbb{R}^n \ltimes Sp(n).$$

3.8.1 The case $n = 3$

Now, let $n = 3$ and $H_4 := U_{12}$, $H_5 := U_{13}$ and $H_6 := U_{23}$. Then,

$$\mathfrak{g} := \mathfrak{f}(2,3) = \operatorname{span}\{H_1, H_2, H_3, U_{12}, U_{13}, U_{23}\} = \operatorname{span}\{H_1, ...H_6\}.$$

Since $[H_j, H_k] \subset \operatorname{span}\{H_i | \, 4 \leq i \leq 6\}$ for all $1 \leq j,k \leq 3$ and $[H_i, \mathfrak{g}] = 0$ for all $4 \leq i \leq 6$, the basis $\mathcal{B} = \{H_1, ..., H_6\}$ is a Jordan-Hölder basis of $\mathfrak{g}$.

Let $\ell = \sum_{i=1}^{6} \alpha_i H_i^*$ for $\alpha_i \in \mathbb{R}$, where H_i^* is the dual basis element of H_i for all $1 \leq i \leq 6$.
Then, one gets the following results for the stabilizer, the Pukanszky index set, the polarization $\mathfrak{p}_\ell^V$ of ℓ in $\mathfrak{g}$, the possible coadjoint orbits $\mathcal{O}$ and the associated $\ell_{\mathcal{O}} \in \mathfrak{g}^*$ (compare Section 3.1 for the notation and approach):

1. If $\alpha_4 = \alpha_5 = \alpha_6 = 0$, the stabilizer $\mathfrak{g}(\ell)$ equals the whole Lie algebra: $\mathfrak{g}(\ell) = \mathfrak{g}$. In this case,
$$I_\ell^{Puk} = \emptyset \quad \text{and} \quad \mathfrak{p}_\ell^V = \mathfrak{g}(\ell) = \mathfrak{g}.$$
In addition,
$$\mathcal{O}^1 := \operatorname{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell = \sum_{i=1}^{3} \alpha_i H_i^*.$$
So, $\mathcal{O}^1$ consists of one single element and obviously, ℓ vanishes on $\{H_i | \; i \in I_\ell^{Puk}\}$, i.e. ℓ is the element $\ell_{\mathcal{O}^1}$ that was chosen in Section 3.2.

2. If $\alpha_4 \neq 0$ or $\alpha_5 \neq 0$ or $\alpha_6 \neq 0$,
$$\mathfrak{g}(\ell) = \operatorname{span}\{\alpha_6 H_1 + \alpha_5 H_2 + \alpha_4 H_3, H_4, H_5, H_6\}.$$
Then, the dimension of the polarization is $\dim(\mathfrak{p}_\ell^V) = \frac{1}{2}\big(\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell))\big) = \frac{6+4}{2} = 5$ and thus, for the construction of the polarization $\mathfrak{p}_\ell^V$, one has to find one further element $Y_1^{\ell,V}$ which is not in the stabilizer.
In this case, one has to regard three subcases:

   (a) If $\alpha_6 \neq 0$, $I_\ell^{Puk} = \{2,3\}$. Now, the largest index in $\{1,...,6\}$ with the property that $H_{j_1(\ell)} \notin \mathfrak{g}(\ell)$ is the index $j_1(\ell) = 3$. Furthermore, the largest index such that $\big\langle \ell, [H_{k_1(\ell)}, H_{j_1(\ell)}]\big\rangle \neq 0$ is $k_1(\ell) = 2$. Hence,
$$Y_1^{V,\ell} = H_3, \; X_1^{V,\ell} = H_2, \; \mathfrak{p}_\ell^V = \operatorname{span}\{H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \operatorname{span}\{H_2\} \oplus \mathfrak{p}_\ell^V.$$
Moreover, in this case, $\mathfrak{g}(\ell)^\perp = \operatorname{span}\{-\alpha_5 H_1^* + \alpha_6 H_2^*, -\alpha_4 H_1^* + \alpha_6 H_3^*\}$ and thus,
$$\begin{aligned}\mathcal{O}^2 \;:=\; \operatorname{Ad}^*(G)\ell \;&=\; \ell + \mathfrak{g}(\ell)^\perp \;=\; \ell + \operatorname{span}\{-\alpha_5 H_1^* + \alpha_6 H_2^*, -\alpha_4 H_1^* + \alpha_6 H_3^*\} \\ &=\; \Big\{(\alpha_1 - \alpha_5 a - \alpha_4 b)H_1^* + (\alpha_2 + \alpha_6 a)H_2^* + (\alpha_3 + \alpha_6 b)H_3^* \\ &\qquad + \sum_{i=4}^{6} \alpha_i H_i^* \Big| \; a, b \in \mathbb{R}\Big\}.\end{aligned}$$

If one chooses $a := -\frac{\alpha_2}{\alpha_6}$ and $b := -\frac{\alpha_3}{\alpha_6}$, the corresponding element vanishes on $\{H_2, H_3\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and therefore, one needs to define

$$\ell_{\mathcal{O}^2} := \Big(\alpha_1 + \frac{\alpha_2\alpha_5}{\alpha_6} + \frac{\alpha_3\alpha_4}{\alpha_6}\Big)H_1^* + \sum_{i=4}^{6} \alpha_i H_i^*.$$

(b) If $\alpha_6 = 0$ and $\alpha_5 \neq 0$, $I_\ell^{Puk} = \{1,3\}$, $j_1(\ell) = 3$ and $k_1(\ell) = 1$. In this case,

$$Y_1^{V,\ell} = H_3,\ X_1^{V,\ell} = H_1,\ \mathfrak{p}_\ell^V = \mathrm{span}\{H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \mathrm{span}\{H_1\} \oplus \mathfrak{p}_\ell^V.$$

Furthermore, $\mathfrak{g}(\ell)^\perp = \mathrm{span}\{H_1^*, -\alpha_4 H_2^* + \alpha_5 H_3^*\}$ and hence,

$$\begin{aligned} \mathcal{O}^3 &:= \mathrm{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell + \mathrm{span}\{H_1^*, -\alpha_4 H_2^* + \alpha_5 H_3^*\} \\ &= \big\{aH_1^* + (\alpha_2 - \alpha_4 b)H_2^* + (\alpha_3 + \alpha_5 b)H_3^* + \alpha_4 H_4^* + \alpha_5 H_5^* |\ a, b \in \mathbb{R}\big\}. \end{aligned}$$

Here, if one chooses $a = 0$ and $b = -\frac{\alpha_3}{\alpha_5}$, the corresponding element vanishes on $\{H_1, H_3\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and thus, one defines

$$\ell_{\mathcal{O}^3} := \Big(\alpha_2 + \frac{\alpha_3\alpha_4}{\alpha_5}\Big)H_2^* + \alpha_4 H_4^* + \alpha_5 H_5^*.$$

(c) If $\alpha_6 = \alpha_5 = 0$ and $\alpha_4 \neq 0$, $I_\ell^{Puk} = \{1,2\}$, $j_1(\ell) = 2$ and $k_1(\ell) = 1$. Here,

$$Y_1^{V,\ell} = H_2,\ X_1^{V,\ell} = H_1,\ \mathfrak{p}_\ell^V = \mathrm{span}\{H_2\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \mathrm{span}\{H_1\} \oplus \mathfrak{p}_\ell^V.$$

In addition, $\mathfrak{g}(\ell)^\perp = \mathrm{span}\{H_1^*, H_2^*\}$ and therefore,

$$\begin{aligned} \mathcal{O}^4 &:= \mathrm{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell + \mathrm{span}\{H_1^*, H_2^*\} \\ &= \big\{aH_1^* + bH_2^* + \alpha_3 H_3^* + \alpha_4 H_4^* |\ a, b \in \mathbb{R}\big\}. \end{aligned}$$

In this last case, for $a = b = 0$ the corresponding element in $\mathcal{O}^4$ vanishes on the set $\{H_1, H_2\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and one has to define

$$\ell_{\mathcal{O}^4} := \alpha_3 H_3^* + \alpha_4 H_4^*.$$

In order to calculate the sets Γ_i for $i \in \{0, ..., r\}$, one has to compute the dimensions d of the Pukanszky index sets for the different $\ell \in \mathfrak{g}^*$. Moreover, one has to find the occuring pairs of sets (J, K), where $J, K \subset \{1, ..., 6\}$, $|J| = |K| = d$ and $J \cap K = \emptyset$ and such that there exists an $\ell \in \mathfrak{g}^*$ that fulfills $J(\ell) = J$ and $K(\ell) = K$.
From the cases distinguished above, one can deduce that the only dimensions that appear are $d = 0$ and $d = 1$, that the only pair of dimension 0 is $(\emptyset, \emptyset)$ and the ones of dimension 1 are $(\{3\}, \{2\})$, $(\{3\}, \{1\})$ and $(\{2\}, \{1\})$.
Since $(\emptyset, \emptyset) < (\{2\}, \{1\}) < (\{3\}, \{1\}) < (\{3\}, \{2\})$, the pair $(\emptyset, \emptyset)$ is assigned the number 0, $(\{2\}, \{1\})$ the number 1, $(\{3\}, \{1\})$ the 2 and the pair $(\{3\}, \{2\})$ is assigned the number 3.

Thus, $r = 3$ and

$$\begin{aligned}
\Gamma_0 &= \Big\{ \big[\pi_\ell^V\big] \Big| \ \exists\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}: \ \ell = \sum_{i=1}^{3} \alpha_i H_i^* \Big\}, \\
\Gamma_1 &= \Big\{ \big[\pi_\ell^V\big] \Big| \ \exists\, \alpha_3 \in \mathbb{R}, \alpha_4 \in \mathbb{R}^*: \ \ell = \alpha_3 H_3^* + \alpha_4 H_4^* \Big\}, \\
\Gamma_2 &= \Big\{ \big[\pi_\ell^V\big] \Big| \ \exists\, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}, \alpha_5 \in \mathbb{R}^*: \ \ell = \Big(\alpha_2 + \frac{\alpha_3\alpha_4}{\alpha_5}\Big) H_2^* + \alpha_4 H_4^* + \alpha_5 H_5^* \Big\} \quad \text{and} \\
\Gamma_3 &= \Big\{ \big[\pi_\ell^V\big] \Big| \ \exists\, \alpha_1, ..., \alpha_5 \in \mathbb{R}, \alpha_6 \in \mathbb{R}^*: \ \ell = \Big(\alpha_1 + \frac{\alpha_2\alpha_5}{\alpha_6} + \frac{\alpha_3\alpha_4}{\alpha_6}\Big) H_1^* + \sum_{i=4}^{6} \alpha_i H_i^* \Big\}.
\end{aligned}$$

In this example, one can also see that the situation in which the polarizations $\mathfrak{p}_\ell^V$ are discontinuous in ℓ on the set $\{\ell_{\mathcal{O}'} | \ \mathcal{O}' \in (\mathfrak{g}^*/G)_{2d}\}$ occurs:
One can choose for every $k \in \mathbb{N}$ the orbit

$$\mathcal{O}_k = \Big\{ bH_1^* + \frac{1}{k} aH_2^* + \frac{1}{k} bH_3^* - H_4^* + \frac{1}{k} H_6^* \Big| \ a, b \in \mathbb{R} \Big\}$$

and the orbit

$$\mathcal{O} = \{ aH_1^* + bH_2^* - H_4^* | \ a, b \in \mathbb{R} \},$$

which both have the dimension 2 and are thus located in the set $\{\ell_{\mathcal{O}'} | \ \mathcal{O}' \in (\mathfrak{g}^*/G)_2\}$. Moreover, $\big[\pi_{\ell_{\mathcal{O}_k}}^V\big] \in \Gamma_3$ and $\big[\pi_{\ell_{\mathcal{O}}}^V\big] \in \Gamma_1$. Now, let

$$\mathcal{O} \ni \tilde{\ell} = a_{\tilde{\ell}} H_1^* + b_{\tilde{\ell}} H_2^* - H_4^*$$

and choose sequences $(\tilde{a}_k)_{k\in\mathbb{N}}$ and $\big(\tilde{b}_k\big)_{k\in\mathbb{N}}$ such that $\tilde{a}_k \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$ and $\tilde{b}_k \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$. If one lets

$$\ell_k := \tilde{a}_k H_1^* + \frac{1}{k}\big(k\tilde{b}_k\big) H_2^* + \frac{1}{k} \tilde{a}_k H_3^* - H_4^* + \frac{1}{k} H_6^* \in \mathcal{O}_k,$$

$\ell_k \overset{k\to\infty}{\longrightarrow} \tilde{\ell}$. Hence, $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$. But on the other hand, for every k,

$$\begin{aligned}
\mathfrak{p}_{\ell_{\mathcal{O}_k}}^V &= \operatorname{span}\{H_3\} \oplus \mathfrak{g}(\ell) = \operatorname{span}\{H_1, H_3, ..., H_6\} \quad \text{and} \\
\mathfrak{p}_{\ell_{\mathcal{O}}}^V &= \operatorname{span}\{H_2\} \oplus \mathfrak{g}(\ell) = \operatorname{span}\{H_2, ..., H_6\}.
\end{aligned}$$

Thus, the sequence of polarizations $\big(\mathfrak{p}_{\ell_{\mathcal{O}_k}}^V\big)_{k\in\mathbb{N}}$ does not converge to the polarization $\mathfrak{p}_{\ell_{\mathcal{O}}}^V$ and the discontinuity is proved.
This shows the necessity of considering the sets $\{\ell_{\mathcal{O}'} | \ \mathcal{O}' \in (\mathfrak{g}^*/G)_{(J,K)}\}$ instead of the sets $\{\ell_{\mathcal{O}'} | \ \mathcal{O}' \in (\mathfrak{g}^*/G)_{2d}\}$.

Now, regard the three cases of Proposition 3.1 that were distinguished in the proof of the three conditions of Definition 1.1.

1. If one chooses the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ as

$$\mathcal{O}_k := \Big\{ aH_1^* + bH_2^* + H_3^* + \Big(1 + \frac{1}{k}\Big) H_4^* \Big| \ a, b \in \mathbb{R} \Big\},$$

then $\left[\pi^V_{\ell_{\mathcal{O}_k}}\right] \in \Gamma_1$ for every k. Furthermore, every $\ell_k \in \mathcal{O}_k$ is of the form

$$\ell_k = a_{\ell_k} H_1^* + b_{\ell_k} H_2^* + H_3^* + \Big(1 + \frac{1}{k}\Big) H_4^*$$

for $a_{\ell_k}, b_{\ell_k} \in \mathbb{R}$. If $a_{\ell_k} \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$ and $b_{\ell_k} \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$, the sequence $(\ell_k)_{k\in\mathbb{N}}$ converges to

$$\tilde{\ell} = a_{\tilde{\ell}} H_1^* + b_{\tilde{\ell}} H_2^* + H_3^* + H_4^* \in \{aH_1^* + bH_2^* + H_3^* + H_4^* |\ a, b \in \mathbb{R}\} =: \mathcal{O}$$

and therefore, $\left[\pi^V_{\ell_{\mathcal{O}}}\right] \in \Gamma_1$.
Here, the orbit $\mathcal{O}$ does not depend on $\tilde{\ell}$ and is thus the only limit point of the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$. Hence,

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \{\mathcal{O}\}, \quad \text{where} \quad 2 = \dim(\mathcal{O}_k) = \dim(\mathcal{O}) \quad \forall k \in \mathbb{N}.$$

This is an example for the first case regarded in the proof of the conditions of Definition 1.1.

2. Now, choose the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ as follows:

$$\mathcal{O}_k := \left\{ aH_1^* + bH_2^* + H_3^* + \frac{1}{k} H_4^* \Big|\ a, b \in \mathbb{R} \right\}.$$

Then, again $\left[\pi^V_{\ell_{\mathcal{O}_k}}\right] \in \Gamma_1$ for every k and every $\ell_k \in \mathcal{O}_k$ is of the form

$$\ell_k = a_{\ell_k} H_1^* + b_{\ell_k} H_2^* + H_3^* + \frac{1}{k} H_4^*$$

for $a_{\ell_k}, b_{\ell_k} \in \mathbb{R}$. If $a_{\ell_k} \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$ and $b_{\ell_k} \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$, the sequence $(\ell_k)_{k\in\mathbb{N}}$ converges to

$$\ell = a_{\tilde{\ell}} H_1^* + b_{\tilde{\ell}} H_2^* + H_3^* =: \mathcal{O}_{\tilde{\ell}}$$

and $\left[\pi^V_{\ell_{\mathcal{O}_{\tilde{\ell}}}}\right] \in \Gamma_0$. Furthermore,

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \{\mathcal{O}_{\tilde{\ell}} |\ a_{\tilde{\ell}}, b_{\tilde{\ell}} \in \mathbb{R}\}, \quad \text{where} \quad \dim(\mathcal{O}_k) = 2,\ \dim(\mathcal{O}_{\tilde{\ell}}) = 0$$
$$\forall k \in \mathbb{N}\ \forall \mathcal{O}_{\tilde{\ell}} \in L((\mathcal{O}_k)_{k\in\mathbb{N}}).$$

This is therefore an example for the second case that was regarded in Proposition 3.1.

3. For an example of the third case mentioned in Proposition 3.1, one has to find a sequence of orbits $(\mathcal{O}_k)_{k\in\mathbb{N}}$ whose limit set $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ consists of orbits of dimension strictly larger than 0 but strictly smaller than $\dim(\mathcal{O}_k)$. But since in this regarded example of $G = F(2,3)$ only orbits of the dimensions 0 and 2 appear, such a sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ does not exist. So, in this example, there are just two cases to be distinguished: the first one and the second one.

3.8.2 The case $n = 4$

Let $n = 4$ and $H_5 := U_{12}$, $H_6 := U_{13}$, $H_7 := U_{14}$, $H_8 := U_{23}$, $H_9 := U_{24}$ and $H_{10} := U_{34}$. Then,

$$\mathfrak{g} := \mathfrak{f}(2,4) = \text{span}\{H_1, H_2, H_3, H_4, U_{12}, U_{13}, U_{14}, U_{23}, U_{24}, U_{34}\} = \text{span}\{H_1, ...H_{10}\}.$$

Since, as in the case $n = 3$, $[H_j, H_k] \subset \text{span}\{H_i | \ 5 \le i \le 10\}$ for all $1 \le j,k \le 4$ and $[H_i, \mathfrak{g}] = 0$ for all $5 \le i \le 10$, the basis $\mathcal{B} = \{H_1, ..., H_{10}\}$ is a Jordan-Hölder basis of $\mathfrak{g}$.

Now, let $\ell = \sum\limits_{i=1}^{10} \alpha_i H_i^*$. One gets the following results:

1. If $\alpha_5 = ... = \alpha_{10} = 0$, $\mathfrak{g}(\ell) = \mathfrak{g}$ and like for $n = 3$,

$$I_\ell^{Puk} = \emptyset \quad \text{and} \quad \mathfrak{p}_\ell^V = \mathfrak{g}(\ell) = \mathfrak{g}.$$

Moreover,

$$\mathcal{O}^1 := \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell = \sum_{i=1}^{4} \alpha_i H_i^*$$

and, as $\mathcal{O}^1$ consists of one single element, ℓ is the element $\ell_{\mathcal{O}^1}$ that was chosen in Section 3.2.

2. If at least one of the α_i for $i \in \{5, ..., 10\}$ is not zero, one has to regard two cases:

   (a) If $\alpha_6\alpha_9 \neq \alpha_5\alpha_{10} + \alpha_7\alpha_8$,

$$\mathfrak{g}(\ell) = \text{span}\{H_5, ..., H_{10}\} = [\mathfrak{g}, \mathfrak{g}]$$

and $I_\ell^{Puk} = \{1, 2, 3, 4\}$.
Hence, the dimension of the polarization is $\dim(\mathfrak{p}_\ell^V) = \frac{10+6}{2} = 8$ and thus, for its construction, one has to find two further elements $Y_1^{\ell,V}$ and $Y_2^{\ell,V}$ which are not in the stabilizer.
Here, the largest index in $\{1, ..., 10\}$ with the property that $H_{j_1(\ell)} \notin \mathfrak{g}(\ell)$ is the index $j_1(\ell) = 4$. To find the largest index $k_1(\ell)$ such that $\langle \ell, [H_{k_1(\ell)}, H_{j_1(\ell)}] \rangle \neq 0$, one has to consider several cases:

   (i) If $\alpha_{10} \neq 0$, one has $k_1(\ell) = 3$ and therefore, $Y_1^{V,\ell} = H_4$ and $X_1^{V,\ell} = H_3$. A basis for the ideal $\mathfrak{g}^1$ which is needed for the construction of $Y_2^{V,\ell}$ and $X_2^{V,\ell}$ is given by the set

$$\left\{ H_1^1 = H_1 - \frac{\alpha_7}{\alpha_{10}} H_3, H_2^1 = H_2 - \frac{\alpha_9}{\alpha_{10}} H_3, H_4^1 = H_4, ..., H_{10}^1 = H_{10} \right\}.$$

The stabilizer $\mathfrak{g}^1(\ell_{|\mathfrak{g}^1})$ can then be calculated to be

$$\mathfrak{g}^1(\ell_{|\mathfrak{g}^1}) = \{H_4^1, ..., H_{10}^1\}.$$

Thus, the largest index $j_2(\ell) \in \{1, 2, 4, ..., 10\}$ such that $H^1_{j_2(\ell)} \notin \mathfrak{g}^1(\ell_{|\mathfrak{g}^1})$, is $j_2(\ell) = 2$. Last, the largest index $k_2(\ell) \in \{1, 2, 4, ..., 10\}$ in such a way that $\langle \ell, [H^1_{k_2(\ell)}, H^1_{j_2(\ell)}] \rangle \neq 0$, is $k_2(\ell) = 1$. So,

$$Y_1^{V,\ell} = H_4, \ Y_2^{V,\ell} = H_2, \ X_1^{V,\ell} = H_3, \ X_2^{V,\ell} = H_1,$$
$$\mathfrak{p}_\ell^V = \text{span}\{H_4, H_2\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_3, H_1\} \oplus \mathfrak{p}_\ell^V.$$

(ii) If $\alpha_{10} = 0$ and $\alpha_9 \neq 0$, one has $k_1(\ell) = 2$, $Y_1^{V,\ell} = H_4$ and $X_1^{V,\ell} = H_2$. In this case, a basis for $\mathfrak{g}^1$ is given by

$$\left\{ H_1^1 = H_1 - \frac{\alpha_7}{\alpha_9} H_2, H_3^1 = H_3, ..., H_{10}^1 = H_{10} \right\}$$

and again the stabilizer $\mathfrak{g}^1\left(\ell_{|\mathfrak{g}^1}\right)$ can be computed to be

$$\mathfrak{g}^1\left(\ell_{|\mathfrak{g}^1}\right) = \left\{H_4^1, ..., H_{10}^1\right\}.$$

Moreover, the largest index $j_2(\ell) \in \{1, 3, ..., 10\}$ such that $H_{j_2(\ell)}^1 \notin \mathfrak{g}^1\left(\ell_{|\mathfrak{g}^1}\right)$, is $j_2(\ell) = 3$ and the largest index $k_2(\ell) \in \{1, 3, ..., 10\}$ in such a way that $\left\langle \ell, [H_{k_2(\ell)}^1, H_{j_2(\ell)}^1]\right\rangle \neq 0$, is $k_2(\ell) = 1$. Therefore,

$$Y_1^{V,\ell} = H_4,\ Y_2^{V,\ell} = H_3,\ X_1^{V,\ell} = H_2,\ X_2^{V,\ell} = H_1,$$

$$\mathfrak{p}_\ell^V = \mathrm{span}\{H_4, H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \mathrm{span}\{H_2, H_1\} \oplus \mathfrak{p}_\ell^V.$$

(iii) If $\alpha_{10} = \alpha_9 = 0$, from the assumption that $\alpha_6\alpha_9 \neq \alpha_5\alpha_{10} + \alpha_7\alpha_8$ follows that $\alpha_8 \neq 0$. Hence, $k_1(\ell) = 1$, $Y_1^{V,\ell} = H_4$ and $X_1^{V,\ell} = H_1$. Here, a basis for $\mathfrak{g}^1$ is given by

$$\left\{H_2^1 = H_2, ..., H_{10}^1 = H_{10}\right\}$$

and again the stabilizer $\mathfrak{g}^1\left(\ell_{|\mathfrak{g}^1}\right)$ can be calculated to be

$$\mathfrak{g}^1\left(\ell_{|\mathfrak{g}^1}\right) = \left\{H_4^1, ..., H_{10}^1\right\}.$$

Furthermore, $j_2(\ell) = 3$ and $k_2(\ell) = 2$ and thus,

$$Y_1^{V,\ell} = H_4,\ Y_2^{V,\ell} = H_3,\ X_1^{V,\ell} = H_1,\ X_2^{V,\ell} = H_2,$$

$$\mathfrak{p}_\ell^V = \mathrm{span}\{H_4, H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \mathrm{span}\{H_1, H_2\} \oplus \mathfrak{p}_\ell^V.$$

Now, in Case (a), $\mathfrak{g}(\ell)^\perp = \mathrm{span}\{H_1^*, H_2^*, H_3^*, H_4^*\}$ and therefore,

$$\begin{aligned} \mathcal{O}^2 \ &:= \ \mathrm{Ad}^*(G)\ell \ = \ \ell + \mathfrak{g}(\ell)^\perp \ = \ \ell + \mathrm{span}\{H_1^*, H_2^*, H_3^*, H_4^*\} \\ &= \ \left\{ aH_1^* + bH_2^* + cH_3^* + dH_4^* + \sum_{i=5}^{10} \alpha_i H_i^* \,\middle|\, a, b, c, d \in \mathbb{R} \right\}. \end{aligned}$$

If one chooses $a = b = c = d = 0$, the corresponding element vanishes on $\{H_1, H_2, H_3, H_4\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and thus, one defines

$$\ell_{\mathcal{O}^2} := \sum_{i=5}^{10} \alpha_i H_i^*.$$

(b) If $\alpha_6\alpha_9 = \alpha_5\alpha_{10} + \alpha_7\alpha_8$, again, there are several cases. In all of them, the dimension of the stabilizer is 8 and hence, $\dim(\mathfrak{p}_\ell^V) = 9$. Thus, for the construction of the polarization $\mathfrak{p}_\ell^V$, one has to find one further element $Y_1^{\ell,V}$ which is not in the stabilizer.

(i) If $\alpha_{10} \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{\alpha_{10}H_1 - \alpha_7H_3 + \alpha_6H_4, \alpha_{10}H_2 - \alpha_9H_3 + \alpha_8H_4, H_5, ..., H_{10}\}$$

and $I_\ell^{Puk} = \{3,4\}$. Now, the largest index in $\{1,...,10\}$ with the property that $H_{j_1(\ell)} \notin \mathfrak{g}(\ell)$, is the index $j_1(\ell) = 4$. Furthermore, the largest index such that $\langle \ell, [H_{k_1(\ell)}, H_{j_1(\ell)}] \rangle \neq 0$, is $k_1(\ell) = 3$. Hence,

$$Y_1^{V,\ell} = H_4,\ X_1^{V,\ell} = H_3,\ \mathfrak{p}_\ell^V = \text{span}\{H_4\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_3\} \oplus \mathfrak{p}_\ell^V.$$

In addition, $\mathfrak{g}(\ell)^\perp = \text{span}\{-\alpha_7H_1^* - \alpha_9H_2^* + \alpha_{10}H_3^*, \alpha_6H_1^* + \alpha_8H_2^* + \alpha_{10}H_4^*\}$ and therefore,

$$\begin{aligned}
\mathcal{O}^3 &:= \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp \\
&= \ell + \text{span}\{-\alpha_7H_1^* - \alpha_9H_2^* + \alpha_{10}H_3^*, \alpha_6H_1^* + \alpha_8H_2^* + \alpha_{10}H_4^*\} \\
&= \Big\{(\alpha_1 - \alpha_7a + \alpha_6b)H_1^* + (\alpha_2 - \alpha_9a + \alpha_8b)H_2^* + (\alpha_3 + \alpha_{10}a)H_3^* \\
&\qquad + (\alpha_4 + \alpha_{10}b)H_4^* + \sum_{i=5}^{10} \alpha_iH_i^* \Big|\ a, b \in \mathbb{R}\Big\}.
\end{aligned}$$

If one chooses $a := -\frac{\alpha_3}{\alpha_{10}}$ and $b := -\frac{\alpha_4}{\alpha_{10}}$, the corresponding element vanishes on $\{H_3, H_4\} = \{H_i|\ i \in I_\ell^{Puk}\}$ and hence, one needs to define

$$\ell_{\mathcal{O}^3} := \Big(\alpha_1 + \frac{\alpha_3\alpha_7}{\alpha_{10}} - \frac{\alpha_4\alpha_6}{\alpha_{10}}\Big)H_1^* + \Big(\alpha_2 + \frac{\alpha_3\alpha_9}{\alpha_{10}} - \frac{\alpha_4\alpha_8}{\alpha_{10}}\Big)H_2^* + \sum_{i=5}^{10} \alpha_iH_i^*.$$

(ii) If $\alpha_{10} = 0$ and $\alpha_9 \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{\alpha_9H_1 - \alpha_7H_2 + \alpha_5H_4, \alpha_9H_3 - \alpha_8H_4, H_5, ..., H_{10}\},$$

$I_\ell^{Puk} = \{2,4\}$, $j_1(\ell) = 4$ and $k_1(\ell) = 2$. Thus,

$$Y_1^{V,\ell} = H_4,\ X_1^{V,\ell} = H_2,\ \mathfrak{p}_\ell^V = \text{span}\{H_4\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_2\} \oplus \mathfrak{p}_\ell^V.$$

Furthermore, $\mathfrak{g}(\ell)^\perp = \text{span}\{-\alpha_7H_1^* + \alpha_9H_2^*, \alpha_5H_1^* + \alpha_8H_3^* + \alpha_9H_4^*\}$ and hence,

$$\begin{aligned}
\mathcal{O}^4 &:= \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp \\
&= \ell + \text{span}\{-\alpha_7H_1^* + \alpha_9H_2^*, \alpha_5H_1^* + \alpha_8H_3^* + \alpha_9H_4^*\} \\
&= \Big\{(\alpha_1 - \alpha_7a + \alpha_5b)H_1^* + (\alpha_2 + \alpha_9a)H_2^* + (\alpha_3 + \alpha_8b)H_3^* \\
&\qquad + (\alpha_4 + \alpha_9b)H_4^* + \sum_{i=5}^{9} \alpha_iH_i^* \Big|\ a, b \in \mathbb{R}\Big\}.
\end{aligned}$$

In this case, if one chooses $a = -\frac{\alpha_2}{\alpha_9}$ and $b = -\frac{\alpha_4}{\alpha_9}$, the corresponding element vanishes on $\{H_2, H_4\} = \{H_i|\ i \in I_\ell^{Puk}\}$ and hence, here one defines

$$\ell_{\mathcal{O}^4} := \Big(\alpha_1 + \frac{\alpha_2\alpha_7}{\alpha_9} - \frac{\alpha_4\alpha_5}{\alpha_9}\Big)H_1^* + \Big(\alpha_3 - \frac{\alpha_4\alpha_8}{\alpha_9}\Big)H_3^* + \sum_{i=5}^{9} \alpha_iH_i^*.$$

(iii) If $\alpha_{10} = \alpha_9 = 0$ and $\alpha_8 \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{\alpha_8 H_1 - \alpha_6 H_2 + \alpha_5 H_3, H_4, ..., H_{10}\},$$

$I_\ell^{Puk} = \{2, 3\}$, $j_1(\ell) = 3$ and $k_1(\ell) = 2$. In this case,

$$Y_1^{V,\ell} = H_3,\ X_1^{V,\ell} = H_2,\ \mathfrak{p}_\ell^V = \text{span}\{H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_2\} \oplus \mathfrak{p}_\ell^V.$$

Here, $\mathfrak{g}(\ell)^\perp = \text{span}\{-\alpha_6 H_1^* + \alpha_8 H_2^*, \alpha_5 H_1^* + \alpha_8 H_3^*\}$ and

$$\begin{aligned} \mathcal{O}^5 &:= \text{Ad}^*(G)\ell \ = \ \ell + \mathfrak{g}(\ell)^\perp \\ &= \ell + \text{span}\{-\alpha_6 H_1^* + \alpha_8 H_2^*, \alpha_5 H_1^* + \alpha_8 H_3^*\} \\ &= \Big\{(\alpha_1 - \alpha_6 a + \alpha_5 b)H_1^* + (\alpha_2 + \alpha_8 a)H_2^* + (\alpha_3 + \alpha_8 b)H_3^* \\ &\qquad + \sum_{i=4}^{8} \alpha_i H_i^* \Big|\ a, b \in \mathbb{R}\Big\}. \end{aligned}$$

For $a = -\frac{\alpha_2}{\alpha_8}$ and $b = -\frac{\alpha_3}{\alpha_8}$, the corresponding element in $\mathcal{O}^5$ vanishes on the set $\{H_2, H_3\} = \{H_i|\ i \in I_\ell^{Puk}\}$ and one has to define

$$\ell_{\mathcal{O}^5} := \Big(\alpha_1 + \frac{\alpha_2\alpha_6}{\alpha_8} - \frac{\alpha_3\alpha_5}{\alpha_8}\Big)H_1^* + \sum_{i=4}^{8} \alpha_i H_i^*.$$

(iv) If $\alpha_{10} = \alpha_9 = \alpha_8 = 0$ and $\alpha_7 \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{\alpha_7 H_2 - \alpha_5 H_4, \alpha_7 H_3 - \alpha_6 H_4, H_5, ..., H_{10}\},$$

$I_\ell^{Puk} = \{1, 4\}$, $j_1(\ell) = 4$ and $k_1(\ell) = 1$. In this case,

$$Y_1^{V,\ell} = H_4,\ X_1^{V,\ell} = H_1,\ \mathfrak{p}_\ell^V = \text{span}\{H_4\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_1\} \oplus \mathfrak{p}_\ell^V.$$

Moreover, $\mathfrak{g}(\ell)^\perp = \text{span}\{H_1^*, -\alpha_5 H_2^* - \alpha_6 H_3^* + \alpha_7 H_4^*\}$ and hence,

$$\begin{aligned} \mathcal{O}^6 &:= \text{Ad}^*(G)\ell \ = \ \ell + \mathfrak{g}(\ell)^\perp \\ &= \ell + \text{span}\{H_1^*, -\alpha_5 H_2^* - \alpha_6 H_3^* + \alpha_7 H_4^*\} \\ &= \Big\{aH_1^* + (\alpha_2 - \alpha_5 b)H_2^* + (\alpha_3 - \alpha_6 b)H_3^* + (\alpha_4 + \alpha_7 b)H_4^* \\ &\qquad + \sum_{i=5}^{7} \alpha_i H_i^* \Big|\ a, b \in \mathbb{R}\Big\}. \end{aligned}$$

Here, for $a = 0$ and $b = -\frac{\alpha_4}{\alpha_7}$, the corresponding element in $\mathcal{O}^6$ vanishes on the set $\{H_1, H_4\} = \{H_i|\ i \in I_\ell^{Puk}\}$ and one has to define

$$\ell_{\mathcal{O}^6} := \Big(\alpha_2 + \frac{\alpha_4\alpha_5}{\alpha_7}\Big)H_2^* + \Big(\alpha_3 + \frac{\alpha_4\alpha_6}{\alpha_7}\Big)H_3^* + \sum_{i=5}^{7} \alpha_i H_i^*.$$

(v) If $\alpha_{10} = ... = \alpha_7 = 0$ and $\alpha_6 \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{\alpha_6 H_2 - \alpha_5 H_3, H_4, ..., H_{10}\},$$

$I_\ell^{Puk} = \{1, 3\}$, $j_1(\ell) = 3$ and $k_1(\ell) = 1$. Hence,

$$Y_1^{V,\ell} = H_3,\ X_1^{V,\ell} = H_1,\ \mathfrak{p}_\ell^V = \text{span}\{H_3\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_1\} \oplus \mathfrak{p}_\ell^V.$$

In this case, $\mathfrak{g}(\ell)^\perp = \text{span}\{H_1^*, -\alpha_5 H_2^* + \alpha_6 H_3^*\}$ and thus,

$$\begin{aligned} \mathcal{O}^7 &:= \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell + \text{span}\{H_1^*, -\alpha_5 H_2^* + \alpha_6 H_3^*\} \\ &= \left\{ aH_1^* + (\alpha_2 - \alpha_5 b)H_2^* + (\alpha_3 + \alpha_6 b)H_3^* + \sum_{i=4}^{6} \alpha_i H_i^* \,\middle|\, a, b \in \mathbb{R} \right\}. \end{aligned}$$

Now, for $a = 0$ and $b = -\frac{\alpha_3}{\alpha_6}$, the corresponding element in $\mathcal{O}^7$ vanishes on the set $\{H_1, H_3\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and one has to define

$$\ell_{\mathcal{O}^7} := \left(\alpha_2 + \frac{\alpha_3 \alpha_5}{\alpha_6}\right) H_2^* + \sum_{i=4}^{6} \alpha_i H_i^*.$$

(vi) If $\alpha_{10} = ... = \alpha_6 = 0$ and $\alpha_5 \neq 0$,

$$\mathfrak{g}(\ell) = \text{span}\{H_3, ..., H_{10}\},$$

$I_\ell^{Puk} = \{1, 2\}$, $j_1(\ell) = 2$ and $k_1(\ell) = 1$. Thus,

$$Y_1^{V,\ell} = H_2,\ X_1^{V,\ell} = H_1,\ \mathfrak{p}_\ell^V = \text{span}\{H_2\} \oplus \mathfrak{g}(\ell) \quad \text{and} \quad \mathfrak{g} = \text{span}\{H_1\} \oplus \mathfrak{p}_\ell^V.$$

Here, $\mathfrak{g}(\ell)^\perp = \text{span}\{H_1^*, H_2^*\}$ and

$$\begin{aligned} \mathcal{O}^8 &:= \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp = \ell + \text{span}\{H_1^*, H_2^*\} \\ &= \left\{ aH_1^* + bH_2^* + \sum_{i=3}^{5} \alpha_i H_i^* \,\middle|\, a, b \in \mathbb{R} \right\}. \end{aligned}$$

In this last case, for $a = b = 0$, the corresponding element in $\mathcal{O}^8$ vanishes on the set $\{H_1, H_2\} = \{H_i |\ i \in I_\ell^{Puk}\}$ and one has to define

$$\ell_{\mathcal{O}^8} := \sum_{i=3}^{5} \alpha_i H_i^*.$$

For $n = 4$, the dimensions $d = 0$, $d = 1$ and $d = 2$ appear. The only pair of dimension 0 is $(\emptyset, \emptyset)$, the ones of dimension 1 are $(\{4\}, \{3\})$, $(\{4\}, \{2\})$, $(\{3\}, \{2\})$, $(\{4\}, \{1\})$, $(\{3\}, \{1\})$ and $(\{2\}, \{1\})$ and the ones of dimension 2 are $(\{4, 2\}, \{3, 1\})$ $(\{4, 3\}, \{2, 1\})$ and $(\{4, 3\}, \{1, 2\})$. Then,

$$\begin{aligned} (\emptyset, \emptyset) &< (\{2\}, \{1\}) < (\{3\}, \{1\}) < (\{3\}, \{2\}) < (\{4\}, \{1\}) < (\{4\}, \{2\}) < (\{4\}, \{3\}) \\ &< (\{4, 3\}, \{1, 2\}) < (\{4, 3\}, \{2, 1\}) < (\{4, 2\}, \{3, 1\}). \end{aligned}$$

Therefore, $(\emptyset,\emptyset)$ is assigned the number 0, $(\{2\},\{1\})$ the number 1, $(\{3\},\{1\})$ the number 2, $(\{3\},\{2\})$ the 3, $(\{4\},\{1\})$ the 4, $(\{4\},\{2\})$ the 5, $(\{4\},\{3\})$ the 6, $(\{4,3\},\{1,2\})$ the 7, $(\{4,3\},\{2,1\})$ the 8 and $(\{4,2\},\{3,1\})$ is assigned the number 9.
Thus, $r=9$ and

$$\begin{aligned}
S_0=\Gamma_0 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_1,...,\alpha_4\in\mathbb{R}:\ \ell=\sum_{i=1}^{4}\alpha_iH_i^*\Big\},\\
\Gamma_1 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_3,\alpha_4\in\mathbb{R},\alpha_5\in\mathbb{R}^*:\ \ell=\sum_{i=3}^{5}\alpha_iH_i^*\Big\},\\
\Gamma_2 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_2,...,\alpha_5\in\mathbb{R},\alpha_6\in\mathbb{R}^*:\ \ell=\Big(\alpha_2+\frac{\alpha_3\alpha_5}{\alpha_6}\Big)H_2^*+\sum_{i=4}^{6}\alpha_iH_i^*\Big\},\\
\Gamma_3 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_1,...,\alpha_7\in\mathbb{R},\alpha_8\in\mathbb{R}^*:\ \ell=\Big(\alpha_1+\frac{\alpha_2\alpha_6}{\alpha_8}-\frac{\alpha_3\alpha_5}{\alpha_8}\Big)H_1^*+\sum_{i=4}^{8}\alpha_iH_i^*\Big\},\\
\Gamma_4 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_2,...,\alpha_6\in\mathbb{R},\alpha_7\in\mathbb{R}^*:\ \ell=\Big(\alpha_2+\frac{\alpha_4\alpha_5}{\alpha_7}\Big)H_2^*+\Big(\alpha_3+\frac{\alpha_4\alpha_6}{\alpha_7}\Big)H_3^*\\
&\qquad +\sum_{i=5}^{7}\alpha_iH_i^*\Big\},\\
\Gamma_5 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_1,...,\alpha_8\in\mathbb{R},\alpha_9\in\mathbb{R}^*:\ \ell=\Big(\alpha_1+\frac{\alpha_2\alpha_7}{\alpha_9}-\frac{\alpha_4\alpha_5}{\alpha_9}\Big)H_1^*\\
&\qquad +\Big(\alpha_3-\frac{\alpha_4\alpha_8}{\alpha_9}\Big)H_3^*+\sum_{i=5}^{9}\alpha_iH_i^*\Big\},\\
\Gamma_6 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_1,...,\alpha_9\in\mathbb{R},\alpha_{10}\in\mathbb{R}^*:\ \ell=\Big(\alpha_1+\frac{\alpha_3\alpha_7}{\alpha_{10}}-\frac{\alpha_4\alpha_6}{\alpha_{10}}\Big)H_1^*\\
&\qquad +\Big(\alpha_2+\frac{\alpha_3\alpha_9}{\alpha_{10}}-\frac{\alpha_4\alpha_8}{\alpha_{10}}\Big)H_2^*+\sum_{i=5}^{10}\alpha_iH_i^*\Big\},\\
\Gamma_7 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_5,\alpha_6,\alpha_7\in\mathbb{R},\alpha_8\in\mathbb{R}^*:\ \ell=\sum_{i=5}^{8}\alpha_iH_i^*\Big\},\\
\Gamma_8 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_5,...,\alpha_8\in\mathbb{R},\alpha_9\in\mathbb{R}^*:\ \ell=\sum_{i=5}^{9}\alpha_iH_i^*\Big\}\quad\text{and}\\
\Gamma_9 &= \Big\{[\pi_\ell^V]\Big|\ \exists\ \alpha_5,...,\alpha_9\in\mathbb{R},\alpha_{10}\in\mathbb{R}^*:\ \ell=\sum_{i=5}^{10}\alpha_iH_i^*\Big\}.
\end{aligned}$$

As for $n=3$, here, one can also see by a similar example that the situation in which the polarizations $\mathfrak{p}_\ell^V$ are discontinuous in ℓ on the set $\{\ell_{\mathcal{O}'}|\ \mathcal{O}'\in(\mathfrak{g}^*/G)_{2d}\}$ occurs.

Now again, regard the three cases of Proposition 3.1 that were distinguished in the proof of the conditions of Definition 1.1.

1. If one chooses the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ as
$$\mathcal{O}_k := \left\{ aH_1^* + \frac{1}{k}bH_2^* + \left(\frac{1}{k}+b\right)H_4^* - \frac{1}{k}H_5^* + H_7^* \middle|\ a,b\in\mathbb{R}\right\},$$
then $\left[\pi^V_{\ell_{\mathcal{O}_k}}\right] \in \Gamma_4$ for every k. Furthermore, every $\ell_k \in \mathcal{O}_k$ is of the form
$$\ell_k = a_{\ell_k}H_1^* + \frac{1}{k}b_{\ell_k}H_2^* + \left(\frac{1}{k}+b_{\ell_k}\right)H_4^* - \frac{1}{k}H_5^* + H_7^*$$
for $a_{\ell_k}, b_{\ell_k} \in \mathbb{R}$. If $a_{\ell_k} \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$ and $b_{\ell_k} \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$, the sequence $(\ell_k)_{k\in\mathbb{N}}$ converges to
$$\tilde{\ell} = a_{\tilde{\ell}}H_1^* + b_{\tilde{\ell}}H_4^* + H_7^* \in \{aH_1^* + bH_4^* + H_7^* |\ a,b\in\mathbb{R}\} =: \mathcal{O}$$
and therefore, $\left[\pi^V_{\ell_{\mathcal{O}}}\right] \in \Gamma_4$.
As the orbit $\mathcal{O}$ does not depend on $\tilde{\ell}$, it is the only limit point of the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$. Hence,
$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \{\mathcal{O}\}, \quad \text{where} \quad 2 = \dim(\mathcal{O}_k) = \dim(\mathcal{O}) \quad \forall k \in \mathbb{N},$$
and this is an example for the first case of Proposition 3.1 regarded in the proof of the conditions of Definition 1.1.

2. Now, choose the sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ as follows:
$$\mathcal{O}_k := \left\{ aH_1^* + H_2^* + \frac{1}{k}bH_3^* + H_4^* + \frac{1}{k}H_6^* \middle|\ a,b\in\mathbb{R}\right\}.$$
Then, $\left[\pi^V_{\ell_{\mathcal{O}_k}}\right] \in \Gamma_2$ for every k, and every $\ell_k \in \mathcal{O}_k$ is of the form
$$\ell_k = a_{\ell_k}H_1^* + H_2^* + \frac{1}{k}b_{\ell_k}H_3^* + H_4^* + \frac{1}{k}H_6^*$$
for $a_{\ell_k}, b_{\ell_k} \in \mathbb{R}$. If $a_{\ell_k} \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$ and $b_{\ell_k} \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$, the sequence $(\ell_k)_{k\in\mathbb{N}}$ converges to
$$\tilde{\ell} = a_{\tilde{\ell}}H_1^* + H_2^* + H_4^* =: \mathcal{O}_{\tilde{\ell}}$$
and $\left[\pi^V_{\ell_{\mathcal{O}_{\tilde{\ell}}}}\right] \in \Gamma_0$. Moreover,
$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \{\mathcal{O}_{\tilde{\ell}} |\ a_{\tilde{\ell}}, b_{\tilde{\ell}} \in \mathbb{R}\}, \quad \text{where} \quad \dim(\mathcal{O}_k) = 2,\ \dim(\mathcal{O}_{\tilde{\ell}}) = 0$$
$$\forall k \in \mathbb{N}\ \forall \mathcal{O}_{\tilde{\ell}} \in L((\mathcal{O}_k)_{k\in\mathbb{N}}).$$
Hence, this is an example for the second case that was regarded in Proposition 3.1.

3. For an example of the third case, choose
$$\mathcal{O}_k := \left\{ aH_1^* + bH_2^* + cH_3^* + dH_4^* + \left(1+\frac{1}{k}\right)H_5^* + \sum_{i=6}^{8}\frac{1}{k}H_i^* \middle|\ a,b,c,d\in\mathbb{R}\right\}.$$
Then, $\left[\pi^V_{\ell_{\mathcal{O}_k}}\right] \in \Gamma_7$ for every k, and every $\ell_k \in \mathcal{O}_k$ is of the form
$$\ell_k = a_{\ell_k}H_1^* + b_{\ell_k}H_2^* + c_{\ell_k}H_3^* + d_{\ell_k}H_4^* + \left(1+\frac{1}{k}\right)H_5^* + \sum_{i=6}^{8}\frac{1}{k}H_i^*$$

for $a_{\ell_k}, b_{\ell_k}, c_{\ell_k}, d_{\ell_k} \in \mathbb{R}$. If $a_{\ell_k} \overset{k\to\infty}{\longrightarrow} a_{\tilde{\ell}}$, $b_{\ell_k} \overset{k\to\infty}{\longrightarrow} b_{\tilde{\ell}}$, $c_{\ell_k} \overset{k\to\infty}{\longrightarrow} c_{\tilde{\ell}}$ and $d_{\ell_k} \overset{k\to\infty}{\longrightarrow} d_{\tilde{\ell}}$, the sequence $(\ell_k)_{k\in\mathbb{N}}$ converges to

$$\tilde{\ell} = a_{\tilde{\ell}}H_1^* + b_{\tilde{\ell}}H_2^* + c_{\tilde{\ell}}H_3^* + d_{\tilde{\ell}}H_4^* + H_5^* \in \{aH_1^* + bH_2^* + c_{\tilde{\ell}}H_3^* + d_{\tilde{\ell}}H_4^* + H_5^* |\ a, b \in \mathbb{R}\} =: \mathcal{O}_{\tilde{\ell}}$$

and thus, $\left[\pi^V_{\ell_{\mathcal{O}_{\tilde{\ell}}}}\right] \in \Gamma_1$. Here,

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \{\mathcal{O}_{\tilde{\ell}} |\ c_{\tilde{\ell}}, d_{\tilde{\ell}} \in \mathbb{R}\}$$

and hence,

$$0 < 2 = \dim(\mathcal{O}_{\tilde{\ell}}) < 4 = \dim(\mathcal{O}_k) \quad \forall k \in \mathbb{N}\ \forall \mathcal{O}_{\tilde{\ell}} \in L((\mathcal{O}_k)_{k\in\mathbb{N}}).$$

Therefore, this represents an example for the third case appearing in the proof of the conditions of Definition 1.1.

4 The C^*-algebra of $SL(2,\mathbb{R})$

In this section, the C^*-algebra of $SL(2,\mathbb{R})$ will be examined. At the beginning, the Lie group $SL(2,\mathbb{R})$ and some of its subgroups will be introduced and some important definitions and results which are needed in order to prove the conditions listed in Definition 1.1 and hence to determine the C^*-algebra of $SL(2,\mathbb{R})$ will be recalled. Section 4.2 is about the spectrum of $SL(2,\mathbb{R})$ and its topology and in the following two subsections, the above specified conditions will be computed for the group $G = SL(2,\mathbb{R})$. Finally, in Section 4.5, a result about the C^*-algebra of $SL(2,\mathbb{R})$ will be presented and the concrete structure of $C^*(G)$ will be given.

For the group $SL(2,\mathbb{R})$, Condition 1 of Definition 1.1 is obvious by the definition of the sets Γ_i and S_i for $i \in \{0,...,r\}$ which will be given in Section 4.2. Condition 2 follows directly as well since $SL(2,\mathbb{R})$ is a connected linear semisimple Lie group. The proof of Condition 3(a) is rather short and straightforward, while the main work of the proof of the conditions of Definition 1.1 consists again in the verification of Condition 3(b). Nevertheless, its proof is less long and less technical than in the case of the two-step nilpotent Lie groups.

4.1 Preliminaries

In this section, let

$$G := SL(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) |\ \det A = 1\}$$

and let

$$K := SO(2) = \left\{ k_\varphi := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi) \right\}$$

be its maximal compact subgroup. Furthermore, define the one-dimensional nilpotent subgroup N of G and the one-dimensional abelian subgroup A of G by

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \quad \text{and} \quad A := \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Let

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) |\ \operatorname{tr} A = 0\}$$

be the Lie algebra of G.

From the Iwasawa decomposition, $G = KAN$ and thus, for every $g \in G$ there exist $\kappa(g) \in K$, $\mu \in N$ and $H(g) \in \mathfrak{a}$, where $\mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$ is the Lie algebra of A, such that

$$g = \kappa(g) e^{H(g)} \mu.$$

Moreover, define on $\mathfrak{a}$ the mappings ρ and ν_u for $u \in \mathbb{C}$ as

$$\rho \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := t \quad \text{and} \quad \nu_u \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := ut \quad \forall t \in \mathbb{R}.$$

Furthermore, let

$$\begin{aligned} L^2(K)_+ &:= \left\{ f \in L^2(K) |\ f(k) = f(-k)\ \forall k \in K \right\} \quad \text{and} \\ L^2(K)_- &:= \left\{ f \in L^2(K) |\ f(k) = -f(-k)\ \forall k \in K \right\} \end{aligned}$$

and define for every $u \in \mathbb{C}$ the representations $\mathcal{P}^{+,u}$ on $\mathcal{H}_{\mathcal{P}^{+,u}} := L^2(K)_+$ and $\mathcal{P}^{-,u}$ on $\mathcal{H}_{\mathcal{P}^{-,u}} := L^2(K)_-$ as

$$\mathcal{P}^{\pm,u}(g)f(k) := e^{-(\nu_u+\rho)H(g^{-1}k)} f\Big(\kappa\big(g^{-1}k\big)\Big) \quad \forall g \in G \; \forall f \in L^2(K)_\pm \; \forall k \in K.$$

Remark 4.1.
The representation $\big(\mathcal{P}^{+,u}, \mathcal{H}_{\mathcal{P}^{+,u}}\big)$ is irreducible if and only if $u \notin 2\mathbb{Z}+1$ and the representation $\big(\mathcal{P}^{-,u}, \mathcal{H}_{\mathcal{P}^{-,u}}\big)$ is irreducible if and only if $u \notin 2\mathbb{Z}$.
Furthermore, $\big(\mathcal{P}^{+,u}, \mathcal{H}_{\mathcal{P}^{+,u}}\big)$ and $\big(\mathcal{P}^{-,u}, \mathcal{H}_{\mathcal{P}^{-,u}}\big)$ are unitary for $u \in i\mathbb{R}$.

For the proof, see [20], Chapter II.5, II.6, VII.1 and VII.2.

For simplicity, within this section, the representations will be identified with their equivalence classes.

Lemma 4.2.
For every function $f \in L^2(K)_\pm$ and every $g \in G$,

$$\int_K e^{-2\rho H(g^{-1}k)} \Big| f\Big(\kappa\big(g^{-1}k\big)\Big)\Big|^2 dk = \|f\|^2_{L^2(K)}.$$

The proof can be found in [20], Chapter VII.2 and is part of the proof of the unitarity of the representations $\mathcal{P}^{\pm,u}$ for $u \in i\mathbb{R}$.

Lemma 4.3.
For every $g \in G$ and every $u \in \mathbb{R}$, the operator $\big(\mathcal{P}^{+,-u}(g)\big)^{-1}$ is the adjoint operator of $\mathcal{P}^{+,u}(g)$ with respect to the usual $L^2(K)$-scalar product.

Proof:
As u and thus also $e^{-(\nu_u+\rho)H(g^{-1}k)}$ are real, with Result (7.4) in [20], Chapter VII.2, which is similar to the equation given in Lemma 4.2, one gets for $f_1, f_2 \in L^2(K)_+$,

$$\begin{aligned}
& \big\langle \mathcal{P}^{+,-u}(g)f_1, \mathcal{P}^{+,u}(g)f_2 \big\rangle_{L^2(K)} \\
= & \int_K \mathcal{P}^{+,-u}(g)f_1(k)\overline{\mathcal{P}^{+,u}(g)f_2(k)}dk \\
= & \int_K e^{-(\nu_{-u}+\rho)H(g^{-1}k)} f_1\Big(\kappa\big(g^{-1}k\big)\Big)\overline{e^{-(\nu_u+\rho)H(g^{-1}k)} f_2\Big(\kappa\big(g^{-1}k\big)\Big)}dk \\
= & \int_K e^{-2\rho H(g^{-1}k)} \big(f_1\overline{f_2}\big)\Big(\kappa\big(g^{-1}k\big)\Big)dk \\
= & \int_K \big(f_1\overline{f_2}\big)(k)dk \\
= & \langle f_1, f_2\rangle_{L^2(K)}.
\end{aligned}$$

□

Definition 4.4 (n-th isotypic component).
For a representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ of K define for every $n \in \mathbb{Z}$ the n-th isotypic component or K-type of $\tilde{\pi}$ as

$$\mathcal{H}_{\tilde{\pi}}(n) := \big\{ v \in \mathcal{H}_{\tilde{\pi}} |\ \tilde{\pi}(k_\varphi)v = e^{in\varphi} v\ \forall \varphi \in [0, 2\pi) \big\}.$$

A representation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ of G is called even (respectively odd), if $\mathcal{H}_{\tilde{\pi}_{|K}}(n) = \{0\}$ for all odd n (respectively for all even n).

Every irreducible unitary representation of G is even or odd.
Furthermore, the algebraic direct sum

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{\tilde{\pi}}(n)$$

is dense in $\mathcal{H}_{\tilde{\pi}}$.

Remark 4.5.
By the definition of the Hilbert spaces $L^2(K)_\pm$ of $\mathcal{P}^{\pm,u}$ for $u \in \mathbb{C}$, it is easy to verify that $\mathcal{P}^{+,u}$ is even for every $u \in \mathbb{C}$ and that $\mathcal{P}^{-,u}$ is odd for every $u \in \mathbb{C}$.

By the definition of the n-th isotypic component, one can remark that

$$\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot} \text{ for all even } n \in \mathbb{Z} \quad \text{and} \quad \mathcal{H}_{\mathcal{P}^{-,u}}(n) = \mathbb{C} \cdot e^{-in\cdot} \text{ for all odd } n \in \mathbb{Z}.$$

In order to prove this, let $n \in \mathbb{Z}$ even, $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ and define $c_f := f(1_G) \in \mathbb{C}$. Since

$$\begin{aligned} \mathcal{H}_{\mathcal{P}^{+,u}}(n) &= \big\{ \tilde{f} \in L^2(K)_+ |\ \mathcal{P}^{+,u}(k_\varphi)\tilde{f} = e^{in\varphi}\tilde{f}\ \forall \varphi \in [0, 2\pi) \big\} \\ &= \big\{ \tilde{f} \in L^2(K)_+ |\ \tilde{f}\big(k_\varphi^{-1} \cdot\big) = e^{in\varphi}\tilde{f}\ \forall \varphi \in [0, 2\pi) \big\}, \end{aligned}$$

one gets

$$f\big(k_\varphi^{-1}\big) = f\big(k_\varphi^{-1} \cdot 1_G\big) = e^{in\varphi} f(1_G) = e^{in\varphi} c_f.$$

Therefore, $f(k_\varphi) = e^{-in\varphi} c_f$ and since one can choose $c_f \in \mathbb{C}$ arbitrarily, $\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot}$.
The second statement follows analogously.

Definition 4.6 (Casimir operator).
Let $\langle \cdot, \cdot \rangle$ be the non-degenerate symmetric and Ad-invariant bilinear form on $\mathfrak{g}$ defined by

$$\langle X, Y \rangle := 2 \operatorname{tr}(XY) \quad \forall X, Y \in \mathfrak{g}.$$

Furthermore, choose the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$, where

$$X_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as well as its dual basis $\big\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\big\}$ with respect to $\langle \cdot, \cdot \rangle$ with $\tilde{X}_1 := \frac{1}{2}X_2$, $\tilde{X}_2 := \frac{1}{2}X_1$ and $\tilde{X}_3 := X_3$.
Then, the Casimir operator $\mathscr{C}$ of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$ is defined as

$$\mathscr{C} := \sum_{i=1}^{3} X_i \tilde{X}_i = \frac{1}{2}(X_1 X_2 + X_2 X_1) + X_3^2.$$

Here, products are regarded in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.
If now $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, one can define $\pi(\mathscr{C})$ as

$$\pi(\mathscr{C}) := \sum_{i=1}^{3} \pi\big(X_i\big)\pi\big(\tilde{X}_i\big) = \frac{1}{2}\big(\pi(X_1)\pi(X_2) + \pi(X_2)\pi(X_1)\big) + \pi(X_3)^2 \in \mathfrak{gl}(V).$$

For every representation $\pi : G \to GL(V)$ of the Lie group G, $\pi(\mathscr{C})$ satisfies

$$\pi(g) \circ \pi(\mathscr{C}) = \pi(\mathscr{C}) \circ \pi(g) \quad \forall g \in G$$

on the space $V^\infty := \big\{v \in V|\ G \ni g \mapsto \pi(g)v \in V \text{ is smooth}\big\}$ of smooth vectors of V.

In general, the definition of the Casimir operator can be made with an arbitrary basis of $\mathfrak{g}$ and an arbitrary associated dual basis. If one takes an arbitrary non-degenerate symmetric and Ad-invariant bilinear form on $\mathfrak{g}$, its definition changes by a scalar multiple.

As $\frac{1}{2}(X_1X_2 + X_2X_1) + X_3^2 = X_3^2 - X_3 + X_1X_2$, one can also write the Casimir operator as

$$\mathscr{C} = X_3^2 - X_3 + X_1X_2,$$

and for a representation π of $\mathfrak{g}$, one then gets

$$\pi(\mathscr{C}) = \pi(X_3)^2 - \pi(X_3) + \pi(X_1)\pi(X_2).$$

Definition 4.7 (p_n)**.**
Denote for $n \in \mathbb{Z}$ by $b_n(f)$ the n-th Fourier coefficient of $f \in L^2(K)_\pm$ which is defined as

$$b_n(f) := \frac{1}{|K|} \int_K f(k_\varphi) e^{-in\varphi} dk_\varphi,$$

and let

$$p_n(f) := b_{-n}(f)e^{-in\cdot}.$$

One can easily show that for every $u \in \mathbb{C}$ and for every $n \in \mathbb{Z}$ the operator p_n is the projection from $\mathcal{H}_{\mathcal{P}^{\pm,u}} = L^2(K)_\pm$ to the n-th isotypic component of the representation $\mathcal{P}^{\pm,u}$.

4.2 The spectrum of $SL(2,\mathbb{R})$

4.2.1 Introduction of the operator K_u

Now, an operator K_u which is needed in order to describe the spectrum of G will be introduced using the Knapp-Stein operator.

Define

$$\begin{aligned} C^\infty(K)_+ &:= \big\{f \in C^\infty(K)|\ f(k) = f(-k)\ \forall k \in K\big\} \quad \text{and} \\ C^\infty(K)_- &:= \big\{f \in C^\infty(K)|\ f(k) = -f(-k)\ \forall k \in K\big\} \end{aligned}$$

and let

$$J_u : C^\infty(K)_+ \to C^\infty(K)_+ \quad \text{for } u \in \mathbb{C} \text{ with } \mathrm{Re}\, u > 0$$

be the Knapp-Stein intertwining operator, which intertwines the representation $\mathcal{P}^{+,u}$ with the representation $\mathcal{P}^{+,-u}$. Furthermore, let $w := k_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and extend $f \in L^2(K)$ to G by using the Iwasawa decomposition $G \ni g = \kappa(g)e^{H(g)}\mu$ for $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $\mu \in N$. Then, defining $\tilde{f}_u\big(\kappa(g)e^{H(g)}\mu\big) := e^{-(\nu_u+\rho)H(g)} f\big(\kappa(g)\big)$, this operator can be written as

$$J_u f(k) = \int_N \tilde{f}_u(k\mu w)d\mu \quad \forall f \in C^\infty(K)_+ \ \forall k \in K. \tag{14}$$

This integral converges for $\operatorname{Re} u > 0$ (see [20], Chapter VII or [33], Chapter 10.1).
The mapping $f \mapsto J_u f$ is continuous and the family of operators $\{J_u | \ u \in \mathbb{C}\}$ is holomorphic in u for $\operatorname{Re} u > 0$ with respect to appropriate topologies (see [20], Chapter VII.7 or [33], Chapter 10.1).
For $u \in \mathbb{R}_{>0}$, the operator J_u is self-adjoint with respect to the usual $L^2(K)$-scalar product. Moreover, one can extend the function $u \mapsto J_u$ meromorphically to $\mathbb{C}$ (see [33], Chapter 10.1). Then, for every $u \in \mathbb{C}$ for which the operator J_u is regular, it is an intertwining operator from $\mathcal{P}^{+,u}$ to $\mathcal{P}^{+,-u}$.

Remark 4.8.
The operator J_u commutes with the projections p_n for all $n \in \mathbb{Z}$ and for every $u \in \mathbb{C}$ for which J_u is regular.

This can be explained in the following way: Since the operator J_u is a G-intertwining operator, it is a K-intertwining operator as well. Thus, one can show as follows that it leaves $\mathcal{H}_{\mathcal{P}^{+,u}}(n)$, the above defined n-th isotypic component of the representation $\mathcal{P}^{+,u}$, invariant:
Let $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(n)$, i.e. $\mathcal{P}^{+,u}\big(k_\varphi\big)f = e^{in\varphi}f$ for every $\varphi \in [0, 2\pi)$. Because of the intertwining property, one gets

$$e^{in\varphi}J_u f = J_u \circ \mathcal{P}^{+,u}\big(k_\varphi\big)f = \mathcal{P}^{+,-u}\big(k_\varphi\big) \circ J_u f.$$

Therefore, $J_u f \in \mathcal{H}_{\mathcal{P}^{+,-u}}(n)$ and since $\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathcal{H}_{\mathcal{P}^{+,\tilde{u}}}(n)$ for every $\tilde{u} \in \mathbb{C}$, one has $J_u f \in \mathcal{H}_{\mathcal{P}^{+,u}}(n)$.
From this, one can easily conclude that $p_n \circ J_u = J_u \circ p_n$ and the assertion follows.

One can now deduce that the operators J_u have the property

$$J_{u|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} = c_n(u) \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{for all even } n \in \mathbb{Z}, \tag{15}$$

as an equality of meromorphic functions, where $c_n : \mathbb{C} \to \mathbb{C}$ is a meromorphic function for every even $n \in \mathbb{Z}$. This follows from the above remark together with the fact that $\mathcal{H}_{\mathcal{P}^{+,u}}(n)$ is one-dimensional.

Using standard integral formulas (see [20], Chapter V.6), one gets by (14) for $u = 1$

$$J_1(f) = c \int_K f(k)dk \quad \forall f \in C^\infty(K)_+$$

for a constant $c > 0$. Hence, and since $\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot}$ for every even $n \in \mathbb{Z}$ which was stated above, one gets

$$c_n(1) \qquad \begin{cases} \neq 0 & \text{for } n = 0 \\ = 0 & \text{for } n \in \mathbb{Z} \setminus \{0\} \text{ even.} \end{cases} \tag{16}$$

For convenience, an explicit formula for the quotients of the functions c_n will now be given. However, this formula will not be used in this work. Instead, further necessary properties of the intertwining operator J_u and the functions c_n will be concluded from the irreducibility of the representations $\mathcal{P}^{\pm,u}$.

Remark 4.9.
The quotients of the functions c_n can be given by

$$\frac{c_n(u)}{c_0(u)} \;=\; \frac{(u-1)(u-3)\cdots\big(u-(|n|-1)\big)}{(u+1)(u+3)\cdots\big(u+(|n|-1)\big)}\cdot(-1)^{\frac{n}{2}} \quad \textit{for } u\in\mathbb{C} \textit{ and all even } n\in\mathbb{Z}.$$

This formula can be deduced from a formula in [7], Chapter 23, Appendix 1, which expresses $c_n(u)$ in terms of gamma functions, together with the gamma function recurrence formula. Here, one has to remark that the definition of $c_n(u)$ in [7] differs by a sign from its definition in this work.

Next, it can be shown that

$$c_0(u)\neq 0 \quad \text{for } u\in(0,1). \tag{17}$$

For this, assume that $c_0(u)=0$ for an element $u\in(0,1)$. Then, by (15), the operator J_u has a non-zero kernel. Furthermore, the representation $\mathcal{P}^{+,u}$ is irreducible on $C^\infty(K)_+$ (see Remark 4.1). As $\ker(J_u)$ is a closed invariant subspace with respect to the representation $\mathcal{P}^{+,u}$, the kernel of J_u has to be the whole space $C^\infty(K)_+$ and hence, the operator J_u is identically zero. But there is no $u\in\mathbb{C}$ with $\operatorname{Re}u>0$ such that the operator J_u is identically zero (see [33], Chapter 10.1). Therefore, $c_0(u)\neq 0$ for every $u\in(0,1)$.

Thus, one can define

$$\tilde{J}_u := \frac{1}{c_0(u)}\, J_u$$

for $u\in\mathbb{C}$, as a meromorphic function.

Lemma 4.10.
$\tilde{J}_u$ is regular at $u=0$ and $\tilde{J}_0 = id$.

Proof:
First, one can observe that on $\mathcal{H}_{\mathcal{P}^{+,u}}(0)$, the operator $\tilde{J}_u$ is always equal to the identity and that in particular $\tilde{J}_{0|\mathcal{H}_{\mathcal{P}^{+,0}}(0)}$ also equals $id_{|\mathcal{H}_{\mathcal{P}^{+,0}}(0)}$.
Now, it has to be shown that $\tilde{J}_u$ is regular at $u=0$.
Since the mapping $u\mapsto\tilde{J}_u$ is an operator-valued meromorphic function for $u\in\mathbb{C}$, one can represent it locally as a Laurent series in 0 with finite principal part and operators as coefficients in the following way:

$$\tilde{J}_u = \sum_{k=k_0}^{\infty} L_k u^k \quad \text{on } C^\infty(K)_+$$

for operators L_k going from $C^\infty(K)_+$ to $C^\infty(K)_+$ for $k\geq k_0$ and where $k_0\in\mathbb{Z}$ is the smallest number such that $L_{k_0}\neq 0$.
Moreover, this gives

$$L_{k_0} = \lim_{u\to 0} u^{-k_0}\tilde{J}_u.$$

Now, for the regularity of $\tilde{J}_u$ at $u = 0$ desired above, it has to be shown that $k_0 \geq 0$.
So, assume that $k_0 < 0$.
As every $\tilde{J}_u$ is an intertwining operator from $\mathcal{P}^{+,u}$ to $\mathcal{P}^{+,-u}$, L_{k_0} is an intertwining operator from $\mathcal{P}^{+,0}$ to itself, i.e. it commutes with $\mathcal{P}^{+,0}$. Furthermore, L_{k_0} vanishes on $\mathcal{H}_{\mathcal{P}^{+,0}}(0)$, the space of all constant functions on K, because $\tilde{J}_u$ equals the identity on $\mathcal{H}_{\mathcal{P}^{+,u}}(0)$ and thus does not have a pole there.
Since L_{k_0} commutes with $\mathcal{P}^{+,0}$, it vanishes on $\mathcal{P}^{+,0}(g)\mathcal{H}_{\mathcal{P}^{+,0}}(0)$ for every $g \in G$. In addition, as Remark 4.1 stays valid if one replaces the space $\mathcal{H}_{\mathcal{P}^{+,0}} = L^2(K)_+$ by $C^\infty(K)_+$, the representation $\mathcal{P}^{+,0}$ is irreducible on the space $C^\infty(K)_+$. Furthermore, for every $0 \neq \xi \in \mathcal{H}_{\mathcal{P}^{+,0}}(0)$, the subspace $\operatorname{span}\{\mathcal{P}^{+,0}(g)\xi | \ g \in G\}$ is G-invariant and hence, by the irreducibility, $C^\infty(K)_+ \subset \overline{\operatorname{span}\{\mathcal{P}^{+,0}(g)\xi | \ g \in G\}}$. Thus, L_{k_0} vanishes on the whole space $C^\infty(K)_+$, which is a contradiction to the choice of k_0.
Therefore, one gets $k_0 \geq 0$, which means that the mapping $u \mapsto \tilde{J}_u$ does not have any poles in $u = 0$. Hence, $\tilde{J}_u$ is regular at $u = 0$ on $C^\infty(K)_+$.
As above, as a limit of intertwining operators, $\tilde{J}_0 = \lim\limits_{u \to 0} \tilde{J}_u$ is an intertwining operator, which intertwines the irreducible unitary representation $\mathcal{P}^{+,0}$ with itself (see Remark 4.1 for the irreducibility and the unitarity of $\mathcal{P}^{+,0}$, again replacing the space $\mathcal{H}_{\mathcal{P}^{+,0}} = L^2(K)_+$ by $C^\infty(K)_+$). Thus, by Schur's lemma (see Theorem 2.6), $\tilde{J}_0$ is a scalar multiple of the identity. Since it equals the identity on $\mathcal{H}_{\mathcal{P}^{+,0}}(0)$, one gets $\tilde{J}_0 = id$.

□

From (17) and Lemma 4.10, one can conclude that

$$\frac{c_n(u)}{c_0(u)} > 0 \quad \text{for } u \in (0,1) \text{ and for all even } n \in \mathbb{Z} : \tag{18}$$

Using the same argumentation for the operator $\tilde{J}_u$ as in the proof of (17), one also gets that $\frac{c_n(u)}{c_0(u)} \neq 0$ for every $u \in (0,1)$.
Furthermore, from Lemma 4.10, one can deduce that $\frac{c_n(0)}{c_0(0)} = 1$, and with the continuity of $\frac{c_n}{c_0}$ on $(0,1)$, one gets $\frac{c_n(u)}{c_0(u)} > 0$ for all $u \in (0,1)$ and all even $n \in \mathbb{Z}$ and (18) is shown.

Now, define a scalar product on $C^\infty(K)_+$ as follows:

$$\langle f_1, f_2 \rangle_u := \langle \tilde{J}_u f_1, f_2 \rangle_{L^2(K)}.$$

Lemma 4.11.
$\langle \cdot, \cdot \rangle_u$ is an invariant positive definite scalar product for $u \in (0,1)$.

Proof:
$\langle \cdot, \cdot \rangle_u$ is hermitian:
By the definition of $\tilde{J}_u$ and as J_u is self-adjoint with respect to the usual $L^2(K)$-scalar product for $u \in (0,1)$, this is straightforward.
$\langle \cdot, \cdot \rangle_u$ is invariant:
By Lemma 4.3, for every $g \in G$, the operator $\left(\mathcal{P}^{+,u}(g)\right)^{-1}$ is the adjoint operator of $\mathcal{P}^{+,-u}(g)$ with respect to the usual $L^2(K)$-scalar product. Now, let $f_1, f_2 \in C^\infty(K)_+$. Then, as $\tilde{J}_u$ intertwines $\mathcal{P}^{+,u}$ and $\mathcal{P}^{+,-u}$, one gets for every $g \in G$,

$$\begin{aligned}\left\langle \mathcal{P}^{+,u}(g)f_1, \mathcal{P}^{+,u}(g)f_2\right\rangle_u &= \left\langle \tilde{J}_u \circ \mathcal{P}^{+,u}(g)f_1, \mathcal{P}^{+,u}(g)f_2\right\rangle_{L^2(K)} \\ &= \left\langle \mathcal{P}^{+,-u}(g) \circ \tilde{J}_u f_1, \mathcal{P}^{+,u}(g)f_2\right\rangle_{L^2(K)} \\ &= \left\langle \tilde{J}_u f_1, f_2\right\rangle_{L^2(K)} = \langle f_1, f_2\rangle_u.\end{aligned}$$

$\langle\cdot,\cdot\rangle_u$ is positive definite:
As $\frac{c_n(u)}{c_0(u)} > 0$ by (18), one gets for every $n \in \mathbb{Z}$ and $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ by (15) above,

$$\langle f, f\rangle_u = \left\langle \tilde{J}_u f, f\right\rangle_{L^2(K)} = \frac{c_n(u)}{c_0(u)} \left\langle id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} f, f\right\rangle_{L^2(K)} = \frac{c_n(u)}{c_0(u)} \langle f, f\rangle_{L^2(K)} \geq 0 \quad \text{and}$$

$$\langle f, f\rangle_u = 0 \iff \langle f, f\rangle_{L^2(K)} = 0 \iff f = 0.$$

Now, it will be shown that the algebraic direct sum

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{\mathcal{P}^{+,u}}(n) \quad \text{is orthogonal with respect to } \langle\cdot,\cdot\rangle_u. \tag{19}$$

As $\mathcal{P}^{+,u}$ is an even representation, only K-types for even $n \in \mathbb{Z}$ appear. So, let $n_1, n_2 \in \mathbb{Z}$ even with $n_1 \neq n_2$ and let $f_1 \in \mathcal{H}_{\mathcal{P}^{+,u}}(n_1)$ and $f_2 \in \mathcal{H}_{\mathcal{P}^{+,u}}(n_2)$, i.e.

$$\mathcal{P}^{+,u}(k_\varphi)f_1 = e^{in_1\varphi}f_1 \quad \text{and} \quad \mathcal{P}^{+,u}(k_\varphi)f_2 = e^{in_2\varphi}f_2 \quad \forall\varphi \in [0, 2\pi).$$

Then, because of the invariance of $\langle\cdot,\cdot\rangle_u$ shown above, one gets for all $\varphi \in [0, 2\pi)$,

$$\begin{aligned}\langle f_1, f_2\rangle_u &= \left\langle \mathcal{P}^{+,u}(k_\varphi)f_1, \mathcal{P}^{+,u}(k_\varphi)f_2\right\rangle_u = \left\langle e^{in_1\varphi}f_1, e^{in_2\varphi}f_2\right\rangle_u \\ &= e^{i(n_1-n_2)\varphi}\langle f_1, f_2\rangle_u.\end{aligned}$$

As this is true for all $\varphi \in [0, 2\pi)$ and as $n_1 \neq n_2$, one gets $\langle f_1, f_2\rangle_u = 0$ and (19) is shown.
Now, since the direct sum $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_u$ and dense in $\mathcal{H}_{\mathcal{P}^{+,u}} = L^2(K)_+$, the above observations hold for all $f \in C^\infty(K)_+$.

□

The completion of $C^\infty(K)_+$ with respect to the scalar product $\langle\cdot,\cdot\rangle_u$ yields a Hilbert space $\mathcal{H}_u$. Considering the restriction of the representation $\mathcal{P}^{+,u}$ to $C^\infty(K)_+$ and then continuously extending it to the space $\mathcal{H}_u$, one gets a unitary representation which will be denoted by $\mathcal{P}^{+,u}$ as well. G acts on $\mathcal{H}_u$ by this unitary representation $\mathcal{P}^{+,u}$.

Furthermore, let $d_n(u) := \sqrt{\frac{c_n(u)}{c_0(u)}} > 0$ for $u \in (0, 1)$. Next, a unitary bijection

$$K_u : \mathcal{H}_u \to L^2(K)_+ \quad \forall u \in (0, 1)$$

shall be defined. On the n-th isotypic component in $\mathcal{H}_u$, define K_u by

$$K_{u|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} := d_n(u) \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{for all even } n \in \mathbb{Z}.$$

Then, one can extend this definition to finite sums of K-types. This operator also is self-adjoint with respect to the usual $L^2(K)$-scalar product and for finite sums of K-types f_1 and f_2

$$\big\langle K_u f_1, K_u f_2\big\rangle_{L^2(K)} = \big\langle K_u^2 f_1, f_2\big\rangle_{L^2(K)} = \big\langle \tilde{J}_u f_1, f_2\big\rangle_{L^2(K)} = \langle f_1, f_2\rangle_u.$$

From this follows directly that it is unitary (if one regards the space $\mathcal{H}_u$ equipped with $\langle\cdot,\cdot\rangle_u$ and $L^2(K)_+$ with $\langle\cdot,\cdot\rangle_{L^2(K)}$) and hence, because of the density of $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ in $\mathcal{H}_u$, one can extend K_u continuously on the whole space $\mathcal{H}_u$.
Moreover, K_u is continuous in u and one also has the property $\lim_{u\to 0} K_u = id$.
By its definition, the operator K_u commutes with the projections p_n for all $n \in \mathbb{Z}$ as well.

Now, by [20], Chapter XIV.4, one gets the identity

$$J_{-u} \circ J_u = \gamma(u) \cdot id, \tag{20}$$

where $u \mapsto \gamma(u)$ is a meromorphic function. One can also obtain this equation by observing that $J_{-u} \circ J_u$ intertwines the representation $\mathcal{P}^{+,u}$, which is irreducible for almost all $u \in \mathbb{C}$ (see Remark 4.1), with itself, and by using a version of Schur's lemma for $(\mathfrak{g}, K)$-modules.
By restricting the above equation to the n-th isotypic component, one thus gets the relation

$$c_n(u)c_n(-u) = \gamma(u) \quad \text{for all even } n \in \mathbb{Z} \tag{21}$$

as meromorphic functions.

Next, another scalar product on the space $C^\infty(K)_{++} := \big\{f \in C^\infty(K)_+ |\ p_n(f) = 0\ \forall n \leq 0\big\}$ is needed.
For this, define

$$\tilde{J}_{(u)} := \frac{1}{c_2(u)}\ J_u$$

for $u \in \mathbb{C}$ as a meromorphic family of operators. Then, on $\mathcal{H}_{\mathcal{P}^{+,u}}(2)$ the operator $\tilde{J}_{(u)}$ is equal to the identity for every $u \in \mathbb{C}$ and in particular $\tilde{J}_{(1)|\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$ equals $id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$.
Moreover, define the space $C^\infty(K)_{+-} := \big\{f \in C^\infty(K)_+ |\ p_n(f) = 0\ \forall n \geq 0\big\}$.
The representation $\mathcal{P}^{+,1}$ is irreducible on the spaces $C^\infty(K)_{++}$ and $C^\infty(K)_{+-}$ (see [32], Chapter 5.6).

Lemma 4.12.

(a) $\tilde{J}_{(-u)} \circ \tilde{J}_{(u)} = \tilde{J}_{(u)} \circ \tilde{J}_{(-u)} = id$ *as a meromorphic family.*

(b) $\tilde{J}_{(u)}$ *is regular at* $u = -1$ *and*

$$ker\big(\tilde{J}_{(-1)}\big) \cap C^\infty(K)_{++} = ker\big(\tilde{J}_{(-1)}\big) \cap C^\infty(K)_{+-} = \{0\}.$$

Furthermore, $\tilde{J}_{(-1)}$ *is an intertwining operator of* $\mathcal{P}^{+,-1}$ *with* $\mathcal{P}^{+,1}$.

(c) $\tilde{J}_{(u)|C^\infty(K)_{++}\oplus C^\infty(K)_{+-}}$ *is regular at* $u = 1$.

Proof:
For all $n \in \mathbb{Z}$ and all $u \in \mathbb{C}$,

$$\tilde{J}_{(-u)} \circ \tilde{J}_{(u)|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \overset{(20)}{=} \frac{\gamma(u)}{c_2(u)c_2(-u)} \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \overset{(21)}{=} id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)}$$

and thus,

$$\tilde{J}_{(-u)} \circ \tilde{J}_{(u)} = \tilde{J}_{(u)} \circ \tilde{J}_{(-u)} = id$$

for $u \in \mathbb{C}$ as a meromorphic family and (a) follows.

Because the mapping $u \mapsto \tilde{J}_{(u)}$ is an operator-valued meromorphic function for $u \in \mathbb{C}$, one can represent it locally as a Laurent series in -1 with finite principal part and operators as coefficients in the following way:

$$\tilde{J}_{(u)} = \sum_{k=k_0}^{\infty} L_k(u+1)^k \quad \text{on } C^\infty(K)_+$$

for operators L_k going from $C^\infty(K)_+$ to $C^\infty(K)_+$ for $k \geq k_0$ and where $k_0 \in \mathbb{Z}$ is the smallest number such that $L_{k_0} \neq 0$.
Then, one gets

$$L_{k_0} = \lim_{u \to 1} (-u+1)^{-k_0} \tilde{J}_{(-u)} = \lim_{u \to 1} (-u+1)^{-k_0} \frac{J_{-u}}{c_2(-u)} \tag{22}$$

and thus, as a limit of intertwining operators, L_{k_0} is an intertwining operator of $\mathcal{P}^{+,-1}$ with $\mathcal{P}^{+,1}$. Now, it shall first be shown that

$$\ker(L_{k_0}) \cap C^\infty(K)_{++} = \ker(L_{k_0}) \cap C^\infty(K)_{+-} = \{0\}. \tag{23}$$

Since $\mathcal{H}_{\mathcal{P}^{+,1}}(n) = \mathcal{H}_{\mathcal{P}^{+,-1}}(n)$,

$$L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(n)} = a_n \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(n)} \quad \text{for} \quad a_n := \lim_{u \to 1} \frac{c_n(-u)}{c_2(-u)}(-u+1)^{-k_0}.$$

As $C^\infty(K)_{++} = \bigoplus_{\substack{n \in \mathbb{Z}_{>0} \\ n \text{ even}}} \mathcal{H}_{\mathcal{P}^{+,1}}(n)$ and $C^\infty(K)_{+-} = \bigoplus_{\substack{n \in \mathbb{Z}_{<0} \\ n \text{ even}}} \mathcal{H}_{\mathcal{P}^{+,1}}(n)$, for (23) it has to be shown that $a_n \neq 0$ for all even $n \in \mathbb{Z} \setminus \{0\}$.
By (21), one has

$$\frac{c_n(u)c_n(-u)}{c_2(-u)} = \frac{\gamma(u)}{c_2(-u)},$$

i.e. the left hand side does not depend on n. So, the order of the pole for the limit for $u \to 1$ has to be the same for every $n \in \mathbb{Z}$. But as $c_n(1) = 0$ for $n \neq 0$ and $c_0(1) \neq 0$ by (16), $c_0(-1)$ has to be 0 in order to get the same order of the pole for $n = 0$ and $n \neq 0$. It follows that $a_0 = 0$. As k_0 was chosen such that $L_{k_0} \neq 0$, there has to exist $0 \neq n \in \mathbb{Z}$ with $a_n \neq 0$.
Furthermore, from (14) one can conclude that $\overline{J_u f} = J_{\overline{u}} \overline{f}$ for all $u \in \mathbb{C}$ with $\operatorname{Re} u > 0$. By extending meromorphically, this holds for all $u \in \mathbb{C}$ and, by regarding the n-th isotypic component, one can deduce that $\overline{c_n(u)} = c_{-n}(\overline{u})$. Hence, one gets for every $n \in \mathbb{Z}$ and every

$u \in \mathbb{R}$ that $\overline{c_n(u)} = c_{-n}(u)$. This means that there also exists an integer $n_1 > 0$ such that $a_{n_1} \neq 0 \neq a_{-n_1}$. Therefore,

$$L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)} = a_{n_1} \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)} \neq 0 \neq a_{-n_1} \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)} = L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)}$$

and thus,

$$\operatorname{Im}\left(L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(n_1)}\right) \subset \mathcal{H}_{\mathcal{P}^{+,1}}(n_1) \subset C^\infty(K)_{++} \quad \text{and}$$
$$\operatorname{Im}\left(L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(-n_1)}\right) \subset \mathcal{H}_{\mathcal{P}^{+,1}}(-n_1) \subset C^\infty(K)_{+-}.$$

So, in particular $\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{++} \neq \{0\} \neq \operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{+-}$. Since $\mathcal{P}^{+,1}$ is irreducible on $C^\infty(K)_{++}$ and on $C^\infty(K)_{+-}$, one gets

$$\overline{\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{++}} = C^\infty(K)_{++} \quad \text{and} \quad \overline{\operatorname{Im}(L_{k_0}) \cap C^\infty(K)_{+-}} = C^\infty(K)_{+-},$$

where the completions are regarded in the spaces $C^\infty(K)_{++}$ and $C^\infty(K)_{+-}$, respectively. Hence, $\mathcal{H}_{\mathcal{P}^{+,1}}(n) \subset \operatorname{Im}(L_{k_0})$ for all even $n \neq 0$. Since $L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(n)} = a_n \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(n)}$, one therefore gets $a_n \neq 0$ for every even $n \neq 0$ and thus, (23) follows.
One can now conclude that $k_0 = 0$:
Because

$$L_{k_0|\mathcal{H}_{\mathcal{P}^{+,1}}(2)} = \lim_{u \to 1} (-u+1)^{-k_0} \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,1}}(2)}$$

does not have any poles, $k_0 \not> 0$, and as it does not have any zeros, $k_0 \not< 0$.
Hence,

$$L_{k_0} = \lim_{u \to 1} \tilde{J}_{(-u)} = \tilde{J}_{(-1)}$$

and thus, (b) follows by (23) and as L_{k_0} intertwines $\mathcal{P}^{+,-1}$ with $\mathcal{P}^{+,1}$.
Now, by (b), $\tilde{J}_{(u)}$ is regular at $u = -1$. As by (a), one has $\tilde{J}_{(-1)} \circ \tilde{J}_{(1)} = \tilde{J}_{(1)} \circ \tilde{J}_{(-1)} = id$ and since by (b) the operator $\tilde{J}_{(-1)}$ is nowhere equal to 0 on $C^\infty(K)_{++} \oplus C^\infty(K)_{+-}$, the operator $\tilde{J}_{(u)}$ has to be regular on $C^\infty(K)_{++} \oplus C^\infty(K)_{+-}$ at $u = 1$ as well and (c) is shown. □

$\tilde{J}_{(1)}$ is not identically 0 on the space $C^\infty(K)_{++}$, because it equals the identity on $\mathcal{H}_{\mathcal{P}^{+,1}}(2)$. Thus, regard the operator $\tilde{J}_{(1)}$ on the space $C^\infty(K)_{++}$. This operator is injective, since for every $f \in C^\infty(K)_{++}$ with $\tilde{J}_{(1)}(f) = 0$, one gets by Lemma 4.12(a) that $0 = \tilde{J}_{(-1)} \circ \tilde{J}_{(1)}(f) = id(f) = f$.

Now, define for all functions $f_1, f_2 \in C^\infty(K)_{++}$,

$$\langle f_1, f_2 \rangle_{(1)} := \langle \tilde{J}_{(1)} f_1, f_2 \rangle_{L^2(K)}.$$

Furthermore, choose $\tilde{f}_1$ in such a way that $\tilde{J}_{(-1)}(\tilde{f}_1) = f_1$. Then, by Lemma 4.12,

$$\langle f_1, f_2 \rangle_{(1)} = \langle \tilde{f}_1, f_2 \rangle_{L^2(K)} \tag{24}$$

since the scalar product does not depend on the choice of $\tilde{f}_1$:
It is possible to add an element $\tilde{f}_{ker} \in \ker(\tilde{J}_{(-1)})$ to the function $\tilde{f}_1$. But by the above Lemma 4.12(b), one has $\ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{++} = \ker(\tilde{J}_{(-1)}) \cap C^\infty(K)_{+-} = \{0\}$ and therefore, $\tilde{f}_{ker} \in \{f \in C^\infty(K)_+ |\, p_n(f) = 0 \ \forall n \neq 0\}$. Thus, the function $\tilde{f}_{ker}$ is orthogonal to $f_2 \in C^\infty(K)_{++} = \{f \in C^\infty(K)_+ |\, p_n(f) = 0 \ \forall n \leq 0\}$ and the scalar product stays the same.

Lemma 4.13.
$\langle \cdot, \cdot \rangle_{(1)}$ is an invariant positive definite scalar product.

Proof:
$\langle \cdot, \cdot \rangle_{(1)}$ is hermitian:
Like above for $\langle \cdot, \cdot \rangle_u$, this is straightforward.
$\langle \cdot, \cdot \rangle_{(1)}$ is invariant:
Let $f_1, f_2 \in C^\infty(K)_{++}$ and choose $\tilde{f}_1$ such that $\tilde{J}_{(-1)}(\tilde{f}_1) = f_1$.
Then, as $\tilde{J}_{(-1)}$ intertwines $\mathcal{P}^{+,-1}$ and $\mathcal{P}^{+,1}$ by Lemma 4.12(b), for every $g \in G$,

$$\tilde{J}_{(-1)}\Big(\mathcal{P}^{+,-1}(g)\tilde{f}_1\Big) = \mathcal{P}^{+,1}(g) \circ \tilde{J}_{(-1)}(\tilde{f}_1) = \mathcal{P}^{+,1}(g) f_1.$$

Hence, one can choose

$$\widetilde{\mathcal{P}^{+,1}(g) f_1} := \mathcal{P}^{+,-1}(g)\tilde{f}_1.$$

Since by Lemma 4.3 for all $g \in G$ the operator $\big(\mathcal{P}^{+,1}(g)\big)^{-1}$ is the adjoint operator of $\mathcal{P}^{+,-1}(g)$ with respect to the usual $L^2(K)$-scalar product, one gets for every $g \in G$,

$$\begin{aligned}
\big\langle \mathcal{P}^{+,1}(g) f_1, \mathcal{P}^{+,1}(g) f_2 \big\rangle_{(1)} &= \Big\langle \widetilde{\mathcal{P}^{+,1}(g) f_1}, \mathcal{P}^{+,1}(g) f_2 \Big\rangle_{L^2(K)} \\
&= \Big\langle \mathcal{P}^{+,-1}(g)\tilde{f}_1, \mathcal{P}^{+,1}(g) f_2 \Big\rangle_{L^2(K)} \\
&= \big\langle \tilde{f}_1, f_2 \big\rangle_{L^2(K)} \\
&= \langle f_1, f_2 \rangle_{(1)}.
\end{aligned}$$

$\langle \cdot, \cdot \rangle_{(1)}$ is positive definite:
$\frac{c_n(u)}{c_2(u)} > 0$ for every $u \in (0,1)$, since

$$\frac{c_n(u)}{c_2(u)} = \frac{c_n(u)}{c_0(u)} \cdot \frac{c_0(u)}{c_2(u)} > 0$$

by (18) of the beginning of this subsection. Hence, its limit $\lim\limits_{u \to 1} \frac{c_n(u)}{c_2(u)}$ is larger or equal to 0 as well. Therefore, similarly as above for $\langle \cdot, \cdot \rangle_u$, for every $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and for every $f_1 \in \mathcal{H}_{\mathcal{P}^{+,1}}(n)$, one gets

$$\langle f_1, f_1 \rangle_{(1)} = \big\langle \tilde{J}_{(1)} f_1, f_1 \big\rangle_{L^2(K)} = \lim_{u \to 1} \frac{c_n(u)}{c_2(u)} \, \langle f_1, f_1 \rangle_{L^2(K)} \geq 0$$

from the argument above and since $\mathcal{H}_{\mathcal{P}^{+,1}}(n) = \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ for every $u \in (0,1)$. Moreover, because of the injectivity of $\tilde{J}_{(1)}$

$$f_1 = 0 \iff \tilde{J}_{(1)} f_1 = 0 \iff \lim_{u \to 1} \frac{c_n(u)}{c_2(u)} \, f_1 = 0,$$

which means that $\lim\limits_{u \to 1} \frac{c_n(u)}{c_2(u)} > 0$. Hence,

$$\langle f_1, f_1 \rangle_{(1)} = 0 \iff \langle f_1, f_1 \rangle_{L^2(K)} = 0 \iff f_1 = 0.$$

Like in (19), one can prove that the algebraic direct sum $\bigoplus\limits_{n\in\mathbb{N}^*} \mathcal{H}_{\mathcal{P}^{+,1}}(n)$ is orthogonal with respect to $\langle\cdot,\cdot\rangle_{(1)}$. Due to this and as the direct sum is dense in $C^\infty(K)_{++}$, the positive definiteness everywhere is shown.

□

The completion of the space $C^\infty(K)_{++}$ with respect to this scalar product gives a Hilbert space which will be called $\mathcal{H}_{(1)}$.

The same procedure can be accomplished for $C^\infty(K)_{+-} = \{f \in C^\infty(K)_+ |\, p_n(f) = 0 \,\forall n \geq 0\}$: Here, define the operator $\tilde{J}_{[u]}$ as

$$\tilde{J}_{[u]} := \frac{1}{c_{-2}(u)}\ J_u$$

for $u \in (0,1)$ as a meromorphic family of operators. Like above, it can be shown that $\tilde{J}_{[u]}$ is regular at $u = 1$ on $C^\infty(K)_{+-}$ and is not identically 0 there.
Thus, define again for all $f_1, f_2 \in C^\infty(K)_{+-}$,

$$\langle f_1, f_2\rangle_{[1]} := \left\langle \tilde{J}_{[1]} f_1, f_2\right\rangle_{L^2(K)}.$$

This is another invariant positive definite scalar product. The completion of the space $C^\infty(K)_{+-}$ with respect to this scalar product gives a Hilbert space called $\mathcal{H}_{[1]}$.

4.2.2 Description of the irreducible unitary representations

Now, some convenient realizations for the spectrum of $SL(2,\mathbb{R})$ shall be provided.

The spectrum $\widehat{G}$ of $G = SL(2,\mathbb{R})$ consists of the following representations:

1. The **principal series** representations:
   (a) $\mathcal{P}^{+,iv}$ for $v \in [0,\infty)$.
   (b) $\mathcal{P}^{-,iv}$ for $v \in (0,\infty)$.

   See Section 4.1 above for the definitions.

2. The **complementary series** representations $\mathcal{C}^u$ for $u \in (0,1)$:
   The Hilbert space $\mathcal{H}_{\mathcal{C}^u}$ is defined by

   $$\mathcal{H}_{\mathcal{C}^u} := L^2(K)_+$$

   and the action is given by

   $$\mathcal{C}^u(g) := K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1}$$

   for all $g \in G$, where here again $\mathcal{P}^{+,u}(g)$ is meant in the following way: One considers the restriction of $\mathcal{P}^{+,u}(g)$ to $C^\infty(K)_+$ and then continuously extends it to the space $\mathcal{H}_u$ (see the definition of $\mathcal{H}_u$ in Section 4.2.1).

3. The **discrete series** representations:

   (a) $\mathcal{D}_m^+$ for odd $m \in \mathbb{N}^*$:

      (i) $\mathcal{D}_1^+$:
      The Hilbert space $\mathcal{H}_{\mathcal{D}_1^+}$ is given by

      $$\mathcal{H}_{\mathcal{D}_1^+} := \mathcal{H}_{(1)}$$

      defined in Section 4.2.1. The action is given by

      $$\mathcal{D}_1^+ := \mathcal{P}^{+,1}.$$

      Here again, as well as in all the definitions in this subsection, the representation $\mathcal{P}^{+,u}$ for the different values $u \in \mathbb{C}$ is meant as described above: One restricts it to the respective subspace of $L^2(K)$ and then continuously extends it to the respective Hilbert space.

      (ii) $\mathcal{D}_m^+$ for odd $m \in \mathbb{N}_{\geq 3}$:
      As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^+}$ for $\mathcal{D}_m^+$ for odd $m \in \mathbb{N}_{\geq 3}$, one can take the completion of the space

      $$\left\{ f \in C^\infty(K)_+ \middle| \; p_n(f) = 0 \; \forall n \leq m-1 \right\}$$

      with respect to an appropriate scalar product, and as the action, one can take

      $$\mathcal{D}_m^+ := \mathcal{P}^{+,m}.$$

      With this realization, the Hilbert spaces $\mathcal{H}_{\mathcal{D}_m^+}$ for odd $m \in \mathbb{N}_{\geq 3}$ depend on m. But as all of them are infinite-dimensional and separable, one can identify them if one conjugates the respective G-action. So, fix an infinite-dimensional separable Hilbert space $\mathcal{H}_{\mathcal{D}}$. The G-action is not needed for the determination of $C^*(G)$.

   (b) $\mathcal{D}_m^-$ for odd $m \in \mathbb{N}^*$:

      (i) $\mathcal{D}_1^-$:
      The Hilbert space $\mathcal{H}_{\mathcal{D}_1^-}$ is given by

      $$\mathcal{H}_{\mathcal{D}_1^-} := \mathcal{H}_{[1]}$$

      defined in Section 4.2.1 above and the action is given by

      $$\mathcal{D}_1^- := \mathcal{P}^{+,1}.$$

      (ii) $\mathcal{D}_m^-$ for odd $m \in \mathbb{N}_{\geq 3}$:
      Similarly as for $\mathcal{D}_m^+$, as a Hilbert space $\mathcal{H}_{\mathcal{D}_m^-}$ for $\mathcal{D}_m^-$ for odd $m \in \mathbb{N}_{\geq 3}$, one can take the completion of the space

      $$\left\{ f \in C^\infty(K)_+ \middle| \; p_n(f) = 0 \; \forall n \geq -m+1 \right\}$$

      with respect to an appropriate scalar product, and as the action, one can take

      $$\mathcal{D}_m^- := \mathcal{P}^{+,m}.$$

      Again, the Hilbert spaces depend on m. One identifies them and takes the common infinite-dimensional separable Hilbert space $\mathcal{H}_{\mathcal{D}}$ fixed in (a)(ii). Again, the G-action is not needed for the determination of $C^*(G)$.

(c) $\mathcal{D}_m^+$ for even $m \in \mathbb{N}^*$:
As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^+}$ for $\mathcal{D}_m^+$ for even $m \in \mathbb{N}^*$, one can take the completion of the space

$$\{f \in C^\infty(K)_- | \, p_n(f) = 0 \; \forall n \leq m-1\}$$

with respect to an appropriate scalar product, and as the action, one can take

$$\mathcal{D}_m^+ := \mathcal{P}^{-,m}.$$

Again, the Hilbert spaces are identified and one takes the common infinite-dimensional separable Hilbert space $\mathcal{H}_\mathcal{D}$, as in (a)(ii).

(d) $\mathcal{D}_m^-$ for even $m \in \mathbb{N}^*$:
As a Hilbert space $\mathcal{H}_{\mathcal{D}_m^-}$ for $\mathcal{D}_m^-$ for even $m \in \mathbb{N}^*$, one can take the completion of the space

$$\{f \in C^\infty(K)_- | \, p_n(f) = 0 \; \forall n \geq -m+1\}$$

with respect to an appropriate scalar product, and as the action, one can take

$$\mathcal{D}_m^- := \mathcal{P}^{-,m}.$$

Here again, the Hilbert spaces are identified and one takes the common Hilbert space $\mathcal{H}_\mathcal{D}$, as in (a)(ii).

4. The **limits of the discrete series**:

(a) $\mathcal{D}_+$:
The Hilbert space $\mathcal{H}_{\mathcal{D}_+}$ is defined by

$$\mathcal{H}_{\mathcal{D}_+} := \{f \in L^2(K)_- | \, p_n(f) = 0 \; \forall n \leq 0\}$$

and the action is given by

$$\mathcal{D}_+ := \mathcal{P}^{-,0}.$$

(b) $\mathcal{D}_-$:
The Hilbert space $\mathcal{H}_{\mathcal{D}_-}$ is defined by

$$\mathcal{H}_{\mathcal{D}_-} := \{f \in L^2(K)_- | \, p_n(f) = 0 \; \forall n \geq 0\}$$

and the action is given by

$$\mathcal{D}_- := \mathcal{P}^{-,0}.$$

5. The **trivial** representation $\mathcal{F}_1$:
Its Hilbert space $\mathcal{H}_{\mathcal{F}_1} = \mathbb{C}$ will be identified with the space of constant functions

$$\{f \in L^2(K)_+ | \, p_n(f) = 0 \; \forall n \neq 0\}.$$

Here, the action is given by

$$\mathcal{F}_1(g) := id$$

for all $g \in G$. One also has

$$\mathcal{F}_1 = \mathcal{P}^{+,-1},$$

as $\nu_{-1} + \rho = 0$ and every $f \in \mathcal{H}_{\mathcal{F}_1}$ is a constant function.

A discussion of all irreducible representations of $SL(2,\mathbb{R})$ without scalar products can be found in [32], Chapter 5.6. But as the scalar products in this thesis are shown to be unitary and invariant and as such scalar products are uniquely determined, they are the correct ones. In [20], Chapter II.5, a different realization of the discrete series representations that is not used in this work can be found. An alternative description of all irreducible unitary representations can be found in [22], Chapter 6.6.

Remark 4.14.

$$\mathcal{H}_{\mathcal{P}^{-,0}} \cong \mathcal{H}_{\mathcal{D}_+} \oplus \mathcal{H}_{\mathcal{D}_-}.$$

Remark 4.15.
By the definition of the operator K_u by means of its value on the space $\mathcal{H}_{\mathcal{P}^{+,u}}$, one can easily verify that p_n also projects $\mathcal{H}_{\mathcal{C}^u}$ to its n-th isotypic component $\mathcal{H}_{\mathcal{C}^u}(n)$ for every $n \in \mathbb{Z}$.
Furthermore, for every irreducible unitary representation π of G, the operator p_n leaves the above defined Hilbert space $\mathcal{H}_\pi$ invariant, since all of the Hilbert spaces are completions of the space $C^\infty(K)$ fulfilling p_n-cancellation properties for certain $n \in \mathbb{Z}$.
Hence, for every $n \in \mathbb{Z}$ and for every irreducible unitary representation π of G, the operator p_n is the projection of $\mathcal{H}_\pi$ to the n-th isotypic component $\mathcal{H}_\pi(n)$.

4.2.3 The $SL(2,\mathbb{R})$-representations applied to the Casimir operator

In order to be able to describe the topology on $\widehat{G}$, the above listed representations applied to the Casimir operator will now be computed.

Lemma 4.16.
Applied to the Casimir operator normalized as in Section 4.1, the representations $\mathcal{P}^{\pm,u}$ for $u \in \mathbb{C}$ give the following:

$$\mathcal{P}^{\pm,u}(\mathscr{C}) = \frac{1}{4}\big(u^2-1\big)\cdot id.$$

Proof:
Following the discussion about the Casimir operator in Section 4.1, one can write $\mathcal{P}^{\pm,u}(\mathscr{C})$ as

$$\mathcal{P}^{\pm,u}(\mathscr{C}) = \mathcal{P}^{\pm,u}(X_3)^2 - \mathcal{P}^{\pm,u}(X_3) + \mathcal{P}^{\pm,u}(X_1)\mathcal{P}^{\pm,u}(X_2).$$

First regard $\mathcal{P}^{\pm,u}(X_3)$. For this, let $f \in C^\infty(K)_\pm$ and let $k = id$ for the moment. Then,

$$\mathcal{P}^{\pm,u}(X_3)(f)(id) = \frac{d}{dt}\Big|_{t=0}\mathcal{P}^{\pm,u}\big(\exp(tX_3)\big)(f)(id),$$

and, as $tX_3 \in \mathfrak{a}$, this gives

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}e^{-(\nu_u+\rho)(-tX_3)}f(id) &= \Big((\nu_u+\rho)(X_3)e^{(\nu_u+\rho)(tX_3)}f(id)\Big)\Big|_{t=0}\\ &= (\nu_u+\rho)(X_3)f(id)\\ &= \frac{1}{2}(u+1)f(id).\end{aligned}$$

Thus, one can deduce that $\mathcal{P}^{\pm,u}(X_3)^2(f)(id) = \frac{1}{4}(u+1)^2 f(id)$.
Next, regard $\mathcal{P}^{\pm,u}(X_1)$. Again, let $f \in C^\infty(K)_\pm$ and $k = id$. Then, since $\exp(tX_1) \in N$ for every $t \in \mathbb{R}$, one gets

$$\begin{aligned}\mathcal{P}^{\pm,u}(X_1)(f)(id) &= \frac{d}{dt}\Big|_{t=0}\mathcal{P}^{\pm,u}\big(\exp(tX_1)\big)(f)(id)\\ &= \frac{d}{dt}\Big|_{t=0} f(id) = 0.\end{aligned}$$

Hence, $\mathcal{P}^{\pm,u}(X_1)\mathcal{P}^{\pm,u}(X_2)(f)(id) = 0$ as well.
Therefore,

$$\mathcal{P}^{\pm,u}(\mathscr{C})(f)(id) \;=\; \frac{1}{4}(u+1)^2 f(id) - \frac{1}{2}(u+1)f(id) \;=\; \frac{1}{4}\big(u^2-1\big)f(id).$$

Now, to show this equality for general $k \in K$, one uses the above stated fact that

$$\mathcal{P}^{\pm,u}(g)\circ\mathcal{P}^{\pm,u}(\mathscr{C}) = \mathcal{P}^{\pm,u}(\mathscr{C})\circ\mathcal{P}^{\pm,u}(g) \quad \forall g \in G.$$

So, let $k \in K$. Then, for every $f \in C^\infty(K)_\pm$, on the one hand,

$$\begin{aligned}\mathcal{P}^{\pm,u}(\mathscr{C})\circ\mathcal{P}^{\pm,u}(k^{-1})(f)(id) &= \mathcal{P}^{\pm,u}\big(k^{-1}\big)\circ\mathcal{P}^{\pm,u}(\mathscr{C})(f)(id)\\ &= \mathcal{P}^{\pm,u}(\mathscr{C})(f)(k).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{P}^{\pm,u}(\mathscr{C})\circ\mathcal{P}^{\pm,u}\big(k^{-1}\big)(f)(id) &= \frac{1}{4}\big(u^2-1\big)\mathcal{P}^{\pm,u}\big(k^{-1}\big)(f)(id)\\ &= \frac{1}{4}\big(u^2-1\big)f(k).\end{aligned}$$

Thus,

$$\mathcal{P}^{\pm,u}(\mathscr{C})(f)(k) = \frac{1}{4}\big(u^2-1\big)f(k),$$

as desired.

□

From Lemma 4.16, one can deduce for the irreducible unitary representations of $SL(2,\mathbb{R})$ listed above:

1. $\mathcal{P}^{\pm,iv}(\mathscr{C}) = \frac{1}{4}\big(-v^2-1\big)\cdot id \quad \forall v \in [0,\infty)$,

2. $\mathcal{C}^u(\mathscr{C}) = K_u \circ \frac{1}{4}\big(u^2-1\big)\cdot id \circ K_u^{-1} = \frac{1}{4}\big(u^2-1\big)\cdot id \quad \forall u \in (0,1)$,

3. $\mathcal{D}^\pm_m(\mathscr{C}) = \frac{1}{4}\big(m^2-1\big)\cdot id \quad \forall m \in \mathbb{N}^*$,

4. $\mathcal{D}_\pm(\mathscr{C}) = \frac{1}{4}(0-1)\cdot id = -\frac{1}{4}\cdot id$ and

5. $\mathcal{F}_1(\mathscr{C}) = \frac{1}{4}\big((-1)^2-1\big)\cdot id = 0.$

4.2.4 The topology on $\widehat{SL(2,\mathbb{R})}$

With the help of the computations above, it is now possible to describe the topology on $\widehat{G}$.

Proposition 4.17.
The topology on $\widehat{G}$ can be characterized in the following way:

1. *For all sequences $(v_j)_{j\in\mathbb{N}}$ and all v in $[0,\infty)$,*

$$\mathcal{P}^{+,iv_j} \stackrel{j\to\infty}{\longrightarrow} \mathcal{P}^{+,iv} \iff v_j \stackrel{j\to\infty}{\longrightarrow} v.$$

2. *For all sequences $(v_j)_{j\in\mathbb{N}}$ and all v in $(0,\infty)$,*

$$\mathcal{P}^{-,iv_j} \stackrel{j\to\infty}{\longrightarrow} \mathcal{P}^{-,iv} \iff v_j \stackrel{j\to\infty}{\longrightarrow} v.$$

   For all sequences $(v_j)_{j\in\mathbb{N}}$ in $[0,\infty)$,

$$\mathcal{P}^{-,iv_j} \stackrel{j\to\infty}{\longrightarrow} \{\mathcal{D}_+,\mathcal{D}_-\} \iff v_j \stackrel{j\to\infty}{\longrightarrow} 0.$$

3. *For all sequences $(u_j)_{j\in\mathbb{N}}$ and all u in $(0,1)$,*

$$\mathcal{C}^{u_j} \stackrel{j\to\infty}{\longrightarrow} \mathcal{C}^{u} \iff u_j \stackrel{j\to\infty}{\longrightarrow} u.$$

   For all sequences $(u_j)_{j\in\mathbb{N}}$ in $(0,1)$,

$$\mathcal{C}^{u_j} \stackrel{j\to\infty}{\longrightarrow} \mathcal{P}^{+,0} \iff u_j \stackrel{j\to\infty}{\longrightarrow} 0.$$

   For all sequences $(u_j)_{j\in\mathbb{N}}$ in $(0,1)$,

$$\mathcal{C}^{u_j} \stackrel{j\to\infty}{\longrightarrow} \{\mathcal{D}_1^+,\mathcal{D}_1^-,\mathcal{F}_1\} \iff u_j \stackrel{j\to\infty}{\longrightarrow} 1.$$

All other sequences $(\pi_j)_{j\in\mathbb{N}}$ can only converge if they fulfill one of the following conditions:

(a) They are constant for large $j\in\mathbb{N}$. In that case, they have one single limit point, namely the value they take for large j.

(b) For large $j\in\mathbb{N}$, they are one of the sequences listed above. Then, they converge in the way described above.

(c) For large $j\in\mathbb{N}$, the sequence consists only of members $\mathcal{P}^{+,iv_j}$ and $\mathcal{C}^{u_j}$. Then, it converges to $\mathcal{P}^{+,0}$ if and only if $v_j \stackrel{j\to\infty}{\longrightarrow} 0$ as well as $u_j \stackrel{j\to\infty}{\longrightarrow} 0$ (compare 1. for $v=0$ and the second part of 3. above).

(d) For large $j\in\mathbb{N}$, the sequence consists only of members $\mathcal{C}^{u_j}$ and $\mathcal{D}_1^+$. Then, it converges to $\mathcal{D}_1^+$ if and only if $u_j \stackrel{j\to\infty}{\longrightarrow} 1$ (compare the third part of 3. above).

(e) For large $j\in\mathbb{N}$, the sequence consists only of members $\mathcal{C}^{u_j}$ and $\mathcal{D}_1^-$. Then, it converges to $\mathcal{D}_1^-$ if and only if $u_j \stackrel{j\to\infty}{\longrightarrow} 1$ (compare the third part of 3. above).

(f) *For large* $j \in \mathbb{N}$, *the sequence consists only of members* $\mathcal{C}^{u_j}$ *and* $\mathcal{F}_1$. *Then, it converges to* $\mathcal{F}_1$ *if and only if* $u_j \overset{j\to\infty}{\longrightarrow} 1$ *(compare the third part of 3. above).*

(g) *For large* $j \in \mathbb{N}$, *the sequence consists only of members* $\mathcal{P}^{-,iv_j}$ *and* $\mathcal{D}_+$. *Then, it converges to* $\mathcal{D}_+$ *if and only if* $v_j \overset{j\to\infty}{\longrightarrow} 0$ *(compare the second part of 2. above).*

(h) *For large* $j \in \mathbb{N}$, *the sequence consists only of members* $\mathcal{P}^{-,iv_j}$ *and* $\mathcal{D}_-$. *Then, it converges to* $\mathcal{D}_-$ *if and only if* $v_j \overset{j\to\infty}{\longrightarrow} 0$ *(compare the second part of 2. above).*

Proof:
If a sequence of representations $(\pi_j)_{j\in\mathbb{N}}$ converges to a representation π, the sequence $\big(\pi_j(\mathscr{C})\big)_{j\in\mathbb{N}}$ has to converge to $\pi(\mathscr{C})$ as well. By the observations of Section 4.2.3, the left hand side thus implies the right hand side in all cases.
Now, for the other implication, it will first be shown that for a sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathbb{C}$, $u \in \mathbb{C}$ and for every compact set $\tilde{K} \subset G$,

$$u_j \overset{j\to\infty}{\longrightarrow} u \implies \sup_{g\in\tilde{K}} \left\|\mathcal{P}^{\pm,u_j}(g) - \mathcal{P}^{\pm,u}(g)\right\|_{op} \overset{j\to\infty}{\longrightarrow} 0. \tag{25}$$

One has

$$\begin{aligned}
&\sup_{g\in\tilde{K}} \left\|\mathcal{P}^{\pm,u_j}(g) - \mathcal{P}^{\pm,u}(g)\right\|_{op}^2 \\
= &\sup_{g\in\tilde{K}} \sup_{\substack{f\in L^2(K)_\pm \\ \|f\|_2=1}} \int_K \left|\mathcal{P}^{\pm,u_j}(g)(f)(k) - \mathcal{P}^{\pm,u}(g)(f)(k)\right|^2 dk \\
= &\sup_{g\in\tilde{K}} \sup_{\substack{f\in L^2(K)_\pm \\ \|f\|_2=1}} \int_K \left|e^{-\nu_{u_j}H(g^{-1}k)} - e^{-\nu_u H(g^{-1}k)}\right|^2 e^{-2\rho H(g^{-1}k)} \left|f\Big(\kappa\big(g^{-1}k\big)\Big)\right|^2 dk.
\end{aligned}$$

For $g \in \tilde{K}$ and $k \in K$, $g^{-1}k$ is also contained in a compact set and therefore, $H\big(g^{-1}k\big)$ is contained in a compact set as well. As

$$H\big(g^{-1}k\big) = \begin{pmatrix} h(g^{-1}k) & 0 \\ 0 & -h(g^{-1}k) \end{pmatrix} \quad \text{for } h(g^{-1}k) \in \mathbb{R},$$

there is a compact set $I \subset \mathbb{R}$ such that $h(g^{-1}k) \in I$ for all $g \in \tilde{K}$ and all $k \in K$. Hence,

$$\begin{aligned}
&\sup_{g\in\tilde{K}} \sup_{\substack{f\in L^2(K)_\pm \\ \|f\|_2=1}} \int_K \left|e^{-\nu_{u_j}H(g^{-1}k)} - e^{-\nu_u H(g^{-1}k)}\right|^2 e^{-2\rho H(g^{-1}k)} \left|f\Big(\kappa\big(g^{-1}k\big)\Big)\right|^2 dk \\
\leq &\sup_{x\in I} \left|e^{-u_j x} - e^{-ux}\right|^2 \sup_{g\in\tilde{K}} \sup_{\substack{f\in L^2(K)_\pm \\ \|f\|_2=1}} \int_K e^{-2\rho H(g^{-1}k)} \left|f\Big(\kappa\big(g^{-1}k\big)\Big)\right|^2 dk \\
\overset{Lemma\ 4.2}{=} &\sup_{x\in I} \left|e^{-u_j x} - e^{-ux}\right|^2 \sup_{\substack{f\in L^2(K)_\pm \\ \|f\|_2=1}} \|f\|_{L^2(K)}^2 \\
= &\sup_{x\in I} \left|e^{-u_j x} - e^{-ux}\right|^2 \overset{j\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, (25) follows.

From this, one can easily deduce that for a sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathbb{C}$ and $u \in \mathbb{C}$,

$$u_j \overset{j\to\infty}{\longrightarrow} u \implies \mathcal{P}^{\pm,u_j} \overset{j\to\infty}{\longrightarrow} \mathcal{P}^{\pm,u} \tag{26}$$

in the sense of convergence of matrix coefficients described in Theorem 2.8. Hence, one has to show that for some $f \in \mathcal{H}_{\mathcal{P}^{\pm,u}}$, there exists for every $j \in \mathbb{N}$ a function $f_j \in \mathcal{H}_{\mathcal{P}^{\pm,u_j}}$ such that

$$\left\langle \mathcal{P}^{\pm,u_j}(\cdot)f_j, f_j \right\rangle_{\mathcal{H}_{\mathcal{P}^{\pm,u_j}}} \overset{j\to\infty}{\longrightarrow} \left\langle \mathcal{P}^{\pm,u}(\cdot)f, f \right\rangle_{\mathcal{H}_{\mathcal{P}^{\pm,u}}}$$

uniformly on compacta.
Let $f \in \mathcal{H}_{\mathcal{P}^{\pm,u}} = L^2(K)_\pm$ and choose $f_j := f \in L^2(K)_\pm = \mathcal{H}_{\mathcal{P}^{\pm,u_j}}$ for all $j \in \mathbb{N}$. Moreover, let $\tilde{K} \subset G$ be compact. Then,

$$\begin{aligned} &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{P}^{\pm,u_j}(g)f_j, f_j \right\rangle_{L^2(K)} - \left\langle \mathcal{P}^{\pm,u}(g)f, f \right\rangle_{L^2(K)} \right| \\ = &\sup_{g\in\tilde{K}} \left| \left\langle \left(\mathcal{P}^{\pm,u_j}(g) - \mathcal{P}^{\pm,u}(g)\right)f, f \right\rangle_{L^2(K)} \right| \\ \leq &\sup_{g\in\tilde{K}} \left\| \mathcal{P}^{\pm,u_j}(g) - \mathcal{P}^{\pm,u}(g) \right\|_{op} \|f\|^2_{L^2(K)} \overset{j\to\infty}{\longrightarrow} 0, \end{aligned}$$

by (25). Therefore, Claim (26) is shown, which directly yields the inverse implications of 1. and 2. (using that $\mathcal{H}_{\mathcal{D}_+}, \mathcal{H}_{\mathcal{D}_-} \subset \mathcal{H}_{\mathcal{P}^{-,iv}}$ for all $v \in (0,\infty)$).

Next, it will be shown that

$$\left\langle K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1} f, f \right\rangle_{L^2(K)} = \left\langle \mathcal{P}^{+,u}(g)f, f \right\rangle_{L^2(K)} \tag{27}$$

for $\tilde{n} \in \mathbb{Z}$, $u \in (0,1)$, $g \in G$ and $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(\tilde{n})$.
So, let $\tilde{n} \in \mathbb{Z}$, $u \in (0,1)$, $g \in G$ and $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(\tilde{n})$. Then,

$$K_u f = d_{\tilde{n}}(u) f \quad \text{and} \quad K_u^{-1} f = \frac{1}{d_{\tilde{n}}(u)} f.$$

Since K_u is self-adjoint with respect to the usual $L^2(K)$-scalar product, one gets

$$\begin{aligned} \left\langle K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1} f, f \right\rangle_{L^2(K)} &= \left\langle \mathcal{P}^{+,u}(g) \circ K_u^{-1} f, K_u \circ f \right\rangle_{L^2(K)} \\ &= \frac{d_{\tilde{n}}(u)}{d_{\tilde{n}}(u)} \left\langle \mathcal{P}^{+,u}(g)f, f \right\rangle_{L^2(K)} = \left\langle \mathcal{P}^{+,u}(g)f, f \right\rangle_{L^2(K)} \end{aligned}$$

and (27) follows.

For the inverse implications of 3., one uses Equality (27) to be able to express the matrix coefficients $\left\langle \pi(g)f, f \right\rangle_{\mathcal{H}_\pi}$ for a representation $\pi \in \widehat{G}$ as $\left\langle \mathcal{P}^{+,u}(g)f, f \right\rangle_{L^2(K)}$. Then, by (26) for $f_j = f$, one gets the convergences needed:

So first, it has to be shown that for a sequence $(u_j)_{j\in\mathbb{N}}$ in $(0,1)$ and $u \in (0,1)$,

$$u_j \overset{j\to\infty}{\longrightarrow} u \implies \mathcal{C}^{u_j} \overset{j\to\infty}{\longrightarrow} \mathcal{C}^u \text{ in } \widehat{G}. \tag{28}$$

For this, let $f \in \mathcal{H}_{\mathcal{P}^{+,u}}(0) = \mathcal{H}_{\mathcal{P}^{+,u_j}}(0)$. Furthermore, choose $f_j := f$ for all $j \in \mathbb{N}$. Then, letting $\tilde{K} \subset G$ compact, by (27) one gets

$$\begin{aligned} &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{C}^{u_j}(g) f_j, f_j \right\rangle_{L^2(K)} - \left\langle \mathcal{C}^u(g) f, f \right\rangle_{L^2(K)} \right| \\ = &\sup_{g\in\tilde{K}} \left| \left\langle K_{u_j} \circ \mathcal{P}^{+,u_j}(g) \circ K_{u_j}^{-1} f, f \right\rangle_{L^2(K)} - \left\langle K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1} f, f \right\rangle_{L^2(K)} \right| \\ = &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{P}^{+,u_j}(g) f, f \right\rangle_{L^2(K)} - \left\langle \mathcal{P}^{+,u}(g) f, f \right\rangle_{L^2(K)} \right| \stackrel{j\to\infty}{\longrightarrow} 0 \end{aligned}$$

by (26). Therefore, (28) follows.

The same reasoning is valid if one regards $u = 0$ and $\mathcal{P}^{+,0}$ instead of $\mathcal{C}^u$. Thus,

$$u_j \stackrel{j\to\infty}{\longrightarrow} 0 \implies \mathcal{C}^{u_j} \stackrel{j\to\infty}{\longrightarrow} \mathcal{P}^{+,0} \text{ in } \widehat{G}.$$

Now, it still needs to be proved that for a sequence $(u_j)_{j\in\mathbb{N}}$ in $(0,1)$,

$$u_j \stackrel{j\to\infty}{\longrightarrow} 1 \implies \mathcal{C}^{u_j} \stackrel{j\to\infty}{\longrightarrow} \left\{ \mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{F}_1 \right\} \text{ in } \widehat{G}. \tag{29}$$

First, the convergence to $\mathcal{D}_1^+$ will be regarded:
For this, let $f \in \mathcal{H}_{\mathcal{P}^{+,1}}(2)$. Then, $f \in \mathcal{H}_{\mathcal{D}_1^+}$ and $\tilde{J}_{(1)} f = f$. By Lemma 4.12(a) and with the notation used in (24), $\tilde{f} = \tilde{J}_{(1)} \circ \tilde{J}_{(-1)} \tilde{f} = \tilde{J}_{(1)} f = f$. Thus, by (24),

$$\begin{aligned} \left\langle \mathcal{D}_1^+(g) f, f \right\rangle_{(1)} &= \left\langle \mathcal{P}^{+,1}(g) f, f \right\rangle_{(1)} = \overline{\left\langle f, \mathcal{P}^{+,1}(g) f \right\rangle_{(1)}} = \overline{\left\langle \tilde{f}, \mathcal{P}^{+,1}(g) f \right\rangle_{L^2(K)}} \\ &= \overline{\left\langle f, \mathcal{P}^{+,1}(g) f \right\rangle_{L^2(K)}} = \left\langle \mathcal{P}^{+,1}(g) f, f \right\rangle_{L^2(K)}. \end{aligned}$$

Now, choose $f_j := f$ for all $j \in \mathbb{N}$. Hence, $f_j = f \in \mathcal{H}_{\mathcal{C}^{u_j}}(2)$.
Using (27) and (26) again, one gets for compact $\tilde{K} \subset G$,

$$\begin{aligned} &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{C}^{u_j}(g) f_j, f_j \right\rangle_{L^2(K)} - \left\langle \mathcal{D}_1^+(g) f, f \right\rangle_{(1)} \right| \\ = &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{P}^{+,u_j}(g) f, f \right\rangle_{L^2(K)} - \left\langle \mathcal{P}^{+,1}(g) f, f \right\rangle_{L^2(K)} \right| \stackrel{j\to\infty}{\longrightarrow} 0. \end{aligned}$$

The proof of the convergence of $\left(\mathcal{C}^{u_j}\right)_{j\in\mathbb{N}}$ to $\mathcal{D}_1^-$ for $u_j \stackrel{j\to\infty}{\longrightarrow} 1$ is similar. One only has to choose $f \in \mathcal{H}_{\mathcal{P}^{+,1}}(-2)$.
For $\mathcal{F}_1$, let $f \in \mathcal{H}_{\mathcal{P}^{+,1}}(0)$. Then, f is a constant function and therefore, with Lemma 4.2,

$$\begin{aligned} \left\langle \mathcal{P}^{+,1}(g) f, f \right\rangle_{L^2(K)} &= \int_K e^{-2\rho H(g^{-1}k)} \left| f\Big(\kappa\big(g^{-1}k\big)\Big) \right|^2 dk = \|f\|_{L^2(K)}^2 \\ &= \int_K |f(k)|^2 dk = \left\langle \mathcal{P}^{+,-1}(g) f, f \right\rangle_{L^2(K)} = \left\langle \mathcal{F}_1(g) f, f \right\rangle_{L^2(K)}. \end{aligned}$$

Again, choose $f_j := f$ for all $j \in \mathbb{N}$. As above, letting $\tilde{K} \subset G$ compact, with (27) and (26) one gets

$$\begin{aligned} &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{C}^{u_j}(g) f_j, f_j \right\rangle_{L^2(K)} - \left\langle \mathcal{F}_1(g) f, f \right\rangle_{L^2(K)} \right| \\ = \; &\sup_{g\in\tilde{K}} \left| \left\langle \mathcal{P}^{+,u_j}(g) f, f \right\rangle_{L^2(K)} - \left\langle \mathcal{P}^{+,1}(g) f, f \right\rangle_{L^2(K)} \right| \overset{j\to\infty}{\longrightarrow} 0, \end{aligned}$$

and (29) is shown.
This yields the inverse implications of 3. and thus, the inverse implication is shown for every case.

At the end, it still has to be shown that there are no other possibilities of convergence than the ones listed in Proposition 4.17. For this, one needs some preliminaries.
First, one can observe that if a sequence of representations $(\pi_j)_{j\in\mathbb{N}}$ converges to a representation π, then for all n with $\mathcal{H}_\pi(n) \neq \{0\}$, there exists an integer $J \in \mathbb{N}$ such that $\mathcal{H}_{\pi_j}(n) \neq \{0\}$ for all $j \geq J$.
To show this, choose $f \in \mathcal{H}_\pi$ in such a way that its projection to the n-th isotypic component $p_n(f) =: f^n$ is non-zero. Furthermore, let $(f_j)_{j\in\mathbb{N}}$ be a sequence with $f_j \in \mathcal{H}_{\pi_j}$ for every $j \in \mathbb{N}$ such that the sequence of matrix coefficients $\Big(\left\langle \pi_j(\cdot) f_j, f_j \right\rangle_{\mathcal{H}_{\pi_j}}\Big)_{j\in\mathbb{N}}$ converges uniformly on compacta to $\left\langle \pi(\cdot) f, f \right\rangle_{\mathcal{H}_\pi}$. Then, there is a $g \in G$ such that $\left\langle \pi(g) f^n, f^n \right\rangle_{\mathcal{H}_\pi}$ is non-zero and, defining f_j^n the projection to the n-th isotypic component of f_j, it can be shown that the sequence $\Big(\left\langle \pi_j(g) f_j^n, f_j^n \right\rangle_{\mathcal{H}_{\pi_j}}\Big)_{j\in\mathbb{N}}$ converges to $\left\langle \pi(g) f^n, f^n \right\rangle_{\mathcal{H}_\pi}$. Therefore, $f_j^n \neq 0$ for large $j \in \mathbb{N}$ and the claim is shown.
So, in particular, if $(\pi_j)_{j\in\mathbb{N}}$ converges to $\pi \neq 0$, then for large $j \in \mathbb{N}$, there has to be a common K-type for $(\pi_j)_{j \text{ large}}$ and π, i.e. an $n \in \mathbb{Z}$ such that $\mathcal{H}_\pi(n) \neq \{0\}$ and $\mathcal{H}_{\pi_j}(n) \neq \{0\}$ for large j. $(\star)$

Now, by Remark 4.5, the representations $\mathcal{F}_1$, $\mathcal{D}_m^\pm$ for odd $m \in \mathbb{N}^*$, $\mathcal{C}^u$ for $u \in (0,1)$ and $\mathcal{P}^{+,iv}$ for $v \in [0,\infty)$ are even, while the representations $\mathcal{D}_+$, $\mathcal{D}_-$, $\mathcal{D}_m^\pm$ for even $m \in \mathbb{N}^*$ and $\mathcal{P}^{-,iv}$ for $v \in (0,\infty)$ are odd. With the result shown above, this leads to the fact that the set of representations $\big\{\mathcal{F}_1,\ \mathcal{D}_m^\pm,\ \mathcal{C}^u,\ \mathcal{P}^{+,iv} |\ \text{odd } m \in \mathbb{N}^*,\ u \in (0,1),\ v \in [0,\infty)\big\}$ is separated from the set $\big\{\mathcal{D}_+,\ \mathcal{D}_-,\ \mathcal{D}_m^\pm,\ \mathcal{P}^{-,iv} |\ \text{even } m \in \mathbb{N}^*,\ v \in (0,\infty)\big\}$.
First, regard the first set of representations.
If a sequence $\big(\mathcal{P}^{+,iv_j}\big)_{j\in\mathbb{N}}$ for $v_j \in [0,\infty)$ converges, $\big(\mathcal{P}^{+,iv_j}(\mathscr{C})\big)_{j\in\mathbb{N}} = \Big(\frac{1}{4}\big(-v_j^2 - 1\big) \cdot id\Big)_{j\in\mathbb{N}}$ converges as well and thus, the sequence $(v_j)_{j\in\mathbb{N}}$ has to converge. This sequence can only converge to $v \in [0,\infty)$, which implies, as seen above, that $\lim\limits_{j\to\infty} \mathcal{P}^{+,iv_j} = \mathcal{P}^{+,iv}$. So, the convergence indicated in 1. is the only possible one for $\big(\mathcal{P}^{+,iv_j}\big)_{j\in\mathbb{N}}$.
With the same reasoning, if a sequence $\big(\mathcal{C}^{u_j}\big)_{j\in\mathbb{N}}$ converges, there are three possibilities for the sequence $(u_j)_{j\in\mathbb{N}}$: It can converge to $u \in (0,1)$, to 0 or to 1, which are the three possibilities listed in 3.
Moreover, the three sets of representations $\{\mathcal{F}_1\}$, $\big\{\mathcal{D}_m^+ |\ m \in \mathbb{N}^*\big\}$ and $\big\{\mathcal{D}_m^- |\ m \in \mathbb{N}^*\big\}$ can be separated from each other, since they do not have any common non-zero K-type. Furthermore, as the values, that the representations $\mathcal{D}_m^+$ and $\mathcal{D}_m^-$ for $m \in \mathbb{N}^*$ take on the Casimir operator, form discrete sets, the representations $\mathcal{D}_{m_1}^+$ and $\mathcal{D}_{m_2}^+$ and the representations $\mathcal{D}_{m_1}^-$

and $\mathcal{D}_{m_2}^-$, respectively, can be separated from each other for every $m_1 \neq m_2 \in \mathbb{N}^*$. Again, due to the values on the Casimir operator, $\{\mathcal{D}_m^+ |\ m \in \mathbb{N}_{>1}\}$ and $\{\mathcal{D}_m^- |\ m \in \mathbb{N}_{>1}\}$ can be isolated from all the other representations.
For these reasons, sequences $(\pi_j)_{j\in\mathbb{N}}$ with elements in $\{\mathcal{D}_m^+ |\ m \in \mathbb{N}^*\}$ or $\{\mathcal{D}_m^- |\ m \in \mathbb{N}^*\}$ can only converge if they are constant for large k and in this case, they have one single limit.
Now, considering other sequences $(\pi_j)_{j\in\mathbb{N}}$ of representations in the set of even representations given above, one has to regard again the value they take on the Casimir operator. Furthermore, taking into account $(\star)$ and the fact that $\mathcal{H}_{\mathcal{D}_1^+} \cap \mathcal{H}_{\mathcal{D}_1^-} = \mathcal{H}_{\mathcal{D}_1^+} \cap \mathcal{H}_{\mathcal{F}_1} = \mathcal{H}_{\mathcal{D}_1^-} \cap \mathcal{H}_{\mathcal{F}_1} = \{0\}$, it is obvious that a sequence which is neither constant for large j nor one of the sequences listed in 1. to 3. for large j, has to fulfill Property (c), (d), (e) or (f) in order to converge. That the convergence is achieved in these cases, can be deduced from the proof given above for 1. and 3.
Cases (a) and (b) are clear.
Moreover, regarding the second set of representations, with the same argumentation as above for $\mathcal{P}^{+,iv}$, if a sequence $\left(\mathcal{P}^{-,iv_j}\right)_{j\in\mathbb{N}}$ converges, then $(v_j)_{j\in\mathbb{N}}$ converges as well, either to $v \in (0,\infty)$ or to 0. This means that $\left(\mathcal{P}^{-,iv_j}\right)_{j\in\mathbb{N}}$ can only converge to a representation $\mathcal{P}^{-,iv}$ or to $\{\mathcal{D}_+, \mathcal{D}_-\}$.
Again, a converging sequence in $\{\mathcal{D}_m^+ |\ m \in \mathbb{N}^* \text{ even}\}$ or $\{\mathcal{D}_m^- |\ m \in \mathbb{N}^* \text{ even}\}$ has to be constant and it has one single limit point.
As for the set of even representations, since $\mathcal{H}_{\mathcal{D}_+} \cap \mathcal{H}_{\mathcal{D}_-} = \{0\}$ and by $(\star)$, the only possibility of convergence for a sequence of representations that is neither constant for large j nor one of the sequences listed in 1. to 3. for large j, is to fulfill Property (g) or (h). Also, it can be deduced from the above proof for 2. that in this case, convergence is indeed achieved.
Again, Cases (a) and (b) are clear.
Hence, the given ones are all possibilities for sequences of irreducible unitary representations of G to converge.

□

4.2.5 Definition of subsets Γ_i of the spectrum

Now, the spectrum will be divided into different subsets which are thereafter proved to meet the requirements of the norm controlled dual limit conditions of Definition 1.1.

Define

$$
\begin{aligned}
\Gamma_0 &:= \{\mathcal{F}_1\},\\
\Gamma_1 &:= \{\mathcal{D}_1^+\},\\
\Gamma_2 &:= \{\mathcal{D}_1^-\},\\
\Gamma_3 &:= \{\mathcal{D}_+\},\\
\Gamma_4 &:= \{\mathcal{D}_-\},\\
\Gamma_5 &:= \{\mathcal{D}_m^\pm |\ m \in \mathbb{N}_{>1}\},\\
\Gamma_6 &:= \{\mathcal{P}^{+,iv} |\ v \in [0,\infty)\},\\
\Gamma_7 &:= \{\mathcal{P}^{-,iv} |\ v \in (0,\infty)\} \quad \text{and}\\
\Gamma_8 &:= \{\mathcal{C}^u |\ u \in (0,1)\}.
\end{aligned}
$$

Obviously, all the sets Γ_i for $i \in \{0, ..., 8\}$ are Hausdorff. The sets

$$S_i := \bigcup_{j \in \{0,...,i\}} \Gamma_j$$

are closed and the set $S_0 = \Gamma_0$ consists of all the characters of $G = SL(2, \mathbb{R})$. In addition, as defined in Section 4.2.2, for every $i \in \{0, ..., 8\}$, there exists one common Hilbert space $\mathcal{H}_i$ that all the representations in Γ_i act on.
Therefore, Condition 1 of Definition 1.1 is fulfilled.

As every connected real linear semisimple Lie group meets the CCR-condition (see [9], Theorem 15.5.6), Condition 2 of Definition 1.1 is fulfilled as well. Thus, Condition 3 remains to be shown.

4.3 Condition 3(a)

For the proof of Condition 3(a) of Definition 1.1, as well as for the proof of Condition 3(b), some preliminaries are needed.

Define the $K \times K$-representation $\pi_{K\times K}$ on the space $V_{\pi_{K\times K}} := C_0^\infty(G)$ of compactly supported $C^\infty(G)$-functions as

$$\begin{aligned} \pi_{K\times K} \quad : \quad & K \times K \to \mathcal{B}\big(C_0^\infty(G)\big), \\ & \pi_{K\times K}(k_1, k_2)h(g) := h(k_1^{-1} g k_2) \quad \forall (k_1, k_2) \in K \times K \ \forall h \in C_0^\infty(G) \ \forall g \in G. \end{aligned}$$

For all $l, n \in \mathbb{Z}$,

$$\begin{aligned} V_{\pi_{K\times K}}(l, n) &= \Big\{ h \in C_0^\infty(G) \Big| \ \pi_{K\times K}\big(k_{\varphi_1}, k_{\varphi_2}\big) h = e^{il\varphi_1 + in\varphi_2} h \ \forall \varphi_1, \varphi_2 \in [0, 2\pi) \Big\} \\ &= \Big\{ h \in C_0^\infty(G) \Big| \ h\big(k_{\varphi_1}^{-1} g k_{\varphi_2}\big) = e^{il\varphi_1 + in\varphi_2} h(g) \ \forall \varphi_1, \varphi_2 \in [0, 2\pi) \Big\}. \end{aligned}$$

Then, the algebraic direct sum $\bigoplus_{l,n\in\mathbb{Z}} V_{\pi_{K\times K}}(l, n)$ is dense in $V_{\pi_{K\times K}} = C_0^\infty(G)$ with respect to the $L^1(G)$-norm and as $\| \cdot \|_{C^*(G)} \leq \| \cdot \|_{L^1(G)}$ on $L^1(G)$, it is dense with respect to the $C^*(G)$-norm as well. $C_0^\infty(G)$ in turn is dense in $C^*(G)$. Hence, the algebraic direct sum $\bigoplus_{l,n\in\mathbb{Z}} V_{\pi_{K\times K}}(l, n)$ is also dense in $C^*(G)$.
Let $p_{l,n}$ be the projection going from $V_{\pi_{K\times K}}$ to $V_{\pi_{K\times K}}(l, n)$ defined in the following way:
For $h \in V_{\pi_{K\times K}} = C_0^\infty(G)$ and $g \in G$,

$$p_{l,n}(h)(g) := \frac{1}{|K|^2} \int_{K\times K} h\big(k_{\varphi_1} g k_{\varphi_2}^{-1}\big) e^{il\varphi_1} e^{in\varphi_2} d\big(k_{\varphi_1}, k_{\varphi_2}\big).$$

Due to the density discussed above and to the Corollaries 7.4 and 7.5 in the appendix, instead of dealing with a general element $a \in C^*(G)$, the calculations in Sections 4.3 and 4.4 can be accomplished with a function $h \in C_0^\infty(G)$ fulfilling $h = p_{l,-n}(h)$ for some integers $l, n \in \mathbb{Z}$.

Lemma 4.18.
For $f \in L^2(K)_+$ and $h \in C_0^\infty(G)$ with $h = p_{l,-n}(h)$ for $l, n \in \mathbb{Z}$, one gets

$$\mathcal{P}^{+,u}(h)(f) = \mathcal{P}^{+,u}(h)\big(p_n(f)\big) = p_l\Big(\mathcal{P}^{+,u}(h)\big(p_n(f)\big)\Big) \quad \forall u \in \mathbb{C}.$$

Furthermore, if $h \neq 0$, the integers l and n must be even.

Proof:
First, one gets for $k_\psi \in K$,

$$\begin{aligned}
\int\limits_K e^{-in\varphi}\mathcal{P}^{+,u}\big(k_\varphi\big)f\big(k_\psi\big)dk_\varphi &= \int\limits_K e^{-in\varphi}f\big(k_\varphi^{-1}k_\psi\big)dk_\varphi \\
&= \int\limits_K e^{in(\varphi-\psi)}f\big(k_\varphi\big)dk_\varphi \\
&= e^{-in\psi}\int\limits_K e^{in\varphi}f\big(k_\varphi\big)dk_\varphi \;=\; |K|\; p_n(f)\big(k_\psi\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{P}^{+,u}(h)(f) &= \int\limits_G h(g)\mathcal{P}^{+,u}(g)fdg \\
&= \int\limits_G p_{l,-n}(h)(g)\mathcal{P}^{+,u}(g)fdg \\
&= \frac{1}{|K|^2}\int\limits_G\int\limits_{K\times K} h\big(k_{\varphi_1}gk_{\varphi_2}^{-1}\big)e^{il\varphi_1}e^{-in\varphi_2}d\big(k_{\varphi_1},k_{\varphi_2}\big)\mathcal{P}^{+,u}(g)fdg \\
&= \frac{1}{|K|^2}\int\limits_{K\times K} e^{il\varphi_1}e^{-in\varphi_2}\int\limits_G h\big(k_{\varphi_1}gk_{\varphi_2}^{-1}\big)\mathcal{P}^{+,u}(g)fdgd\big(k_{\varphi_1},k_{\varphi_2}\big) \\
&= \frac{1}{|K|^2}\int\limits_{K\times K} e^{il\varphi_1}e^{-in\varphi_2}\int\limits_G h(g)\mathcal{P}^{+,u}\big(k_{\varphi_1}^{-1}gk_{\varphi_2}\big)fdgd\big(k_{\varphi_1},k_{\varphi_2}\big) \\
&= \frac{1}{|K|^2}\int\limits_K e^{il\varphi_1}\mathcal{P}^{+,u}\big(k_{\varphi_1}^{-1}\big)\int\limits_G h(g)\mathcal{P}^{+,u}(g)\int\limits_K e^{-in\varphi_2}\mathcal{P}^{+,u}\big(k_{\varphi_2}\big)fdk_{\varphi_2}dgdk_{\varphi_1} \\
&= \frac{1}{|K|}\int\limits_K e^{il\varphi_1}\mathcal{P}^{+,u}\big(k_{\varphi_1}^{-1}\big)\int\limits_G h(g)\mathcal{P}^{+,u}(g)\big(p_n(f)\big)dgdk_{\varphi_1} \\
&= \frac{1}{|K|}\int\limits_K e^{il\varphi_1}\mathcal{P}^{+,u}\big(k_{\varphi_1}^{-1}\big)\mathcal{P}^{+,u}(h)\big(p_n(f)\big)dk_{\varphi_1} \\
&= p_l\Big(\mathcal{P}^{+,u}(h)\big(p_n(f)\big)\Big).
\end{aligned}$$

From this equality, one can also immediately deduce that $\mathcal{P}^{+,u}(h)(f) = \mathcal{P}^{+,u}(h)\big(p_n(f)\big)$ by replacing f by $p_n(f)$. Hence, the equalities of Lemma 4.18 are shown.
By these equalities and as f and $\mathcal{P}^{+,u}(h)\big(p_n(f)\big)$ are in $L^2(K)_+$, for $h \neq 0$, the integers l and n must be even.

$\square$

Now, the proof of Condition 3(a) of Definition 1.1 can be executed.
Condition 3(a) is obvious for the sets Γ_i for $i \in \{0, ..., 5\}$, as these are discrete sets.

So, let $v_j, v \in [0,\infty)$ and $\left(\mathcal{P}^{+,iv_j}\right)_{j\in\mathbb{N}}$ a sequence in Γ_6 converging to $\mathcal{P}^{+,iv}$. Then, $v_j \overset{j\to\infty}{\longrightarrow} v$. Let $h \in C_0^\infty(G)$ be supported in the compact set $\tilde{K} \in G$. Hence, similarly as in the proof of (25), there is a compact set $I \subset \mathbb{R}$ such that

$$\begin{aligned}
&\left\|\mathcal{P}^{+,iv_j}(h) - \mathcal{P}^{+,iv}(h)\right\|_{op}^2 \\
= \quad & \sup_{\substack{f\in L^2(K)_+ \\ \|f\|_2=1}} \int\limits_K \left| \int\limits_{\tilde{K}} h(g)\Big(\mathcal{P}^{+,iv_j}(g)(f)(k) - \mathcal{P}^{+,iv}(g)(f)(k)\Big)dg\right|^2 dk \\
\leq \quad & \sup_{x\in I}\left|e^{-iv_jx} - e^{-ivx}\right|^2 \sup_{\substack{f\in L^2(K)_+ \\ \|f\|_2=1}} \int\limits_K \left(\int\limits_{\tilde{K}} |h(g)|\; e^{-\rho H(g^{-1}k)}\left|f\Big(\kappa\big(g^{-1}k\big)\Big)\right|dg\right)^2 dk \\
\overset{\text{Hölder}}{\leq} \quad & \sup_{x\in I}\left|e^{-iv_jx} - e^{-ivx}\right|^2 \; \|h\|_{L^2(G)}^2 \sup_{\substack{f\in L^2(K)_+ \\ \|f\|_2=1}} \int\limits_K \int\limits_{\tilde{K}} e^{-2\rho H(g^{-1}k)}\left|f\Big(\kappa\big(g^{-1}k\big)\Big)\right|^2 dgdk \\
\overset{Lemma\ 4.2}{=} \quad & \left|\tilde{K}\right| \sup_{x\in I}\left|e^{-iv_jx} - e^{-ivx}\right|^2 \; \|h\|_{L^2(G)}^2 \overset{j\to\infty}{\longrightarrow} 0, \qquad (30)
\end{aligned}$$

as $v_j \overset{j\to\infty}{\longrightarrow} v$.
Because of the density of $C_0^\infty(G)$ in $C^*(G)$ and by Corollary 7.4 which can be found in the appendix, one gets the desired convergence for $a \in C^*(G)$.
The reasoning is the same for Γ_7.

For Γ_8, let $u_j, u \in (0,1)$ and $\left(\mathcal{C}^{u_j}\right)_{j\in\mathbb{N}}$ a sequence in Γ_8 converging to $\mathcal{C}^u \in \Gamma_8$. Then, $u_j \overset{j\to\infty}{\longrightarrow} u$. Moreover, let $h \in C^*(G)$ and let $l, n \in \mathbb{Z}$ such that $h = p_{l,-n}(h)$, as discussed at the beginning of this subsection.
Let $f \in L^2(K)_+$ with $\|f\|_{L^2(K)} = 1$.
Since $K_{\tilde{u}}^{-1}$ commutes with p_n, by Lemma 4.18, one has for every $\tilde{u} \in (0,1)$,

$$\begin{aligned}
\mathcal{C}^{\tilde{u}}(h)(f) \quad &= \quad K_{\tilde{u}} \circ \mathcal{P}^{+,\tilde{u}}(h) \circ K_{\tilde{u}}^{-1}(f) \;=\; K_{\tilde{u}} \circ p_l\Big(\mathcal{P}^{+,\tilde{u}}(h)\Big(p_n\big(K_{\tilde{u}}^{-1}(f)\big)\Big)\Big) \\
&= \quad d_l\big(\tilde{u}\big)\; p_l\Big(\mathcal{P}^{+,\tilde{u}}(h)\Big(K_{\tilde{u}}^{-1}\big(p_n(f)\big)\Big)\Big) \;=\; \frac{d_l\big(\tilde{u}\big)}{d_n\big(\tilde{u}\big)}\; p_l\Big(\mathcal{P}^{+,\tilde{u}}(h)\big(p_n(f)\big)\Big) \\
&= \quad \sqrt{\frac{c_l\big(\tilde{u}\big)}{c_n\big(\tilde{u}\big)}}\; \mathcal{P}^{+,\tilde{u}}(h)(f). \qquad (31)
\end{aligned}$$

Hence,

$$\left\|\mathcal{C}^{u_j}(h) - \mathcal{C}^u(h)\right\|_{op}^2 \quad = \quad \left\|\sqrt{\frac{c_l(u_j)}{c_n(u_j)}}\; \mathcal{P}^{+,u_j}(h) - \sqrt{\frac{c_l(u)}{c_n(u)}}\; \mathcal{P}^{+,u}(h)\right\|_{op}^2 \overset{j\to\infty}{\longrightarrow} 0, \qquad (32)$$

with the same reasoning as in (30) and since $u_j \overset{j\to\infty}{\longrightarrow} u$.
Because of the density of $C_0^\infty(G)$ in $C^*(G)$ and with Corollary 7.4, one gets the desired convergence for $a \in C^*(G)$ and thus, the claim is also shown for Γ_8.

4.4 Condition 3(b)

Now, only Condition 3(b) remains to be shown. This is the most complicated part of the proof of the conditions listed in Definition 1.1.

Remark 4.19.
The setting of a sequence $(\gamma_j)_{j\in\mathbb{N}}$ *in* Γ_i *converging to a limit set contained in* $S_{i-1} = \bigcup\limits_{l<i}\Gamma_l$ *regarded in Condition 3(b) can only occur in the following cases:*

(i) $(\gamma_j)_{j\in\mathbb{N}} = \big(\mathcal{P}^{-,iv_j}\big)_{j\in\mathbb{N}}$ *is a sequence in* Γ_7 *whose limit set is* $\Gamma_3 \cup \Gamma_4 = \big\{\mathcal{D}_+, \mathcal{D}_-\big\}$.

(ii) $(\gamma_j)_{j\in\mathbb{N}} = \big(\mathcal{C}^{u_j}\big)_{j\in\mathbb{N}}$ *is a sequence in* Γ_8 *whose limit set is* $\big\{\mathcal{P}^{+,0}\big\} \subset \Gamma_7$.

(iii) $(\gamma_j)_{j\in\mathbb{N}} = \big(\mathcal{C}^{u_j}\big)_{j\in\mathbb{N}}$ *is a sequence in* Γ_8 *whose limit set is* $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 = \big\{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\big\}$.

For $i \in \{0, ..., 6\}$, the sets Γ_i are closed and thus, the regarded situation cannot appear for sequences $(\gamma_j)_{j\in\mathbb{N}}$ in Γ_i for $i \in \{0, ..., 6\}$.

Since all the sequences regarded in the Cases (i), (ii) and (iii) of Remark 4.19 are properly converging, the transition to subsequences will be omitted.

First, regard Case (i) mentioned in Remark 4.19.
Let $\big(\mathcal{P}^{-,iv_j}\big)_{j\in\mathbb{N}}$ be a sequence in Γ_7 whose limit set is $\big\{\mathcal{D}_+, \mathcal{D}_-\big\}$. As $\mathcal{P}^{-,iv_j} \stackrel{j\to\infty}{\longrightarrow} \big\{\mathcal{D}_+, \mathcal{D}_-\big\}$, it follows that $v_j \stackrel{j\to\infty}{\longrightarrow} 0$.
Now, a bounded, linear and involutive mapping

$$\tilde{\nu}_j : CB(S_6) \to \mathcal{B}\big(L^2(K)_-\big)$$

fulfilling

$$\lim_{j\to\infty} \Big\|\mathcal{P}^{-,iv_j}(a) - \tilde{\nu}_j\big(\mathcal{F}(a)_{|S_6}\big)\Big\|_{op} = 0 \quad \forall a \in C^*(G)$$

has to be defined.
Since in this and in the following cases, this mapping will not depend on j, it will from now on be denoted by $\tilde{\nu}$ instead of $\tilde{\nu}_j$.
Let p_+ be the projection from $L^2(K)_-$ to the space $\mathcal{H}_{\mathcal{D}_+}$ and p_- the projection from $L^2(K)_-$ to $\mathcal{H}_{\mathcal{D}_-}$. Then, by Remark 4.14, one has $L^2(K)_- = \mathcal{H}_{\mathcal{D}_+} \oplus \mathcal{H}_{\mathcal{D}_-}$, i.e. $id_{|L^2(K)_-} = p_+ + p_-$.
Now, let

$$\tilde{\nu}(\psi) := \tilde{\nu}_{\{\mathcal{D}_+,\mathcal{D}_-\}}(\psi) := \psi(\mathcal{D}_+) \circ p_+ + \psi(\mathcal{D}_-) \circ p_- \quad \forall \psi \in CB(S_6).$$

This is well-defined, as $\mathcal{D}_+, \mathcal{D}_- \in S_6$ and furthermore, $\tilde{\nu}(\psi) \in \mathcal{B}\big(L^2(K)_-\big)$.
The linearity of the mapping $\tilde{\nu}$ is clear. For the involutivity, let $\psi \in CB(S_6)$. Then, since p_+ and p_- equal the identity on the image of $\psi(\mathcal{D}_+)$ and $\psi(\mathcal{D}_-)$, respectively,

$$\begin{aligned}
\big(\tilde{\nu}(\psi)\big)^* &= \Big(\psi(\mathcal{D}_+) \circ p_+\Big)^* + \Big(\psi(\mathcal{D}_-) \circ p_-\Big)^* \\
&= \Big(p_+ \circ \psi(\mathcal{D}_+) \circ p_+\Big)^* + \Big(p_- \circ \psi(\mathcal{D}_-) \circ p_-\Big)^* \\
&= p_+^* \circ \psi^*(\mathcal{D}_+) \circ p_+^* + p_-^* \circ \psi^*(\mathcal{D}_-) \circ p_-^* \\
&= p_+ \circ \psi^*(\mathcal{D}_+) \circ p_+ + p_- \circ \psi^*(\mathcal{D}_-) \circ p_- \\
&= \psi^*(\mathcal{D}_+) \circ p_+ + \psi^*(\mathcal{D}_-) \circ p_- \;=\; \tilde{\nu}(\psi^*).
\end{aligned}$$

To show that $\tilde{\nu}$ is bounded, again let $\psi \in CB(S_6)$:

$$\begin{aligned}\big\|\tilde{\nu}(\psi)\big\|_{op} &= \big\|\psi(\mathcal{D}_+)\circ p_+ + \psi(\mathcal{D}_-)\circ p_-\big\|_{op} = \max\Big\{\big\|\psi(\mathcal{D}_+)\big\|_{op}, \big\|\psi(\mathcal{D}_-)\big\|_{op}\Big\} \\ &\leq \sup_{\gamma\in S_6} \|\psi(\gamma)\|_{op} = \|\psi\|_{S_6}.\end{aligned}$$

Now, only the demanded convergence remains to be shown:
For $h \in C_0^\infty(G)$, one has

$$\begin{aligned}\Big\|\mathcal{P}^{-,iv_j}(h) - \tilde{\nu}\big(\mathcal{F}(h)_{|S_6}\big)\Big\|_{op}^2 &= \Big\|\mathcal{P}^{-,iv_j}(h) - \Big(\mathcal{F}(h)(\mathcal{D}_+)\circ p_+ + \mathcal{F}(h)(\mathcal{D}_-)\circ p_-\Big)\Big\|_{op}^2 \\ &= \Big\|\mathcal{P}^{-,iv_j}(h) - \Big(\mathcal{D}_+(h)\circ p_+ + \mathcal{D}_-(h)\circ p_-\Big)\Big\|_{op}^2 \\ &= \Big\|\mathcal{P}^{-,iv_j}(h) - \Big(\mathcal{P}^{-,0}(h)\circ p_+ + \mathcal{P}^{-,0}(h)\circ p_-\Big)\Big\|_{op}^2 \\ &= \big\|\mathcal{P}^{-,iv_j}(h) - \mathcal{P}^{-,0}(h)\big\|_{op}^2 \overset{j\to\infty}{\longrightarrow} 0\end{aligned}$$

as in (30) and since $v_j \overset{j\to\infty}{\longrightarrow} 0$.
Again, because of the density of $C_0^\infty(G)$ in $C^*(G)$ and with Corollary 7.5, one gets the desired convergence for $a \in C^*(G)$.

Now, regard Case (ii) mentioned in Remark 4.19.
Let $\big(\mathcal{C}^{u_j}\big)_{j\in\mathbb{N}}$ be a sequence in Γ_8 whose limit set is $\big\{\mathcal{P}^{+,0}\big\}$. Thus, $u_j \overset{j\to\infty}{\longrightarrow} 0$.
Here, a bounded, linear and involutive mapping

$$\tilde{\nu} : CB(S_7) \to \mathcal{B}\big(L^2(K)_+\big)$$

fulfilling

$$\lim_{j\to\infty} \Big\|\mathcal{C}^{u_j}(a) - \tilde{\nu}\big(\mathcal{F}(a)_{|S_7}\big)\Big\|_{op} = 0 \quad \forall a \in C^*(G)$$

is needed.
Define

$$\tilde{\nu}(\psi) := \tilde{\nu}_{\mathcal{P}^{+,0}}(\psi) := \psi\big(\mathcal{P}^{+,0}\big) \quad \forall \psi \in CB(S_7).$$

$\tilde{\nu}(\psi) \in \mathcal{B}\big(L^2(K)_+\big)$ for every $\psi \in CB(S_7)$ and $\tilde{\nu}$ is well-defined, as $\mathcal{P}^{+,0} \in S_7$.
The linearity and the involutivity of $\tilde{\nu}$ are clear.
For the boundedness of $\tilde{\nu}$, let $\psi \in CB(S_7)$. Then,

$$\big\|\tilde{\nu}(\psi)\big\|_{op} = \big\|\psi\big(\mathcal{P}^{+,0}\big)\big\|_{op} \leq \sup_{\gamma\in S_7} \|\psi(\gamma)\|_{op} = \|\psi\|_{S_7}.$$

Again, it remains to show the demanded convergence:
Let $h \in C_0^\infty(G)$. Then, one can assume again that there exist $l, n \in \mathbb{Z}$ such that $h = p_{l,-n}(h)$.
Since $\lim\limits_{u\to 0} \frac{c_{\tilde{n}}(u)}{c_{n'}(u)} = 1$ for all $\tilde{n}, n' \in \mathbb{Z}$ by (15) and Lemma 4.10, one gets with (31),

$$\begin{aligned}\Big\|\mathcal{C}^{u_j}(h) - \tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big)\Big\|_{op}^2 &= \Bigg\|\sqrt{\frac{c_l(u_j)}{c_n(u_j)}}\,\mathcal{P}^{+,u_j}(h) - \mathcal{F}(h)\big(\mathcal{P}^{+,0}\big)\Bigg\|_{op}^2 \\ &= \Bigg\|\sqrt{\frac{c_l(u_j)}{c_n(u_j)}}\,\mathcal{P}^{+,u_j}(h) - \mathcal{P}^{+,0}(h)\Bigg\|_{op}^2 \overset{j\to\infty}{\longrightarrow} 0,\end{aligned}$$

since $u_j \overset{j\to\infty}{\longrightarrow} 0$ and with the same arguments as in the subsection above.
The desired convergence for $a \in C^*(G)$ follows.

Last, regard Case (iii) of Remark 4.19.
Let $\left(\mathcal{C}^{u_j}\right)_{j\in\mathbb{N}}$ be a sequence in Γ_8 whose limit set is $\{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\}$. This means that $u_j \overset{j\to\infty}{\longrightarrow} 1$.
Again, a bounded, linear and involutive mapping

$$\tilde{\nu} : CB(S_7) \to \mathcal{B}\big(L^2(K)_+\big)$$

fulfilling

$$\lim_{j\to\infty} \left\| \mathcal{C}^{u_j}(a) - \tilde{\nu}\big(\mathcal{F}(a)_{|S_7}\big) \right\|_{op} = 0 \quad \forall a \in C^*(G)$$

is needed.
For this, let p_+ be the projection from $L^2(K)_+$ to the space $\{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \leq 0\}$ and p_- the projection from $L^2(K)_+$ to $\{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \geq 0\}$. Then, since

$$\begin{aligned} L^2(K)_+ &= \{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \leq 0\} + \{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \geq 0\} \\ &\quad + \{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \neq 0\}, \end{aligned}$$

every $f \in L^2(K)_+$ can be written as $f = p_+(f) + p_-(f) + p_0(f)$.
Furthermore, let

$$\begin{aligned} d_{n,2}(1) &:= \lim_{u\to 1} \sqrt{\frac{c_n(u)}{c_2(u)}} \quad \text{for all even } n > 0 \quad \text{and} \\ d_{n,-2}(1) &:= \lim_{u\to 1} \sqrt{\frac{c_n(u)}{c_{-2}(u)}} \quad \text{for all even } n < 0. \end{aligned}$$

The existence of these limits follows with Lemma 4.12(c) in Section 4.2.1 and the analogous statement for $\tilde{J}_{[1]}$.
Now, define the operators

$$K_{(1)} : \mathcal{H}_{(1)} \to L^2(K)_+ \quad \text{by} \quad K_{(1)|\mathcal{H}_{\mathcal{P}+,1}(n)} := d_{n,2}(1) \cdot id_{|\mathcal{H}_{\mathcal{P}+,1}(n)} \quad \text{for all even } n > 0$$

and

$$K_{[1]} : \mathcal{H}_{[1]} \to L^2(K)_+ \quad \text{by} \quad K_{[1]|\mathcal{H}_{\mathcal{P}+,1}(n)} := d_{n,-2}(1) \cdot id_{|\mathcal{H}_{\mathcal{P}+,1}(n)} \quad \text{for all even } n < 0.$$

By definition, these operators are linear and they are unitary by the construction of the scalar products defined on $\mathcal{H}_{(1)}$ and $\mathcal{H}_{[1]}$. Moreover, like $\tilde{J}_{(1)}$ and $\tilde{J}_{(-1)}$, they are injective and, as proved in Remark 4.8 for the operator J_u, $K_{(1)}$ and $K_{[1]}$ commute with the projections p_n for all $n \in \mathbb{Z}$.
One can easily see that

$$\begin{aligned} K_{(1)}\big(\mathcal{H}_{(1)}\big) &= \{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \leq 0\} = p_+\big(L^2(K)_+\big) \quad \text{and} \\ K_{[1]}\big(\mathcal{H}_{[1]}\big) &= \{f \in L^2(K)_+ |\ p_n(f) = 0\ \forall n \geq 0\} = p_-\big(L^2(K)_+\big). \end{aligned}$$

Hence, one can build the inverse of the operators $K_{(1)}$ and $K_{[1]}$ on the image of p_+ and p_-, respectively. Therefore, one can define

$$\begin{aligned}\tilde{\nu}(\psi) := \tilde{\nu}_{\{\mathcal{F}_1,\mathcal{D}_1^+,\mathcal{D}_1^-\}}(\psi) \quad &:= \quad K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ \; + \; K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \\ &\quad + \psi(\mathcal{F}_1) \circ p_0 \quad \forall \psi \in CB(S_7).\end{aligned}$$

The mapping $\tilde{\nu}$ is well-defined, since $\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{F}_1$ are contained in S_7, and $\tilde{\nu}(\psi) \in \mathcal{B}\big(L^2(K)_+\big)$ for every $\psi \in CB(S_7)$. In addition, its linearity is clear again.
For the involutivity and the boundedness, let $\psi \in CB(S_7)$. Then, as $K_{(1)}$ and $K_{[1]}$ are unitary and as p_+, p_- and p_0 equal the identity on the image of $K_{(1)}$, $K_{[1]}$ and $\psi(\mathcal{F}_1)$, respectively,

$$\begin{aligned} & \big(\tilde{\nu}(\psi)\big)^* \\ = \; & \Big(K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ \; + \; K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \; + \; \psi(\mathcal{F}_1) \circ p_0\Big)^* \\ = \; & \Big(p_+ \circ K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ \; + \; p_- \circ K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \; + \; p_0 \circ \psi(\mathcal{F}_1) \circ p_0\Big)^* \\ = \; & p_+ \circ K_{(1)} \circ \psi^*(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ \; + \; p_- \circ K_{[1]} \circ \psi^*(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \; + \; p_0 \circ \psi^*(\mathcal{F}_1) \circ p_0 \\ = \; & \tilde{\nu}\big(\psi^*\big). \end{aligned}$$

Furthermore, since $\big\|K_{(1)}\big\|_{op}\big\|K_{(1)}^{-1}\big\|_{op} = \big\|K_{[1]}\big\|_{op}\big\|K_{[1]}^{-1}\big\|_{op} = 1$, one gets

$$\begin{aligned} \big\|\tilde{\nu}(\psi)\big\|_{op} \; &= \; \Big\|K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+ \; + \; K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_- \; + \; \psi(\mathcal{F}_1) \circ p_0\Big\|_{op} \\ &= \; \max\Big\{\Big\|K_{(1)} \circ \psi(\mathcal{D}_1^+) \circ K_{(1)}^{-1}\Big\|_{op}, \; \Big\|K_{[1]} \circ \psi(\mathcal{D}_1^-) \circ K_{[1]}^{-1}\Big\|_{op}, \; \big\|\psi(\mathcal{F}_1)\big\|_{op}\Big\} \\ &= \; \max\Big\{\big\|\psi(\mathcal{D}_1^+)\big\|_{op}, \; \big\|\psi(\mathcal{D}_1^-)\big\|_{op}, \; \big\|\psi(\mathcal{F}_1)\big\|_{op}\Big\} \\ &\leq \; \sup_{\gamma \in S_7} \|\psi(\gamma)\|_{op} \; = \; \|\psi\|_{S_7}. \end{aligned}$$

For the demanded convergence, let $h \in C_0^\infty(G)$. Like in the proof of (ii) and Condition 3(a), one can assume that there exist $l, n \in \mathbb{Z}$ such that $h = p_{l,-n}(h)$.
Let $f \in L^2(K)_+$ with $\|f\|_{L^2(K)} = 1$.
Since $K_{(1)}^{-1}$ and $K_{[1]}^{-1}$ commute with p_n, similarly as in the proof of (31), by Lemma 4.18, one gets

$$\begin{aligned} & \tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big)(f) \\ = \; & K_{(1)} \circ \mathcal{F}(h)(\mathcal{D}_1^+) \circ K_{(1)}^{-1} \circ p_+(f) \; + \; K_{[1]} \circ \mathcal{F}(h)(\mathcal{D}_1^-) \circ K_{[1]}^{-1} \circ p_-(f) \\ & + \mathcal{F}(h)(\mathcal{F}_1) \circ p_0(f) \\ = \; & K_{(1)} \circ \mathcal{D}_1^+(h) \circ K_{(1)}^{-1} \circ p_+(f) \; + \; K_{[1]} \circ \mathcal{D}_1^-(h) \circ K_{[1]}^{-1} \circ p_-(f) \; + \; \mathcal{F}_1(h) \circ p_0(f) \\ = \; & K_{(1)} \circ \mathcal{P}^{+,1}(h) \circ K_{(1)}^{-1} \circ p_+(f) \; + \; K_{[1]} \circ \mathcal{P}^{+,1}(h) \circ K_{[1]}^{-1} \circ p_-(f) \; + \; \mathcal{P}^{+,-1}(h) \circ p_0(f) \\ = \; & K_{(1)} \circ p_l\Bigg(\mathcal{P}^{+,1}(h)\Big(p_n\Big(K_{(1)}^{-1} \circ p_+(f)\Big)\Big)\Bigg) \\ & + K_{[1]} \circ p_l\Bigg(\mathcal{P}^{+,1}(h)\Big(p_n\Big(K_{[1]}^{-1} \circ p_-(f)\Big)\Big)\Bigg) + \; p_l\Big(\mathcal{P}^{+,-1}(h)\Big(p_n\big(p_0(f)\big)\Big)\Big) \end{aligned}$$

$$= d_{l,2}(1)\; p_l\Bigg(\mathcal{P}^{+,1}(h)\Big(K^{-1}_{(1)}\big(p_n\circ p_+(f)\big)\Big)\Bigg)$$

$$+\, d_{l,-2}(1)\; p_l\Bigg(\mathcal{P}^{+,1}(h)\Big(K^{-1}_{[1]}\big(p_n\circ p_-(f)\big)\Big)\Bigg) +\; p_l\Big(\mathcal{P}^{+,-1}(h)\big(p_n\circ p_0(f)\big)\Big).$$

There are three cases to consider: In the first case, $n>0$, in the second case, $n<0$ and in the third case, $n=0$.

So, first let $n>0$. Then,

$$\begin{aligned}\tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big)(f) &= d_{l,2}(1)\; p_l\Bigg(\mathcal{P}^{+,1}(h)\Big(K^{-1}_{(1)}\big(p_n\circ p_+(f)\big)\Big)\Bigg)\\ &= \frac{d_{l,2}(1)}{d_{n,2}(1)}\; p_l\Big(\mathcal{P}^{+,1}(h)\big(p_n(f)\big)\Big) \;=\; \lim_{u\to1}\sqrt{\frac{c_l(u)}{c_n(u)}}\;\mathcal{P}^{+,1}(h)(f).\end{aligned}$$

Therefore, joining this result with (31), one gets

$$\Big\|\mathcal{C}^{u_j}(h)\;-\;\tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big)\Big\|^2_{op} \;=\; \left\|\sqrt{\frac{c_l(u_j)}{c_n(u_j)}}\;\mathcal{P}^{+,u_j}(h)\;-\;\lim_{u\to1}\sqrt{\frac{c_l(u)}{c_n(u)}}\;\mathcal{P}^{+,1}(h)\right\|^2_{op} \overset{j\to\infty}{\longrightarrow} 0,$$

with the same reasoning as in (30) and since $u_j \overset{j\to\infty}{\longrightarrow} 1$.

The proof for the case $n<0$ is the same as the one for the first case. Hence, only the case $n=0$ remains:
Since by (16)

$$\frac{c_l(1)}{c_0(1)} \;=\; \begin{cases}1 & \text{if } l=0\\ 0 & \text{if } l\neq 0,\end{cases}$$

one gets for $l\neq 0$ with (31),

$$\Big\|\mathcal{C}^{u_j}(h)\;-\;\tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big)\Big\|^2_{op} \;=\; \left\|\sqrt{\frac{c_l(u_j)}{c_0(u_j)}}\;\mathcal{P}^{+,u_j}(h)\;-\;0\right\|^2_{op} \overset{j\to\infty}{\longrightarrow} 0,$$

since $u_j \overset{j\to\infty}{\longrightarrow} 1$.
So, let $l=0$ and define $C_h := \int_G h(g)dg$.
Then,

$$\mathcal{F}_1(h)\circ p_0(f) = \Bigg(\int_G h(g)\mathcal{F}_1(g)dg\Bigg)\big(p_0(f)\big) = id\big(p_0(f)\big)\cdot C_h = C_h p_0(f).$$

Furthermore, with Lemma 4.2, one gets for $g\in G$,

$$\begin{aligned}p_0\circ\mathcal{P}^{+,1}(g)\circ p_0(f) &= \frac{1}{|K|}\int_K \mathcal{P}^{+,1}(g)p_0(f)(k)dk \;=\; \frac{p_0(f)}{|K|}\int_K e^{-2\rho H(g^{-1}k)}dk\\ &= \frac{p_0(f)}{|K|}\int_K e^{-2\rho H(g^{-1}k)}\Big|1\Big(\kappa\big(g^{-1}k\big)\Big)\Big|^2 dk \;=\; \frac{p_0(f)}{|K|}\;\|1\|_{L^2(K)} \;=\; p_0(f).\end{aligned}$$

Thus, for h one has

$$p_0 \circ \mathcal{P}^{+,1}(h) \circ p_0(f) = \int_G h(g) p_0(f) dg = C_h p_0(f) = \mathcal{F}_1(h) \circ p_0(f).$$

Therefore, for $l = n = 0$, by (31),

$$\left\| \mathcal{C}^{u_j}(h) \; - \; \tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big) \right\|_{op}^2 \; = \; \left\| p_0 \circ \mathcal{P}^{+,u_j}(h) \circ p_0 \; - \; p_0 \circ \mathcal{P}^{+,1}(h) \circ p_0 \right\|_{op}^2 \overset{j \to \infty}{\longrightarrow} 0,$$

since $u_j \overset{j \to \infty}{\longrightarrow} 1$.
Hence, the claim is also shown in the case $n = 0$ and thus,

$$\left\| \mathcal{C}^{u_j}(h) \; - \; \tilde{\nu}\big(\mathcal{F}(h)_{|S_7}\big) \right\|_{op}^2 \overset{j \to \infty}{\longrightarrow} 0,$$

as demanded.
Again, the desired convergence for $a \in C^*(G)$ follows.

□

4.5 Result for the Lie group $SL(2, \mathbb{R})$

Now, after having verified all the conditions listed in Definition 1.1, Theorem 1.2 is proved for the Lie group $SL(2, \mathbb{R})$. Therefore, with the sets Γ_i and S_i and the Hilbert spaces $\mathcal{H}_i$ for $i \in \{0, ..., 8\}$ defined in Section 4.2.5 and Section 4.2.2 and the mappings $\tilde{\nu}$ constructed in Section 4.4, the C^*-algebra of $G = SL(2, \mathbb{R})$ can be characterized by the following theorem which was already stated in Section 1:

Theorem 4.20.
The C^*-algebra $C^*(G)$ of $G = SL(2, \mathbb{R})$ is isomorphic (under the Fourier transform) to the set of all operator fields φ defined over $\widehat{G}$ such that:

1. *$\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ for every $i \in \{1, ..., 8\}$ and every $\gamma \in \Gamma_i$.*
2. *$\varphi \in l^\infty(\widehat{G})$.*
3. *The mappings $\gamma \mapsto \varphi(\gamma)$ are norm continuous on the different sets Γ_i.*
4. *For any sequence $(\gamma_j)_{j \in \mathbb{N}} \subset \widehat{G}$ going to infinity, $\lim\limits_{j \to \infty} \|\varphi(\gamma_j)\|_{\text{op}} = 0$.*
5. *For every $i \in \{1, ..., 8\}$ and any properly converging sequence $\overline{\gamma} = (\gamma_j)_{j \in \mathbb{N}} \subset \Gamma_i$ whose limit set is contained in S_{i-1} (taking a subsequence if necessary) and for the mappings $\tilde{\nu} = \tilde{\nu}_{\overline{\gamma}, j} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ constructed in the preceding subsections, one has*

$$\lim_{j \to \infty} \left\| \varphi(\gamma_j) - \tilde{\nu}\big(\varphi_{|S_{i-1}}\big) \right\|_{op} = 0.$$

Let for a topological Hausdorff space V and a C^*-algebra B, $C_\infty(V, B)$ be the C^*-algebra of all continuous functions defined on V with values in B that are vanishing at infinity. Then, from this theorem one can deduce more concretely the following result for $G = SL(2, \mathbb{R})$:

Theorem 4.21.
Let the operator p_+ be the projection from $L^2(K)_\pm$ to $\big\{f \in L^2(K)_\pm|\ p_n(f) = 0\ \forall n \leq 0\big\}$, p_- the projection from $L^2(K)_\pm$ to the space $\big\{f \in L^2(K)_\pm|\ p_n(f) = 0\ \forall n \geq 0\big\}$ and p_0 the projection from $L^2(K)_+$ to $\big\{f \in L^2(K)_+|\ p_n(f) = 0\ \forall n \neq 0\big\} = \mathbb{C}$.
Then, the C^*-algebra $C^*(G)$ of $G = SL(2,\mathbb{R})$ is isomorphic to the direct sum of C^*-algebras

$$\Big\{F \in C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big)\Big|\ F(1) \text{ commutes with } p_+,\ p_- \text{ and } p_0\Big\}$$
$$\oplus\quad \Big\{F \in C_\infty\Big(i[0,\infty), \mathcal{K}\big(L^2(K)_-\big)\Big)\Big|\ F(0) \text{ commutes with } p_+ \text{ and } p_-\Big\}$$
$$\oplus\quad C_\infty\Big(\mathbb{Z} \setminus \{-1,0,1\}, \mathcal{K}\big(\mathcal{H}_\mathcal{D}\big)\Big)$$

for the infinite-dimensional and separable Hilbert space $\mathcal{H}_\mathcal{D}$ fixed in Section 4.2.2.

Proof:
The spectrum of $C^*(G)$ or equivalently of G is given by the disjoint union

$$\widehat{G}_{even}\ \dot\cup\ \widehat{G}_{odd}\ \dot\cup\ \widehat{G}_{discrete},$$

where the set $\widehat{G}_{even}$ consists of the even representations $\mathcal{P}^{+,iv}$ for $v \in [0,\infty)$, $\mathcal{C}^u$ for $u \in (0,1)$, $\mathcal{D}_1^+$, $\mathcal{D}_1^-$ and $\mathcal{F}_1$, the set $\widehat{G}_{odd}$ consists of the odd representations $\mathcal{P}^{-,iv}$ for $v \in (0,\infty)$, $\mathcal{D}_+$ and $\mathcal{D}_-$ and the set $\widehat{G}_{discrete}$ consists of the even or respectively odd representations $\mathcal{D}_m^+$ for $m \in \mathbb{N}_{>1}$ and $\mathcal{D}_m^-$ for $m \in \mathbb{N}_{>1}$.
These three listed sets of representations are topologically separated from each other (see Section 4.2.4).
Mapping $\mathcal{P}^{+,iv}$ for $v \in [0,\infty)$ to iv, $\mathcal{C}^u$ for $u \in (0,1)$ to u and $\mathcal{D}_1^+$, $\mathcal{D}_1^-$ and $\mathcal{F}_1$ to 1, one gets a surjection from $\widehat{G}_{even}$ to the set $i[0,\infty) \cup [0,1] =: I_1$.
Furthermore, mapping $\mathcal{P}^{-,iv}$ for $v \in (0,\infty)$ to iv and $\mathcal{D}_+$ and $\mathcal{D}_-$ to 0, one gets a surjection from $\widehat{G}_{odd}$ to $i[0,\infty) =: I_2$.
Last, mapping $\mathcal{D}_m^+$ for $m \in \mathbb{N}_{>1}$ to m and $\mathcal{D}_m^-$ for $m \in \mathbb{N}_{>1}$ to $-m$, one gets a surjection from $\widehat{G}_{discrete}$ to $\mathbb{Z} \setminus \{-1,0,1\} =: I_3$.
Hence, one regards the three sets

$$I_1 = i[0,\infty) \cup [0,1], \quad I_2 = i[0,\infty) \quad \text{and} \quad I_3 = \mathbb{Z} \setminus \{-1,0,1\}.$$

In order to prove this theorem, it has to be shown that for every operator field $\varphi = \mathcal{F}(a)$ for $a \in C^*(G)$ that fulfills the properties listed in Theorem 4.20, there exists a mapping
$F_a^1 \in \Big\{F \in C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big)\Big|\ F(1) \text{ commutes with } p_+,\ p_- \text{ and } p_0\Big\} =: P_1$,
a mapping $F_a^2 \in \Big\{F \in C_\infty\Big(i[0,\infty), \mathcal{K}\big(L^2(K)_-\big)\Big)\Big|\ F(0) \text{ commutes with } p_+ \text{ and } p_-\Big\} =: P_2$
and a mapping $F_a^3 \in C_\infty\Big(\mathbb{Z} \setminus \{-1,0,1\}, \mathcal{K}\big(\mathcal{H}_\mathcal{D}\big)\Big) =: P_3$.
On the other hand, for every $F_1 \in P_1$, every $F_2 \in P_2$ and every $F_3 \in P_3$, one has to construct an operator field φ_{F_1,F_2,F_3} over $\widehat{G}$ that fulfills the properties of Theorem 4.20. Since the above-mentioned sets of representations $\widehat{G}_{even}$, $\widehat{G}_{odd}$ and $\widehat{G}_{discrete}$ are topologically separated from each other, it suffices to define three different operator fields φ_{F_1} over $\widehat{G}_{even}$, φ_{F_2} over $\widehat{G}_{odd}$ and φ_{F_3} over $\widehat{G}_{discrete}$.

For every $a \in C^*(G)$, define a function $F_a^1 : I_1 \to \mathcal{B}\big(L^2(K)_+\big)$ by

$$\begin{aligned} F_a^1(x) &:= \mathcal{F}(a)\big(\mathcal{P}^{+,x}\big) \quad \forall x \in i[0,\infty), \\ F_a^1(x) &:= \mathcal{F}(a)\big(\mathcal{C}^x\big) \quad \forall x \in (0,1) \quad \text{and} \\ F_a^1(1) &:= K_{(1)} \circ \mathcal{F}(a)\big(\mathcal{D}_1^+\big) \circ K_{(1)}^{-1} \circ p_+ \ + \ K_{[1]} \circ \mathcal{F}(a)\big(\mathcal{D}_1^-\big) \circ K_{[1]}^{-1} \circ p_- \ + \ \mathcal{F}(a)\big(\mathcal{F}_1\big) \circ p_0. \end{aligned}$$

By Property 1 of Theorem 4.20, $F_a^1(x) \in \mathcal{K}\big(L^2(K)_+\big)$ for all $x \in I_1 \setminus \{1\}$. Furthermore, since $\mathcal{F}(a)\big(\mathcal{D}_1^+\big)$, $\mathcal{F}(a)\big(\mathcal{D}_1^-\big)$ and $\mathcal{F}(a)\big(\mathcal{F}_1\big)$ are also compact, their composition with the bounded operators $K_{(1)}$, $K_{(1)}^{-1}$, $K_{[1]}$, $K_{[1]}^{-1}$, p_+, p_- and p_0 is compact as well. Therefore, $F_a^1(1) \in \mathcal{K}\big(L^2(K)_+\big)$, too.
By Property 4 of Theorem 4.20, F_a^1 vanishes at infinity. Moreover, for all $x \in I_1 \setminus \{0,1\}$, F_a^1 is obviously continuous in x.
For the continuity in 0, let $\overline{u} = (u_j)_{j\in\mathbb{N}}$ be a sequence in $(0,1)$ converging to 0. Then,

$$\begin{aligned} \lim_{j\to\infty} \big\|F_a^1(u_j) - F_a^1(0)\big\|_{op} &= \lim_{j\to\infty} \big\|\mathcal{F}(a)\big(\mathcal{C}^{u_j}\big) - \mathcal{F}(a)\big(\mathcal{P}^{+,0}\big)\big\|_{op} \\ &= \lim_{j\to\infty} \big\|\mathcal{C}^{u_j}(a) - \mathcal{P}^{+,0}(a)\big\|_{op} \\ &= \lim_{j\to\infty} \big\|K_{u_j} \circ \mathcal{P}^{+,u_j}(a) \circ K_{u_j}^{-1} - id \circ \mathcal{P}^{+,0}(a) \circ id^{-1}\big\|_{op} = 0, \end{aligned}$$

as $K_{u_j} \stackrel{j\to\infty}{\longrightarrow} id$ and with the same arguments as in the proof of (32) in Section 4.3.
For the continuity in 1, let $\overline{u} = (u_j)_{j\in\mathbb{N}}$ be a sequence in $(0,1)$ converging to 1. By Property 5 of Theorem 4.20 and since $F_a^1(1) = \tilde{\nu}_{\overline{u},j}\big(\mathcal{F}(a)\big) = \tilde{\nu}\big(\mathcal{F}(a)\big)$ by the definition of $\tilde{\nu}$ in Case (iii) of Remark 4.19 in Section 4.4,

$$\lim_{j\to\infty} \big\|F_a^1(u_j) - F_a^1(1)\big\|_{op} = \lim_{j\to\infty} \big\|\mathcal{F}(a)\big(\mathcal{C}^{u_j}\big) - \tilde{\nu}\big(\mathcal{F}(a)\big)\big\|_{op} = 0.$$

Hence, F_a^1 is also continuous in 0 and in 1 and thus, $F_a^1 \in C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big)$.
Since p_+, p_- and p_0 equal the identity on the image of $K_{(1)}$, $K_{[1]}$ and $\mathcal{F}(a)\big(\mathcal{F}_1\big)$, respectively, as discovered in Section 4.4, and since $p_+ \circ p_- = p_- \circ p_+ = p_+ \circ p_0 = p_0 \circ p_+ = p_- \circ p_0 = p_0 \circ p_- = 0$, $F_a^1(1)$ commutes with p_+, p_- and p_0.

On the other hand, taking a function $F_1 \in C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big)$ that commutes with p_+, p_- and p_0, it has to be shown that there is an operator field φ_{F_1} over $\widehat{G}_{even}$ that meets the Properties 1 to 5 of Theorem 4.20.
Define

$$\begin{aligned} \varphi_{F_1}\big(\mathcal{P}^{+,x}\big) &:= F_1(x) \in \mathcal{B}\big(L^2(K)_+\big) \quad \forall x \in i[0,\infty), \\ \varphi_{F_1}\big(\mathcal{C}^{x}\big) &:= F_1(x) \in \mathcal{B}\big(L^2(K)_+\big) \quad \forall x \in (0,1), \\ \varphi_{F_1}\big(\mathcal{D}_1^+\big) &:= K_{(1)}^{-1} \circ p_+ \circ F_1(1) \circ K_{(1)} \in \mathcal{B}\big(\mathcal{H}_{(1)}\big), \\ \varphi_{F_1}\big(\mathcal{D}_1^-\big) &:= K_{[1]}^{-1} \circ p_- \circ F_1(1) \circ K_{[1]} \in \mathcal{B}\big(\mathcal{H}_{[1]}\big) \quad \text{and} \\ \varphi_{F_1}\big(\mathcal{F}_1\big) &:= p_0 \circ F_1(1) \circ p_0 \in \mathbb{C}. \end{aligned}$$

By the definition of $C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big)$ and as the composition of the compact operator $F_1(1)$ with bounded operators is compact again, the Properties 1 to 4 are obviously fulfilled.

For Property 5, there are two cases to consider: a sequence in $(0,1)$ converging to 0 and a sequence in $(0,1)$ converging to 1.
First, let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $(0,1)$ converging to 0. Then, by the definition of $\tilde{\nu}$ in Case (ii) of Remark 4.19 in Section 4.4,

$$\left\|\varphi_{F_1}\big(\mathcal{C}^{u_j}\big) - \tilde{\nu}\big(\varphi_{F_1}\big)\right\|_{op} = \left\|F_1(u_j) - \varphi_{F_1}\big(\mathcal{P}^{+,0}\big)\right\|_{op} = \left\|F_1(u_j) - F_1(0)\right\|_{op} \overset{j\to\infty}{\longrightarrow} 0,$$

since F_1 is continuous in 0.
Now, let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $(0,1)$ converging to 1. As $F_1(1)$ commutes with p_+, p_- and p_0, by the definition of $\tilde{\nu}$ in Case (iii) of Remark 4.19 and since $p_+ + p_- + p_0 = id_{L^2(K)_+\to L^2(K)_+}$,

$$\begin{aligned}
& \left\|\varphi_{F_1}\big(\mathcal{C}^{u_j}\big) - \tilde{\nu}\big(\varphi_{F_1}\big)\right\|_{op} \\
= & \left\|F_1(u_j) - K_{(1)}\circ\varphi_{F_1}\big(\mathcal{D}_1^+\big)\circ K_{(1)}^{-1}\circ p_+ - K_{[1]}\circ\varphi_{F_1}\big(\mathcal{D}_1^-\big)\circ K_{[1]}^{-1}\circ p_- - \varphi_{F_1}\big(\mathcal{F}_1\big)\circ p_0\right\|_{op} \\
= & \Big\|F_1(u_j) - K_{(1)}\circ K_{(1)}^{-1}\circ p_+\circ F_1(1)\circ K_{(1)}\circ K_{(1)}^{-1}\circ p_+ \\
& - K_{[1]}\circ K_{[1]}^{-1}\circ p_-\circ F_1(1)\circ K_{[1]}\circ K_{[1]}^{-1}\circ p_- - p_0\circ F_1(1)\circ p_0\circ p_0\Big\|_{op} \\
= & \left\|F_1(u_j) - p_+\circ F_1(1)\circ p_+ - p_-\circ F_1(1)\circ p_- - p_0\circ F_1(1)\circ p_0\right\|_{op} \\
= & \left\|F_1(u_j) - F_1(1)\circ\big(p_+ + p_- + p_0\big)\right\|_{op} \\
= & \left\|F_1(u_j) - F_1(1)\right\|_{op} \overset{j\to\infty}{\longrightarrow} 0
\end{aligned}$$

because of the continuity of F_1 in 1.
Therefore, Property 5 is fulfilled as well.

One defines for all $a \in C^*(G)$ the function $F_a^2 : I_2 \to \mathcal{B}\big(L^2(K)_-\big)$ by

$$\begin{aligned}
F_a^2(x) &:= \mathcal{F}(a)\big(\mathcal{P}^{-,x}\big) \quad \forall x \in i(0,\infty) \quad \text{and} \\
F_a^2(0) &:= \mathcal{F}(a)\big(\mathcal{D}_+\big)\circ p_+ \;+\; \mathcal{F}(a)\big(\mathcal{D}_-\big)\circ p_-.
\end{aligned}$$

As for I_1, the compactness of $F_a^2(x)$ for all $x \in I_2$ follows from Property 1 of Theorem 4.20 and with the fact that the compositions of the compact operators $\mathcal{F}(a)\big(\mathcal{D}_+\big)$ and $\mathcal{F}(a)\big(\mathcal{D}_-\big)$ with the bounded operators p_+ and p_-, respectively, are compact. The continuity of F_a^2 outside of 0 follows from Property 3 and F_a^2 vanishes at infinity by Property 4. Using again Property 5 of Theorem 4.20 and the fact that $F_a^2(0) = \tilde{\nu}\big(\mathcal{F}(a)\big)$ by the definition of $\tilde{\nu}$ in Case (i) of Remark 4.19 in Section 4.4, one also obtains the continuity of F_a^2 in 0. Hence, $F_a^2 \in C_\infty\Big(i[0,\infty), \mathcal{K}\big(L^2(K)_-\big)\Big)$.
As

$$\begin{aligned}
\mathcal{F}(a)\big(\mathcal{D}_+\big) &= \mathcal{D}_+(a) \in \mathcal{H}_{\mathcal{D}_+} = \big\{f \in L^2(K)_- \,|\, p_n(f) = 0 \;\forall n \leq 0\big\} \quad \text{and} \\
\mathcal{F}(a)\big(\mathcal{D}_-\big) &= \mathcal{D}_-(a) \in \mathcal{H}_{\mathcal{D}_-} = \big\{f \in L^2(K)_- \,|\, p_n(f) = 0 \;\forall n \geq 0\big\},
\end{aligned}$$

one gets

$$p_+\circ\mathcal{F}(a)\big(\mathcal{D}_+\big) = \mathcal{F}(a)\big(\mathcal{D}_+\big), \quad p_-\circ\mathcal{F}(a)\big(\mathcal{D}_-\big) = \mathcal{F}(a)\big(\mathcal{D}_-\big) \quad \text{and}$$
$$p_-\circ\mathcal{F}(a)\big(\mathcal{D}_+\big) = 0 = p_+\circ\mathcal{F}(a)\big(\mathcal{D}_-\big).$$

Therefore, $F(0)$ commutes with p_+ and p_-.

Next, take a function $F_2 \in C_\infty\Big(i[0,\infty), \mathcal{K}\big(L^2(K)_-\big)\Big)$ that commutes with p_+ and p_-. An operator field φ_{F_2} over $\widehat{G}_{odd}$ meeting the Properties 1 to 5 of Theorem 4.20 has to be constructed.
Define

$$\begin{aligned}
\varphi_{F_2}\big(\mathcal{P}^{-,x}\big) &:= F_2(x) \in \mathcal{B}\big(L^2(K)_-\big) \quad \forall x \in i(0,\infty), \\
\varphi_{F_2}\big(\mathcal{D}_+\big) &:= p_+ \circ F_2(0) \in \mathcal{B}\big(\mathcal{H}_{\mathcal{D}_+}\big) \quad \text{and} \\
\varphi_{F_2}\big(\mathcal{D}_-\big) &:= p_- \circ F_2(0) \in \mathcal{B}\big(\mathcal{H}_{\mathcal{D}_-}\big).
\end{aligned}$$

Here again, the Properties 1 to 4 are obviously fulfilled. For Property 5, let $(u_j)_{j\in\mathbb{N}}$ be a sequence in $i(0,\infty)$ converging to 0. Then, as $F_2(0)$ commutes with p_+ and p_-, by the definition of $\tilde{\nu}$ in Case (i) of Remark 4.19 in Section 4.4 and since $p_+ + p_- = id_{L^2(K)_- \to L^2(K)_-}$,

$$\begin{aligned}
\big\|\varphi_{F_2}\big(\mathcal{P}^{-,u_j}\big) - \tilde{\nu}\big(\varphi_{F_2}\big)\big\|_{op} &= \big\|F_2(u_j) - \varphi_{F_2}\big(\mathcal{D}_+\big) \circ p_+ - \varphi_{F_2}\big(\mathcal{D}_-\big) \circ p_-\big\|_{op} \\
&= \big\|F_2(u_j) - p_+ \circ F_2(0) \circ p_+ - p_- \circ F_2(0) \circ p_-\big\|_{op} \\
&= \big\|F_2(u_j) - F_2(0) \circ \big(p_+ + p_-\big)\big\|_{op} \\
&= \|F_2(u_j) - F_2(0)\|_{op} \overset{j\to\infty}{\longrightarrow} 0
\end{aligned}$$

because of the continuity of F_2 in 0.

Now, take the infinite-dimensional and separable Hilbert space $\mathcal{H}_{\mathcal{D}}$ for the representations $\mathcal{D}_m^+$ and $\mathcal{D}_m^-$ for $m > 1$, fixed in Section 4.2.2. Then, define for every $a \in C^*(G)$ the function $F_a^3 : I_3 \to \mathcal{B}\big(\mathcal{H}_{\mathcal{D}}\big)$ by

$$\begin{aligned}
F_a^3(x) &:= \mathcal{F}(a)\big(\mathcal{D}_x^+\big) \quad \forall x \in \mathbb{Z}_{>1} \quad \text{and} \\
F_a^3(x) &:= \mathcal{F}(a)\big(\mathcal{D}_{-x}^-\big) \quad \forall x \in \mathbb{Z}_{<-1}.
\end{aligned}$$

Here, Property 5 of Theorem 4.20 does not emerge and the Properties 1 to 4 are obvious. Taking a function $F_3 \in C_\infty\Big(\mathbb{Z} \setminus \{-1,0,1\}, \mathcal{K}\big(\mathcal{H}_{\mathcal{D}}\big)\Big)$, one has to choose

$$\begin{aligned}
\varphi_{F_3}\big(\mathcal{D}_x^+\big) &:= F_3(x) \in \mathcal{B}\big(\mathcal{H}_{\mathcal{D}}\big) \quad \forall x \in \mathbb{Z}_{>1} \quad \text{and} \\
\varphi_{F_3}\big(\mathcal{D}_x^-\big) &:= F_3(-x) \in \mathcal{B}\big(\mathcal{H}_{\mathcal{D}}\big) \quad \forall x \in \mathbb{Z}_{>1}
\end{aligned}$$

and it is again easy to verify that φ_F complies with the properties of Theorem 4.20.

□

5 On the dual topology of the groups $\mathbf{U(n)} \ltimes \mathbb{H}_\mathbf{n}$

In this section, the semidirect product $G_n = U(n) \ltimes \mathbb{H}_n$ for $n \in \mathbb{N}^*$ will be analyzed and the topology of its spectrum shall be described.
The first part contains some preliminary definitions and results about the spectrum and the space of all admissible coadjoint orbits of G_n. In Section 5.2, the topology of the orbit space will be examined and thereafter, in Section 5.3, the topology of the spectrum will be discussed. Finally, in Section 5.4, the results of the two preceding subsections will be combined and in that way, the topology of the spectrum of G_n shall be linked to the one of the space of its admissible coadjoint orbits. Until now, this succeeds only partly, as the characterization of the spectrum of G_n is not yet entirely finished.
As already mentioned in the introduction, this section is based on a preprint by M.Elloumi and J.Ludwig which can be found in the doctoral thesis of M.Elloumi (see [11], Chapter 3). In the present thesis, it has been elaborated, completed and several important changes have been made.
Besides numerous minor changes, the main modifications of this work compared to the preprint by M.Elloumi and J.Ludwig can be found in the proofs of Theorem 5.4(5) and Theorem 5.10(1) and (2). Furthermore, there were some details added in the proof of Theorem 5.16. Theorem 5.11 and Proposition 5.12 with their proofs and Conjecture 5.13 were appended in this thesis.
In the second implication of Theorem 5.4(5)(i) and (ii) (see Section 5.2), there were mistakes in the preprint which caused the existing proof only to be valid in a particular case, namely for $\lambda_n^k \overset{k\to\infty}{\longrightarrow} -\infty$ (compare the second implication of the proof of Theorem 5.4(5)(i)) and for $\lambda_1^k \overset{k\to\infty}{\longrightarrow} \infty$, respectively (compare the second implication of the proof of Theorem 5.4(5)(ii)). Hence, the proofs for the case $\lambda_n^k \overset{k\to\infty}{\not\longrightarrow} -\infty$ and $\lambda_1^k \overset{k\to\infty}{\not\longrightarrow} \infty$, respectively, had to be elaborated and for this reason, Lemma 5.3(2) and its proof were also added in this thesis.
Concerning Theorem 5.10, there were further mistakes in both parts, (i) and (ii), of its proof which lead to the necessity to completely revise it and to construct another proof.

5.1 Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex vector space equipped with the standard scalar product $\langle .,. \rangle_{\mathbb{C}^n}$ given by

$$\langle x, y \rangle_{\mathbb{C}^n} = \sum_{j=1}^{n} x_j \overline{y_j} \quad \forall x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{C}^n.$$

Moreover, let $(.,.)_{\mathbb{C}^n}$ and $\omega(.,.)_{\mathbb{C}^n}$ denote the real and the imaginary part of $\langle .,. \rangle_{\mathbb{C}^n}$, respectively, i.e.

$$\langle .,. \rangle_{\mathbb{C}^n} = (.,.)_{\mathbb{C}^n} + i\omega(.,.)_{\mathbb{C}^n}.$$

The bilinear forms $(.,.)_{\mathbb{C}^n}$ and $\omega(.,.)_{\mathbb{C}^n}$ define a positive definite inner product and a symplectic structure on the underlying real vector space $\mathbb{R}^{2n}$ of $\mathbb{C}^n$, respectively. The associated Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ of dimension $2n+1$ over $\mathbb{R}$ is given by the group multiplication

$$(z,t)(z',t') := \Big(z + z', t + t' - \frac{1}{2}\omega(z,z')_{\mathbb{C}^n}\Big) \quad \forall (z,t),(z',t') \in \mathbb{H}_n.$$

Furthermore, consider the unitary group $U(n)$ of automorphisms of $\mathbb{H}_n$ preserving $\langle .,. \rangle_{\mathbb{C}^n}$ on $\mathbb{C}^n$ which embeds into $\mathrm{Aut}(\mathbb{H}_n)$ via

$$A.(z,t) := (Az,t) \quad \forall A \in U(n)\ \forall (z,t) \in \mathbb{H}_n.$$

Then, $U(n)$ yields a maximal compact connected subgroup of $\mathrm{Aut}(\mathbb{H}_n)$ (see [14], Theorem 1.22 and [20], Chapter I.1). Moreover, $G_n = U(n) \ltimes \mathbb{H}_n$ denotes the semidirect product of $U(n)$ with the Heisenberg group $\mathbb{H}_n$ equipped with the group law

$$(A,z,t)(B,z',t') := \Big(AB, z + Az', t + t' - \frac{1}{2}\omega(z, Az')_{\mathbb{C}^n}\Big) \quad \forall (A,z,t),(B,z',t') \in G_n.$$

The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ will be identified with $\mathbb{H}_n$ itself via the exponential map. The Lie bracket of $\mathfrak{h}_n$ is defined as

$$\big[(z,t),(w,s)\big] := \big(0, -\omega(z,w)_{\mathbb{C}^n}\big) \quad \forall (z,t),(w,s) \in \mathfrak{h}_n$$

and the derived action of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ on $\mathfrak{h}_n$ is

$$A.(z,t) := (Az, 0) \quad \forall A \in \mathfrak{u}(n)\ \forall (z,t) \in \mathfrak{h}_n.$$

Denoting by $\mathfrak{g}_n = \mathfrak{u}(n) \ltimes \mathfrak{h}_n$ the Lie algebra of G_n, for all $(A,z,t) \in G_n$ and all $(B,w,s) \in \mathfrak{g}_n$, one gets

$$\begin{aligned} \mathrm{Ad}\,(A,z,t)(B,w,s) &= \frac{d}{dy}\Big|_{y=0} \mathrm{Ad}\,(A,z,t)\big(e^{yB}, yw, ys\big) \qquad (33) \\ &= \Big(ABA^*, -ABA^*z + Aw, s - \omega(z, Aw)_{\mathbb{C}^n} + \frac{1}{2}\omega(A^*z, BA^*z)_{\mathbb{C}^n}\Big), \end{aligned}$$

where A^* is the adjoint matrix of A. In particular

$$\mathrm{Ad}\,(A)(B,w,s) = (ABA^*, Aw, s). \qquad (34)$$

From Identity (33), one can deduce the Lie bracket

$$\begin{aligned} \big[(A,z,t),(B,w,s)\big] &= \frac{d}{dy}\Big|_{y=0} \mathrm{Ad}\,\big(e^{yA}, yz, yt\big)(B,w,s) \\ &= \big(AB - BA, Aw - Bz, -\omega(z,w)_{\mathbb{C}^n}\big) \end{aligned}$$

for all $(A,z,t),(B,w,s) \in \mathfrak{g}_n$.

5.1.1 The coadjoint orbits of G_n

In this subsection, the coadjoint orbit space of G_n will be described according to [3], Section 2.5.

In the following, $\mathfrak{u}(n)$ will be identified with its vector dual space $\mathfrak{u}^*(n)$ with the help of the $U(n)$-invariant inner product

$$\langle A, B \rangle_{\mathfrak{u}(n)} := \mathrm{tr}\,(AB) \quad \forall A, B \in \mathfrak{u}(n).$$

For $z \in \mathbb{C}^n$ define the linear form z^* in $(\mathbb{C}^n)^*$ by

$$z^*(w) := \omega(z, w)_{\mathbb{C}^n} \quad \forall w \in \mathbb{C}^n.$$

Furthermore, one defines a map $\times : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathfrak{u}^*(n)$, $(z, w) \mapsto z \times w$ by

$$z \times w(B) := w^*(Bz) = \omega(w, Bz)_{\mathbb{C}^n} \quad \forall B \in \mathfrak{u}(n).$$

One can verify that for $A \in U(n)$, $B \in \mathfrak{u}(n)$ and $z, w \in \mathbb{C}^n$,

$$\begin{aligned} Az^* := z^* \circ A^{-1} &= (Az)^*, \\ z^* \circ B &= -(Bz)^*, \\ z \times w &= w \times z \quad \text{and} \\ A(z \times w)A^* &= (Az) \times (Aw). \end{aligned} \tag{35}$$

Hence, the dual $\mathfrak{g}_n^* = \big(\mathfrak{u}(n) \ltimes \mathfrak{h}_n\big)^*$ will be identified with $\mathfrak{u}(n) \oplus \mathfrak{h}_n$, i.e. each element $\ell \in \mathfrak{g}_n^*$ can be identified with an element $(U, u, x) \in \mathfrak{u}(n) \times \mathbb{C}^n \times \mathbb{R}$ such that

$$\big\langle (U, u, x), (B, w, s) \big\rangle_{\mathfrak{g}_n} = \langle U, B \rangle_{\mathfrak{u}(n)} + u^*(w) + xs \quad \forall (B, w, s) \in \mathfrak{g}_n.$$

From (34) and (35), one obtains for all $A \in U(n)$,

$$\mathrm{Ad}^*(A)(U, u, x) = (AUA^*, Au, x) \quad \forall (U, u, x) \in \mathfrak{u}(n) \times \mathbb{C}^n \times \mathbb{R} \tag{36}$$

and for all $(A, z, t) \in G_n$ and all $(U, u, x) \in \mathfrak{u}(n) \times \mathbb{C}^n \times \mathbb{R}$,

$$\mathrm{Ad}^*(A, z, t)(U, u, x) = \Big(AUA^* + z \times (Au) + \frac{x}{2} z \times z, Au + xz, x\Big), \tag{37}$$

where $z \times w(B) = w^*(Bz) = \omega(w, Bz)_{\mathbb{C}^n}$.
Letting A and z vary over $U(n)$ and $\mathbb{C}^n$, respectively, the coadjoint orbit $\mathcal{O}_{(U,u,x)}$ of the linear form (U, u, x) can then be written as

$$\mathcal{O}_{(U,u,x)} = \Big\{ \Big(AUA^* + z \times (Au) + \frac{x}{2} z \times z, Au + xz, x\Big) \Big| \; A \in U(n), z \in \mathbb{C}^n \Big\}$$

or equivalently, by replacing z by Az and using Identity (36),

$$\mathcal{O}_{(U,u,x)} = \Big\{ \mathrm{Ad}^*(A)\Big(U + z \times u + \frac{x}{2} z \times z, u + xz, x\Big) \Big| \; A \in U(n), z \in \mathbb{C}^n \Big\}.$$

Here, z is regarded as a column vector $z = (z_1, \dots, z_n)^T$ and $z^* := \overline{z}^t$.
One can show as follows that $z \times u \in \mathfrak{u}^*(n) \cong \mathfrak{u}(n)$ is the $n \times n$ skew-Hermitian matrix $\frac{i}{2}(uz^* + zu^*)$:
For all $B \in \mathfrak{u}(n)$

$$\big\langle uz^* + zu^*, B \big\rangle_{\mathfrak{u}(n)} = \mathrm{tr}\,\big((uz^* + zu^*)B\big) = \sum_{1 \le i,j \le n} B_{ji} z_i \overline{u}_j - \sum_{1 \le i,j \le n} u_i \overline{B}_{ij} \overline{z}_j = -2iz \times u(B).$$

In particular, $z \times z$ is the skew-Hermitian matrix izz^* whose entries are determined by $(izz^*)_{lj} = iz_l \overline{z}_j$.

5.1.2 The spectrum of $U(n)$

As in the preceding sections, within Section 5, the representations are identified with their equivalence classes again.

First of all, as $U(n)$ is a compact group, one knows that its spectrum is discrete and that every representation of $U(n)$ is finite-dimensional.

Now, let

$$\mathbb{T}_n = \left\{ T = \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} \middle| \; \theta_j \in \mathbb{R} \; \forall j \in \{1, ..., n\} \right\}$$

be a maximal torus of the unitary group $U(n)$ and let $\mathfrak{t}_n$ be its Lie algebra. By complexification of $\mathfrak{u}(n)$ and $\mathfrak{t}_n$, one gets the complex Lie algebras $\mathfrak{u}^{\mathbb{C}}(n) = \mathfrak{gl}(n, \mathbb{C}) = M(n, \mathbb{C})$ and

$$\mathfrak{t}_n^{\mathbb{C}} = \left\{ H = \begin{pmatrix} h_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & h_n \end{pmatrix} \middle| \; h_j \in \mathbb{C} \; \forall j \in \{1, ..., n\} \right\},$$

respectively, which is a Cartan subalgebra of $\mathfrak{u}^{\mathbb{C}}(n)$. For $j \in \{1, ..., n\}$, define a linear functional e_j by

$$e_j \begin{pmatrix} h_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & h_n \end{pmatrix} := h_j.$$

Let P_n be the set of all dominant integral forms λ for $U(n)$ which may be written in the form $\sum_{j=1}^{n} i\lambda_j e_j$, or simply as $\lambda = (\lambda_1, \cdots, \lambda_n)$, where λ_j are integers for every $j \in \{1, ..., n\}$ such that $\lambda_1 \geq \cdots \geq \lambda_n$. P_n is a lattice in the vector dual space $\mathfrak{t}_n^*$ of $\mathfrak{t}_n$, fulfilling $P_n \cong \mathbb{Z}^n$. Since each irreducible unitary representation $(\tau_\lambda, \mathcal{H}_\lambda)$ of $U(n)$ is determined by its highest weight $\lambda \in P_n$, the spectrum $\widehat{U(n)}$ of $U(n)$ is in bijection with the set P_n.

For each λ in P_n, the highest vector ϕ^λ in the corresponding Hilbert space $\mathcal{H}_\lambda$ of τ_λ, verifies $\tau_\lambda(T)\phi^\lambda = \chi_\lambda(T)\phi^\lambda$, where χ_λ is the character of $\mathbb{T}_n$ associated to the linear functional λ and defined by

$$\chi_\lambda \left(T = \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} \right) := e^{-i\lambda_1\theta_1} \cdots e^{-i\lambda_n\theta_n}.$$

Moreover, for two irreducible unitary representations $(\tau_\lambda, \mathcal{H}_\lambda)$ and $(\tau_{\lambda'}, \mathcal{H}_{\lambda'})$, the Schur orthogonality relation states that for all $\xi, \eta \in \mathcal{H}_\lambda$ and all $\xi', \eta' \in \mathcal{H}_{\lambda'}$,

$$\int_{U(n)} \langle \tau_\lambda(g)\xi, \eta \rangle_{\mathcal{H}_\lambda} \overline{\langle \tau_{\lambda'}(g)\xi', \eta' \rangle_{\mathcal{H}_{\lambda'}}} \, dg = \begin{cases} 0 & \text{if } \lambda \neq \lambda', \\ \frac{\langle \xi, \xi' \rangle_{\mathcal{H}_\lambda} \langle \eta', \eta \rangle_{\mathcal{H}_\lambda}}{d_\lambda} & \text{if } \lambda = \lambda', \end{cases} \tag{38}$$

where d_λ denotes the dimension of the representation τ_λ.

According to [18], Chapter 1, if ρ_μ is an irreducible representation of $U(n-1)$ with highest weight $\mu = (\mu_1, ..., \mu_{n-1})$, the induced representation $\pi_\mu := \mathrm{ind}_{U(n-1)}^{U(n)} \rho_\mu$ of $U(n)$ decomposes with multiplicity one, and the representations of $U(n)$ that appear in this decomposition are exactly those with the highest weight $\lambda = (\lambda_1, ..., \lambda_n)$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n. \tag{39}$$

5.1.3 The spectrum and the admissible coadjoint orbits of G_n

The description of the spectrum of G_n is based on a method by Mackey (see [28], Chapter 10), which states that one has to determine the irreducible unitary representations of the subgroup $\mathbb{H}_n$ in order to construct representations of G_n.

First, regard the infinite-dimensional irreducible representations of the Heisenberg group $\mathbb{H}_n$, which are parametrized by $\mathbb{R}^*$:
For each $\alpha \in \mathbb{R}^*$, the coadjoint orbit $\mathcal{O}_\alpha$ of the irreducible representation σ_α is the hyperplane $\mathcal{O}_\alpha = \big\{(z, \alpha) |\, z \in \mathbb{C}^n\big\}$. Since for every α, this orbit is invariant under the action of $U(n)$, the unitary group $U(n)$ preserves the equivalence class of σ_α.
The representation σ_α can be realized in the Fock space

$$\mathcal{F}_\alpha(n) = \left\{ f : \mathbb{C}^n \longrightarrow \mathbb{C} \text{ entire} \;\Big|\; \int_{\mathbb{C}^n} |f(w)|^2 e^{-\frac{|\alpha|}{2}|w|^2} dw < \infty \right\}$$

as

$$\begin{aligned} \sigma_\alpha(z,t) f(w) &:= e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w, z\rangle_{\mathbb{C}^n}} f(w+z) \quad \text{for } \alpha > 0 \qquad \text{and} \\ \sigma_\alpha(z,t) f(\overline{w}) &:= e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w}, \overline{z}\rangle_{\mathbb{C}^n}} f(\overline{w}+\overline{z}) \quad \text{for } \alpha < 0. \end{aligned}$$

See [14], Chapter 1.6 or [19], Section 1.7 for a discussion of the Fock space.

For each $A \in U(n)$, the operator $W_\alpha(A)$ defined by

$$W_\alpha(A) : \mathcal{F}_\alpha(n) \to \mathcal{F}_\alpha(n),\ W_\alpha(A) f(z) := f\big(A^{-1} z\big) \quad \forall f \in \mathcal{F}_\alpha(n)\ \forall z \in \mathbb{C}^n$$

intertwines σ_α and $(\sigma_\alpha)_A$ given by $(\sigma_\alpha)_A(z,t) := \sigma_\alpha(Az, t)$. W_α is called the projective intertwining representation of $U(n)$ on the Fock space. Then, by [28], Chapter 10, for each $\alpha \in \mathbb{R}^*$ and each element τ_λ in $\widehat{U(n)}$,

$$\pi_{(\lambda,\alpha)}(A, z, t) := \tau_\lambda(A) \otimes \big(\sigma_\alpha(z,t) \circ W_\alpha(A)\big) \quad \forall (A, z, t) \in G_n$$

is an irreducible unitary representation of G_n realized in $\mathcal{H}_{(\lambda,\alpha)} := \mathcal{H}_\lambda \otimes \mathcal{F}_\alpha(n)$, where $\mathcal{H}_\lambda$ is the Hilbert space of τ_λ.
Associate to $\pi_{(\lambda,\alpha)}$ the linear functional $\ell_{\lambda,\alpha} := (J_\lambda, 0, \alpha) \in \mathfrak{g}_n^*$ given by

$$J_\lambda := \begin{pmatrix} i\lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i\lambda_n \end{pmatrix}.$$

Denote by $G_n[\ell_{\lambda,\alpha}]$, $U(n)[\ell_{\lambda,\alpha}]$ and $\mathbb{H}_n[\ell_{\lambda,\alpha}]$ the stabilizers of $\ell_{\lambda,\alpha}$ in G_n, $U(n)$ and $\mathbb{H}_n$, respectively. By Formula (37),

$$\begin{aligned} G_n[\ell_{\lambda,\alpha}] &= \Big\{(A,z,t)\in G_n \Big| \Big(AJ_\lambda A^* + \frac{i}{2}\alpha z z^*, \alpha z, \alpha\Big) = (J_\lambda, 0, \alpha)\Big\} \\ &= \big\{(A,0,t)\in G_n |\ AJ_\lambda A^* = J_\lambda\big\}, \end{aligned}$$

$$\begin{aligned} U(n)[\ell_{\lambda,\alpha}] &= \big\{A\in U(n) |\ (AJ_\lambda A^*, 0, \alpha) = (J_\lambda, 0, \alpha)\big\} \\ &= \big\{A\in U(n) |\ AJ_\lambda A^* = J_\lambda\big\} \quad \text{and} \\ \mathbb{H}_n[\ell_{\lambda,\alpha}] &= \Big\{(z,t)\in \mathbb{H}_n \Big| \Big(J_\lambda + \frac{i}{2}\alpha z z^*, \alpha z, \alpha\Big) = (J_\lambda, 0, \alpha)\Big\} \ = \ \{0\}\times\mathbb{R}. \end{aligned}$$

It follows that $G_n[\ell_{\lambda,\alpha}] = U(n)[\ell_{\lambda,\alpha}] \ltimes \mathbb{H}_n[\ell_{\lambda,\alpha}]$. Hence, $\ell_{\lambda,\alpha}$ is aligned in the sense of Lipsman (see [25], Lemma 4.2).

The finite-dimensional irreducible representations of $\mathbb{H}_n$ are the characters χ_v for $v \in \mathbb{C}^n$, defined by

$$\chi_v(z,t) := e^{-i(v,z)_{\mathbb{C}^n}} \qquad \forall (z,t)\in\mathbb{H}_n.$$

Denote by $U(n)_v$ the stabilizer of the character χ_v, or equivalently of the vector v, under the action of $U(n)$. Then, for every irreducible unitary representation ρ of $U(n)_v$, the tensor product $\rho\otimes\chi_v$ is an irreducible representation of $U(n)_v \ltimes \mathbb{H}_n$ whose restriction to $\mathbb{H}_n$ is a multiple of χ_v, and the induced representation

$$\pi_{(\rho,v)} := \mathrm{ind}_{U(n)_v\ltimes\mathbb{H}_n}^{U(n)\ltimes\mathbb{H}_n} \rho\otimes\chi_v$$

is an irreducible representation of G_n. Furthermore, the restriction of $\pi_{(\rho,v)}$ to $U(n)$ is equivalent to $\mathrm{ind}_{U(n)_v}^{U(n)}\rho$.
For any $v' = Av$ for $A\in U(n)$ (i.e. v and v' belong to the same sphere centered at 0 and of radius $r = \|v\|_{\mathbb{C}^n}$), one has $U(n)_{v'} = AU(n)_v A^*$ and thus, the representation $\pi_{(\rho,v)}$ is equivalent with $\pi_{(\rho',v')}$ for any $\rho'\in\widehat{U(n)_{v'}}$ such that $\rho'(B) = \rho(A^*BA)$ for each $B\in U(n)_{v'}$. Hence, one can regard the character χ_r associated to the linear form v_r which is identified with the vector $(0,\dots,0,r)^T$ in $\mathbb{C}^n$. Throughout this text, denote by ρ_μ the representation of the subgroup $U(n-1) = U(n)_{v_r}$ with highest weight μ and by $\pi_{(\mu,r)}$ the representation $\pi_{(\rho_\mu,v_r)} = \mathrm{ind}_{U(n-1)\ltimes\mathbb{H}_n}^{G_n}\rho_\mu\otimes\chi_r$. Its Hilbert space $\mathcal{H}_{(\mu,r)}$ is given by

$$\mathcal{H}_{(\mu,r)} = L^2\Big(G_n/\big(U(n-1)\ltimes\mathbb{H}_n\big), \rho_\mu\otimes\chi_r\Big).$$

Again, $\pi_{(\mu,r)}$ is linked to the linear functional $\ell_{\mu,r} := (J_\mu, v_r, 0)\in\mathfrak{g}_n^*$ for

$$J_\mu := \begin{pmatrix} i\mu_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & i\mu_{n-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

By (37), one can check that

$$\begin{aligned} G_n[\ell_{\mu,r}] &= \Big\{(A,z,t)\in G_n \Big|\ \big(AJ_\mu A^* + z\times(Av_r), Av_r, 0\big) = \big(J_\mu, v_r, 0\big)\Big\} \\ &= \Big\{(A,z,t)\in G_n \Big|\ A\in U(n-1),\ AJ_\mu A^* + \frac{i}{2}\big(v_r z^* + z v_r^*\big) = J_\mu\Big\} \\ &= \big\{(A,z,t)\in G_n |\ z\in i\mathbb{R}v_r, A\in U(n-1),\ AJ_\mu A^* = J_\mu\big\}, \end{aligned}$$

since $AJ_\mu A^*\in\mathfrak{u}(n-1)$ and

$$v_r z^* + z v_r^* = \begin{pmatrix} 0 & \dots & 0 & rz_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & rz_{n-1} \\ r\overline{z}_1 & \dots & r\overline{z}_{n-1} & 2r\,\mathrm{Re}\,(z_n) \end{pmatrix}.$$

In addition,

$$\begin{aligned} U(n)[\ell_{\mu,r}] &= \big\{A\in U(n-1) |\ AJ_\mu A^* = J_\mu\big\} \quad \text{and} \\ \mathbb{H}_n[\ell_{\mu,r}] &= i\mathbb{R}v_r\times\mathbb{R}. \end{aligned}$$

Thus, similarly to $\ell_{\lambda,\alpha}$, the linear functional $\ell_{\mu,r}$ is aligned.

One obtains in this way all the finite-dimensional irreducible unitary representations of G_n which are not trivial on $\mathbb{H}_n$.

On the other hand, the trivial extension of each element τ_λ of $\widehat{U(n)}$ to the entire group G_n is an irreducible representation which will also be denoted by τ_λ. The corresponding linear functional is $\ell_\lambda := (J_\lambda, 0, 0)$.

Therefore, by Mackey's theory, the spectrum $\widehat{G_n}$ consists of the following families of representations:

(i) $\pi_{(\lambda,\alpha)}$ for $\lambda\in P_n$ and $\alpha\in\mathbb{R}^* = \mathbb{R}\setminus\{0\}$,

(ii) $\pi_{(\mu,r)}$ for $\mu\in P_{n-1}$ and $r\in\mathbb{R}_{>0}$ and

(iii) τ_λ for $\lambda\in P_n$.

Hence, $\widehat{G_n}$ is in bijection with the set

$$(P_n\times\mathbb{R}^*)\cup\big(P_{n-1}\times\mathbb{R}_{>0}\big)\cup P_n.$$

A linear functional ℓ in $\mathfrak{g}_n^*$ is defined to be admissible if there exists a unitary character χ of the connected component of $G_n[\ell]$ such that $d\chi = i\ell_{|\mathfrak{g}_n[\ell]}$. A calculation shows that all the linear functionals $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ are admissible. Then, according to [25], the representations $\pi_{(\lambda,\alpha)}$, $\pi_{(\mu,r)}$ and τ_λ described above are equivalent to the representations of G_n obtained by holomorphic induction from their respective linear functionals $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ.
Denote by $\mathcal{O}_{(\lambda,\alpha)}$, $\mathcal{O}_{(\mu,r)}$ and $\mathcal{O}_\lambda$ the coadjoint orbits associated to the linear forms $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ and ℓ_λ, respectively. Let $\mathfrak{g}_n^\ddagger\subset\mathfrak{g}_n^*$ be the union of all the elements in $\mathcal{O}_{(\lambda,\alpha)}$, $\mathcal{O}_{(\mu,r)}$ and $\mathcal{O}_\lambda$ and denote by $\mathfrak{g}_n^\ddagger/G_n$ the corresponding set in the orbit space. Now, from [25] follows that $\mathfrak{g}_n^\ddagger$ is the set of all admissible linear functionals of $\mathfrak{g}_n$.

5.2 Convergence in the quotient space $\mathfrak{g}_n^\ddagger/G_n$

According to the last subsection, the spectrum of G_n is parametrized by the dominant integral forms λ for $U(n)$ and μ for $U(n-1)$, the non-zero $\alpha \in \mathbb{R}$ attached to the generic orbits $\mathcal{O}_\alpha$ in $\mathfrak{h}_n^*$ and the positive real r derived from the natural action of the unitary group $U(n)$ on the characters of the Heisenberg group $\mathbb{H}_n$.
Moreover, it has been elaborated that the quotient space $\mathfrak{g}_n^\ddagger/G_n$ of admissible coadjoint orbits is in bijection with $\widehat{G_n}$.
Now, the convergence of the admissible coadjoint orbits will be linked to the convergence in the parameter space $\Big\{\alpha \in \mathbb{R}^*, r > 0, \rho_\mu \in \widehat{U(n-1)}, \tau_\lambda \in \widehat{U(n)}\Big\}$.

Letting $\mathcal{W}$ be the subspace of $\mathfrak{u}(n)$ generated by the matrices $z \times v_r = \frac{i}{2}(v_r z^* + z v_r^*)$ for $z \in \mathbb{C}^n$, the space $\mathfrak{g}_n^\ddagger/G_n$ is the set of all orbits

$$\begin{aligned}
\mathcal{O}_{(\lambda,\alpha)} &= \Big\{\Big(AJ_\lambda A^* + \frac{i\alpha}{2} zz^*, \alpha z, \alpha\Big)\Big|\ z \in \mathbb{C}^n,\ A \in U(n)\Big\},\\
\mathcal{O}_{(\mu,r)} &= \Big\{\big(A(J_\mu + \mathcal{W})A^*, Av_r, 0\big)\big|\ A \in U(n)\Big\} \quad \text{and}\\
\mathcal{O}_\lambda &= \big\{(AJ_\lambda A^*, 0, 0)\big|\ A \in U(n)\big\}
\end{aligned}$$

for $\alpha \in \mathbb{R}^*$, $r \in \mathbb{R}_{>0}$, $\mu \in P_{n-1}$ and $\lambda \in P_n$.

Before beginning the discussion on the convergence of the admissible coadjoint orbits, the following preliminary lemmas are needed:

Lemma 5.1.
For $n \in \mathbb{N}^*$ and for any scalars $X_1, ..., X_n$ and $Y_1, ..., Y_{n-1}$ fulfilling $Y_i \neq Y_j$ for $i \neq j$, one has

$$\sum_{j=1}^{n-1} \frac{\prod\limits_{\substack{i=1\\ i\neq k}}^{n} (X_i - Y_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n-1} (Y_i - Y_j)} = \sum_{\substack{j=1\\ j\neq k}}^{n} X_j - \sum_{j=1}^{n-1} Y_j$$

for each $k \in \{1, ..., n\}$.

Proof:
For $n = 1$, the formula is trivial.
So, let $n > 1$ and assume that the assertion is true for this n.
For $k = n+1$, a simple calculation gives the result. If $k \neq n+1$, one gets

$$\sum_{j=1}^{n} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n+1}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n}(Y_i - Y_j)}$$

$$= \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n+1}(X_i - Y_n)}{\prod\limits_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n+1}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n}(Y_i - Y_j)}$$

$$= (X_{n+1} - Y_n) \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_n)}{\prod\limits_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n-1}(Y_i - Y_j)} \cdot \frac{(X_{n+1} - Y_j)}{Y_n - Y_j}$$

$$= (X_{n+1} - Y_n) \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_n)}{\prod\limits_{i=1}^{n-1}(Y_i - Y_n)} + \sum_{j=1}^{n-1} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n-1}(Y_i - Y_j)} \cdot \frac{(X_{n+1} - Y_n)}{Y_n - Y_j} + \underbrace{\sum_{j=1}^{n-1} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n-1}(Y_i - Y_j)}}_{=\sum\limits_{\substack{j=1\\j\neq k}}^{n} X_j - \sum\limits_{j=1}^{n-1} Y_j}$$

$$= (X_{n+1} - Y_n) \underbrace{\sum_{j=1}^{n} \frac{\prod\limits_{\substack{i=1\\i\neq k}}^{n}(X_i - Y_j)}{\prod\limits_{\substack{i=1\\i\neq j}}^{n}(Y_i - Y_j)}}_{=1 \text{ by [12], Lemma 5.3}} + \sum_{\substack{j=1\\j\neq k}}^{n} X_j - \sum_{j=1}^{n-1} Y_j = \sum_{\substack{j=1\\j\neq k}}^{n+1} X_j - \sum_{j=1}^{n} Y_j$$

and the claim is shown.

□

Lemma 5.2.
Let $\mu \in P_{n-1}$ and $\lambda \in P_n$. Then, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{n-1} \geq \lambda_n$ if and only if there is a skew-Hermitian matrix

$$B = \begin{pmatrix} 0 & 0 & \dots & 0 & -z_1 \\ 0 & 0 & \dots & 0 & -z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -z_{n-1} \\ \overline{z}_1 & \overline{z}_2 & \dots & \overline{z}_{n-1} & ix \end{pmatrix}$$

in $\mathcal{W}$ such that $A(J_\mu + B)A^* = J_\lambda$ for an element $A \in U(n)$.

Proof:
For $y \in \mathbb{R}$, a computation shows that $\det(J_\mu + B - iy\mathbb{I}) = (-i)^n P(y)$, where

$$P(y) := (y-x)\prod_{i=1}^{n-1}(y-\mu_i) \;-\; \sum_{j=1}^{n-1}\Bigg(|z_j|^2 \prod_{\substack{i=1\\ i\neq j}}^{n-1}(y-\mu_i)\Bigg).$$

Furthermore, one can observe that $P(y) \overset{y\to\infty}{\longrightarrow} \infty$ and that $P(\mu_j) \leq 0$ if j is odd and $P(\mu_j) \geq 0$ if j is even.
Now, if $A(J_\mu + B)A^* = J_\lambda$ for an element $A \in U(n)$, by the spectral theorem, $i\lambda_1, i\lambda_2, \cdots, i\lambda_n$ are all the elements of the spectrum of $J_\mu + B$ fulfilling $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{n-1} \geq \lambda_n$.
Conversely, suppose first that all the μ_j for $j \in \{1, ..., n-1\}$ are pairwise distinct. In this case, let B be the skew-Hermitian matrix with the entries $z_1, \cdots, z_{n-1}, x$ satisfying

$$|z_j|^2 = -\frac{\prod_{i=1}^{n}(\lambda_i - \mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i - \mu_j)} \quad \text{for every } j \in \{1, ..., n-1\} \ \text{ and}$$

$$x = \sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\mu_j.$$

From Lemma 5.1,

$$\begin{aligned} P(\lambda_k) &= \Bigg(\sum_{j=1}^{n-1}\mu_j - \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j\Bigg)\prod_{i=1}^{n-1}(\lambda_k-\mu_i) + \sum_{j=1}^{n-1}\left(\frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\prod_{i=1}^{n-1}(\lambda_k-\mu_i)\right) \\ &= \prod_{i=1}^{n-1}(\lambda_k-\mu_i)\left(\sum_{j=1}^{n-1}\mu_j - \sum_{\substack{j=1\\ j\neq k}}^{n}\lambda_j + \sum_{j=1}^{n-1}\frac{\prod_{\substack{i=1\\ i\neq k}}^{n}(\lambda_i-\mu_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\right) = 0. \end{aligned}$$

Hence, the spectrum of the matrix $J_\mu + B$ is the set $\{i\lambda_1, i\lambda_2, \cdots, i\lambda_n\}$ and thus, the spectral theorem implies that $A(J_\mu + B)A^* = J_\lambda$ for an element $A \in U(n)$.
Now, if the μ_j for $j \in \{1, ..., n-1\}$ are not pairwise distinct, there exist two families of integers $\{p_l |\ 1 \leq l \leq s\}$ and $\{q_l |\ 1 \leq l \leq s\}$ such that $1 \leq p_1 < q_1 < p_2 < q_2 < \cdots < p_s < q_s \leq n-1$ and $\mu_{p_l} = \mu_{p_l+1} = \cdots = \mu_{q_l-1} = \mu_{q_l}$, $\mu_{q_l} \neq \mu_{q_l+1}$ and $\mu_{p_l-1} \neq \mu_{p_l}$ for all $l \in \{1, ..., s\}$. Let

$$Q(y) := \prod_{i=1}^{p_1}\prod_{i=q_1+1}^{p_2}\cdots\prod_{i=q_s+1}^{n-1}(y-\mu_i), \quad \tilde{Q}_l(y) := \prod_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod_{i=q_s+1}^{n-1}(y-\mu_i)$$

$$\text{and} \quad Q_j(y) := \prod_{\substack{i=1\\ i\neq j}}^{p_1}\prod_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(y-\mu_i).$$

Then, for

$$P(y) := (y-x)Q(y) \; - \; \sum_{l=1}^{s}\Big(\sum_{j=p_l}^{q_l}|z_j|^2\Big)\tilde{Q}_l(y) \; - \; \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}...\sum_{j=q_s+1}^{n-1}\Big(|z_j|^2 Q_j(y)\Big),$$

one gets $\det(J_\mu + B - iy\mathbb{I}) = (-i)^n \prod_{l=1}^{s}(y-\mu_{p_l})^{q_l-p_l}P(y)$.
Now, the entries z_j of the skew-Hermitian matrix B can be chosen as follows:

$$|z_j|^2 := -\frac{\prod\limits_{i=1}^{n}(\lambda_i-\mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)} \;=\; -\frac{\prod\limits_{i=1}^{p_1}\prod\limits_{i=q_1+1}^{p_2}\cdots\prod\limits_{i=q_s+1}^{n}(\lambda_i-\mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{p_1}\prod\limits_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod\limits_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}$$

for each $j \in \{1, ..., p_1-1, q_1+1, ..., p_s-1, q_s+1, ..., n-1\}$ and

$$|z_{p_l}|^2 + ... + |z_{q_l-1}|^2 + |z_{q_l}|^2 := -\frac{\prod\limits_{i=1}^{p_1}\prod\limits_{i=q_1+1}^{p_2}\cdots\prod\limits_{i=q_s+1}^{n}(\lambda_i-\mu_{p_l})}{\prod\limits_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod\limits_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod\limits_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod\limits_{i=q_s+1}^{n-1}(\mu_i-\mu_{p_l})}$$

for each $l \in \{1, ..., s\}$. The entry x can be defined as

$$x := \sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\mu_j = \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}...\sum_{j=q_s+1}^{n}\lambda_j \; - \; \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}...\sum_{j=q_s+1}^{n-1}\mu_j.$$

Then, if $\lambda_k = \mu_{p_l}$, one obviously has $P(\lambda_k) = Q(\lambda_k) = 0$. Otherwise, one gets

$$\begin{aligned} P(\lambda_k) \;&=\; \Big(\lambda_k - \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n}\lambda_j + \sum_{j=1}^{p_1}\sum_{j=q_1+1}^{p_2}\cdots\sum_{j=q_s+1}^{n-1}\mu_j\Big)Q(\lambda_k) \\ &\quad + \sum_{j=1}^{p_1-1}\sum_{j=q_1+1}^{p_2-1}\cdots\sum_{j=q_s+1}^{n-1}\left(\frac{\prod\limits_{i=1}^{p_1}\prod\limits_{i=q_1+1}^{p_2}\cdots\prod\limits_{i=q_s+1}^{n}(\lambda_i-\mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{p_1}\prod\limits_{\substack{i=q_1+1\\ i\neq j}}^{p_2}\cdots\prod\limits_{\substack{i=q_s+1\\ i\neq j}}^{n-1}(\mu_i-\mu_j)}\,Q_j(\lambda_k)\right) \\ &\quad + \sum_{l=1}^{s}\left(\frac{\prod\limits_{i=1}^{p_1}\prod\limits_{i=q_1+1}^{p_2}\cdots\prod\limits_{i=q_s+1}^{n}(\lambda_i-\mu_{p_l})}{\prod\limits_{\substack{i=1\\ i\neq p_l}}^{p_1}\prod\limits_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2}\cdots\prod\limits_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s}\prod\limits_{i=q_s+1}^{n-1}(\mu_i-\mu_{p_l})}\,\tilde{Q}_l(\lambda_k)\right) \end{aligned}$$

$$= Q(\lambda_k)\Bigg(\sum_{j=1}^{p_1} \sum_{j=q_1+1}^{p_2} \cdots \sum_{j=q_s+1}^{n-1} \mu_j - \sum_{\substack{j=1\\ j\neq k}}^{p_1} \sum_{\substack{j=q_1+1\\ j\neq k}}^{p_2} \cdots \sum_{\substack{j=q_s+1\\ j\neq k}}^{n} \lambda_j$$

$$+ \sum_{j=1}^{p_1-1} \sum_{j=q_1+1}^{p_2-1} \cdots \sum_{j=q_s+1}^{n-1} \frac{\prod\limits_{\substack{i=1\\ i\neq k}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq k}}^{p_2} \cdots \prod\limits_{\substack{i=q_s+1\\ i\neq k}}^{n} (\lambda_i - \mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq j}}^{p_2} \cdots \prod\limits_{\substack{i=q_s+1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)}$$

$$+ \sum_{l=1}^{s} \frac{\prod\limits_{\substack{i=1\\ i\neq k}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq k}}^{p_2} \cdots \prod\limits_{\substack{i=q_s+1\\ i\neq k}}^{n} (\lambda_i - \mu_{p_l})}{\prod\limits_{\substack{i=1\\ i\neq p_l}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq p_l}}^{p_2} \cdots \prod\limits_{\substack{i=q_{s-1}+1\\ i\neq p_l}}^{p_s} \prod\limits_{i=q_s+1}^{n-1} (\mu_i - \mu_{p_l})} \Bigg)$$

$$= Q(\lambda_k)\Bigg(\sum_{j=1}^{p_1} \sum_{j=q_1+1}^{p_2} \cdots \sum_{j=q_s+1}^{n-1} \mu_j - \sum_{\substack{j=1\\ j\neq k}}^{p_1} \sum_{\substack{j=q_1+1\\ j\neq k}}^{p_2} \cdots \sum_{\substack{j=q_s+1\\ j\neq k}}^{n} \lambda_j$$

$$+ \sum_{j=1}^{p_1} \sum_{j=q_1+1}^{p_2} \cdots \sum_{j=q_s+1}^{n-1} \frac{\prod\limits_{\substack{i=1\\ i\neq k}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq k}}^{p_2} \cdots \prod\limits_{\substack{i=q_s+1\\ i\neq k}}^{n} (\lambda_i - \mu_j)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{p_1} \prod\limits_{\substack{i=q_1+1\\ i\neq j}}^{p_2} \cdots \prod\limits_{\substack{i=q_s+1\\ i\neq j}}^{n-1} (\mu_i - \mu_j)} \Bigg) = 0.$$

Hence, the spectrum of the matrix $J_\mu + B$ equals the set $\{i\lambda_1, i\lambda_2, \cdots, i\lambda_n\}$. As above, this completes the proof.

□

Lemma 5.3.

1. *Let $\lambda \in P_n$, $\alpha \in \mathbb{R}^*$ and $z \in \mathbb{C}^n$. Then, the matrix $J_\lambda + \frac{i}{\alpha} zz^*$ admits n eigenvalues $i\beta_1, i\beta_2, \ldots, i\beta_n$ in such a way that $\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq \cdots \geq \beta_n \geq \lambda_n$ if $\alpha > 0$ and $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \cdots \geq \lambda_n \geq \beta_n$ if $\alpha < 0$.*
2. *Let $\lambda, \beta \in P_n$. If $\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq ... \geq \beta_n \geq \lambda_n$, there exists a number $z \in \mathbb{C}^n$ in such a way that the matrix $J_\lambda + izz^*$ admits the n eigenvalues $i\beta_1, ..., i\beta_n$. If $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq ... \geq \lambda_n \geq \beta_n$, there exists $z \in \mathbb{C}^n$ such that the matrix $J_\lambda - izz^*$ admits the n eigenvalues $i\beta_1, ..., i\beta_n$.*

Proof:
1) One can prove by induction that the characteristic polynomial of the matrix $\frac{1}{i} J_\lambda + \frac{zz^*}{\alpha}$ is equal to $Q_n^{\lambda,z,\alpha}$ defined by

$$Q_n^{\lambda,z,\alpha}(x) := \prod_{i=1}^{n} (x - \lambda_i) \;-\; \sum_{j=1}^{n} \frac{|z_j|^2}{\alpha} \prod_{\substack{i=1\\ i\neq j}}^{n} (x - \lambda_i).$$

Assume that α is negative. Then, $Q_n^{\lambda,z,\alpha}(x) \overset{x\to\infty}{\longrightarrow} \infty$ and $Q_n^{\lambda,z,\alpha}(\lambda_j) \geq 0$ if j is odd and $Q_n^{\lambda,z,\alpha}(\lambda_j) \leq 0$ if j is even. Furthermore, $Q_n^{\lambda,z,\alpha}(x) \overset{x\to\infty}{\longrightarrow} -\infty$ if n is odd and $Q_n^{\lambda,z,\alpha}(x) \overset{x\to-\infty}{\longrightarrow} \infty$ if n is even and therefore, one can deduce that $\frac{1}{i}J_\lambda + \frac{zz^*}{\alpha}$ admits n eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$ verifying $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \cdots \geq \lambda_n \geq \beta_n$. Hence, $J_\lambda + \frac{i}{\alpha}zz^*$ admits the n eigenvalues $i\beta_1, i\beta_2, \ldots, i\beta_n$ fulfilling $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq \cdots \geq \lambda_n \geq \beta_n$.
The same reasoning applies when α is positive.

2) Let $\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq ... \geq \beta_n \geq \lambda_n$.
For any $z \in \mathbb{C}^n$, the characteristic polynomial of $\frac{1}{i}J_\lambda + zz^*$ is equal to $Q_n^{\lambda,z,1}$ with $Q_n^{\lambda,z,1} =: Q_n^{\lambda,z}$ like above.
First, assume that $\beta_1 > \lambda_1 > ... > \beta_n > \lambda_n$.
Let

$$|z_j|^2 := -\frac{\prod\limits_{i=1}^{n}(\lambda_j - \beta_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(\lambda_j - \lambda_i)}.$$

Then, as $\lambda_j < \beta_i$ for all $i \in \{1, ..., j\}$, as $\lambda_j > \beta_i$ for all $i \in \{j+1, ..., n\}$, as $\lambda_j < \lambda_i$ for all $i \in \{1, ..., j-1\}$ and as $\lambda_j > \lambda_i$ for all $i \in \{j+1, ..., n\}$, one gets $\operatorname{sgn}(|z_j|^2) = (-1)\frac{(-1)^j}{(-1)^{j-1}} = 1$ and thus, this definition is reasonable.
One now has to show that $Q_n^{\lambda,z}(\beta_\ell) = 0$ for all $\ell \in \{1, ..., n\}$.

$$\begin{aligned}
Q_n^{\lambda,z}(\beta_\ell) &= \prod_{i=1}^{n}(\beta_\ell - \lambda_i) + \sum_{j=1}^{n} \frac{\prod\limits_{i=1}^{n}(\lambda_j - \beta_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(\lambda_j - \lambda_i)} \prod_{\substack{i=1\\ i\neq j}}^{n}(\beta_\ell - \lambda_i) \\
&= \prod_{i=1}^{n}(\beta_\ell - \lambda_i) \left(1 + \sum_{j=1}^{n} \frac{\prod\limits_{i=1}^{n}(\lambda_j - \beta_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(\lambda_j - \lambda_i)(\beta_\ell - \lambda_j)}\right) \\
&= \prod_{i=1}^{n}(\beta_\ell - \lambda_i) \left(1 - \sum_{j=1}^{n} \frac{\prod\limits_{\substack{i=1\\ i\neq \ell}}^{n}(\lambda_j - \beta_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(\lambda_j - \lambda_i)}\right) = 0,
\end{aligned}$$

as by [12], Lemma 5.3, one obtains $\sum\limits_{j=1}^{n} \frac{\prod\limits_{\substack{i=1\\ i\neq \ell}}^{n}(\lambda_j - \beta_i)}{\prod\limits_{\substack{i=1\\ i\neq j}}^{n}(\lambda_j - \lambda_i)} = 1.$

Now, regard arbitrary $\beta_1 \geq \lambda_1 \geq \beta_2 \geq \lambda_2 \geq ... \geq \beta_n \geq \lambda_n$.
For $n = 1$, one can choose $|z_1|^2 := (\beta_1 - \lambda_1) \geq 0$ and the claim is shown.
Let $n > 1$ and assume that the assertion is true for $n - 1$.

If $\lambda_{\ell-1} \neq \beta_\ell \neq \lambda_\ell$ for all $\ell \in \{1, ..., n\}$, the claim is already shown above. So, without restriction let $\ell \in \{1, ..., n\}$ with $\beta_\ell = \lambda_\ell$. The case $\lambda_{\ell-1} = \beta_\ell$ is very similar.
Hence, for $\lambda^\ell := (\lambda_1, ..., \lambda_{\ell-1}, \lambda_{\ell+1}, ..., \lambda_n)$ and $\beta^\ell := (\beta_1, ..., \beta_{\ell-1}, \beta_{\ell+1}, ..., \beta_n)$,

$$\beta_1 \geq \lambda_1 \geq ... \geq \beta_{\ell-1} \geq \lambda_{\ell-1} \geq \beta_{\ell+1} \geq \lambda_{\ell+1} \geq ... \geq \beta_n \geq \lambda_n$$

and thus, by the induction hypothesis, there exists $\mathbb{C}^{n-1} \ni z^\ell := (z_1, ..., z_{\ell-1}, z_{\ell+1}, ..., z_n)$ such that $Q_{n,\ell}^{\lambda^\ell, z^\ell}(\beta_i) = 0$ for all $i \in \{1, ..., \ell-1, \ell+1, ..., n\}$, where

$$Q_{n,\ell}^{\lambda^\ell, z^\ell}(x) := \prod_{\substack{i=1 \\ i \neq \ell}}^{n} (x - \lambda_i) \ - \ \sum_{\substack{j=1 \\ j \neq \ell}}^{n} |z_j|^2 \prod_{\substack{i=1 \\ i \neq j, i \neq \ell}}^{n} (x - \lambda_i).$$

Now, let $z_\ell := 0$, i.e. $z := (z_1, ..., z_{\ell-1}, 0, z_{\ell+1}, ..., z_n)$. Then,

$$\begin{aligned} Q_n^{\lambda,z}(x) &= (x - \lambda_\ell) \prod_{\substack{i=1 \\ i \neq \ell}}^{n} (x - \lambda_i) \ - \ \sum_{\substack{j=1 \\ j \neq \ell}}^{n} |z_j|^2 (x - \lambda_\ell) \prod_{\substack{i=1 \\ i \neq j, i \neq \ell}}^{n} (x - \lambda_i) \ - \ |z_\ell|^2 \prod_{\substack{i=1 \\ i \neq \ell}}^{n} (x - \lambda_i) \\ &= (x - \lambda_\ell) Q_{n,\ell}^{\lambda^\ell, z^\ell}(x) \ - \ |z_\ell|^2 \prod_{\substack{i=1 \\ i \neq \ell}}^{n} (x - \lambda_i) \\ &= (x - \lambda_\ell) Q_{n,\ell}^{\lambda^\ell, z^\ell}(x). \end{aligned}$$

If $i \in \{1, ..., \ell-1, \ell+1, ..., n\}$, then $Q_{n,\ell}^{\lambda^\ell, z^\ell}(\beta_i) = 0$ and thus, $Q_n^{\lambda,z}(\beta_i) = 0$. Furthermore, $Q_n^{\lambda,z}(\beta_\ell) = 0$, as $\beta_\ell = \lambda_\ell$.
Therefore, $Q_n^{\lambda,z}(\beta_i) = 0$ for all $i \in \{1, ..., n\}$ and the claim is shown.

Next, let $\lambda_1 \geq \beta_1 \geq \lambda_2 \geq \beta_2 \geq ... \geq \lambda_n \geq \beta_n$.
Then, for any $z \in \mathbb{C}^n$, the characteristic polynomial of $\frac{1}{i} J_\lambda - zz^*$ is equal to $Q_n^{\lambda,z,-1}$.
If $\lambda_1 > \beta_1 > ... > \lambda_n > \beta_n$, let

$$|z_j|^2 := \frac{\prod_{i=1}^{n} (\lambda_j - \beta_i)}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_j - \lambda_i)}.$$

Here, $\mathrm{sgn}(|z_j|^2) = \frac{(-1)^{j-1}}{(-1)^{j-1}} = 1$ and hence, this definition is reasonable.
The rest of the proof is the same as in the first part of (2).

□

With these lemmas, one can now prove the following theorem which describes the topology of the space of admissible coadjoint orbits of G_n.

Theorem 5.4.
Let $\alpha \in \mathbb{R}^*$, $r > 0$, $\mu \in P_{n-1}$ and $\lambda \in P_n$. Then, the following holds:

1. *A sequence of coadjoint orbits $\left(\mathcal{O}_{(\mu^k, r_k)}\right)_{k \in \mathbb{N}}$ converges to $\mathcal{O}_{(\mu,r)}$ in $\mathfrak{g}_n^\ddagger / G_n$ if and only if $\lim\limits_{k \to \infty} r_k = r$ and $\mu^k = \mu$ for large k.*

2. *A sequence of coadjoint orbits* $\big(\mathcal{O}_{(\mu^k,r_k)}\big)_{k\in\mathbb{N}}$ *converges to* $\mathcal{O}_\lambda$ *in* $\mathfrak{g}_n^\ddagger/G_n$ *if and only if* $(r_k)_{k\in\mathbb{N}}$ *tends to* 0 *and* $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_{n-1} \geq \mu_{n-1}^k \geq \lambda_n$ *for* k *large enough.*

3. *A sequence of coadjoint orbits* $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *converges to the orbit* $\mathcal{O}_{(\lambda,\alpha)}$ *in* $\mathfrak{g}_n^\ddagger/G_n$ *if and only if* $\lim\limits_{k\to\infty}\alpha_k = \alpha$ *and* $\lambda^k = \lambda$ *for large* k.

4. *A sequence of coadjoint orbits* $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *converges to the orbit* $\mathcal{O}_{(\mu,r)}$ *in* $\mathfrak{g}_n^\ddagger/G_n$ *if and only if* $\lim\limits_{k\to\infty}\alpha_k = 0$ *and the sequence* $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *satisfies one of the following conditions:*

   (i) *For* k *large enough,* $\alpha_k > 0$, $\lambda_j^k = \mu_j$ *for all* $j \in \{1, ..., n-1\}$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_n^k = -\frac{r^2}{2}$.

   (ii) *For* k *large enough,* $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ *for all* $j \in \{2, ..., n\}$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = -\frac{r^2}{2}$.

5. *A sequence of coadjoint orbits* $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *converges to the orbit* $\mathcal{O}_\lambda$ *in* $\mathfrak{g}_n^\ddagger/G_n$ *if and only if* $\lim\limits_{k\to\infty}\alpha_k = 0$ *and the sequence* $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *satisfies one of the following conditions:*

   (i) *For* k *large enough,* $\alpha_k > 0$, $\lambda_1 \geq \lambda_1^k \geq ... \geq \lambda_{n-1} \geq \lambda_{n-1}^k \geq \lambda_n \geq \lambda_n^k$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_n^k = 0$.

   (ii) *For* k *large enough,* $\alpha_k < 0$, $\lambda_1^k \geq \lambda_1 \geq \lambda_2^k \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = 0$.

6. *A sequence of coadjoint orbits* $\big(\mathcal{O}_{\lambda^k}\big)_{k\in\mathbb{N}}$ *converges to the orbit* $\mathcal{O}_\lambda$ *in* $\mathfrak{g}_n^\ddagger/G_n$ *if and only if* $\lambda^k = \lambda$ *for large* k.

Proof:
Examining the shape of the coadjoint orbits listed at the beginning of this subsection, 3) and 6) are clear and Assertion 2) follows immediately from Lemma 5.2. Furthermore, the proof of 1) is similar to that of [12], Theorem 4.2.

4) Assume that $\big(\mathcal{O}_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_{(\mu,r)}$. Then, there exist a sequence $(A_k)_{k\in\mathbb{N}}$ in $U(n)$ and a sequence of vectors $\big(z(k)\big)_{k\in\mathbb{N}}$ in $\mathbb{C}^n$ such that

$$\lim_{k\to\infty}\bigg(A_k\Big(J_{\lambda^k} + \frac{i}{\alpha_k}z(k)z(k)^*\Big)A_k^*, \sqrt{2}A_k z(k), \alpha_k\bigg) = (J_\mu, v_r, 0).$$

Let $A = (a_{mj})_{1\leq m,j\leq n}$ be the limit of a subsequence $(A_s)_{s\in I}$ for $I \subset \mathbb{N}$. Then,

$$\lim_{s\to\infty} J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^* = A^*J_\mu A, \quad \lim_{s\to\infty} z_j(s) = \frac{r}{\sqrt{2}}\,\overline{a}_{nj} \text{ for } j \in \{1, ..., n\} \quad \text{and} \quad \lim_{s\to\infty}\alpha_s = 0.$$

On the other hand, one has $(A^*J_\mu A)_{mj} = i\sum\limits_{l=1}^{n-1}\mu_l\overline{a}_{lm}a_{lj}$ and

$$J_{\lambda^s} + \frac{i}{\alpha_s}z(s)z(s)^* = \begin{pmatrix} i\lambda_1^s + i\frac{|z_1(s)|^2}{\alpha_s} & i\frac{z_1(s)\overline{z}_2(s)}{\alpha_s} & \dots & i\frac{z_1(s)\overline{z}_n(s)}{\alpha_s} \\ i\frac{z_2(s)\overline{z}_1(s)}{\alpha_s} & i\lambda_2^s + i\frac{|z_2(s)|^2}{\alpha_s} & \dots & i\frac{z_2(s)\overline{z}_n(s)}{\alpha_s} \\ \vdots & \vdots & \ddots & \vdots \\ i\frac{z_n(s)\overline{z}_1(s)}{\alpha_s} & i\frac{z_n(s)\overline{z}_2(s)}{\alpha_s} & \dots & i\lambda_n^s + i\frac{|z_n(s)|^2}{\alpha_s} \end{pmatrix}.$$

Hence, for $m \neq j$, $\lim_{s\to\infty} \left|\frac{z_m(s)\overline{z}_j(s)}{\alpha_s}\right| = \left|\sum_{l=1}^{n-1} \mu_l \overline{a}_{lm} a_{lj}\right| < \infty$, and since $\lim_{s\to\infty} \|z(s)\|_{\mathbb{C}^n} = \frac{r}{\sqrt{2}} \neq 0$, there is a unique $i_0 \in \{1, ..., n\}$ such that $\lim_{s\to\infty} z_{i_0}(s) = \frac{r}{\sqrt{2}} e^{i\theta}$ for a certain number $\theta \in \mathbb{R}$ and $\lim_{s\to\infty} z_j(s) = 0$ for $j \neq i_0$. One obtains $a_{ni_0} = e^{-i\theta}$ and $a_{nj} = 0$ for $j \neq i_0$, i.e. the matrices A and $A^* J_\mu A$ can be written in the following way:

$$A = \begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & \underbrace{e^{-i\theta}}_{i_0-\text{th position}} & 0 & \cdots & 0 \end{pmatrix} \quad \text{and}$$

$$A^* J_\mu A = \begin{pmatrix} * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & \underbrace{0}_{i_0-\text{th position}} & * & \cdots & * \end{pmatrix} \}i_0 - \text{th position}$$

since $\overline{(A^* J_\mu A)}_{i_0 j} = -(A^* J_\mu A)_{j i_0} = -i \sum_{l=1}^{n-1} \mu_l \overline{a}_{lj} a_{li_0} = 0$ for $j \in \{1, ..., n\}$. It follows that $\lim_{s\to\infty} \lambda^s_{i_0} + \frac{|z_{i_0}(s)|^2}{\alpha_s} = 0$, which in turn implies that $\lim_{s\to\infty} \left|\lambda^s_{i_0}\right| = \infty$ and that for each $j \neq i_0$

$$\lim_{s\to\infty} \lambda^s_j = \sum_{l=1}^{n-1} \mu_l |a_{lj}|^2, \quad \lim_{s\to\infty} \frac{z_j(s)\overline{z}_{i_0}(s)}{\alpha_s} = 0, \quad \lim_{s\to\infty} \frac{z_j(s)}{\alpha_s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{|z_j(s)|^2}{\alpha_s} = 0.$$

This proves that i_0 can only take the value 1 if $\alpha_s < 0$ and n if $\alpha_s > 0$. Otherwise, since $\lambda^s_{i_0-1} \geq \lambda^s_{i_0} \geq \lambda^s_{i_0+1}$, one gets $\lim_{s\to\infty} \lambda^s_{i_0-1} = \infty$ if $\alpha_s < 0$ and $\lim_{s\to\infty} \lambda^s_{i_0+1} = -\infty$ if $\alpha_s > 0$ which contradicts the fact that $\lim_{s\to\infty} \lambda^s_j$ is finite for all $j \neq i_0$.

Case $i_0 = n$:

In this case, one has $\lim_{s\to\infty} \alpha_s \lambda^s_n = -\frac{r^2}{2}$ and $\lim_{s\to\infty} \lambda^s_j = \sum_{l=1}^{n-1} \mu_l |a_{lj}|^2$ for all $j \in \{1, ..., n-1\}$. Furthermore, the matrices A and $A^* J_\mu A$ have the form

$$A = \begin{pmatrix} & & & 0 \\ & \tilde{A} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & e^{-i\theta} \end{pmatrix} \quad \text{and} \quad A^* J_\mu A = \begin{pmatrix} * & \ldots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \ldots & * & 0 \\ 0 & \ldots & 0 & 0 \end{pmatrix},$$

where $\tilde{A} \in U(n-1)$. However, the limit matrix of the subsequence $\left(J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^*\right)_{s\in I}$ has to be diagonal because $\lim\limits_{s\to\infty} \frac{z_m(s)\bar{z}_j(s)}{\alpha_s} = 0$ for all $m \neq j$. This implies that

$$A^* J_\mu A = \begin{pmatrix} i\mu_1 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & i\mu_{n-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

and consequently, $\lambda_j^s = \mu_j$ for large s and $j \in \{1, ..., n-1\}$.

Case $i_0 = 1$:
In this case, $\lim\limits_{s\to\infty} \alpha_s \lambda_1^s = -\frac{r^2}{2}$ and $\lim\limits_{s\to\infty} \lambda_j^s = \sum\limits_{l=1}^{n-1} \mu_l |a_{lj}|^2$ for every $j \in \{2, ..., n\}$. Moreover, there is an element $\tilde{A} \in U(n-1)$ such that the matrix A is given by

$$A = \begin{pmatrix} 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \\ e^{-i\theta} & 0 & \cdots & 0 \end{pmatrix} \quad \text{and hence,} \quad A^* J_\mu A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

Using the same arguments as above, one has $\lambda_{j+1}^s = \mu_j$ for s large enough and for every $j \in \{1, ..., n-1\}$.

Conversely, suppose that $\lim\limits_{k\to\infty} \alpha_k = 0$. If the regarded sequence of orbits satisfies the first condition, one can take $z(k) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{-\alpha_k \lambda_n^k} \end{pmatrix}$ and $A_k := \mathbb{I}$ for $k \geq N$ and $N \in \mathbb{N}$ large enough. In the other case, one lets

$$z(k) := \begin{pmatrix} \sqrt{-\alpha_k \lambda_1^k} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad A_k := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{for } k \geq N.$$

Thus, $\lim\limits_{k\to\infty} \left(A_k \left(J_{\lambda^k} + \frac{i}{\alpha_k} z(k)z(k)^* \right) A_k^*, \sqrt{2} A_k z(k), \alpha_k \right) = (J_\mu, v_r, 0)$.

5) Suppose that $\left(\mathcal{O}_{(\lambda^k, \alpha_k)}\right)_{k\in\mathbb{N}}$ converges to the orbit $\mathcal{O}_\lambda$. Then, there exist a sequence $(A_k)_{k\in\mathbb{N}}$ in $U(n)$ and a sequence $\left(z(k)\right)_{k\in\mathbb{N}}$ in $\mathbb{C}^n$ such that

$$\lim_{k\to\infty} \left(A_k \left(J_{\lambda^k} + \frac{i}{\alpha_k} z(k)z(k)^* \right) A_k^*, \sqrt{2} A_k z(k), \alpha_k \right) = (J_\lambda, 0, 0).$$

It follows that $\lim_{k\to\infty} \alpha_k = 0$ and that $\big(z(k)\big)_{k\in\mathbb{N}}$ tends to 0 in $\mathbb{C}^n$. Denote by $A = (a_{mj})_{1\le m,j\le n}$ the limit matrix of a subsequence $(A_s)_{s\in I}$ for an index set $I \subset \mathbb{N}$. Then,

$$\lim_{s\to\infty} J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^* = A^* J_\lambda A \quad \text{with} \quad (A^* J_\lambda A)_{mj} = i \sum_{l=1}^{n} \lambda_l \overline{a}_{lm} a_{lj}.$$

Since $\lim_{k\to\infty} \alpha_k = 0$, one can assume that α_s is either strictly positive for all $s \in I$ or strictly negative for all $s \in I$.
Let $\sqrt{|\alpha_s|}$ be the square root of $|\alpha_s|$. The fact that $\lim_{s\to\infty} \frac{z_m(s)\overline{z}_j(s)}{\alpha_s}$ is finite for all $m, j \in \{1, ..., n\}$ implies that there exists at most one integer $1 \le i_0 \le n$ such that $\lim_{s\to\infty} \frac{z_{i_0}(s)}{\sqrt{|\alpha_s|}} = \infty$. Therefore,

$$\lim_{s\to\infty} \frac{z_j(s)}{\sqrt{|\alpha_s|}}$$

exists for all j distinct from i_0. Hence, for the same reasons as in the proof of 4), necessarily $i_0 \in \{1, n\}$.

If there is no such i_0, then there exists for all $j \in \{1, ..., n\}$ an integer $\lambda'_j \in \mathbb{Z}$ such that $\lambda'_j = \lambda^s_j$ for all $s \in I$ (by passing to a subsequence if necessary) and $\tilde{z}_j := \lim_{s\to\infty} \frac{z_j(s)}{\sqrt{|\alpha_s|}}$ is finite for all $j \in \{1, \dots, n\}$. Thus, one gets

$$\begin{cases} A^* J_\lambda A = J_{\lambda'} + i\tilde{z}(\tilde{z})^*, & \text{if } \alpha_s > 0 \ \forall s \in I \\ A^* J_\lambda A = J_{\lambda'} - i\tilde{z}(\tilde{z})^*, & \text{if } \alpha_s < 0 \ \forall s \in I. \end{cases}$$

It follows by Lemma 5.3 applied to $\tilde{z}$ and $\alpha = 1$ or $\alpha = -1$, respectively, that

$$\begin{cases} \lambda_1 \ge \lambda'_1 = \lambda^s_1 \ge \lambda_2 \ge \lambda'_2 = \lambda^s_2 \ge ... \ge \lambda_n \ge \lambda'_n = \lambda^s_n, & \text{if } \alpha_s > 0 \ \forall s \in I \\ \lambda'_1 = \lambda^s_1 \ge \lambda_1 \ge \lambda'_2 = \lambda^s_2 \ge \lambda_2 \ge ... \ge \lambda'_n = \lambda^s_n \ge \lambda_n, & \text{if } \alpha_s < 0 \ \forall s \in I. \end{cases}$$

Case $i_0 = n$:
In this case, $\lim_{s\to\infty} \alpha_s \lambda^s_n = 0$, as $\lim_{s\to\infty} \big|\lambda^s_n + \frac{|z_n(s)|^2}{\alpha_s}\big| < \infty$. Furthermore, $\lim_{s\to\infty} \lambda^s_n = -\infty$ and $\lim_{s\to\infty} \lambda^s_j = \sum_{l=1}^{n} \lambda_l |a_{lj}|^2$ for every $j \in \{1, ..., n-1\}$ and α_s has to be positive for large s. Since $\lim_{s\to\infty} \frac{z_j(s)}{\alpha_s}$ exists and $\lim_{s\to\infty} \frac{|z_n(s)|}{\alpha_s} = \infty$, it follows that $\lim_{s\to\infty} \frac{z_j(s)}{\alpha_s} = 0$ for all $j \in \{1, ..., n-1\}$.
Now, choose

$$x := \lim_{s\to\infty} \lambda^s_n + \frac{|z_n(s)|^2}{\alpha_s}, \quad \lambda'_j := \lim_{s\to\infty} \lambda^s_j \quad \text{and} \quad w_j := -i \lim_{s\to\infty} \frac{z_j(s)\overline{z}_n(s)}{\alpha_s} \quad \forall j \in \{1, ..., n-1\}.$$

Then, the limit matrix $A^* J_\lambda A$ of the sequence $\big(J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^*\big)_{s\in I}$ has the form

$$\begin{pmatrix} i\lambda'_1 & 0 & \dots & 0 & -w_1 \\ 0 & i\lambda'_2 & \dots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \dots & \overline{w}_{n-1} & ix \end{pmatrix}.$$

By Lemma 5.2, one obtains $\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq ... \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n$, and thus, $\lambda_1 \geq \lambda_1^s \geq \lambda_2 \geq \lambda_2^s \geq ... \geq \lambda_{n-1} \geq \lambda_{n-1}^s \geq \lambda_n \geq \lambda_n^s$ for large s.

Case $i_0 = 1$:
Here, $\lim_{s\to\infty} \alpha_s \lambda_1^s = 0$, since $\lim_{s\to\infty} \left|\lambda_1^s + \frac{|z_1(s)|^2}{\alpha_s}\right| < \infty$. Moreover,

$$\lim_{s\to\infty} \lambda_1^s = \infty, \quad \lim_{s\to\infty} \lambda_j^s = \sum_{l=1}^{n} \lambda_l |a_{lj}|^2 \quad \text{and} \quad \lim_{s\to\infty} \frac{z_j(s)}{\alpha_s} = 0 \quad \forall j \in \{2, ..., n\}.$$

Hence, $\alpha_s < 0$ for s large enough. If one sets

$$x := \lim_{s\to\infty} \lambda_1^s + \frac{|z_1(s)|^2}{\alpha_s}, \quad \lambda'_j := \lim_{s\to\infty} \lambda_{j+1}^s \quad \text{and} \quad w_j := -i \lim_{s\to\infty} \frac{\overline{z}_1(s) z_{j+1}(s)}{\alpha_s} \quad \forall j \in \{1, ..., n-1\},$$

the limit matrix $A^* J_\lambda A$ of $\left(J_{\lambda^s} + \frac{i}{\alpha_s} z(s) z(s)\right)_{s\in I}$ can be written as follows:

$$\begin{pmatrix} ix & \overline{w}_1 & \overline{w}_2 & \dots & \overline{w}_{n-1} \\ -w_1 & i\lambda'_1 & 0 & \dots & 0 \\ -w_2 & 0 & i\lambda'_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n-1} & 0 & 0 & \dots & i\lambda'_{n-1} \end{pmatrix} = \tilde{A}^* \begin{pmatrix} i\lambda'_1 & 0 & \dots & 0 & -w_1 \\ 0 & i\lambda'_2 & \dots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \dots & \overline{w}_{n-1} & ix \end{pmatrix} \tilde{A}, \tag{40}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This proves that $\lambda_1^s \geq \lambda_1 \geq \lambda_2^s \geq \lambda_2 \geq ... \geq \lambda_{n-1}^s \geq \lambda_{n-1} \geq \lambda_n^s \geq \lambda_n$ for large s.

Conversely, suppose that the sequence $\left(\mathcal{O}_{(\lambda^k, \alpha_k)}\right)_{k\in\mathbb{N}}$ satisfies the first condition.
First, consider the case $\lambda_n^k \overset{k\to\infty}{\longrightarrow} -\infty$.
Then, there is a subsequence $(\lambda^s)_{s\in I}$ for an index set $I \subset \mathbb{N}$ fulfilling $\lambda_j^s = \lambda'_j$ for every $j \in \{1, ..., n-1\}$ and all $s \in I$. By Lemma 5.2, there exist $w_1, w_2, \dots, w_{n-1} \in \mathbb{C}$, $x \in \mathbb{R}$ and $A \in U(n)$ such that

$$A^* J_\lambda A = \begin{pmatrix} i\lambda'_1 & 0 & \dots & 0 & -w_1 \\ 0 & i\lambda'_2 & \dots & 0 & -w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & i\lambda'_{n-1} & -w_{n-1} \\ \overline{w}_1 & \overline{w}_2 & \dots & \overline{w}_{n-1} & ix \end{pmatrix}.$$

In this case, $\lambda^k \neq \lambda$ for large k, as $\lambda_n^k \overset{k\to\infty}{\longrightarrow} -\infty$. Choose $x := \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n-1} \lambda'_j$. (Compare the proof of Lemma 5.2.) It follows that

$$\alpha_s(x - \lambda_n^s) = \sum_{j=1}^{n} \alpha_s(\lambda_j - \lambda_j^s) > 0.$$

Furthermore, define the sequence $\big(z(s)\big)_{s\in I}$ in $\mathbb{C}^n$ by

$$z_n(s) := \sqrt{\alpha_s(x-\lambda_n^s)} \quad \text{and} \quad z_j(s) := i\frac{\alpha_s w_j}{\sqrt{\alpha_s(x-\lambda_n^s)}} \quad \forall j \in \{1,2,...,n-1\}.$$

Then, one gets

$$\begin{aligned}
\lim_{s\to\infty} z(s) &= 0,\\
\lambda_n^s + \frac{|z_n(s)|^2}{\alpha_s} &= x,\\
\lim_{s\to\infty}\frac{|z_j(s)|^2}{\alpha_s} &= \lim_{s\to\infty}\frac{|w_j|^2}{x-\lambda_n^s} = 0 \quad \forall j\in\{1,...,n-1\},\\
\lim_{s\to\infty}\frac{z_m(s)\overline{z_j(s)}}{\alpha_s} &= \lim_{s\to\infty}\frac{w_m\overline{w}_j}{x-\lambda_n^s} = 0 \quad \forall m\neq j\in\{1,...,n-1\} \quad \text{and}\\
\lim_{s\to\infty}\frac{z_j(s)\overline{z_n(s)}}{\alpha_s} &= iw_j \quad \forall j\in\{1,...,n-1\}.
\end{aligned}$$

Hence, $\Big(A\big(J_{\lambda^s}+\frac{i}{\alpha_s}z(s)z(s)^*\big)A^*\Big)_{s\in I}$ converges to J_λ and $\big(z(s)\big)_{s\in I}$ to 0.

If $\lim\limits_{k\to\infty}\lambda_n^k \neq -\infty$, there is a subsequence $(\lambda^s)_{s\in I}$ for an index set $I\subset\mathbb{N}$ fulfilling $\lambda_j^s = \lambda'_j$ for all $j\in\{1,...,n\}$ and all $s\in I$. Therefore,

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq ... \geq \lambda_n \geq \lambda'_n$$

and thus, by Lemma 5.3(2), there exists $\tilde{z}\in\mathbb{C}^n$ such that $i\lambda_1,...,i\lambda_n$ are the eigenvalues of $J_{\lambda'}+i\tilde{z}(\tilde{z})^*$.
Let $z(s) := \tilde{z}\sqrt{\alpha_s}$, which is reasonable since $\alpha_s>0$ in this case.
As the matrices $J_{\lambda'}+i\tilde{z}(\tilde{z})^*$ and J_λ are both skew-Hermitian and have the same eigenvalues, they are unitarily conjugated. Therefore, there exists an element $A\in U(n)$ in such a way that $J_{\lambda'}+i\tilde{z}(\tilde{z})^* = A^*J_\lambda A$. Hence,

$$A^*J_\lambda A = J_{\lambda'}+i\tilde{z}(\tilde{z})^* = \lim_{s\to\infty} J_{\lambda'}+i\tilde{z}(\tilde{z})^* = \lim_{s\to\infty} J_{\lambda'}+i\frac{z(s)z(s)^*}{\alpha_s},$$

i.e. $\Big(A\big(J_{\lambda'}+i\frac{z(s)z(s)^*}{\alpha_s}\big)A^*\Big)_{s\in I}$ converges to J_λ. Furthermore,

$$z(s) = \tilde{z}\sqrt{\alpha_s} \overset{s\to\infty}{\longrightarrow} 0,$$

as $\alpha_k \overset{k\to\infty}{\longrightarrow} 0$.
Thus, the claim is shown in this case.

Suppose now that for k large enough $\alpha_k<0$, $\lambda_1^k \geq \lambda_1 \geq ... \geq \lambda_{n-1}^k \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ and $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = 0$. First consider the case $\alpha_1^k \overset{k\to\infty}{\longrightarrow} \infty$.
In this case, there is a subsequence $(\lambda^s)_{s\in I}$ for an index set $I\subset\mathbb{N}$ such that $\lambda_j^s = \lambda'_{j-1}$ for all

$j \in \{2, ..., n\}$ and all $s \in I$. By Identity (40) and Lemma 5.2 , there exist $w_1, w_2, \ldots, w_{n-1} \in \mathbb{C}$, $x \in \mathbb{R}$ and $A \in U(n)$ such that

$$A^* J_\lambda A = \begin{pmatrix} ix & \overline{w}_1 & \overline{w}_2 & \ldots & \overline{w}_{n-1} \\ -w_1 & i\lambda'_1 & 0 & \ldots & 0 \\ -w_2 & 0 & i\lambda'_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -w_{n-1} & 0 & 0 & \ldots & i\lambda'_{n-1} \end{pmatrix}.$$

Similarly to the last case, one takes $x := \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{n-1} \lambda'_j$ and thus gets

$$\alpha_s(x - \lambda_1^s) = \sum_{j=1}^{n} \alpha_s(\lambda_j - \lambda_j^s) > 0.$$

Hence, one can define the sequence $\big(z(s)\big)_{s \in I}$ in $\mathbb{C}^n$ by

$$z_1(s) := \sqrt{\alpha_s(x - \lambda_1^s)} \quad \text{and} \quad z_j(s) := -i\frac{\alpha_s w_{j-1}}{\sqrt{\alpha_s(x - \lambda_1^s)}} \quad \forall j \in \{2, ..., n\}.$$

Here again, one gets

$$\begin{aligned} \lim_{s\to\infty} z(s) &= 0, \\ \lambda_1^s + \frac{|z_1(s)|^2}{\alpha_s} &= x, \\ \lim_{s\to\infty} \frac{|z_j(s)|^2}{\alpha_s} &= \lim_{s\to\infty} \frac{|w_{j-1}|^2}{x - \lambda_1^s} = 0 \quad \forall j \in \{2, ..., n\}, \\ \lim_{s\to\infty} \frac{z_m(s)\overline{z_j(s)}}{\alpha_s} &= \lim_{s\to\infty} \frac{w_{m-1}\overline{w}_{j-1}}{x - \lambda_1^s} = 0 \quad \forall m \neq j \in \{2, ..., n\} \quad \text{and} \\ \lim_{s\to\infty} \frac{z_j(s)\overline{z_1(s)}}{\alpha_s} &= iw_{j-1} \quad \forall j \in \{2, ..., n\}. \end{aligned}$$

Again, one can conclude that $\left(\left(A\big(J_{\lambda^s} + \frac{i}{\alpha_s} z(s)z(s)^*\big)A^*, \sqrt{2}Az(s), \alpha_s\right)\right)_{s \in I}$ converges to $(J_\lambda, 0, 0)$.

If $\lim_{k\to\infty} \lambda_1^k \neq \infty$, there is a subsequence $(\lambda^s)_{s \in I}$ for an index set $I \subset \mathbb{N}$ fulfilling $\lambda_j^s = \lambda'_j$ for all $j \in \{1, ..., n\}$ and all $s \in I$. Hence,

$$\lambda'_1 \geq \lambda_1 \geq \lambda'_2 \geq \lambda_2 \geq ... \geq \lambda'_n \geq \lambda_n$$

and therefore, by Lemma 5.3(2), there exists $\tilde{z} \in \mathbb{C}^n$ such that $i\lambda_1, ..., i\lambda_n$ are the eigenvalues of $J_{\lambda'} - i\tilde{z}(\tilde{z})^*$.
Let now $z(s) := \tilde{z}\sqrt{-\alpha_s}$, which is reasonable since this time $\alpha_s < 0$.
As above, there exists an element $A \in U(n)$ such that $J_{\lambda'} - i\tilde{z}(\tilde{z})^* = A^* J_\lambda A$ and thus,

$$A^* J_\lambda A = \lim_{s\to\infty} J_{\lambda'} - i\frac{z(s)z(s)^*}{-\alpha_s} = \lim_{s\to\infty} J_{\lambda'} + i\frac{z(s)z(s)^*}{\alpha_s},$$

i.e. $\left(A\left(J_{\lambda'}+i\frac{z(s)z(s)^*}{\alpha_s}\right)A^*\right)_{s\in I}$ converges to J_λ. Furthermore,

$$z(s)=\tilde{z}\sqrt{-\alpha_s}\overset{s\to\infty}{\longrightarrow}0,$$

as $\alpha_k\overset{k\to\infty}{\longrightarrow}0$.
Therefore, the assertion is also shown in this case.

□

5.3 The topology of the spectrum of G_n

In this subsection, the topology of $G_n=U(n)\ltimes\mathbb{H}_n$ will be analyzed. The aim is to show that it is determined by the topology of its admissible quotient space.
For this, some results on the topology of the spectrum of the semidirect product $U(n)\ltimes\mathbb{H}_n$ in terms of the Mackey data will be given.

5.3.1 The representation $\pi_{(\mu,r)}$

First, examine the representation $\pi_{(\mu,r)}=\mathrm{ind}_{U(n-1)\ltimes\mathbb{H}_n}^{G_n}\rho_\mu\otimes\chi_r$. Its Hilbert space $\mathcal{H}_{(\mu,r)}$ is given by the space

$$L^2\Big(G_n/\big(U(n-1)\ltimes\mathbb{H}_n\big),\rho_\mu\otimes\chi_r\Big)\cong L^2\big(U(n)/U(n-1),\rho_\mu\big).$$

Let ξ be a unit vector in $\mathcal{H}_{(\mu,r)}$. For all $(z,t)\in\mathbb{H}_n$, and all $A,B\in U(n)$,

$$\pi_{(\mu,r)}(A,z,t)\xi(B)=e^{-i(Bv_r,z)_{\mathbb{C}^n}}\xi\big(A^{-1}B\big).$$

Therefore,

$$\begin{aligned}C_\xi^{\pi_{(\mu,r)}}(A,z,t)&=\Big\langle\pi_{(\mu,r)}(A,z,t)\xi,\xi\Big\rangle_{L^2\big(U(n)/U(n-1),\rho_\mu\big)}\\&=\int\limits_{U(n)}e^{-i(Bv_r,z)_{\mathbb{C}^n}}\Big\langle\xi\big(A^{-1}B\big),\xi(B)\Big\rangle_{\mathcal{H}_{\rho_\mu}}dB.\end{aligned}$$

By (39) in Section 5.1.2, one has

$$\pi_\mu:=\pi_{(\mu,r)|U(n)}\cong\mathrm{ind}_{U(n-1)}^{U(n)}\rho_\mu=\sum_{\substack{\tau_\lambda\in\widehat{U(n)}\\ \lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\ldots\geq\lambda_{n-1}\geq\mu_{n-1}\geq\lambda_n}}\tau_\lambda.$$

Every irreducible representation τ_λ of $U(n)$ can be realized as a subrepresentation of the left regular representation on $L^2\big(U(n)\big)$ via the intertwining operator

$$U_\lambda:\mathcal{H}_\lambda\to L^2\big(U(n)\big),\ U_\lambda(\xi)(A):=\big\langle\xi,\tau_\lambda(A)\xi_\lambda\big\rangle_{\mathcal{H}_\lambda}\quad\forall A\in U(n)\ \forall\xi\in\mathcal{H}_\lambda$$

for a fixed unit vector $\xi_\lambda\in\mathcal{H}_\lambda$.

For $\tau_\lambda\in\widehat{U(n)}$, consider the orthonormal basis $\mathcal{B}^\lambda=\big\{\phi_j^\lambda|\ j\in\{1,...,d_\lambda\}\big\}$ of $\mathcal{H}_\lambda$ consisting of eigenvectors for $\mathbb{T}_n$ of $\mathcal{H}_\lambda$.

Moreover, as a basis of the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group, one can take the left invariant vector fields $\{Z_1, Z_2, \ldots, Z_n, \overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_n, T\}$, where

$$Z_j := 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \quad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t} \quad \text{and} \quad T := \frac{\partial}{\partial t}$$

and gets the Lie brackets $[Z_j, \overline{Z}_j] = -2iT$ for $j \in \{1, ..., n\}$.
Now, regard the Heisenberg sub-Laplacian differential operator which is given by

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_jZ_j\right).$$

This operator is $U(n)$-invariant.

Lemma 5.5.
For every representation $\pi_{(\mu,r)}$ for $r > 0$ and $\rho_\mu \in \widehat{U(n-1)}$,

$$d\pi_{(\mu,r)}(\mathcal{L}) = -r^2\mathbb{I}.$$

Proof:
Since the representation $\pi_{(\mu,r)}$ is trivial on the center of $\mathfrak{h}_n$, one has

$$d\pi_{(\mu,r)}(\mathcal{L})\xi(B) = 2\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial z_j\partial\overline{z}_j} + \frac{\partial^2}{\partial\overline{z}_j\partial z_j}\right)\left(e^{-i(Bv_r,z)_{\mathbb{C}^n}}\right)\xi(B).$$

Let $\mathcal{D} = \{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{C}^n$. By writing

$$(Bv_r, z)_{\mathbb{C}^n} = \frac{1}{2}\left(\langle Bv_r, z\rangle_{\mathbb{C}^n} + \overline{\langle Bv_r, z\rangle}_{\mathbb{C}^n}\right),$$

one gets

$$d\pi_{(\mu,r)}(\mathcal{L})\xi(B) = -\sum_{j=1}^{n}\left|\langle Bv_r, e_j\rangle_{\mathbb{C}^n}\right|^2\xi(B) = -r^2\xi(B).$$

□

In addition, the following theorem describes the convergence of sequences of representations $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$:

Theorem 5.6.
Let $r > 0$, $\rho_\mu \in \widehat{U(n-1)}$ and $\tau_\lambda \in \widehat{U(n)}$.

1. *A sequence $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ of irreducible unitary representations of G_n converges to $\pi_{(\mu,r)}$ in $\widehat{G}_n$ if and only if $\lim_{k\to\infty} r_k = r$ and $\mu^k = \mu$ for k large enough.*
2. *A sequence $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ of irreducible unitary representations of G_n converges to τ_λ in $\widehat{G}_n$ if and only if $\lim_{k\to\infty} r_k = 0$ and τ_λ occurs in π_{μ^k} for k large enough.*

These are all possibilities for a sequence $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ of irreducible unitary representations of G_n to converge.

The proof of 1) and 2) of this theorem can be found in [1], Theorem 6.2.A.
Furthermore, since the representations $\pi_{(\mu,r)}$ and τ_λ are trivial on $\{(\mathbb{I}, 0, t)|\ t \in \mathbb{R}\}$, the center of G_n, while the representations $\pi_{(\lambda,\alpha)}$ are non-trivial there, the possibilities of convergence of a sequence $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ listed above are the only ones that are possible.

5.3.2 The representation $\pi_{(\lambda,\alpha)}$

Next, regard the representations $\pi_{(\lambda,\alpha)}$.

Consider the unit vector $\xi := \sum_{j=1}^{d_\lambda} \phi_j^\lambda \otimes f_j$ in the Hilbert space $\mathcal{H}_{(\lambda,\alpha)} = \mathcal{H}_\lambda \otimes \mathcal{F}_\alpha(n)$ of $\pi_{(\lambda,\alpha)}$, where $f_1, \ldots, f_{d_\lambda}$ belong to the Fock space $\mathcal{F}_\alpha(n)$. Then, for all $A \in U(n)$ and $(z,t) \in \mathbb{H}_n$,

$$\pi_{(\lambda,\alpha)}(A,z,t)\xi(w) = \sum_{j=1}^{d_\lambda} \tau_\lambda(A)\phi_j^\lambda \otimes e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w,z\rangle_{\mathbb{C}^n}} f_j\big(A^{-1}w + A^{-1}z\big) \quad \text{if } \alpha > 0 \quad \text{and}$$

$$\pi_{(\lambda,\alpha)}(A,z,t)\xi(\overline{w}) = \sum_{j=1}^{d_\lambda} \tau_\lambda(A)\phi_j^\lambda \otimes e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w},\overline{z}\rangle_{\mathbb{C}^n}} f_j\Big(\overline{A^{-1}w} + \overline{A^{-1}z}\Big) \quad \text{if } \alpha < 0.$$

It follows that

$$C_\xi^{\pi_{(\lambda,\alpha)}}(A,z,t) = \big\langle \pi_{(\lambda,\alpha)}(A,z,t)\xi, \xi \big\rangle_{\mathcal{H}_{(\lambda,\alpha)}} =$$

$$\begin{cases} \sum\limits_{j,j'=1}^{d_\lambda} \big\langle \tau_\lambda(A)\phi_j^\lambda, \phi_{j'}^\lambda \big\rangle_{\mathcal{H}_\lambda} \int\limits_{\mathbb{C}^n} e^{i\alpha t - \frac{\alpha}{4}|z|^2 - \frac{\alpha}{2}\langle w,z\rangle_{\mathbb{C}^n}} f_j\big(A^{-1}w + A^{-1}z\big)\overline{f_{j'}(w)} e^{-\frac{\alpha}{2}|w|^2} dw \text{ if } \alpha > 0, \\ \sum\limits_{j,j'=1}^{d_\lambda} \big\langle \tau_\lambda(A)\phi_j^\lambda, \phi_{j'}^\lambda \big\rangle_{\mathcal{H}_\lambda} \int\limits_{\mathbb{C}^n} e^{i\alpha t + \frac{\alpha}{4}|z|^2 + \frac{\alpha}{2}\langle \overline{w},\overline{z}\rangle_{\mathbb{C}^n}} f_j\Big(\overline{A^{-1}w} + \overline{A^{-1}z}\Big)\overline{f_{j'}(\overline{w})} e^{\frac{\alpha}{2}|w|^2} dw \text{ if } \alpha < 0. \end{cases}$$

Lemma 5.7.
For each representation $\pi_{(\lambda,\alpha)}$ for $\alpha \in \mathbb{R}^*$ and $\tau_\lambda \in \widehat{U(n)}$, one has

$$d\pi_{(\lambda,\alpha)}(T) = i\alpha\mathbb{I}.$$

Proof:
Let $\xi = \sum_{j=1}^{d_\lambda} \phi_j^\lambda \otimes f_j$ be a unit vector in $\mathcal{H}_{(\lambda,\alpha)}$. Then,

$$\big\langle d\pi_{(\lambda,\alpha)}(T)\xi, \xi \big\rangle_{\mathcal{H}_{(\lambda,\alpha)}} = \frac{d}{dt}\Big|_{t=0} \big\langle \pi_{(\lambda,\alpha)}(\mathbb{I},0,t)\xi, \xi \big\rangle_{\mathcal{H}_{(\lambda,\alpha)}} = \frac{d}{dt}\Big|_{t=0} e^{i\alpha t} \sum_{j=1}^{d_\lambda} \|f_j\|^2_{\mathcal{F}_\alpha(n)} = i\alpha.$$

□

If α is positive, the polynomials $\mathbb{C}[\mathbb{C}^n]$ are dense in $\mathcal{F}_\alpha(n)$ and its multiplicity free decomposition is

$$\mathbb{C}[\mathbb{C}^n] = \sum_{m=0}^{\infty} \mathcal{P}_m,$$

where $\mathcal{P}_m$ is the space of homogeneous polynomials of degree m. Thus, $p_m(z) = z_1^m$ is the highest weight vector in $\mathcal{P}_m$ with weight $(m,0,...,0) =: [m]$. Applying the classical Pieri's rule (see [15], Proposition 15.25), one obtains

$$\big(\tau_\lambda \otimes W_\alpha\big)_{|U(n)} = \sum_{m=0}^{\infty} \tau_\lambda \otimes \tau_{[m]} = \sum_{\substack{\lambda' \in P_n \\ \lambda_1' \geq \lambda_1 \geq \ldots \geq \lambda_n' \geq \lambda_n}} \tau_{\lambda'}. \tag{41}$$

If α is negative, one gets

$$\left(\tau_\lambda \otimes W_\alpha\right)_{|U(n)} = \sum_{m=0}^{\infty} \tau_\lambda \otimes \tau_{[m]} = \sum_{\substack{\lambda' \in P_n \\ \lambda_1 \geq \lambda'_1 \geq \ldots \geq \lambda_n \geq \lambda'_n}} \tau_{\lambda'}.$$

Both of the sums again are multiplicity free. This follows from [20], Chapter IV.11, since W_α is multiplicity free.
Furthermore, let $\mathcal{R}_\alpha := \left\{h_{m,\alpha} | \ m = (m_1, \ldots, m_n) \in \mathbb{N}^n\right\}$ be the orthonormal basis of the Fock space $\mathcal{F}_\alpha(n)$ defined by the Hermite functions

$$h_{m,\alpha}(z) = \left(\frac{|\alpha|}{2\pi}\right)^{\frac{n}{2}} \sqrt{\frac{|\alpha|^{|m|}}{2^{|m|} m!}} \, z^m$$

with $|m| = m_1 + ... + m_n$, $m! = m_1! \cdots m_n!$ and $z^m = z_1^{m_1} \cdots z_n^{m_n}$ (see [14], Chapter 1.7).

Now, one obtains the following theorem about the convergence of sequences of representations $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$:

Theorem 5.8.
Let $\alpha \in \mathbb{R}^*$ and $\tau_\lambda \in \widehat{U(n)}$. Then, a sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ of elements in $\widehat{G_n}$ converges to $\pi_{(\lambda,\alpha)}$ if and only if $\lim\limits_{k\to\infty} \alpha_k = \alpha$ and $\lambda^k = \lambda$ for large k.

Proof:
First, consider the case where α is positive. Assume that $\alpha_k \stackrel{k\to\infty}{\longrightarrow} \alpha$ and that $\lambda^k = \lambda$ for k large enough. Moreover, let $f \in C_0^\infty(G_n)$ and let ξ be a unit vector in $\mathcal{H}_\lambda$. Then,

$$\begin{aligned}&\left\langle C^{\pi_{(\lambda^k,\alpha_k)}}_{\xi\otimes h_{0,\alpha_k}}, f\right\rangle_{\left(L^\infty(G_n),L^1(G_n)\right)} \\ = &\int\limits_{U(n)} \int\limits_{\mathbb{H}_n} f(A,z,t) \langle \tau_{\lambda^k}(A)\xi,\xi\rangle_{\mathcal{H}_{\lambda^k}} e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \int\limits_{\mathbb{C}^n} \left(\frac{1}{2\pi}\right)^n e^{-\frac{1}{2}\sqrt{\alpha_k}\langle w,z\rangle_{\mathbb{C}^n} - \frac{1}{2}|w|^2} dw d(z,t) dA\end{aligned}$$

tends to $\left\langle C^{\pi_{(\lambda,\alpha)}}_{\xi\otimes h_{0,\alpha}}, f\right\rangle_{\left(L^\infty(G_n),L^1(G_n)\right)}$. Hence, $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ converges to $\pi_{(\lambda,\alpha)}$.
The same reasoning applies when α is negative.

Conversely, the fact that the sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ converges to the representation $\pi_{(\lambda,\alpha)}$ implies by Corollary 2.10 that for $\xi \in \mathcal{H}^\infty_{(\lambda,\alpha)}$ of length 1, there is for every $k \in \mathbb{N}$ a unit vector $\xi_k \in \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ such that $\left(\left\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k, \xi_k\right\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}}\right)_{k\in\mathbb{N}}$ converges to $\left\langle d\pi_{(\lambda,\alpha)}(T)\xi,\xi\right\rangle_{\mathcal{H}_{(\lambda,\alpha)}}$. Thus, by Lemma 5.7, $\lim\limits_{k\to\infty} \alpha_k = \alpha$.
Hence, it remains to show that $\lambda^k = \lambda$ for k large enough.
Let ξ be a unit vector in $\mathcal{H}_\lambda$. Then, by Theorem 2.8, for every $k \in \mathbb{N}$, there exists a vector $\xi_k = \sum\limits_{m\in\mathbb{N}^n} \zeta^k_m \otimes h_{m,\alpha_k} \in \mathcal{H}_{(\lambda^k,\alpha_k)}$ of length 1 such that $\left(C^{\pi_{(\lambda^k,\alpha_k)}}_{\xi_k}\right)_{k\in\mathbb{N}}$ converges uniformly on compacta to $C^{\pi_{(\lambda,\alpha)}}_{\xi\otimes h_{0,\alpha}}$.

Now, take $\delta \in \mathbb{R}_{>0}$ such that $0 \notin I_{\alpha,\delta} = (\alpha - \delta, \alpha + \delta)$, as well as a Schwartz function φ on $\mathbb{R}$ fulfilling $\varphi_{|I_{\alpha,\delta}} \equiv 1$ and $\varphi \equiv 0$ in a neighbourhood of 0. Then, there is a Schwartz function ψ on $\mathbb{H}_n$ with the property

$$\sigma_\beta(\psi) = \varphi(\beta) P_\beta \quad \forall \beta \in \mathbb{R}^*,$$

where $P_\beta : \mathcal{F}_\beta(n) \to \mathbb{C}$ is the orthogonal projection onto the one-dimensional subspace $\mathbb{C}h_{0,\beta}$ of all constant functions in $\mathcal{F}_\beta(n)$. On the other hand, there exists $k_\delta \in \mathbb{N}$ such that $\alpha_k \in I_{\alpha,\delta}$ for all $k \geq k_\delta$. One obtains $\sigma_\alpha(\psi)h_{0,\alpha} = h_{0,\alpha}$ and $\sigma_{\alpha_k}(\psi)h_{0,\alpha_k} = h_{0,\alpha_k}$ for all $k \geq k_\delta$ and thus, it follows that

$$\begin{aligned}
\lim_{k\to\infty} \|\zeta_0^k\|^2_{\mathcal{H}_{\lambda^k}} &= \lim_{k\to\infty} \sum_{m,m' \in \mathbb{N}^n} \langle \zeta_m^k, \zeta_{m'}^k \rangle_{\mathcal{H}_{\lambda^k}} \langle \sigma_{\alpha_k}(\psi) h_{m,\alpha_k}, h_{m',\alpha_k} \rangle_{\mathcal{F}_{\alpha_k}(n)} \\
&= \lim_{k\to\infty} \left\langle C^{\pi_{(\lambda^k,\alpha_k)}}_{\sum\limits_{m\in\mathbb{N}^n} \zeta_m^k \otimes h_{m,\alpha_k}}(\mathbb{I}, .,.), \overline{\psi} \right\rangle_{\left(L^\infty(\mathbb{H}_n), L^1(\mathbb{H}_n)\right)} \\
&= \langle \sigma_\alpha(\psi) h_{0,\alpha}, h_{0,\alpha} \rangle_{\mathcal{F}_\alpha(n)} = 1.
\end{aligned}$$

Hence, one gets

$$\lim_{k\to\infty} \left\| \xi_k - \zeta_0^k \otimes h_{0,\alpha_k} \right\|_{\mathcal{H}_{(\lambda^k,\alpha_k)}} = 0$$

and one can deduce that

$$\lim_{k\to\infty} \langle \tau_{\lambda^k}(A)\zeta_0^k, \zeta_0^k \rangle_{\mathcal{H}_{\lambda^k}} = \langle \tau_\lambda(A)\xi, \xi \rangle_{\mathcal{H}_\lambda}$$

uniformly in $A \in U(n)$. Therefore, for all $k \in \mathbb{N}$, one can take the unit vector $\phi_k = \frac{\zeta_0^k}{\|\zeta_0^k\|_{\mathcal{H}_{\lambda^k}}}$ in $\mathcal{H}_{\lambda^k}$ to finally obtain the uniform convergence on compacta of $\left(C^{\tau_{\lambda^k}}_{\phi_k} \right)_{k\in\mathbb{N}}$ to $C^{\tau_\lambda}_\xi$. Thus, $\lambda^k = \lambda$ for k large enough.

□

Lemma 5.9.
For each representation $\pi_{(\lambda,\alpha)}$ for $\alpha \in \mathbb{R}^*$ and $\tau_\lambda \in \widehat{U(n)}$,

$$\langle d\pi_{(\lambda,\alpha)}(\mathcal{L}) h_{m,\alpha}, h_{m,\alpha} \rangle_{\mathcal{F}_\alpha(n)} = -|\alpha|\left(n + 2|m|\right) \quad \forall m \in \mathbb{N}^n.$$

The proof follows from [4], Proposition 3.20 together with [5], Lemma 3.4.

Theorem 5.10.
Let $r > 0$, $\rho_\mu \in \widehat{U(n-1)}$ and $\tau_\lambda \in \widehat{U(n)}$.

1. *If a sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ of elements of $\widehat{G}_n$ converges to the representation $\pi_{(\mu,r)}$ in $\widehat{G}_n$, then $\lim\limits_{k\to\infty} \alpha_k = 0$ and the sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ satisfies one of the following conditions:*
   - *(i) For k large enough, $\alpha_k > 0$, $\lambda_j^k = \mu_j$ for all $j \in \{1, ..., n-1\}$ and $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = -\frac{r^2}{2}$.*
   - *(ii) For k large enough, $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ for all $j \in \{2, ..., n\}$ and $\lim\limits_{k\to\infty} \alpha_k \lambda_1^k = -\frac{r^2}{2}$.*

2. *If a sequence* $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *of elements of* $\widehat{G}_n$ *converges to the representation* τ_λ *in* $\widehat{G}_n$*, then* $\lim\limits_{k\to\infty}\alpha_k = 0$ *and the sequence* $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ *satisfies one of the following conditions:*

   (i) For k large enough, $\alpha_k > 0$*,* $\lambda_1 \geq \lambda_1^k \geq ... \geq \lambda_{n-1} \geq \lambda_{n-1}^k \geq \lambda_n \geq \lambda_n^k$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_n^k = 0$*.*

   (ii) For k large enough, $\alpha_k < 0$*,* $\lambda_1^k \geq \lambda_1 \geq \lambda_2^k \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ *and* $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = 0$*.*

Proof:
1) Let $\tilde{\mu}^s = (\mu_1,\dots,\mu_s,\mu_s,\mu_{s+1},\dots,\mu_{n-1})$ for $s\in\{1,...,n-1\}$. By hypothesis, the sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ converges to the representation $\pi_{(\mu,r)}$ in $\widehat{G}_n$. Thus, by Corollary 2.10, for the unit vector $\xi^s = \sqrt{d_{\tilde{\mu}^s}}\overline{C^{\tilde{\mu}^s}_{\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}}} \in \mathcal{H}^\infty_{(\mu,r)}$, there is a sequence of unit vectors $(\xi_k^s)_{k\in\mathbb{N}} \subset \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ such that

$$\big\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overset{k\to\infty}{\longrightarrow} \big\langle d\pi_{(\mu,r)}(T)(\xi^s),\xi^s\big\rangle_{\mathcal{H}_{(\mu,r)}} = 0 \quad \forall T\in\mathfrak{t}_n \qquad \text{and}$$

$$\big\langle d\pi_{(\lambda^k,\alpha_k)}(\mathcal{L})\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overset{k\to\infty}{\longrightarrow} \big\langle d\pi_{(\mu,r)}(\mathcal{L})(\xi^s),\xi^s\big\rangle_{\mathcal{H}_{(\mu,r)}} = -r^2.$$

Since by Lemma 5.7 one gets $\big\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} = \big\langle i\alpha_k\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}}$, it follows that $\lim\limits_{k\to\infty}\alpha_k = 0$. Therefore, one can assume without restriction that $\alpha_k > 0$ for large k (by passing to a subsequence if necessary). The case $\alpha_k < 0$ is very similar.

On the other hand, the sequence $\Big(\big\langle\tau_{\lambda^k}\otimes W_{\alpha_k}(A)\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}}\Big)_{k\in\mathbb{N}}$ converges to the matrix coefficient $C^{\pi_{(\mu,r)}}_{\xi^s}(A,0,0) = C^{\tilde{\mu}^s}_{\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}}(A)$ uniformly in each $A\in U(n)$. Hence, from this convergence, Orthogonality Relation (38) and the fact that $\Big\|C^{\tilde{\mu}^s}_{\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}}\Big\|_{L^2(U(n))} = \frac{1}{\sqrt{d_{\tilde{\mu}^s}}}$ follows

$$\lim_{k\to\infty}\int\limits_{U(n)} \big\langle\tau_{\lambda^k}\otimes W_{\alpha_k}(A)\xi_k^s,\xi_k^s\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}}\overline{\big\langle\tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s},\phi_1^{\tilde{\mu}^s}\big\rangle_{\mathcal{H}_{(\mu,r)}}}\,dA = \frac{1}{d_{\tilde{\mu}^s}} \neq 0. \tag{42}$$

By (41), one can write the expression $\big(\tau_{\lambda^k}\otimes W_{\alpha_k}\big)_{|U(n)}$ as

$$\big(\tau_{\lambda^k}\otimes W_{\alpha_k}\big)_{|U(n)} = \sum_{\substack{\tilde{\lambda}^k\in P_n\\ \tilde{\lambda}_1^k\geq\lambda_1^k\geq...\geq\tilde{\lambda}_n^k\geq\lambda_n^k}} \tau_{\tilde{\lambda}^k}$$

and, since for k large enough the above integral is not 0, again by the orthogonality relation, there has to be one $\tilde{\lambda}^k\in P_n$ with $\tilde{\lambda}_1^k\geq\lambda_1^k\geq...\geq\tilde{\lambda}_n^k\geq\lambda_n^k$ such that $\tilde{\lambda}^k = \tilde{\mu}^s$. But as $\tilde{\lambda}_s^k = \tilde{\mu}_s^s = \tilde{\mu}_{s+1}^s = \tilde{\lambda}_{s+1}^k$, one obtains that $\lambda_s^k = \tilde{\lambda}_s^k = \tilde{\mu}_s^s = \mu_s$ for k large enough. As this is true for all $s\in\{1,...,n-1\}$, one gets $\lambda_j^k = \mu_j$ for all $j\in\{1,...,n-1\}$.

So, it remains to show that $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = -\frac{r^2}{2}$.

Again, by the decomposition of $\left(\tau_{\lambda^k} \otimes W_{\alpha_k}\right)_{|U(n)}$ in (41), one can decompose $\mathcal{H}_{(\lambda^k,\alpha_k)}$ as follows

$$\mathcal{H}_{(\lambda^k,\alpha_k)} = \sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \mathcal{H}_{\tilde{\lambda}^k}$$

and thus, for every $k \in \mathbb{N}$, the vector ξ_k^s can be written as

$$\xi_k^s = \sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \xi_{\tilde{\lambda}^k}^s \quad \text{for } \xi_{\tilde{\lambda}^k}^s \in \mathcal{H}_{\tilde{\lambda}^k} \quad \forall k \in \mathbb{N}.$$

Then, with Orthogonality Relation (38),

$$\begin{aligned}
&\int\limits_{U(n)} \left\langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k^s, \xi_k^s \right\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overline{\left\langle \tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s}, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{(\mu,r)}}} \, dA \\
= &\sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \sum_{\substack{\tilde{\gamma}^k \in P_n \\ \tilde{\gamma}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\gamma}_n^k \geq \lambda_n^k}} \int\limits_{U(n)} \left\langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_{\tilde{\lambda}^k}^s, \xi_{\tilde{\gamma}^k}^s \right\rangle_{\mathcal{H}_{\tilde{\lambda}^k}} \\
&\qquad \overline{\left\langle \tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s}, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{(\mu,r)}}} \, dA \\
= &\sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \sum_{\substack{\tilde{\gamma}^k \in P_n \\ \tilde{\gamma}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\gamma}_n^k \geq \lambda_n^k}} \int\limits_{U(n)} \Bigg\langle \sum_{\substack{\tilde{\nu}^k \in P_n \\ \tilde{\nu}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\nu}_n^k \geq \lambda_n^k}} \tau_{\tilde{\nu}^k}(A)\xi_{\tilde{\lambda}^k}^s, \xi_{\tilde{\gamma}^k}^s \Bigg\rangle_{\mathcal{H}_{\tilde{\lambda}^k}} \\
&\qquad \overline{\left\langle \tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s}, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{(\mu,r)}}} \, dA \\
= &\sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \sum_{\substack{\tilde{\gamma}^k \in P_n \\ \tilde{\gamma}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\gamma}_n^k \geq \lambda_n^k}} \int\limits_{U(n)} \left\langle \tau_{\tilde{\lambda}^k}(A)\xi_{\tilde{\lambda}^k}^s, \xi_{\tilde{\gamma}^k}^s \right\rangle_{\mathcal{H}_{\tilde{\lambda}^k}} \overline{\left\langle \tau_{\tilde{\mu}^s}(A)\phi_1^{\tilde{\mu}^s}, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{(\mu,r)}}} \, dA \\
= &\sum_{\substack{\tilde{\gamma}^k \in P_n \\ \tilde{\gamma}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\gamma}_n^k \geq \lambda_n^k}} \frac{\left\langle \xi_{\tilde{\mu}^s}^s, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}} \left\langle \phi_1^{\tilde{\mu}^s}, \xi_{\tilde{\gamma}^k}^s \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}}}{d_{\tilde{\mu}^s}} \\
= &\frac{\left\langle \xi_{\tilde{\mu}^s}^s, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}} \left\langle \phi_1^{\tilde{\mu}^s}, \xi_{\tilde{\mu}^s}^s \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}}}{d_{\tilde{\mu}^s}} = \frac{\left|\left\langle \xi_{\tilde{\mu}^s}^s, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}}\right|^2}{d_{\tilde{\mu}^s}}.
\end{aligned}$$

From (42) follows that

$$\left|\left\langle \xi_{\tilde{\mu}^s}^s, \phi_1^{\tilde{\mu}^s} \right\rangle_{\mathcal{H}_{\tilde{\mu}^s}}\right|^2 \stackrel{k\to\infty}{\longrightarrow} 1.$$

As

$$1 = \|\xi_k^s\|_{\mathcal{H}_{(\lambda^k,\alpha_k)}}^2 = \sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq \ldots \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \left\|\xi_{\tilde{\lambda}^k}^s\right\|_{\mathcal{H}_{\tilde{\lambda}^k}}^2,$$

one can assume that $\xi_k^s = \phi_1^{\tilde{\mu}^s}$ for large $k \in \mathbb{N}$. Since $\lambda_j^k = \mu_j$ for all $k \in \mathbb{N}$ and for all $j \in \{1, ..., n-1\}$, one gets for $s = n-1$

$$\tilde{\mu}^{n-1} = \lambda^k + m_k \quad \text{for } m_k = \big(0, ..., 0, \mu_{n-1} - \lambda_n^k\big).$$

From now on, consider only k large enough such that $\xi_k^{n-1} = \phi_1^{\tilde{\mu}^{n-1}}$. Then, ξ_k^{n-1} is the highest weight vector with weight $\tilde{\mu}^{n-1}$. Moreover,

$$\sum_{\substack{\tilde{\lambda}^k \in P_n \\ \tilde{\lambda}_1^k \geq \lambda_1^k \geq ... \geq \tilde{\lambda}_n^k \geq \lambda_n^k}} \mathcal{H}_{\tilde{\lambda}^k} = \mathcal{H}_{(\lambda^k, \alpha_k)} = \mathcal{H}_{\lambda^k} \otimes \mathcal{F}_{\alpha_k}(n) = \sum_{m=0}^{\infty} \mathcal{H}_{\lambda^k} \otimes \mathcal{P}_m.$$

Every weight in the decomposition on the left hand side has multiplicity one, as mentioned in (41), and therefore, this is the case for every weight appearing in the sum on the right hand side as well. From this, one can deduce that there exists one unique M_k such that $\tilde{\mu}^{n-1}$, the weight of $\xi_k^{n-1} \in \mathcal{H}_{(\lambda^k, \alpha_k)}$, appears in $\mathcal{H}_{\lambda^k} \otimes \mathcal{P}_{M_k}$.
By [20], Chapter IV.11, every highest weight appearing in $\mathcal{H}_{\lambda^k} \otimes \mathcal{P}_{M_k}$ is the sum of the highest weight of $\mathcal{H}_{\lambda^k}$ and a weight of $\mathcal{P}_{M_k}$. Hence, $\tilde{\mu}^{n-1}$ is the sum of λ^k and a weight of $\mathcal{P}_{M_k}$. From this follows that the mentioned weight of $\mathcal{P}_{M_k}$ has the same length as $\tilde{\mu}^{n-1} - \lambda^k = m_k$. Therefore, $M_k = |m_k|$, i.e. $\mathcal{P}_{M_k} = \mathcal{P}_{|m_k|}$.
Taking an orthonormal basis of $\mathcal{H}_{\lambda^k}$ consisting of eigenvectors for $\mathbb{T}_n$ as at the beginning of this subsection, then, due to the above considerations, one can write ξ_k^{n-1} as

$$\xi_k^{n-1} = \sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \phi^{\gamma^k} \otimes h_{\tilde{m}_k, \alpha_k},$$

where ϕ^{γ^k} is a uniquely determined eigenvector for $\mathbb{T}_n$ of $\mathcal{H}_{\lambda^k}$ with weight γ^k and $\Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}$ is the set of all pairs $(\gamma^k, \tilde{m}_k)$ such that $\tilde{m}_k \in \mathbb{N}^n$ with $|\tilde{m}_k| = |m_k|$ and γ^k is a weight that appears in the representation τ_{λ^k} fulfilling $\gamma^k + \tilde{m}_k = \tilde{\mu}^{n-1}$. Furthermore,

$$\sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \big|\phi^{\gamma^k}\big|^2 = 1.$$

Then, from Lemma 5.9 and the $U(n)$-invariance of $\mathcal{L}$,

$$\begin{aligned}
-r^2 \quad &\overset{k\to\infty}{\longleftarrow} \quad \big\langle d\pi_{(\lambda^k, \alpha_k)}(\mathcal{L}) \xi_k^{n-1}, \xi_k^{n-1} \big\rangle_{\mathcal{H}_{(\lambda^k, \alpha_k)}} \\
&= \sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \; \sum_{(\tilde{\gamma}^k, \tilde{\tilde{m}}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \Big\langle d\pi_{(\lambda^k, \alpha_k)}(\mathcal{L}) \phi^{\gamma^k} \otimes h_{\tilde{m}_k, \alpha_k}, \phi^{\tilde{\gamma}^k} \otimes h_{\tilde{\tilde{m}}_k, \alpha_k} \Big\rangle_{\mathcal{H}_{(\lambda^k, \alpha_k)}} \\
&= \sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \big|\phi^{\gamma^k}\big|^2 \, \big\langle d\pi_{(\lambda^k, \alpha_k)}(\mathcal{L}) h_{\tilde{m}_k, \alpha_k}, h_{\tilde{m}_k, \alpha_k} \big\rangle_{\mathcal{F}_{\alpha_k}(n)} \\
&= \sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \big|\phi^{\gamma^k}\big|^2 \Big(- \alpha_k \big(n + 2|\tilde{m}_k|\big)\Big) \\
&= -\alpha_k\big(n + 2|m_k|\big) \sum_{(\gamma^k, \tilde{m}_k) \in \Omega_{\lambda^k}^{\tilde{\mu}^{n-1}}} \big|\phi^{\gamma^k}\big|^2 \\
&= -\alpha_k\big(n + 2|m_k|\big) \;=\; -\alpha_k\big(n + 2\mu_{n-1} - 2\lambda_n^k\big).
\end{aligned}$$

As $\alpha_k \overset{k\to\infty}{\longrightarrow} 0$, also $\alpha_k(n+2\mu_{n-1}) \overset{k\to\infty}{\longrightarrow} 0$ and thus, $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = -\frac{r^2}{2}$.

2) The fact that the sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ converges to τ_λ in $\widehat{G}_n$ implies that for the unit vector $\phi_1^\lambda \in \mathcal{H}_\lambda^\infty$, there is a sequence of unit vectors $(\xi_k)_{k\in\mathbb{N}} \subset \mathcal{H}^\infty_{(\lambda^k,\alpha_k)}$ such that

$$\big\langle d\pi_{(\lambda^k,\alpha_k)}(T)\xi_k, \xi_k\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overset{k\to\infty}{\longrightarrow} \big\langle d\tau_\lambda(T)\phi_1^\lambda, \phi_1^\lambda\big\rangle_{\mathcal{H}_\lambda} \quad \forall T \in \mathfrak{t}_n \qquad \text{and}$$

$$\big\langle d\pi_{(\lambda^k,\alpha_k)}(\mathcal{L})\xi_k, \xi_k\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overset{k\to\infty}{\longrightarrow} \big\langle d\tau_\lambda(\mathcal{L})\phi_1^\lambda, \phi_1^\lambda\big\rangle_{\mathcal{H}_\lambda} = 0.$$

As above in the first part, by Lemma 5.7, from the first convergence it follows that $\lim\limits_{k\to\infty} \alpha_k = 0$ and one can assume without restriction that $\alpha_k > 0$ for large k.
On the other hand, $\Big(\big\langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k, \xi_k\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}}\Big)_{k\in\mathbb{N}}$ converges to $C^\lambda_{\phi_1^\lambda,\phi_1^\lambda}(A)$ uniformly in each $A \in U(n)$. Hence, as above one gets

$$\lim_{k\to\infty} \int\limits_{U(n)} \big\langle \tau_{\lambda^k} \otimes W_{\alpha_k}(A)\xi_k, \xi_k\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \overline{\big\langle \tau_\lambda(A)\phi_1^\lambda, \phi_1^\lambda\big\rangle_{\mathcal{H}_\lambda}}\, dA = \frac{1}{d_\lambda} \neq 0.$$

Again, as in the first part above, by (41) and the orthogonality relation, one can deduce that $\lambda_1 \geq \lambda_1^k \geq .... \geq \lambda_n \geq \lambda_n^k$ for large k.

So again, it remains to show that $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = 0$.
In the same manner as above, by replacing $\tilde{\mu}^{n-1}$ by λ, one can now show that for large $k \in \mathbb{N}$, it is possible to assume $\xi_k = \phi_1^\lambda$. So consider k large enough in order for this equality to be true. Then ξ_k is the highest weight vector with weight λ.
Now,

$$\lambda = \lambda^k + m_k \quad \text{for } m_k = \big(\lambda_1 - \lambda_1^k, ..., \lambda_n - \lambda_n^k\big),$$

where the sequences $\big(\lambda_1 - \lambda_1^k\big)_{k\in\mathbb{N}}, ..., \big(\lambda_{n-1} - \lambda_{n-1}^k\big)_{k\in\mathbb{N}}$ are bounded, because $\lambda_1 \geq \lambda_1^k \geq .... \geq \lambda_n \geq \lambda_n^k$ for large k.
Again, by replacing $\tilde{\mu}^{n-1}$ by λ in the proof of the first part above, one can also write ξ_k as

$$\xi_k = \sum_{(\gamma^k,\tilde{m}_k)\in\Omega^\lambda_{\lambda^k}} \phi^{\gamma^k} \otimes h_{\tilde{m}_k,\alpha_k},$$

where ϕ^{γ^k} is a uniquely determined eigenvector for $\mathbb{T}_n$ of $\mathcal{H}_{\lambda^k}$ with weight γ^k and $\Omega^\lambda_{\lambda^k}$ is the set of all pairs $(\gamma^k, \tilde{m}_k)$ such that $\tilde{m}_k \in \mathbb{N}^n$ with $|\tilde{m}_k| = |m_k|$ and γ^k is a weight that appears in the representation τ_{λ^k} fulfilling $\gamma^k + \tilde{m}_k = \lambda$. Furthermore, again

$$\sum_{(\gamma^k,\tilde{m}_k)\in\Omega^\lambda_{\lambda^k}} \big|\phi^{\gamma^k}\big|^2 = 1.$$

Now, like in the first part above, by Lemma 5.9 and the $U(n)$-invariance of $\mathcal{L}$,

$$\begin{aligned} 0 \overset{k\to\infty}{\longleftarrow} &\ \big\langle d\pi_{(\lambda^k,\alpha_k)}(\mathcal{L})\xi_k, \xi_k\big\rangle_{\mathcal{H}_{(\lambda^k,\alpha_k)}} \\ = &\ -\alpha_k\big(n + 2|m_k|\big) \\ = &\ -\alpha_k\Big(n + 2\big(\lambda_1 - \lambda_1^k\big) + ... + 2\big(\lambda_{n-1} - \lambda_{n-1}^k\big) + 2\lambda_n - 2\lambda_n^k\Big). \end{aligned}$$

As $\alpha_k \overset{k\to\infty}{\longrightarrow} 0$, also $\alpha_k\Big(n + 2\big(\lambda_1 - \lambda_1^k\big) + ... + 2\big(\lambda_{n-1} - \lambda_{n-1}^k\big) + 2\lambda_n\Big) \overset{k\to\infty}{\longrightarrow} 0$ because of the boundedness of the sequences $\big(\lambda_1 - \lambda_1^k\big)_{k\in\mathbb{N}}$,...,$\big(\lambda_{n-1} - \lambda_{n-1}^k\big)_{k\in\mathbb{N}}$. Therefore, $\lim\limits_{k\to\infty} \alpha_k\lambda_n^k = 0$.
□

Theorem 5.11.
Let $r > 0$, $\rho_\mu \in \widehat{U(n-1)}$ and $\tau_\lambda \in \widehat{U(n)}$.
If $\lim\limits_{k\to\infty} \alpha_k = 0$ and the sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ of elements of $\widehat{G}_n$ satisfies one of the following conditions:

(i) for k large enough, $\alpha_k > 0$, $\lambda_j^k = \mu_j$ for all $j \in \{1, ..., n-1\}$ and $\lim\limits_{k\to\infty} \alpha_k\lambda_n^k = -\frac{r^2}{2}$,

(ii) for k large enough, $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ for all $j \in \{2, ..., n\}$ and $\lim\limits_{k\to\infty} \alpha_k\lambda_1^k = -\frac{r^2}{2}$,

then the sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ converges to the representation $\pi_{(\mu,r)}$ in $\widehat{G}_n$.

In order to prove this theorem, one needs the following proposition:

Proposition 5.12.
Let $r > 0$, $\rho_\mu \in \widehat{U(n-1)}$ and $\tau_\lambda \in \widehat{U(n)}$.
Furthermore, let $\lim\limits_{k\to\infty} \alpha_k = 0$, $\alpha_k > 0$ for large k and consider the sequence $\big(\lambda^k\big)_{k\in\mathbb{N}}$ in P_n fulfilling $\lambda_j^k = \mu_j$ for all $j \in \{1, ..., n-1\}$ and $\lim\limits_{k\to\infty} \alpha_k\lambda_n^k = -\frac{r^2}{2}$.
Denote $\tilde{\mu} := \tilde{\mu}^{n-1} = (\mu_1, ..., \mu_{n-1}, \mu_{n-1})$, $N_k := \mu_{n-1} - \lambda_n^k$ and let $\overline{\mathcal{P}_{N_k}}$ be the space of conjugated homogeneous polynomials of degree N_k.
Define the representation $\pi^{(\tilde{\mu},\alpha_k)}$ of G_n on the subspace $\mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{P}_{N_k}} \otimes \mathcal{P}_{N_k}$ of the Hilbert space $\mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{F}_{\alpha_k}(n)} \otimes \mathcal{F}_{\alpha_k}(n)$ by

$$\pi^{(\tilde{\mu},\alpha_k)}(A, z, t) := \tau_{\tilde{\mu}}(A) \otimes \overline{W_{\alpha_k}}(A) \otimes \big(\sigma_{\alpha_k}(z, t) \circ W_{\alpha_k}(A)\big) \quad \forall (A, z, t) \in G_n.$$

Then, there exists a vector $\xi \in \mathcal{H}_{\tilde{\mu}} \otimes \mathcal{H}_{(1,r)}$ and for each $k \in \mathbb{N}$ vectors $\xi_k \in \mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{P}_{N_k}} \otimes \mathcal{P}_{N_k}$ such that for all $(A, z, t) \in G_n$,

$$\Big\langle \pi^{(\tilde{\mu},\alpha_k)}(A, z, t)\xi_k, \xi_k \Big\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)} \overset{k\to\infty}{\longrightarrow} \Big\langle \big(\tau_{\tilde{\mu}} \otimes \pi_{(1,r)}\big)(A, z, t)\xi, \xi \Big\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\mathcal{H}_{(1,r)}}.$$

Proof:
Let $m_k := (0, ..., 0, N_k)$. Then, $\lambda^k = \tilde{\mu} + m_k$. Moreover, since $\lim\limits_{k\to\infty} \alpha_k\lambda_n^k = -\frac{r^2}{2}$, $\lim\limits_{k\to\infty} \alpha_k = 0$ and $\alpha_k > 0$, one gets $N_k \overset{k\to\infty}{\longrightarrow} \infty$.
Let $\phi^{\tilde{\mu}}$ be the highest weight vector with weight $\tilde{\mu}$ in $\mathcal{H}_{\tilde{\mu}}$.
Now, define

$$\xi_k := \phi^{\tilde{\mu}} \otimes \Bigg(\frac{1}{N_k^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\ |q|=N_k}} \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k}\Bigg) = \frac{1}{N_k^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\ |q|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k} \in \mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{P}_{N_k}} \otimes \mathcal{P}_{N_k}.$$

Since ${N_k}^{\frac{1}{2}}$ is the norm of $\sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k}$, the vector $\frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k}$ has norm 1. Let $(A,z,t) \in G_n$. One has

$$\begin{aligned}
&\left\langle \pi^{(\tilde{\mu},\alpha_k)}(A,z,t)\xi_k, \xi_k \right\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)} \\
= &\left\langle \pi^{(\tilde{\mu},\alpha_k)}(A,z,t)\left(\frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k}\right),\right. \\
&\left. \frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k} \right\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)} \\
= &\left\langle \pi^{(\tilde{\mu},\alpha_k)}\big((\mathbb{I},z,t)(A,0,0)\big)\left(\frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k}\right),\right. \\
&\left. \frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k} \right\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)} \\
= &\left\langle \frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{W_{\alpha_k}(A)h_{q,\alpha_k}} \otimes \Big(\sigma_{\alpha_k}(z,t)\circ W_{\alpha_k}(A)h_{q,\alpha_k}\Big),\right. \\
&\left. \frac{1}{{N_k}^{\frac{1}{2}}}\, \tau_{\tilde{\mu}}(A^{-1})\phi^{\tilde{\mu}} \otimes \left(\sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k}\right) \right\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)}.
\end{aligned}$$

Now, one can write

$$\begin{aligned}
W_{\alpha_k}(A)h_{q,\alpha_k} &= \sum_{\substack{m\in\mathbb{N}^n:\\|m|=N_k}} w^k_{m,q}(A)h_{m,\alpha_k} \quad \text{and} \\
\overline{W_{\alpha_k}(A)h_{q,\alpha_k}} &= \sum_{\substack{m\in\mathbb{N}^n:\\|m|=N_k}} \overline{w^k_{m,q}(A)h_{m,\alpha_k}}
\end{aligned}$$

with $w^k_{m,q}(A) \in \mathbb{C}$. Because of the unitarity of the matrix $W_{\alpha_k}(A)$, one gets for $m, m' \in \mathbb{N}^n$ with $|m| = |m'| = N_k$,

$$\sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} w^k_{m,q}(A)\overline{w^k_{m',q}(A)} = \begin{cases} 0 & \text{if } m \neq m', \\ 1 & \text{if } m = m'. \end{cases}$$

Hence,

$$\Bigg\langle \frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{W_{\alpha_k}(A)h_{q,\alpha_k}} \otimes \Big(\sigma_{\alpha_k}(z,t)\circ W_{\alpha_k}(A)h_{q,\alpha_k}\Big),$$

$$\frac{1}{{N_k}^{\frac{1}{2}}} \tau_{\tilde{\mu}}(A^{-1})\phi^{\tilde{\mu}} \otimes \Bigg(\sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k}\Bigg)\Bigg\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)}$$

$$= \Bigg\langle \frac{1}{{N_k}^{\frac{1}{2}}} \sum_{\substack{m\in\mathbb{N}^n:\\|m|=N_k}} \phi^{\tilde{\mu}} \otimes \overline{h_{m,\alpha_k}} \otimes \Big(\sigma_{\alpha_k}(z,t)h_{m,\alpha_k}\Big),$$

$$\frac{1}{{N_k}^{\frac{1}{2}}} \tau_{\tilde{\mu}}(A^{-1})\phi^{\tilde{\mu}} \otimes \Bigg(\sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k}\Bigg)\Bigg\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)}$$

$$= \Big\langle \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}}, \phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}}$$

$$\frac{1}{N_k} \Bigg\langle \sum_{\substack{m\in\mathbb{N}^n:\\|m|=N_k}} \overline{h_{m,\alpha_k}} \otimes \Big(\sigma_{\alpha_k}(z,t)h_{m,\alpha_k}\Big), \sum_{\substack{\tilde{q}\in\mathbb{N}^n:\\|\tilde{q}|=N_k}} \overline{h_{\tilde{q},\alpha_k}} \otimes h_{\tilde{q},\alpha_k}\Bigg\rangle_{\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)}$$

$$= \Big\langle \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}}, \phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}} \frac{1}{N_k} \sum_{\substack{m,\tilde{q}\in\mathbb{N}^n:\\|m|=|\tilde{q}|=N_k}} \Big\langle \overline{h_{m,\alpha_k}}, \overline{h_{\tilde{q},\alpha_k}}\Big\rangle_{\overline{\mathcal{F}_{\alpha_k}(n)}} \Big\langle \sigma_{\alpha_k}(z,t)h_{m,\alpha_k}, h_{\tilde{q},\alpha_k}\Big\rangle_{\mathcal{F}_{\alpha_k}(n)}$$

$$= \Big\langle \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}}, \phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}} \frac{1}{N_k} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \Big\langle \sigma_{\alpha_k}(z,t)h_{q,\alpha_k}, h_{q,\alpha_k}\Big\rangle_{\mathcal{F}_{\alpha_k}(n)}$$

$$= \Big\langle \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}}, \phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}} \frac{1}{N_k} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \int_{\mathbb{C}^n} e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2} e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}^n}} h_{q,\alpha_k}(z+w)\overline{h_{q,\alpha_k}(w)}e^{-\frac{\alpha_k}{2}|w|^2}dw$$

$$= \Big\langle \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}}, \phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}} \frac{1}{N_k} \sum_{\substack{q\in\mathbb{N}^n:\\|q|=N_k}} \left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!}\, e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2}$$

$$\int_{\mathbb{C}^n} e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}^n}} (z+w)^q\overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2}dw.$$

Now, by the binomial theorem, letting $\binom{q}{l} := \binom{q_1}{l_1}\cdots\binom{q_n}{l_n}$ for $q=(q_1,...,q_n)\in\mathbb{N}^n$ and $l=(l_1,...,l_n)\in\mathbb{N}^n$,

$$(z+w)^q = \sum_{l_1=0}^{q_1}\binom{q_1}{l_1} z_1^{q_1-l_1}w_1^{l_1}\cdots\sum_{l_n=0}^{q_n}\binom{q_n}{l_n} z_n^{q_n-l_n}w_n^{l_n} = \sum_{\substack{l:=(l_1,...,l_n)\in\mathbb{N}^n:\\ l_1\le q_1,...,l_n\le q_n}} \binom{q}{l} z^{q-l}w^l.$$

Thus, one gets for $q \in \mathbb{N}^n$ with $|q| = N_k$,

$$\left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \int\limits_{\mathbb{C}^n} e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}^n}} (z+w)^q \overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2} dw$$

$$= \left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{l:=(l_1,\dots,l_n)\in\mathbb{N}^n: \\ l_1\le q_1,\dots,l_n\le q_n}} \binom{q}{l} z^{q-l} \int\limits_{\mathbb{C}^n} e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}^n}} w^l \overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2} dw.$$

The integrals in w_m for $m \in \{1, ..., n\}$ can be written as follows:

$$\sum_{j_m=0}^{\infty} \int\limits_{\mathbb{C}} \frac{{w_m}^{j_m} (-\overline{z_m})^{j_m}}{j_m!} \left(\frac{\alpha_k}{2}\right)^{j_m} e^{-\frac{\alpha_k}{2}|w_m|^2} {w_m}^{l_m} \overline{w_m}^{q_m} dw_m.$$

Therefore,

$$\left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{l:=(l_1,\dots,l_n)\in\mathbb{N}^n: \\ l_1\le q_1,\dots,l_n\le q_n}} \binom{q}{l} z^{q-l} \int\limits_{\mathbb{C}^n} e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}^n}} w^l \overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2} dw$$

$$= \left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{l:=(l_1,\dots,l_n)\in\mathbb{N}^n: \\ l_1\le q_1,\dots,l_n\le q_n}} \sum_{j\in\mathbb{N}^n} \left(\frac{\alpha_k}{2}\right)^{|j|} \binom{q}{l} z^{q-l} \frac{(-\overline{z})^j}{j!}$$

$$\int\limits_{\mathbb{C}^n} w^{j+l} \overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2} dw.$$

Because of the orthogonality of the functions $\mathbb{C}^n \to \mathbb{C}^n, x \mapsto x^a$ and $\mathbb{C}^n \to \mathbb{C}^n, x \mapsto x^b$ for $a, b \in \mathbb{N}^n$ with respect to the scalar product of the Fock space, $j + l = q$, i.e. $l = q - j$. Moreover, as $\|h_{q,\alpha_k}\|^2_{\mathcal{F}_{\alpha_k}(n)} = 1$,

$$\| \cdot^{\,q} \|^2_{\mathcal{F}_{\alpha_k}(n)} = \frac{1}{\left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k} q!}}.$$

Hence,

$$\left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{l:=(l_1,\dots,l_n)\in\mathbb{N}^n: \\ l_1\le q_1,\dots,l_n\le q_n}} \sum_{j\in\mathbb{N}^n} \left(\frac{\alpha_k}{2}\right)^{|j|} \binom{q}{l} z^{q-l} \frac{(-\overline{z})^j}{j!}$$

$$\int\limits_{\mathbb{C}^n} w^{j+l} \overline{w}^q e^{-\frac{\alpha_k}{2}|w|^2} dw$$

$$= \left(\frac{\alpha_k}{2\pi}\right)^n \frac{\alpha_k^{N_k}}{2^{N_k}} \frac{1}{q!} \; e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{j:=(j_1,\dots,j_n)\in\mathbb{N}^n: \\ j_1\le q_1,\dots,j_n\le q_n}} \left(\frac{\alpha_k}{2}\right)^{|j|} \binom{q}{q-j} z^j \frac{(-\overline{z})^j}{j!} \| \cdot^{\,q} \|^2_{\mathcal{F}_{\alpha_k}(n)}$$

$$= e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \sum_{\substack{j:=(j_1,\dots,j_n)\in\mathbb{N}^n: \\ j_1\le q_1,\dots,j_n\le q_n}} \left(\frac{\alpha_k}{2}\right)^{|j|} \frac{q!}{(q-j)!} \frac{z^j (-\overline{z})^j}{(j!)^2}.$$

Thus,

$$\Big\langle \pi^{(\tilde{\mu},\alpha_k)}(A,z,t)\xi_k,\xi_k\Big\rangle_{\mathcal{H}_{\tilde{\mu}}\otimes\overline{\mathcal{F}_{\alpha_k}(n)}\otimes\mathcal{F}_{\alpha_k}(n)}$$
$$= \Big\langle \tau_{\tilde{\mu}}(A)\big(\phi^{\tilde{\mu}}\big),\phi^{\tilde{\mu}}\Big\rangle_{\mathcal{H}_{\tilde{\mu}}} e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2}\frac{1}{N_k}\sum_{\substack{q\in\mathbb{N}^n:\\ |q|=N_k}}\sum_{\substack{j:=(j_1,...,j_n)\in\mathbb{N}^n:\\ j_1\le q_1,...,j_n\le q_n}}\Big(\frac{\alpha_k}{2}\Big)^{|j|}\frac{q!}{(q-j)!}\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}.$$

Now, regard

$$\frac{1}{N_k}\sum_{\substack{q\in\mathbb{N}^n:\\ |q|=N_k}}\sum_{\substack{j:=(j_1,...,j_n)\in\mathbb{N}^n:\\ j_1\le q_1,...,j_n\le q_n}}\Big(\frac{\alpha_k}{2}\Big)^{|j|}\frac{q!}{(q-j)!}\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}$$
$$= \frac{1}{N_k}\sum_{\substack{q_1,...,q_n\in\mathbb{N}:\\ q_1+...+q_n=N_k}}\sum_{\substack{j:=(j_1,...,j_n)\in\mathbb{N}^n:\\ j_1\le q_1,...,j_n\le q_n}}\Big(\frac{\alpha_k}{2}\Big)^{j_1+...+j_n}\Big(q_1(q_1-1)\cdots(q_1-j_1+1)\Big)$$
$$\cdots\Big(q_n(q_n-1)\cdots(q_n-j_n+1)\Big)\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}.$$

Then, fixing large $k\in\mathbb{N}$, one gets for $j=(j_1,...,j_n)\in\mathbb{N}^n$,

$$\Bigg|\frac{1}{N_k}\sum_{\substack{q_1\in\mathbb{N}_{\ge j_1},...,q_n\in\mathbb{N}_{\ge j_n}:\\ q_1+...+q_n=N_k}}\Big(\frac{\alpha_k}{2}\Big)^{j_1+...+j_n}\Big(q_1(q_1-1)\cdots(q_1-j_1+1)\Big)$$
$$\cdots\Big(q_n(q_n-1)\cdots(q_n-j_n+1)\Big)\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}\Bigg| \tag{43}$$

$$= \Bigg|\frac{1}{N_k}\sum_{\substack{q_1\in\mathbb{N}_{\ge j_1},...,q_n\in\mathbb{N}_{\ge j_n}:\\ q_1+...+q_n=N_k}}\Big(\frac{\alpha_k N_k}{2}\Big)^{j_1+...+j_n}\frac{q_1(q_1-1)\cdots(q_1-j_1+1)}{{N_k}^{j_1}}$$
$$\cdots\frac{q_n(q_n-1)\cdots(q_n-j_n+1)}{{N_k}^{j_n}}\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}\Bigg|$$
$$\le \Bigg|\frac{1}{N_k}\sum_{\substack{q_1\in\mathbb{N}_{\ge j_1},...,q_n\in\mathbb{N}_{\ge j_n}:\\ q_1+...+q_n=N_k}}\Big(\frac{r^2}{4}+1\Big)^{j_1+...+j_n}\frac{z^j(-\overline{z})^j}{\big(j!\big)^2}\Bigg| = \bigg(\Big(\frac{r^2}{4}+1\Big)z\bigg)^j\overline{z}^j\frac{1}{(j!)},$$

since $\lim\limits_{k\to\infty}\alpha_k N_k=\frac{r^2}{2}$. The above expression does not depend on k and

$$\sum_{j:=(j_1,...,j_n)\in\mathbb{N}^n}\bigg(\Big(\frac{r^2}{4}+1\Big)z\bigg)^j\overline{z}^j\frac{1}{(j!)} = \exp\bigg(\Big(\frac{r^2}{4}+1\Big)z\overline{z}\bigg)<\infty.$$

So, by the theorem of Lebesgue, the sum in (43) converges and it suffices to regard the limit of each summand by itself. Hence, for $j = (j_1, ..., j_n) \in \mathbb{N}^n$, one has

$$\frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1}, ..., q_n \in \mathbb{N}_{\geq j_n}: \\ q_1 + ... + q_n = N_k}} \left(\frac{\alpha_k}{2}\right)^{j_1 + ... + j_n} \Big(q_1(q_1 - 1) \cdots (q_1 - j_1 + 1)\Big) \cdots \Big(q_n(q_n - 1) \cdots (q_n - j_n + 1)\Big) \frac{z^j (-\overline{z})^j}{(j!)^2}$$

$$\cong \frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1}, ..., q_n \in \mathbb{N}_{\geq j_n}: \\ q_1 + ... + q_n = N_k}} \left(\frac{r^2}{4}\right)^{j_1 + ... + j_n} \frac{q_1(q_1 - 1) \cdots (q_1 - j_1 + 1)}{{N_k}^{j_1}} \cdots \frac{q_n(q_n - 1) \cdots (q_n - j_n + 1)}{{N_k}^{j_n}} \frac{z^j (-\overline{z})^j}{(j!)^2}$$

$$= \frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1}, ..., q_n \in \mathbb{N}_{\geq j_n}: \\ q_1 + ... + q_n = N_k}} \left(\frac{r^2}{4}\right)^{j_1 + ... + j_n} \frac{q_1}{N_k}\Big(\frac{q_1}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_1}{N_k} - \frac{j_1 - 1}{N_k}\Big) \cdots \frac{q_n}{N_k}\Big(\frac{q_n}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_n}{N_k} - \frac{j_n - 1}{N_k}\Big) \frac{z^j (-\overline{z})^j}{(j!)^2}$$

$$= \frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1}, ..., q_{n-1} \in \mathbb{N}_{\geq j_{n-1}}: \\ q_1 + ... + q_{n-1} \leq N_k - j_n}} \left(\frac{r^2}{4}\right)^{j_1 + ... + j_n} \frac{q_1}{N_k}\Big(\frac{q_1}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_1}{N_k} - \frac{j_1 - 1}{N_k}\Big) \cdots \frac{q_{n-1}}{N_k}\Big(\frac{q_{n-1}}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_{n-1}}{N_k} - \frac{j_{n-1} - 1}{N_k}\Big)$$
$$\cdot \Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big)\Big(\Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big) - \frac{1}{N_k}\Big) \cdots \Big(\Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big) - \frac{j_n - 1}{N_k}\Big) \frac{z^j (-\overline{z})^j}{(j!)^2}.$$

Now, define for $k \in \mathbb{N}$ the function $F_k : [0, 1]^{n-1} \to \mathbb{R}$ by

$$F_k(s_1, ..., s_{n-1}) :=$$
$$\left(\frac{r^2}{4}\right)^{j_1 + ... + j_n} s_1\Big(s_1 - \frac{1}{N_k}\Big) \cdots \Big(s_1 - \frac{j_1 - 1}{N_k}\Big) \cdots s_{n-1}\Big(s_{n-1} - \frac{1}{N_k}\Big) \cdots \Big(s_{n-1} - \frac{j_{n-1} - 1}{N_k}\Big)$$
$$\cdot \big(1 - (s_1 + ... + s_{n-1})\big)\Big(\big(1 - (s_1 + ... + s_{n-1})\big) - \frac{1}{N_k}\Big)$$
$$\cdots \Big(\big(1 - (s_1 + ... + s_{n-1})\big) - \frac{j_n - 1}{N_k}\Big) \frac{z^j (-\overline{z})^j}{(j!)^2}.$$

Then, for $\varepsilon > 0$ and large $k \in \mathbb{N}$,

$$\left| \frac{1}{N_k} \sum_{q_1,...,q_{n-1} \in \mathbb{N}_{\leq N_k}} F_k\Big(\frac{q_1}{N_k},...,\frac{q_{n-1}}{N_k}\Big) \; - \; \int_0^1 \cdots \int_0^1 F_k(s_1,...,s_{n-1}) ds_1...ds_{n-1} \right| < \varepsilon.$$

Since $F_k\Big(\frac{q_1}{N_k},...,\frac{q_{n-1}}{N_k}\Big) = 0$, if $q_1 < j_1$, $q_2 < j_2$ or ... or $q_{n-1} < j_{n-1}$ or $q_1 + ... + q_{n-1} > N_k - j_n$, it follows that

$$\left| \frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1},...,q_{n-1} \in \mathbb{N}_{\geq j_{n-1}}: \\ q_1+...+q_{n-1} \leq N_k - j_n}} F_k\Big(\frac{q_1}{N_k},...,\frac{q_{n-1}}{N_k}\Big) \; - \; \int_0^1 \cdots \int_0^1 F_k(s_1,...,s_{n-1}) ds_1...ds_{n-1} \right| < \varepsilon.$$

Furthermore, F_k converges pointwise to the function $F : [0,1]^{n-1} \to \mathbb{R}$ defined by

$$F(s_1,...,s_{n-1}) := \Big(\frac{r^2}{4}\Big)^{j_1+...+j_n} {s_1}^{j_1} \cdots {s_{n-1}}^{j_{n-1}} \cdot \big(1 - (s_1 + ... + s_{n-1})\big)^{j_n} \; \frac{z^j(-\overline{z})^j}{(j!)^2}$$

and thus, by the theorem of Lebesgue for integrals,

$$\lim_{k\to\infty} \int_0^1 \cdots \int_0^1 F_k(s_1,...,s_{n-1}) ds_1...ds_{n-1} \; = \; \int_0^1 \cdots \int_0^1 F(s_1,...,s_{n-1}) ds_1...ds_{n-1}.$$

From these observations, one can now deduce that

$$\begin{aligned}
&\frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1},...,q_{n-1} \in \mathbb{N}_{\geq j_{n-1}}: \\ q_1+...+q_{n-1} \leq N_k - j_n}} \Big(\frac{r^2}{4}\Big)^{j_1+...+j_n} \frac{q_1}{N_k}\Big(\frac{q_1}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_1}{N_k} - \frac{j_1 - 1}{N_k}\Big) \\
&\qquad \cdots \frac{q_{n-1}}{N_k}\Big(\frac{q_{n-1}}{N_k} - \frac{1}{N_k}\Big) \cdots \Big(\frac{q_{n-1}}{N_k} - \frac{j_{n-1} - 1}{N_k}\Big) \\
&\qquad \cdot \Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big)\Big(\Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big) - \frac{1}{N_k}\Big) \\
&\qquad \cdots \Big(\Big(1 - \frac{q_1 + ... + q_{n-1}}{N_k}\Big) - \frac{j_n - 1}{N_k}\Big) \; \frac{z^j(-\overline{z})^j}{(j!)^2} \\
= \; &\frac{1}{N_k} \sum_{\substack{q_1 \in \mathbb{N}_{\geq j_1},...,q_{n-1} \in \mathbb{N}_{\geq j_{n-1}}: \\ q_1+...+q_{n-1} \leq N_k - j_n}} F_k\Big(\frac{q_1}{N_k},...,\frac{q_{n-1}}{N_k}\Big) \; \overset{k\to\infty}{\longrightarrow} \; \int_0^1 \cdots \int_0^1 F(s_1,...,s_{n-1}) ds_1...ds_{n-1} \\
= \; &\int_0^1 \cdots \int_0^1 \Big(\frac{r^2}{4}\Big)^{j_1+...+j_n} {s_1}^{j_1} \cdots {s_{n-1}}^{j_{n-1}} \big(1 - (s_1 + ... + s_{n-1})\big)^{j_n} \; \frac{z^j(-\overline{z})^j}{(j!)^2} \; ds_1...ds_{n-1}.
\end{aligned}$$

Therefore,

$$\left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} e^{i\alpha_k t - \frac{\alpha_k}{4}|z|^2} \frac{1}{N_k} \sum_{\substack{q \in \mathbb{N}^n: \\ |q| = N_k}} \sum_{\substack{j := (j_1,\ldots,j_n) \in \mathbb{N}^n: \\ j_1 \le q_1, \ldots, j_n \le q_n}} \left(\frac{\alpha_k}{2}\right)^{|j|} \frac{q!}{(q-j)!} \frac{z^j (-\overline{z})^j}{\left(j!\right)^2}$$

$$\stackrel{k\to\infty}{\longrightarrow} \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \sum_{j := (j_1,\ldots,j_n) \in \mathbb{N}^n} \int_0^1 \cdots \int_0^1 \left(\frac{r^2}{4}\right)^{j_1 + \ldots + j_n} {s_1}^{j_1} \cdots {s_{n-1}}^{j_{n-1}}$$

$$\left((1 - (s_1 + \ldots + s_{n-1})\right)^{j_n} \frac{z^j (-\overline{z})^j}{\left(j!\right)^2} \, ds_1 \ldots ds_{n-1}$$

$$= \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \int_0^1 \cdots \int_0^1 \underbrace{\left(\sum_{j_1 \in \mathbb{N}} \frac{\left(\frac{-|z_1|^2 s_1 r^2}{4}\right)^{j_1}}{\left(j_1!\right)^2} \right)}_{\text{Bessel function}}$$

$$\cdots \underbrace{\left(\sum_{j_{n-1} \in \mathbb{N}} \frac{\left(\frac{-|z_{n-1}|^2 s_{n-1} r^2}{4}\right)^{j_{n-1}}}{\left(j_{n-1}!\right)^2} \right)}_{\text{Bessel function}} \underbrace{\left(\sum_{j_n \in \mathbb{N}} \frac{\left(\frac{-|z_n|^2 \left(1 - (s_1 + \ldots + s_{n-1})\right) r^2}{4}\right)^{j_n}}{\left(j_n!\right)^2} \right)}_{\text{Bessel function}} ds_1 \ldots ds_{n-1}$$

$$= \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \left(\frac{1}{2\pi}\right)^n \int_0^1 \cdots \int_0^1 \int_0^{2\pi} \cdots \int_0^{2\pi} e^{-ir \operatorname{Re}\left(e^{ia_1} \sqrt{s_1} \, \overline{z_1}\right)}$$

$$\cdots e^{-ir \operatorname{Re}\left(e^{ia_{n-1}} \sqrt{s_{n-1}} \, \overline{z_{n-1}}\right)} e^{-ir \operatorname{Re}\left(e^{ia_n} \sqrt{1 - (s_1 + \ldots + s_{n-1})} \, \overline{z_n}\right)} da_1 \ldots da_n ds_1 \ldots ds_{n-1}$$

$$= \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \left(\frac{1}{2\pi}\right)^n \int_0^1 \cdots \int_0^1 \int_0^{2\pi} \cdots \int_0^{2\pi}$$

$$e^{-ir\left(\left(\sqrt{s_1} e^{ia_1}, \ldots, \sqrt{s_{n-1}} e^{ia_{n-1}}, \sqrt{1 - (s_1 + \ldots + s_{n-1})} e^{ia_n}\right), (z_1, \ldots, z_n)\right)_{\mathbb{C}^n}} da_1 \ldots da_n ds_1 \ldots ds_{n-1}$$

$$= \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \int_{S^n} e^{-i(sr, z)_{\mathbb{C}^n}} d\sigma(s) \tag{44}$$

$$= \left\langle \tau_{\tilde{\mu}}(A)\left(\phi^{\tilde{\mu}}\right), \phi^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{\tilde{\mu}}} \int_{U(n)} e^{-i(Bv_r, z)_{\mathbb{C}^n}} dB$$

$$= \left\langle \left(\tau_{\tilde{\mu}} \otimes \pi_{(1,r)}\right)(A, z, t)\left(\phi^{\tilde{\mu}} \otimes 1\right), \phi^{\tilde{\mu}} \otimes 1 \right\rangle_{\mathcal{H}_{\tilde{\mu}} \otimes \mathcal{H}_{(1,r)}}.$$

Equality (44) can be explained in the following way:
By substitution, for a function f on the n-dimensional complex unit ball B^n,

$$\int_{B^n} f(x) dx = \int_0^1 \rho^{2n-1} \int_{S^n} f(\rho\sigma) d\mu(\sigma) d\rho, \tag{45}$$

S^n being the n-dimensional complex sphere and $d\mu$ its invariant measure. Now, regard the integral

$$\int_0^1 \rho^{2n-1} \int_{[0,1]^{n-1}\times[0,2\pi)^n} f\big(\psi(s_1,...s_{n-1},t_1,...,t_n,\rho)\big)d(s_1,...s_{n-1},t_1,...,t_n,\rho)$$

for any continuous function f on $\overline{B^n}$ and

$$\begin{aligned}&\psi(s_1,...,s_{n-1},t_1,...,t_n,\rho) := \\ &\rho\Big(\sqrt{s_1}\cos(t_1),\sqrt{s_1}\sin(t_1),...,\sqrt{s_{n-1}}\cos(t_{n-1}),\sqrt{s_{n-1}}\sin(t_{n-1}), \\ &\quad \sqrt{1-s}\cos(t_n),\sqrt{1-s}\cos(t_n)\Big)\end{aligned}$$

with $s = \sum_{i=1}^{n-1} s_i$. By substitution and Lemma 7.7 in the appendix, up to a positive constant, this integral equals $\int_{B^n} f(x)dx$. Together with (45), this shows that the measure used on $[0,1]^{n-1}\times[0,2\pi)^n$ coincides with the invariant measure $d\sigma$ on the sphere S^n and hence, Equality (44) is proved.
Choosing $\xi := \phi^{\tilde{\mu}} \otimes 1 \in \mathcal{H}_{\tilde{\mu}} \otimes \mathcal{H}_{(1,r)}$, the claim is shown.

$\square$

Proof of Theorem 5.11:
Without restriction, one can assume that the sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ fulfills Condition (i). The case of a sequence $\big(\pi_{(\lambda^k,\alpha_k)}\big)_{k\in\mathbb{N}}$ fulfilling the second condition is very similar.

For $\tilde{n} \in \mathbb{N}$ and $\nu \in P_{\tilde{n}}$, let ϕ^ν be the heighest weight vector of τ_ν in the Hilbert space $\mathcal{H}_\nu$. Let $\tilde{\mu} := (\mu_1,...,\mu_{n-1},\mu_{n-1})$ and define the representation $\sigma_{(\tilde{\mu},r)}$ of G_n by

$$\sigma_{(\tilde{\mu},r)} := \tau_{\tilde{\mu}} \otimes \pi_{(1,r)}.$$

The Hilbert space $\mathcal{H}_{\sigma_{(\tilde{\mu},r)}}$ of the representation $\sigma_{(\tilde{\mu},r)}$ is the space

$$\mathcal{H}_{\sigma_{(\tilde{\mu},r)}} = L^2\big(U(n)/U(n-1),\mathcal{H}_{\tilde{\mu}}\big)$$

and G_n acts on $\mathcal{H}_{\sigma_{(\tilde{\mu},r)}}$ by

$$\sigma_{(\tilde{\mu},r)}(A,z,t)(\xi)(B) = e^{-i(Bv_r,z)_{\mathbb{C}^n}}\tau_{\tilde{\mu}}(A)\big(\xi(A^{-1}B)\big) \quad \forall A,B\in U(n)\ \forall(z,t)\in\mathbb{H}_n\ \forall\xi\in\mathcal{H}_{\sigma_{(\tilde{\mu},r)}}.$$

One decomposes the representation $\tau_{\tilde{\mu}|U(n-1)}$ into the direct sum of irreducible representations of the group $U(n-1)$ as follows:

$$\tau_{\tilde{\mu}|U(n-1)} = \sum_{\nu\in S(\tilde{\mu})} \rho_\nu,$$

where $S(\tilde{\mu})$ denotes the support of $\tau_{\tilde{\mu}|U(n-1)}$ in $\widehat{U(n-1)}$. Furthermore, let P_ν be the orthogonal projection of $\mathcal{H}_{\tilde{\mu}}$ onto its $U(n-1)$-invariant component $\mathcal{H}_\nu$.

The representation ρ_μ is one of the representations appearing in this sum, since the highest weight vector $\phi^{\tilde{\mu}}$ of $\tau_{\tilde{\mu}}$ is also the highest weight vector of the representation ρ_μ.

Defining for $\tilde{\nu} \in P_n$ the function $c^{\tilde{\nu}}_{\eta,\phi^{\tilde{\nu}}}$ by

$$c^{\tilde{\nu}}_{\eta,\phi^{\tilde{\nu}}}(A) := \big\langle \tau_{\tilde{\nu}}(A^{-1})\eta, \phi^{\tilde{\nu}} \big\rangle_{\mathcal{H}_{\tilde{\nu}}} \quad \forall A \in U(n)\ \forall \eta \in \mathcal{H}_{\tilde{\nu}},$$

one can identify for any $\tau_{\tilde{\nu}} \in \widehat{U(n)}$ the Hilbert space $\mathcal{H}_{\tilde{\nu}}$ with the subspace $L^2_{\tilde{\nu}}$ of $L^2\big(U(n)\big)$ given by

$$L^2_{\tilde{\nu}} = \big\{ c^{\tilde{\nu}}_{\eta,\phi^{\tilde{\nu}}} |\ \eta \in \mathcal{H}_{\tilde{\nu}} \big\}.$$

Now, it will be shown that

$$\sigma_{(\tilde{\mu},r)} \cong \sum_{\nu \in S(\tilde{\mu})} \pi_{(\nu,r)}. \tag{46}$$

In particular, one then gets

$$\mathcal{H}_{\sigma_{(\tilde{\mu},r)}} \cong \sum_{\nu \in S(\tilde{\mu})} L^2\big(U(n)/U(n-1), \rho_\nu\big).$$

Let for $\nu \in S(\tilde{\mu})$,

$$U^\nu_{\tilde{\mu}}(\xi)(A)(A') := \sqrt{d_\nu} \big\langle \tau_{\tilde{\mu}}(A^{-1})\xi(A), \rho_\nu(A')\phi^\nu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} \quad \forall \xi \in \mathcal{H}_{\tilde{\mu}}\ \forall A \in U(n)\ \forall A' \in U(n-1).$$

Then,

$$\begin{aligned} \big\langle \tau_{\tilde{\mu}}(A^{-1})\xi(A), \rho_\nu(A')\phi^\nu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} &= \Big\langle \tau_{\tilde{\mu}}(A^{-1})\xi(A), P_\nu\big(\rho_\nu(A')\phi^\nu\big) \Big\rangle_{\mathcal{H}_{\tilde{\mu}}} \\ &= \Big\langle P_\nu\big(\tau_{\tilde{\mu}}(A^{-1})\xi(A)\big), \rho_\nu(A')\phi^\nu \Big\rangle_{\mathcal{H}_{\tilde{\mu}}}, \end{aligned}$$

i.e. one has a scalar product on the space $\mathcal{H}_\nu$. Now, define

$$U_{\tilde{\mu}} : L^2\big(U(n)/U(n-1), \mathcal{H}_{\tilde{\mu}}\big) \to \sum_{\nu \in S(\tilde{\mu})} L^2\big(U(n)/U(n-1), \rho_\nu\big),\ U_{\tilde{\mu}}(\xi) := \sum_{\nu \in S(\tilde{\mu})} U^\nu_{\tilde{\mu}}(\xi).$$

For all $\xi \in L^2\big(U(n)/U(n-1), \mathcal{H}_{\tilde{\mu}}\big)$, all $A \in U(n)$ and all $A', B' \in U(n-1)$,

$$\begin{aligned} U_{\tilde{\mu}}\xi(AB')(A') &= \sum_{\nu \in S(\tilde{\mu})} \sqrt{d_\nu} \big\langle \tau_{\tilde{\mu}}\big(B'^{-1}A^{-1}\big)\xi(A), \rho_\nu(A')\phi^\nu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} \\ &= \sum_{\nu \in S(\tilde{\mu})} \sqrt{d_\nu} \big\langle \tau_{\tilde{\mu}}(A^{-1})\xi(A), \rho_\nu(B'A')\phi^\nu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} \\ &= \rho_\nu(B')^{-1} U_{\tilde{\mu}}\xi(A)(A'). \end{aligned}$$

Hence, the vector $U_{\tilde{\mu}}\xi$ fulfills the covariance condition of $L^2\big(U(n)/U(n-1), \rho_\nu\big)$ and is thus contained in the space $\sum_{\nu \in S(\tilde{\mu})} L^2\big(U(n)/U(n-1), \rho_\nu\big)$. Furthermore,

$$
\begin{aligned}
\|U_{\tilde\mu}\xi\|_2^2 &= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \|U_{\tilde\mu}^{\nu}(\xi)(A)\|^2_{\mathcal{H}_{\tilde\mu}} dA \\
&= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \int\limits_{U(n-1)} \sqrt{d_\nu}\big\langle \tau_{\tilde\mu}(A^{-1})\xi(A), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}} \\
&\qquad\qquad \overline{\sqrt{d_\nu}\big\langle \tau_{\tilde\mu}(A^{-1})\xi(A), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}}} dA' dA \\
&= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \int\limits_{U(n-1)} d_\nu\big\langle P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}} \\
&\qquad\qquad \overline{\big\langle P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}}} dA' dA \\
&= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \int\limits_{U(n-1)} d_\nu\Big|\big\langle P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}}\Big|^2 dA' dA \\
&= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \big\|P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big)\big\|^2_{\mathcal{H}_{\tilde\mu}} \ \big|\phi^\nu\big|^2 dA \\
&= \sum_{\nu\in S(\tilde\mu)} \int\limits_{U(n)} \big\|P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big)\big\|^2_{\mathcal{H}_{\tilde\mu}} dA \\
&= \int\limits_{U(n)} \Big\| \sum_{\nu\in S(\tilde\mu)} P_\nu\big(\tau_{\tilde\mu}(A^{-1})\xi(A)\big)\Big\|^2_{\mathcal{H}_{\tilde\mu}} dA \\
&= \big\|\tau_{\tilde\mu}(\cdot^{-1})\xi(\cdot)\big\|_2^2 \ = \ \|\xi\|_2^2.
\end{aligned}
$$

Moreover, for all $(A, z, t) \in G_n$, all $\xi \in L^2\big(U(n)/U(n-1), \mathcal{H}_{\tilde\mu}\big)$, all $B \in U(n)$ and all $A' \in U(n-1)$, one gets

$$
\begin{aligned}
&\sum_{\nu\in S(\tilde\mu)} \pi_{(\nu,r)}(A, z, t)(U_{\tilde\mu}\xi)(B)(A') \\
&= e^{-i(Bv_r, z)_{\mathbb{C}^n}} \sum_{\nu\in S(\tilde\mu)} (U^\nu_{\tilde\mu}\xi)(A^{-1}B)(A') \\
&= e^{-i(Bv_r, z)_{\mathbb{C}^n}} \sum_{\nu\in S(\tilde\mu)} \sqrt{d_\nu}\big\langle \tau_{\tilde\mu}(B^{-1}A)\xi(A^{-1}B), \rho_\nu(A')\phi^\nu\big\rangle_{\mathcal{H}_{\tilde\mu}} \\
&= e^{-i(Bv_r, z)_{\mathbb{C}^n}} \sum_{\nu\in S(\tilde\mu)} \sqrt{d_\nu}\Big\langle \tau_{\tilde\mu}(B^{-1})\big(\tau_{\tilde\mu}(A)\xi(A^{-1}B)\big), \rho_\nu(A')\phi^\nu\Big\rangle_{\mathcal{H}_{\tilde\mu}} \\
&= e^{-i(Bv_r, z)_{\mathbb{C}^n}} \sum_{\nu\in S(\tilde\mu)} U^\nu_{\tilde\mu}\Big(\tau_{\tilde\mu}(A)\xi\big(A^{-1}\,\cdot\,\big)\Big)(B)(A') \\
&= U_{\tilde\mu}\big(\tau_{\tilde\mu}\otimes\pi_{(1,r)}(A, z, t)\xi\big)(B)(A') \ = \ U_{\tilde\mu}\big(\sigma_{(\tilde\mu,r)}(A, z, t)\xi\big)(B)(A').
\end{aligned}
$$

Therefore, (46) holds.

For the element $\xi = \phi^{\tilde{\mu}} \otimes 1 \in L^2\big(U(n)/U(n-1), \mathcal{H}_{\tilde{\mu}}\big)$, one has for all $A \in U(n)$ and all $A' \in U(n-1)$,

$$U_{\tilde{\mu}}\xi(A)(A') = \sum_{\nu \in S(\tilde{\mu})} \sqrt{d_\nu} \big\langle \tau_{\tilde{\mu}}(A^{-1})\phi^{\tilde{\mu}}, \rho_\nu(A')\phi^\nu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} =: \sum_{\nu \in S(\tilde{\mu})} \Phi^\nu_{\tilde{\mu}}(A)(A').$$

In particular, since $\phi^\mu = \phi^{\tilde{\mu}}$,

$$\Phi^\mu_{\tilde{\mu}}(A)(I) = \sqrt{d_\mu} \big\langle \tau_{\tilde{\mu}}(A^{-1})\phi^{\tilde{\mu}}, \phi^\mu \big\rangle_{\mathcal{H}_{\tilde{\mu}}} = \sqrt{d_\mu} \big\langle \phi^{\tilde{\mu}}, \tau_{\tilde{\mu}}(A)\phi^{\tilde{\mu}} \big\rangle_{\mathcal{H}_{\tilde{\mu}}} \neq 0.$$

From Theorem 5.6 follows that the subset $\big\{\pi_{(\nu,r)} | \ \mu \neq \nu \in P_{n-1}\big\}$ is closed in $\widehat{G_n}$. Hence, there exists $F_\mu = (F_\mu)^*$ in $C^*(G_n)$ whose Fourier transform at $\pi_{(\nu,r)}$ is 0 if $\mu \neq \nu \in P_{n-1}$ and for which

$$\pi_{(\mu,r)}(F_\mu) =: P_{\Phi^\mu_{\tilde{\mu}}}$$

is the orthogonal projection onto the space $\mathbb{C}\Phi^\mu_{\tilde{\mu}} \subset \mathcal{H}_{(\mu,r)}$. In particular,

$$\begin{aligned} U_{\tilde{\mu}}\big(\sigma_{(\tilde{\mu},r)}(F_\mu)(\phi^{\tilde{\mu}} \otimes 1)\big) &= \sum_{\nu \in S(\tilde{\mu})} \pi_{(\nu,r)}(F_\mu)\big(U_{\tilde{\mu}}(\phi^{\tilde{\mu}} \otimes 1)\big) = \pi_{(\mu,r)}(F_\mu)\big(U_{\tilde{\mu}}(\phi^{\tilde{\mu}} \otimes 1)\big) \\ &= P_{\Phi^\mu_{\tilde{\mu}}}\big(U_{\tilde{\mu}}(\phi^{\tilde{\mu}} \otimes 1)\big) = c \cdot \Phi^\mu_{\tilde{\mu}} \end{aligned} \tag{47}$$

for a constant $c \neq 0$. Without restriction, one can assume that $c = 1$.
Define the coefficient c_μ of $L^1(G_n)$ by

$$\begin{aligned} c_\mu(F) &:= \big\langle \sigma_{(\tilde{\mu},r)}(F_\mu * F * F_\mu)(\phi^{\tilde{\mu}} \otimes 1), \phi^{\tilde{\mu}} \otimes 1 \big\rangle_{\mathcal{H}_{\sigma_{(\tilde{\mu},r)}}} \\ &= \big\langle \sigma_{(\tilde{\mu},r)}(F * F_\mu)(\phi^{\tilde{\mu}} \otimes 1), \sigma_{(\tilde{\mu},r)}(F_\mu)(\phi^{\tilde{\mu}} \otimes 1) \big\rangle_{\mathcal{H}_{\sigma_{(\tilde{\mu},r)}}} \\ &= \Big\langle U_{\tilde{\mu}}\big(\sigma_{(\tilde{\mu},r)}(F * F_\mu)(\phi^{\tilde{\mu}} \otimes 1)\big), U_{\tilde{\mu}}\big(\sigma_{(\tilde{\mu},r)}(F_\mu)(\phi^{\tilde{\mu}} \otimes 1)\big) \Big\rangle_{\mathcal{H}_{(\mu,r)}} \\ &\overset{(47)}{=} \Big\langle \sum_{\nu \in S(\tilde{\mu})} \pi_{(\nu,r)}(F) \circ \pi_{(\nu,r)}(F_\mu)\big(U_{\tilde{\mu}}(\phi^{\tilde{\mu}} \otimes 1)\big), \Phi^\mu_{\tilde{\mu}} \Big\rangle_{\mathcal{H}_{(\mu,r)}} \\ &= \big\langle \pi_{(\mu,r)}(F) \circ \pi_{(\mu,r)}(F_\mu)\big(U_{\tilde{\mu}}(\phi^{\tilde{\mu}} \otimes 1)\big), \Phi^\mu_{\tilde{\mu}} \big\rangle_{\mathcal{H}_{(\mu,r)}} \\ &\overset{(47)}{=} \big\langle \pi_{(\mu,r)}(F)\Phi^\mu_{\tilde{\mu}}, \Phi^\mu_{\tilde{\mu}} \big\rangle_{\mathcal{H}_{(\mu,r)}} \end{aligned} \tag{48}$$

for all $F \in L^1(G_n)$.
Let $X(\tilde{\mu} \times \overline{\alpha_k})$ be the collection of all $\tilde{\nu} = (\nu_1, ..., \nu_n) \in P_n$ such that $\chi_{\tilde{\nu}}$ is a character of $\mathbb{T}_n$ appearing in $\mathcal{H}_{\tilde{\mu}}$ and such that $\tau_{\tilde{\nu}_k}$ is contained in the representation $\tau_{\tilde{\mu}} \otimes \overline{W_{\alpha_k}}$ for $\tilde{\nu}_k := (\nu_1, \nu_2, ..., \nu_{n-1}, \lambda^k_n)$.
Then, for $\pi^{(\tilde{\mu},\alpha_k)}$ defined as in Proposition 5.12, by [20], Chapter IV.11,

$$\pi^{(\tilde{\mu},\alpha_k)} = \sum_{\tilde{\nu} \in X(\tilde{\mu} \times \overline{\alpha_k})} \pi_{(\tilde{\nu}_k,\alpha_k)}.$$

Furthermore, decompose the vector

$$\xi_k = \phi^{\tilde{\mu}} \otimes \left(\frac{1}{N_k^{\frac{1}{2}}} \sum_{\substack{q \in \mathbb{N}^n: \\ |q| = N_k}} \overline{h_{q,\alpha_k}} \otimes h_{q,\alpha_k} \right)$$

for every $k \in \mathbb{N}$ into the orthogonal sum

$$\xi_k = \sum_{\tilde{\nu} \in X(\tilde{\mu} \times \overline{\alpha_k})} \xi_k^{\tilde{\nu}}$$

for $\xi_k^{\tilde{\nu}} \in \mathcal{H}_{(\tilde{\nu}_k, \alpha_k)}$. This gives a decomposition

$$\langle \pi^{(\tilde{\mu}, \alpha_k)}(\cdot)\xi_k, \xi_k \rangle_{\mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{F}_{\alpha_k}(n)} \otimes \mathcal{F}_{\alpha_k}(n)} = c_{\xi_k}^{\pi^{(\tilde{\mu}, \alpha_k)}} = \sum_{\tilde{\nu} \in X(\tilde{\mu} \times \overline{\alpha_k})} c_{\xi_k^{\tilde{\nu}}}^{\tilde{\nu}}.$$

Let $c_{\xi^{\tilde{\nu}}}$ be the weak*-limit of a subsequence of $\left(c_{\xi_k^{\tilde{\nu}}}\right)_{k \in \mathbb{N}}$ and let for $c_{\xi^{\tilde{\nu}}} \neq 0$ the representation $\pi \in \widehat{G}_n$ be an element of the support of $c_{\xi^{\tilde{\nu}}}$. From Theorem 5.10 follows that $\pi = \lim\limits_{k \to \infty} \pi_{(\tilde{\nu}_k, \alpha_k)} = \pi_{(\nu, r)}$ for $\nu = (\nu_1, ..., \nu_{n-1})$. Furthermore, one observes that for $\tilde{\mu} \neq \tilde{\nu} := (\nu_1, ..., \nu_{n-1}, \nu_n) \in X(\tilde{\mu} \times \overline{\alpha}_k)$, one has $\nu \neq \mu$. Hence, $\pi_{(\nu, r)}(F_\mu) = 0$. Thus,

$$\lim_{k \to \infty} \langle \pi_{(\tilde{\nu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\nu}}, \pi_{(\tilde{\nu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\nu}} \rangle_{\mathcal{H}_{(\tilde{\nu}_k, \alpha_k)}} = \langle \pi_{(\nu, r)}(F_\mu)\xi^{\tilde{\nu}}, \pi_{(\nu, r)}(F_\mu)\xi^{\tilde{\nu}} \rangle_{\mathcal{H}_{(\nu, r)}} = 0$$

and therefore,

$$\lim_{k \to \infty} \langle \pi_{(\tilde{\nu}_k, \alpha_k)}(F) \circ \pi_{(\tilde{\nu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\nu}}, \pi_{(\tilde{\nu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\nu}} \rangle_{\mathcal{H}_{(\tilde{\nu}_k, \alpha_k)}} = 0 \quad \forall F \in C^*(G_n). \tag{49}$$

Now, since $\tilde{\mu}_k = (\mu_1, ..., \mu_{n-1}, \lambda_n^k) = \lambda^k$, by Proposition 5.12 and its proof, for all $F \in C^*(G_n)$,

$$\begin{aligned}
& \langle \pi_{(\mu, r)}(F)\Phi_{\tilde{\mu}}^{\mu}, \Phi_{\tilde{\mu}}^{\mu} \rangle_{\mathcal{H}_{(\mu, r)}} \\
= \quad & c_\mu(F_\mu) \\
\overset{(48)}{=} \quad & \langle \tau_{\tilde{\mu}} \otimes \pi_{(1, r)}(F_\mu * F * F_\mu)(\phi^{\tilde{\mu}} \otimes 1), \phi^{\tilde{\mu}} \otimes 1 \rangle_{\mathcal{H}_{\sigma(\tilde{\mu}, r)}} \\
\underset{5.12}{\overset{Proposition}{=}} \quad & \lim_{k \to \infty} \langle \pi^{(\tilde{\mu}, \alpha_k)}(F_\mu * F * F_\mu)\xi_k, \xi_k \rangle_{\mathcal{H}_{\tilde{\mu}} \otimes \overline{\mathcal{F}_{\alpha_k}(n)} \otimes \mathcal{F}_{\alpha_k}(n)} \\
= \quad & \lim_{k \to \infty} \sum_{\tilde{\nu} \in X(\tilde{\mu} \times \overline{\alpha_k})} \langle \pi_{(\tilde{\nu}_k, \alpha_k)}(F_\mu * F * F_\mu)\xi_k^{\tilde{\nu}}, \xi_k^{\tilde{\nu}} \rangle_{\mathcal{H}_{(\tilde{\nu}_k, \alpha_k)}} \\
\overset{(49)}{=} \quad & \lim_{k \to \infty} \langle \pi_{(\tilde{\mu}_k, \alpha_k)}(F_\mu * F * F_\mu)\xi_k^{\tilde{\mu}}, \xi_k^{\tilde{\mu}} \rangle_{\mathcal{H}_{(\tilde{\mu}_k, \alpha_k)}} + 0 \\
= \quad & \lim_{k \to \infty} \left\langle \pi_{(\tilde{\mu}_k, \alpha_k)}(F)\big(\pi_{(\tilde{\mu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\mu}}\big), \pi_{(\tilde{\mu}_k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{(\tilde{\mu}_k, \alpha_k)}} \\
= \quad & \lim_{k \to \infty} \left\langle \pi_{(\lambda^k, \alpha_k)}(F)\big(\pi_{(\lambda^k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\mu}}\big), \pi_{(\lambda^k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\mu}} \right\rangle_{\mathcal{H}_{(\lambda^k, \alpha_k)}}.
\end{aligned}$$

Choosing $\tilde{\xi} := \Phi_{\tilde{\mu}}^{\mu} \in \mathcal{H}_{(\mu, r)}$ and $\tilde{\xi}_k := \pi_{(\lambda^k, \alpha_k)}(F_\mu)\xi_k^{\tilde{\mu}} \in \mathcal{H}_{(\lambda^k, \alpha_k)}$, one has

$$\lim_{k \to \infty} \langle \pi_{(\lambda^k, \alpha_k)}(F)\tilde{\xi}_k, \tilde{\xi}_k \rangle_{\mathcal{H}_{(\lambda^k, \alpha_k)}} = \langle \pi_{(\mu, r)}(F)\tilde{\xi}, \tilde{\xi} \rangle_{\mathcal{H}_{(\mu, r)}}$$

and hence,

$$\pi_{(\mu, r)} = \lim_{k \to \infty} \pi_{(\lambda^k, \alpha_k)}.$$

□

The inverse implication of 2) of Theorem 5.10 is formulated in the following conjecture:

Conjecture 5.13.
Let $r > 0$ and $\tau_\lambda \in \widehat{U(n)}$.
If $\lim\limits_{k\to\infty} \alpha_k = 0$ and the sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ of elements of $\widehat{G}_n$ satisfies one of the following conditions:

(i) for k large enough, $\alpha_k > 0$, $\lambda_1 \geq \lambda_1^k \geq ... \geq \lambda_{n-1} \geq \lambda_{n-1}^k \geq \lambda_n \geq \lambda_n^k$ and $\lim\limits_{k\to\infty} \alpha_k \lambda_n^k = 0$,

(ii) for k large enough, $\alpha_k < 0$, $\lambda_1^k \geq \lambda_1 \geq \lambda_2^k \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ and $\lim\limits_{k\to\infty} \alpha_k \lambda_1^k = 0$,

then the sequence $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ converges to the representation τ_λ in $\widehat{G}_n$.

5.3.3 The representation τ_λ

As τ_λ only acts on $U(n)$ and $\widehat{U(n)}$ is discrete, every converging sequence $(\tau_{\lambda^k})_{k\in\mathbb{N}}$ has to be constant for large k. Hence,

$$\tau_{\lambda^k} \overset{k\to\infty}{\longrightarrow} \tau_\lambda \iff \lambda^k = \lambda \text{ for large } k.$$

5.4 Results

Combining Theorem 5.4 in Section 5.2 that describes the topology of the space of all admissible coadjoint orbits of G_n with the Theorems 5.6, 5.8 and 5.10 in Section 5.3 and the result in Subsection 5.3.3 which characterize the spectrum $\widehat{G_n}$ of G_n, one gets the following result:

Theorem 5.14.
The mapping

$$\widehat{G}_n \longrightarrow \mathfrak{g}_n^\ddagger/G_n,\ \pi \mapsto \mathcal{O}_\pi$$

is continuous.

Once one has succeeded to prove Conjecture 5.13, together with the Theorems 5.6, 5.8, 5.10 and 5.11 and the result in Subsection 5.3.3, one gets the result below:

Conjecture 5.15.
For general $n \in \mathbb{N}^*$, the spectrum of the group $G_n = U(n) \ltimes \mathbb{H}_n$ is homeomorphic to its space of admissible coadjoint orbits $\mathfrak{g}_n^\ddagger/G_n$.

For $n = 1$, the situation is a lot easier than for general $n \in \mathbb{N}^*$ and therefore, this conjecture could directly be proved in this case:

Theorem 5.16.
The spectrum of the semidirect product $U(1) \ltimes \mathbb{H}_1$ is homeomorphic to its admissible coadjoint orbit space.

Proof:
Assume that $(\alpha_k)_{k\in\mathbb{N}}$ tends to 0 and that $\lim\limits_{k\to\infty} \lambda_k \alpha_k = -\frac{r^2}{2}$. If α_k is positive (respectively negative) for k large enough, one can define the sequence $(f_k)_{k\in\mathbb{N}}$ of elements in the Fock space $\mathcal{F}_{\alpha_k}(1)$ by $f_k(w) := c_{\alpha_k,\lambda_k} w^{-\lambda_k}$ (respectively $f_k(w) := c_{\alpha_k,\lambda_k} w^{\lambda_k}$) with $c_{\alpha_k,\lambda_k} \in \mathbb{C}$ such that $\|f_k\|_{\mathcal{F}_{\alpha_k}(1)} = 1$. Then, for $f \in C_0^\infty(G_1)$, one has

$$
\begin{aligned}
&\left\langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\right\rangle_{\left(L^\infty(G_1),L^1(G_1)\right)} \\
= &\int\limits_{G_1} f(\theta,z,t)\chi_{\lambda_k}\big(e^{i\theta}\big)e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2} \\
&\qquad \int\limits_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2 e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}}}\Big(e^{-i\theta}w+e^{-i\theta}z\Big)^{-\lambda_k}\overline{w}^{-\lambda_k}e^{-\frac{\alpha_k}{2}|w|^2}dwd(\theta,z,t) \\
= &\int\limits_{G_1} f(\theta,z,t)e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2}\int\limits_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2 e^{-\frac{\alpha_k}{2}\langle w,z\rangle_{\mathbb{C}}}(w+z)^{-\lambda_k}\overline{w}^{-\lambda_k}e^{-\frac{\alpha_k}{2}|w|^2}dwd(\theta,z,t) \\
= &\int\limits_{G_1} f(\theta,z,t)e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2} \\
&\qquad \sum_{j=0}^{\infty}\sum_{l=0}^{-\lambda_k}\left(\int\limits_{\mathbb{C}} |c_{\alpha_k,\lambda_k}|^2\frac{\binom{-\lambda_k}{l}}{j!}\Big(\frac{\alpha_k}{2}\Big)^j w^{j+l}\overline{w}^{-\lambda_k}(-\overline{z})^j z^{(-\lambda_k-l)}e^{-\frac{\alpha_k}{2}|w|^2}dw\right)d(\theta,z,t) \\
= &\int\limits_{G_1} f(\theta,z,t)e^{i\alpha_k t-\frac{\alpha_k}{4}|z|^2}\left(\sum_{j=0}^{-\lambda_k}\frac{(-\lambda_k)!}{(-\lambda_k-j)!(j!)^2}\Big(\frac{\alpha_k}{2}\Big)^j\big(-|z|^2\big)^j\right)d(\theta,z,t).
\end{aligned}
$$

Now, fixing large k, one gets for every $j \in \{0, ..., -\lambda_k\}$,

$$
\begin{aligned}
&\left|\frac{(-\lambda_k)!}{(-\lambda_k-j)!(j!)^2}\Big(\frac{\alpha_k}{2}\Big)^j\big(-|z|^2\big)^j\right| \\
= &\left|\Big(\frac{-\alpha_k\lambda_k}{2}\Big)^j\frac{(-\lambda_k)!}{(-\lambda_k)^j(-\lambda_k-j)!}\frac{1}{(j!)^2}\big(-|z|^2\big)^j\right| \\
\leq &\left|\Big(\frac{r^2}{4}+1\Big)^j\Big(\frac{-\lambda_k}{-\lambda_k}\Big)\cdot\Big(\frac{-\lambda_k-1}{-\lambda_k}\Big)\cdots\Big(\frac{-\lambda_k-j+1}{-\lambda_k}\Big)\frac{1}{(j!)^2}\big(-|z|^2\big)^j\right| \\
\leq &\left(\Big(\frac{r^2}{4}+1\Big)|z|^2\right)^j\frac{1}{(j!)}.
\end{aligned}
$$

This expression does not depend on k and

$$
\sum_{j=0}^{\infty}\left(\Big(\frac{r^2}{4}+1\Big)|z|^2\right)^j\frac{1}{(j!)} = \exp\left(\Big(\frac{r^2}{4}+1\Big)|z|^2\right) < \infty.
$$

So, by the theorem of Lebesgue, this sum converges and it suffices to regard the limit of each summand by itself:

$$
\Big(\frac{-\alpha_k\lambda_k}{2}\Big)^j\frac{(-\lambda_k)!}{(-\lambda_k)^j(-\lambda_k-j)!}\frac{1}{(j!)^2}\big(-|z|^2\big)^j \overset{k\to\infty}{\longrightarrow} \frac{\big(\frac{-r^2|z|^2}{4}\big)^j}{(j!)^2}.
$$

Therefore, one gets

$$\begin{aligned}
\lim_{k\to\infty}\left\langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\right\rangle_{\left(L^\infty(G_1),L^1(G_1)\right)} &= \int_{G_1} f(\theta,z,t)\underbrace{\left(\sum_{j=0}^{\infty}\frac{\left(\frac{-r^2|z|^2}{4}\right)^j}{(j!)^2}\right)}_{\text{Bessel function}} d(\theta,z,t) \\
&= \int_{G_1} f(\theta,z,t)\,\frac{1}{2\pi}\int_0^{2\pi} e^{-ir\,\mathrm{Re}\,(e^{i\beta}\overline{z})} d\beta d(\theta,z,t) \\
&= \int_{G_1} f(\theta,z,t)\,\frac{1}{2\pi}\int_0^{2\pi} e^{-i(e^{i\beta}r,z)_{\mathbb{C}^n}} d\beta d(\theta,z,t) \\
&= \int_{G_1} f(\theta,z,t)\int_{U(1)} e^{-i(Bv_r,z)_{\mathbb{C}^n}} dB d(\theta,z,t) \\
&= \int_{G_1} f(\theta,z,t)\left\langle\left(\mathrm{ind}_{\mathbb{H}_1}^{G_1}\chi_r\right)(\theta,z,t)(1),1\right\rangle_{L^2(G_1/\mathbb{H}_1,\chi_r)}.
\end{aligned}$$

Hence, $\left(\pi_{(\lambda_k,\alpha_k)}\right)_{k\in\mathbb{N}}$ converges to the irreducible unitary representation $\pi_r := \mathrm{ind}_{\mathbb{H}_1}^{G_1}\chi_r$.

Assume now that $\lim_{k\to\infty}\lambda_k\alpha_k = 0$. For $\lambda \geq \lambda_k$ (respectively for $\lambda \leq \lambda_k$) for k large enough, define the sequence $(f_k)_{k\in\mathbb{N}}$ by $f_k(w) := c_{\alpha_k,\lambda_k,\lambda} w^{\lambda-\lambda_k}$ (respectively $f_k(w) := c_{\alpha_k,\lambda_k,\lambda} w^{\lambda_k-\lambda}$) for $c_{\alpha_k,\lambda_k,\lambda} \in \mathbb{C}$ such that $\|f_k\|_{\mathcal{F}_{\alpha_k}(1)} = 1$. Then, by the same computation as above, one gets

$$\lim_{k\to\infty}\left\langle C_{f_k}^{\pi(\lambda_k,\alpha_k)}, f\right\rangle_{\left(L^\infty(G_1),L^1(G_1)\right)} = \int_{G_1} f(\theta,z,t)\chi_\lambda(\theta)d(\theta,z,t) = \langle\chi_\lambda, f\rangle_{\left(L^\infty(G_1),L^1(G_1)\right)}.$$

Thus, $\left(\pi_{(\lambda_k,\alpha_k)}\right)_{k\in\mathbb{N}}$ converges to the character χ_λ of $U(1)$.

□

6 Résumé étendu de cette thèse en français

6.1 Introduction

Dans la présente thèse de doctorat, la structure des C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux et la structure de la C^*-algèbre du groupe de Lie $SL(2,\mathbb{R})$ sont analysées. En outre, en vue de la détermination de sa C^*-algèbre, la topologie du spectre du produit semi-direct $U(n) \ltimes \mathbb{H}_n$ est décrite, où $\mathbb{H}_n$ dénote le groupe de Lie de Heisenberg et $U(n)$ le groupe unitaire qui agit sur $\mathbb{H}_n$ par automorphismes.

La recherche des C^*-algèbres – une abstraction des algèbres d'opérateurs linéaires bornés sur des espaces de Hilbert – a commencé dans les années 1930 en raison d'un besoin dans la mécanique quantique et plus précisément pour servir de modèles mathématiques pour des algèbres d'observables physiques. Des systèmes quantiques sont décris à l'aide d'opérateurs auto-adjoints sur des espaces de Hilbert. Par conséquent, des algèbres d'opérateurs bornés sur ces espaces étaient considérées. Le terme "C^*-algèbre" a premièrement été introduit dans les années 1940 par I.Segal pour décrire des sous-algèbres fermées pour la norme de l'algèbre d'opérateurs linéaires bornés sur un espace de Hilbert. Entre-temps, les C^*-algèbres représentent un outil important dans la théorie des représentations unitaires des groupes localement compacts et dans la description mathématique de la mécanique quantique.
Les groupes de Lie ont été introduits dans les années 1870 par S.Lie dans le cadre de la théorie de Lie pour examiner des symétries continues dans des équations différentielles et leur théorie a évolué pendant le 20ième siècle. Aujourd'hui, les groupes de Lie sont utilisés dans plusieurs domaines mathématiques et dans la physique théorique, par exemple dans la physique des particules.
Puisque des groupes de Lie et d'autres groupes de symétries jouent un rôle capital dans la physique, l'examination de leurs C^*-algèbres de groupes – des C^*-algèbres qui sont construites en formant une fermeture de l'espace de toutes les fonctions L^1 sur ces groupes respectifs – a été poursuivie et avancée. Cette tâche représente l'objectif de cette thèse de doctorat.
La méthode d'analyse d'une C^*-algèbre d'un groupe utilisée dans ce travail a été initiée par G.M.Fell dans les années 1960. On utilise la transformation de Fourier non-abélienne afin d'écrire les C^*-algèbres comme des algèbres de champs d'opérateurs et d'étudier ainsi leurs éléments. Une exigence afin d'utiliser cette méthode, et ainsi de comprendre ces C^*-algèbres, est la connaissance de leur spectre et de sa topologie, lesquels sont inconnus pour beaucoup de groupes. Par conséquent, cela représente un problème majeur de la procédure de Fell. L'avancement de la méthode utilisée dans cette thèse a été élaborée pendant les dernières années (voir [24] et [26]) et cela rend l'étude d'une large classe de C^*-algèbres de groupes possible.

Afin de comprendre les C^*-algèbres, la transformation de Fourier est un outil important. Si on note $\widehat{A}$ le dual unitaire ou le spectre d'une C^*-algèbre A, c.à.d. l'ensemble de toutes les classes d'équivalence des représentations irréductibles unitaires de A, la transformée de Fourier $\mathcal{F}(a) = \hat{a}$ d'un élément $a \in A$ est définie de la manière suivante: On choisit dans chaque classe d'équivalence $\gamma \in \widehat{A}$ une représentation $(\pi_\gamma, \mathcal{H}_\gamma)$ et définit

$$\mathcal{F}(a)(\gamma) := \pi_\gamma(a) \in \mathcal{B}(\mathcal{H}_\gamma),$$

où $\mathcal{B}(\mathcal{H}_\gamma)$ désigne la C^*-algèbre des opérateurs linéaires bornés sur $\mathcal{H}_\gamma$. Alors, $\mathcal{F}(a)$ est contenu dans l'algèbre de tous les champs d'opérateurs bornés sur $\widehat{A}$

$$l^\infty(\widehat{A}) = \Big\{\phi = \big(\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma)\big)_{\gamma\in\widehat{A}} \mid \|\phi\|_\infty := \sup_{\gamma\in\widehat{A}} \|\phi(\pi_\gamma)\|_{op} < \infty\Big\}$$

et l'application

$$\mathcal{F} : A \to l^\infty(\widehat{A}),\ a \mapsto \hat{a}$$

est un $*$-homomorphisme isométrique.
Soit dx la mesure de Haar du groupe localement compact G. On peut définir sur $L^1(G) := L^1(G, \mathbb{C})$ le produit de convolution $*$ comme suit:

$$f * f'(g) := \int_G f(x)f'\big(x^{-1}g\big)dx \quad \forall f, f' \in L^1(G)\ \forall g \in G$$

et cela donne une algèbre de Banach $\big(L^1(G), *\big)$. En outre, on obtient une involution isométrique en définissant

$$f^*(g) := \Delta_G(g)^{-1}\overline{f\big(g^{-1}\big)} \quad \forall g \in G,$$

où Δ_G est la fonction modulaire.
Pour toute représentation irréductible unitaire $(\pi, \mathcal{H})$ de G, une représentation $(\tilde{\pi}, \mathcal{H})$ de $L^1(G)$ peut être définie de la manière suivante:

$$\tilde{\pi}(f) := \int_G f(g)\pi(g)dg \quad \forall f \in L^1(G).$$

Cette représentation est irréductible et unitaire aussi.
Maintenant, la C^*-algèbre de G est définie comme la fermeture de l'algèbre de convolution $L^1(G)$ par rapport à la C^*-norme de $L^1(G)$, c.à.d.

$$C^*(G) := \overline{L^1(G, dx)}^{\|\cdot\|_{C^*(G)}} \quad \text{avec} \quad \|f\|_{C^*(G)} := \sup_{[\pi]\in\widehat{G}} \|\pi(f)\|_{op},$$

où $\widehat{G}$ est le spectre de G qui est défini comme en haut comme l'ensemble de toutes les classes d'équivalence des représentations irréductibles unitaires de G.
De plus, toute représentation irréductible unitaire $(\tilde{\pi}, \mathcal{H})$ de $L^1(G)$ peut être écrite de façon unique comme une intégrale de la façon montrée en haut et alors, on obtient une représentation irréductible unitaire $(\pi, \mathcal{H})$ de G. C'est pourquoi on a le résultat suivant, bien connu, qui peut être trouvé dans [9], Chapitre 13.9, et qui dit que le spectre de $C^*(G)$ se correspond avec le spectre de G:

$$\widehat{C^*(G)} = \widehat{G}.$$

La structure des C^*-algèbres des groupes est déjà connue pour certaines classes de groupes de Lie, comme le groupe de Heisenberg et les groupes de Lie filiformes (voir [26]) et les groupes “ax+b” (voir [24]). En outre, les C^*-algèbres des groupes de Lie nilpotents de dimension 5 ont été déterminées dans [31], pendant que celles de tous les groupes de Lie nilpotents de

dimension 6 ont été caractérisées par H.Regeiba dans [30].
Dans cette thèse, un résultat démontré par H.Regeiba et J.Ludwig (voir [31], Théorème 3.5) est utilisé pour la caractérisation des C^*-algèbres mentionnées en haut.
Pour les C^*-algèbres des groupes de Lie nilpotents de pas deux, l'approche est en partie similaire, mais plus complexe, que celle qui est utilisée pour la caractérisation de la C^*-algèbre du groupe de Lie de Heisenberg (voir [26]), qui est, lui aussi, nilpotent de pas deux et sert ainsi d'exemple.
Dans le cas du groupe de Lie de Heisenberg, la situation est beaucoup plus facile que pour des groupes de Lie nilpotents de pas deux généraux, parce que dans ce cas spécial les orbites coadjointes qui apparaissent ne peuvent qu'avoir deux différentes dimensions, tandis que dans le cas général il y a beaucoup de différentes dimensions qui peuvent apparaître. Par conséquent, des cas beaucoup plus difficiles se produisent.
En revanche, les groupes de Lie nilpotents de dimension 5 et 6 ont une structure plus compliquée que les groupes de Lie nilpotents de pas deux. Ici, des orbites coadjointes non-plates apparaissent, alors que dans le cas des groupes de Lie nilpotents de pas deux, on est justement obligé de traiter les orbites coadjointes plates. Pour cette raison, dans ces cas des groupes de Lie nilpotents de dimension 5 et 6, on examine tout groupe cas par cas, tandis que pour les groupes de Lie nilpotents de pas deux, on traite toute cette classe de groupes sans connaître leurs structures exactes.
Comme mentionné en haut, l'approche pour le groupe $SL(2,\mathbb{R})$ est complètement différente. Puisqu'on traite un seul groupe qui est explicitement donné, dans ce cas, on peut effectuer des calculs plus concrets.
Pour des groupes de Lie semi-simples en général, il n'y a pas de description explicite de leurs C^*-algèbres de groupe dans la littérature. Cependant, pour les groupes de Lie semi-simples dont le spectre est classifié, la procédure de détermination du C^*-algèbre de groupe utilisée dans ce travail pourrait être appliquée avec succès d'une façon similaire. Une description de la C^*-algèbre du groupe de Lie $SL(2,\mathbb{C})$ est donnée dans [13], Théorème 5.3 et Théorème 5.4, et une caractérisation des C^*-algèbres réduites des groupes de Lie semi-simples peut être trouvée dans [34].
Dans la présente thèse de doctorat, les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux et de $SL(2,\mathbb{R})$ sont décrites très explicitement. Elles sont caractérisées par des conditions qui s'appellent "limites duales sous contrôle normique" et qui seront données ci-dessous. L'objectif principal est de calculer ces conditions d'une façon concrète et de construire les "contrôles normiques" nécessaires. Dans un résultat abstrait d'existence (voir [2], Théorème 4.6), ces conditions sont vérifiées pour tous les groupes de Lie connexes simplement connexes. Elles sont explicitement calculées pour tous les groupes de Lie nilpotents de dimension 5 et 6 (voir [31]), pour les groupes de Lie de Heisenberg et les groupes de Lie filiformes (voir [26]).
Les résultats sur les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux ont été publiés dans la "Revista Matemática Complutense", article en commun avec J.Ludwig (voir [17]). Un autre article sur la C^*-algèbre du groupe $SL(2,\mathbb{R})$ a été soumis au "Journal of Lie Theory" et peut être trouvé sur arXiv (voir [16]).

Puisque pour la détermination d'une C^*-algèbre de groupe le spectre du groupe et sa topologie sont nécessaires, dans cette thèse de doctorat, la topologie du spectre des groupes $G_n = U(n) \ltimes \mathbb{H}_n$ pour $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ est analysée. Ce travail a été initié par M.Elloumi et J.Ludwig (voir [11], Chapitre 3) et leur prépublication est complétée dans cette thèse.

Comme pour plusieurs autres classes de groupes de Lie, par exemple les groupes de Lie exponentiels résoluble et les groupes de mouvements euclidiens, le spectre $\widehat{G_n}$ peut être paramétré par l'espace des orbites coadjointes admissibles de l'espace dual $\mathfrak{g}_n^*$ de l'algèbre de Lie $\mathfrak{g}_n$ de G_n.
Le but est de démontrer que cette bijection entre l'espace des orbites coadjointes admissibles muni de la topologie quotient et le spectre $\widehat{G_n}$ de G_n est un homéomorphisme. Cela a déjà été démontré pour les groupes de mouvements classiques $SO(n) \ltimes \mathbb{R}^n$ pour $n \geq 2$ dans [12].
Pour les groupes $G_n = U(n) \ltimes \mathbb{H}_n$, cette assertion a été prouvée pour $n = 1$. Dans le cas général, il a pu être démontré que l'application entre le spectre $\widehat{G_n}$ et l'espace des orbites coadjointes admissibles de G_n est continue. Les parties restantes de la preuve de l'assertion sont en progrès.

Dans cette sous-section, la définition d'une C^*-algèbre à "limites duales sous contrôle normique" sera donnée. Les conditions dont on a besoin pour cette définition caractérisent des C^*-algèbres de groupes. Après cela, dans la Section 6.3, ces conditions seront analysées pour les groupes de Lie nilpotents de pas deux. Dans la Section 6.4, elles seront vérifiées dans le cas du groupe de Lie $SL(2, \mathbb{R})$, dans une approche très différente à celle utilisée pour les groupes de Lie nilpotents de pas deux.

Définition 6.1.
Une C^*-algèbre A est appelée C^*-algèbre à "limites duales sous contrôle normique" si elle satisfait les conditions suivantes:

- **Condition 1**: *Stratification du spectre:*

  (a) *Il existe une famille finie croissante $S_0 \subset S_1 \subset \ldots \subset S_r = \widehat{A}$ de sous-ensembles fermés du spectre $\widehat{A}$ de A telle que pour $i \in \{1, \cdots, r\}$ les sous-ensembles $\Gamma_0 = S_0$ et $\Gamma_i := S_i \setminus S_{i-1}$ sont Hausdorff dans leurs topologies relatives et telle que S_0 constitue l'ensemble de tous les caractères, c.à.d. les représentations de dimension 1, de A.*

  (b) *Pour tout $i \in \{0, \cdots, r\}$ il y a un espace de Hilbert $\mathcal{H}_i$ et pour tout $\gamma \in \Gamma_i$ il existe une réalisation concrète $(\pi_\gamma, \mathcal{H}_i)$ de γ sur l'espace de Hilbert $\mathcal{H}_i$.*

- **Condition 2**: *CCR C^*-algèbre:*
  A est une CCR C^*-algèbre (ou C^*-algèbre liminaire) séparable, c.à.d. une C^*-algèbre séparable telle que l'image de toute représentation irréductible $(\pi, \mathcal{H})$ de A est contenue dans l'algèbre d'opérateurs compacts $\mathcal{K}(\mathcal{H})$ (ce qui implique que l'image est égale à $\mathcal{K}(\mathcal{H})$).

- **Condition 3**: *Changement de couches:*
  Soit $a \in A$.

  (a) *Sur chacun des différents ensembles Γ_i, l'application $\gamma \mapsto \mathcal{F}(a)(\gamma)$ est continue par rapport à la norme.*

  (b) *Pour tout $i \in \{0, \cdots, r\}$ et pour toute suite convergente contenue dans Γ_i dont l'ensemble limite se trouve au dehors de Γ_i (alors dans S_{i-1}), il existe une sous-suite proprement convergente $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}}$ (c.à.d. les sous-suites de $\overline{\gamma}$ ont toutes le même ensemble limite – voir la Définition 6.7), ainsi qu'une constante $C > 0$ et pour tout $k \in \mathbb{N}$ une application linéaire involutive $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma},k} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$, qui est bornée par $C\| \cdot \|_{S_{i-1}}$, telles que*

$$\lim_{k\to\infty} \left\| \mathcal{F}(a)(\gamma_k) - \tilde{\nu}_k\big(\mathcal{F}(a)_{|S_{i-1}}\big) \right\|_{op} = 0.$$

Ici, $CB(S_{i-1})$ est la $*$-algèbre de tous les champs d'opérateurs uniformément bornés $\big(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_j)\big)_{\gamma\in\Gamma_j, j=0,\cdots,i-1}$, qui sont continus pour la norme d'opérateurs sur les sous-ensembles Γ_j pour tout $j \in \{0,\cdots,i-1\}$, munie avec la norme infinie

$$\|\psi\|_{S_{i-1}} := \sup_{\gamma\in S_{i-1}} \|\psi(\gamma)\|_{\mathrm{op}}.$$

Théorème 6.2 (Théorème principal)**.**
Les C^*-algèbres des groupes de Lie nilpotents de pas deux G et du groupe de Lie $G = SL(2,\mathbb{R})$ sont des C^*-algèbres à limites duales sous contrôle normique.

Les propriétés en haut caractérisent complètement la structure d'une C^*-algèbre (voir [31], Théorème 3.5).

6.2 Définitions et résultats généraux

6.2.1 Les groupes de Lie nilpotents – Préliminaires

Soit $\mathfrak{g}$ une algèbre de Lie nilpotente.
On prend sur $\mathfrak{g}$ un produit scalaire $\langle\cdot,\cdot\rangle$ et la multiplication de Campbell-Baker-Hausdorff

$$u\cdot v = u+v+\frac{1}{2}[u,v]+\frac{1}{12}\big[u,[u,v]\big]+\frac{1}{12}\big[v,[v,u]\big]+\ldots \quad \forall u,v\in\mathfrak{g},$$

laquelle est une somme finie comme $\mathfrak{g}$ est nilpotente.
Cela donne le groupe de Lie connexe simplement connexe $G = (\mathfrak{g},\cdot)$ avec l'algèbre de Lie $\mathfrak{g}$.
L'application exponentielle $\exp : \mathfrak{g} \to G = (\mathfrak{g},\cdot)$ est en ce cas l'identité.
La mesure de Haar de ce groupe est une mesure de Lebesgue qui est dénommée dx.

Convention 6.3.
Dans cette section, tous les espaces de fonctions sont des espaces de fonctions à valeurs complexes.

Maintenant, pour une fonctionnelle linéaire ℓ de $\mathfrak{g}$, on considère la forme antisymétrique

$$B_\ell(X,Y) := \langle \ell, [X,Y]\rangle$$

sur $\mathfrak{g}$. De plus, soit

$$\mathfrak{g}(\ell) := \big\{X\in\mathfrak{g} \,|\, \langle \ell,[X,\mathfrak{g}]\rangle = \{0\}\big\}$$

le radical de B_ℓ et le stabilisateur de la fonctionnelle linéaire ℓ.

Définition 6.4 (Polarisation)**.**
Une sous-algèbre $\mathfrak{p}$ de $\mathfrak{g}$ qui est subordonnée à ℓ (c.à.d. qui satisfait $\langle \ell,[\mathfrak{p},\mathfrak{p}]\rangle = \{0\}$) et qui a la dimension

$$dim(\mathfrak{p}) = \frac{1}{2}\Big(dim(\mathfrak{g}) + dim(\mathfrak{g}(\ell))\Big),$$

ce qui veut dire que $\mathfrak{p}$ est maximal isotrope pour B_ℓ, s'appelle une polarisation de ℓ.

Si $\mathfrak{p} \subset \mathfrak{g}$ est une sous-algèbre quelconque de $\mathfrak{g}$ qui est subordonnée à ℓ, la fonctionnelle linéaire ℓ définit un caractère unitaire χ_ℓ de $P := \exp(\mathfrak{p})$:

$$\chi_\ell(x) := e^{-2\pi i\langle \ell, \log(x)\rangle} = e^{-2\pi i\langle \ell, x\rangle} \quad \forall x \in P.$$

Définition 6.5 (L'action coadjointe)**.**
On définit pour tout $g \in G$ les applications

$$\alpha_g : G \to G,\ x \mapsto gxg^{-1} \quad \text{et} \quad \mathrm{Ad}\,(g) := d(\alpha_g)_e : \mathfrak{g} \to \mathfrak{g}.$$

Alors, $\mathrm{Ad} : G \to GL(\mathfrak{g})$ *est une représentation du groupe. L'action coadjointe* Ad^* *peut être définie comme suit:*

$$\mathrm{Ad}^*(g) : \mathfrak{g}^* \to \mathfrak{g}^* \quad \forall g \in G, \quad \mathrm{Ad}^*(g)\ell(X) = \ell\Big(\mathrm{Ad}\,\big(g^{-1}\big)X\Big) \quad \forall \ell \in \mathfrak{g}^*\ \forall X \in \mathfrak{g}.$$

$\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ *est une autre représentation du groupe.*
L'espace d'orbites coadjointes $\{\mathrm{Ad}^*(G)\ell |\ \ell \in \mathfrak{g}^*\}$ *est dénommé* $\mathfrak{g}^*/G$.
En outre, on définit l'application

$$\mathrm{ad}^*(X) : \mathfrak{g}^* \to \mathfrak{g}^* \quad \forall X \in \mathfrak{g}, \quad \mathrm{ad}^*(X)\ell(Y) = \ell([Y, X]).$$

$\mathrm{ad}^* : \mathfrak{g} \to GL(\mathfrak{g}^*)$ *est une représentation de l'algèbre de Lie* $\mathfrak{g}$.

Définition 6.6 (Représentation induite)**.**
Soit H un sous-groupe fermé de G et soit

$$L^2(G/H, \chi_\ell) \ := \ \Big\{\xi : G \to \mathbb{C} \ \Big|\ \xi \text{ mesurable},\ \xi(gh) = \overline{\chi_\ell(h)}\xi(g)\ \forall g \in G\ \forall h \in H,$$
$$\|\xi\|_2^2 := \int_{G/H} |\xi(g)|^2 d\dot{g} < \infty\Big\},$$

où $d\dot{g}$ est une mesure invariante sur G/H qui existe toujours pour un groupe G nilpotent.
Alors, $L^2(G/H, \chi_\ell)$ est un espace de Hilbert et on peut définir la représentation induite

$$\mathrm{ind}_H^G \chi_\ell(g)\xi(u) := \xi\big(g^{-1}u\big) \quad \forall g, u \in G\ \forall \xi \in L^2(G/H, \chi_\ell).$$

Cela donne une représentation unitaire. Si l'algèbre de Lie associée $\mathfrak{h}$ de H est une polarisation de ℓ, cette représentation est irréductible.

6.2.2 Les groupes de Lie nilpotents – La méthode des orbites

D'après la théorie de Kirillov (voir [8], Section 2.2), pour toute classe de représentations $\gamma \in \widehat{G}$, il existe un élément $\ell \in \mathfrak{g}^*$ et une polarisation $\mathfrak{p}$ de ℓ dans $\mathfrak{g}$ tels que $\gamma = \big[\mathrm{ind}_P^G \chi_\ell\big]$, où $P := \exp(\mathfrak{p})$. En outre, si $\ell, \ell' \in \mathfrak{g}^*$ se trouvent dans la même orbite coadjointe $\mathcal{O} \in \mathfrak{g}^*/G$ et si $\mathfrak{p}$ et $\mathfrak{p}'$ sont des polarisations de ℓ et de ℓ' respectivement, les représentations induites $\mathrm{ind}_P^G \chi_\ell$ et $\mathrm{ind}_{P'}^G \chi_{\ell'}$ sont équivalentes et l'application de Kirillov qui va de l'espace des orbites coadjointes $\mathfrak{g}^*/G$ dans le spectre $\widehat{G}$ de G

$$K : \mathfrak{g}^*/G \to \widehat{G},\ \mathrm{Ad}^*(G)\ell \mapsto \big[\mathrm{ind}_P^G \chi_\ell\big]$$

est un homéomorphisme (voir [6] où [23], Chapitre 3). Par conséquent,

$$\mathfrak{g}^*/G \cong \widehat{G}$$

en tant qu'espaces.

Définition 6.7.
Une suite $(t_k)_{k\in\mathbb{N}}$ *d'un espace topologique séparable* T *est dite proprement convergente si* $(t_k)_{k\in\mathbb{N}}$ *a des points limites et si toute sous-suite de* $(t_k)_{k\in\mathbb{N}}$ *a le même ensemble limite que* $(t_k)_{k\in\mathbb{N}}$.

Il est bien connu que toute suite convergente dans T possède une sous-suite qui est proprement convergente.

6.2.3 Les groupes de Lie nilpotents – La C^*-algèbre $C^*(G/U, \chi_\ell)$

Soit $\mathfrak{u} \subset \mathfrak{g}$ un idéal de $\mathfrak{g}$ contenant $[\mathfrak{g}, \mathfrak{g}]$, $U := \exp(\mathfrak{u})$ et soit $\ell \in \mathfrak{g}^*$ vérifiant $\langle \ell, [\mathfrak{g}, \mathfrak{u}]\rangle = \{0\}$ et $\mathfrak{u} \subset \mathfrak{g}(\ell)$. Alors, le caractère χ_ℓ du groupe $U = \exp(\mathfrak{u})$ est G-invariant. Par conséquent, on peut définir l'algèbre de Banach involutive $L^1(G/U, \chi_\ell)$ comme suit:

$$\begin{aligned} L^1(G/U, \chi_\ell) \quad := \quad & \Big\{ f : G \to \mathbb{C} \ \Big| \ f \text{ mesurable, } f(gu) = \chi_\ell(u^{-1}) f(g) \ \forall g \in G \ \forall u \in U, \\ & \|f\|_1 := \int\limits_{G/U} |f(g)| d\dot{g} < \infty \Big\}. \end{aligned}$$

La convolution

$$f * f'(g) := \int\limits_{G/U} f(x) f'(x^{-1}g) d\dot{x} \quad \forall g \in G$$

et l'involution

$$f^*(g) := \overline{f(g^{-1})} \quad \forall g \in G$$

sont bien définies pour $f, f' \in L^1(G/U, \chi_\ell)$ et

$$\|f * f'\|_1 \leq \|f\|_1 \ \|f'\|_1.$$

En outre, l'application linéaire

$$p_{G/U} : L^1(G) \to L^1(G/U, \chi_\ell), \ p_{G/U}(F)(g) := \int\limits_U F(gu) \chi_\ell(u) du \quad \forall F \in L^1(G) \ \ \forall g \in G$$

est un $*$-homomorphisme surjectif entre les algèbres $L^1(G)$ et $L^1(G/U, \chi_\ell)$.
Soit

$$\widehat{G}_{\mathfrak{u},\ell} := \Big\{ [\pi] \in \widehat{G} \Big| \ \pi_{|U} = \chi_{\ell|U} \mathbb{I}_{\mathcal{H}_\pi} \Big\}.$$

Alors $\widehat{G}_{\mathfrak{u},\ell}$ est un sous-ensemble fermé de $\widehat{G}$ qui peut être identifié avec le spectre de l'algèbre $L^1(G/U, \chi_\ell)$. En fait, il est facile de voir que toute représentation irréductible unitaire $(\pi, \mathcal{H}_\pi)$ dont la classe d'équivalence est contenue dans $\widehat{G}_{\mathfrak{u},\ell}$ définit une représentation irréductible $(\tilde{\pi}, \mathcal{H}_\pi)$ de l'algèbre $L^1(G/U, \chi_\ell)$ comme suit:

$$\tilde{\pi}\big(p_{G/U}(F)\big) := \pi(F) \quad \forall F \in L^1(G).$$

De façon similaire, si $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ est une représentation irréductible unitaire de $L^1(G/U, \chi_\ell)$, alors $[\pi]$ donné par

$$\pi := \tilde{\pi} \circ p_{G/U}$$

est un élément de $\widehat{G}_{\mathfrak{u},\ell}$.
Soit $\mathfrak{s} \subset \mathfrak{g}$ un sous-espace de $\mathfrak{g}$ tel que $\mathfrak{g} = \mathfrak{g}(\ell) \oplus \mathfrak{s}$. Comme $\mathfrak{u}$ contient $[\mathfrak{g}, \mathfrak{g}]$, il est facile de voir que

$$\widehat{G}_{\mathfrak{u},\ell} = \big\{ \big[\chi_q \otimes \pi_\ell\big] \big|\ q \in (\mathfrak{u} + \mathfrak{s})^\perp \big\},$$

si on note $\pi_\ell := \mathrm{ind}_P^G \chi_\ell$, où $\mathfrak{p}$ désigne une polarisation de ℓ et $P := \exp(\mathfrak{p})$.
On dénomme $C^*(G/U, \chi_\ell)$ la C^*-algèbre associée à $L^1(G/U, \chi_\ell)$ dont le spectre peut être identifié avec $\widehat{G}_{\mathfrak{u},\ell}$.
En définissant $\pi_{\ell+q} := \mathrm{ind}_P^G \chi_{\ell+q}$, la transformation de Fourier $\mathcal{F}$ définie par

$$\mathcal{F}(a)(q) := \pi_{\ell+q}(a) \quad \forall q \in (\mathfrak{u} + \mathfrak{s})^\perp,$$

envoie la C^*-algèbre $C^*(G/U, \chi_\ell)$ sur l'algèbre $C_\infty\big((\mathfrak{u} + \mathfrak{s})^\perp, \mathcal{K}(\mathcal{H}_{\pi_\ell})\big)$ des applications continues $\varphi : (\mathfrak{u} + \mathfrak{s})^\perp \to \mathcal{K}(\mathcal{H}_{\pi_\ell})$ qui tendent vers zéro à l'infini et à valeurs dans l'algèbre des opérateurs compacts sur l'espace de Hilbert $\mathcal{H}_{\pi_\ell}$ de la représentation π_ℓ.

Si on restreint $p_{G/U}$ à l'algèbre de Fréchet $\mathcal{S}(G) \subset L^1(G)$, son image est l'algèbre de Fréchet

$$\begin{aligned} \mathcal{S}(G/U, \chi_\ell) \;=\; & \big\{ f \in L^1(G/U, \chi_\ell) |\ f \text{ lisse et pour tout sous-espace } \mathfrak{s}' \subset \mathfrak{g} \text{ avec } \mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{u} \\ & \text{et pour } S' = \exp(\mathfrak{s}'),\ f_{|S'} \in \mathcal{S}(S') \big\}. \end{aligned}$$

6.2.4 Résultats généraux

Théorème 6.8 (La topologie du spectre)**.**
Soit $(\pi_k, \mathcal{H}_{\pi_k})_{k\in\mathbb{N}}$ une famille de représentations irréductibles unitaires d'un groupe de Lie G. Alors $(\pi_k)_{k\in\mathbb{N}}$ converge vers π dans $\widehat{G}$ si et seulement si pour un vecteur non nul (respectivement pour tous les vecteurs non nul) ξ dans $\mathcal{H}_\pi$ et pour tout $k \in \mathbb{N}$, il existe $\xi_k \in \mathcal{H}_{\pi_k}$ tel que la suite des coefficients matriciels $\big(\langle \pi_k(\cdot)\xi_k, \xi_k \rangle\big)_{k\in\mathbb{N}}$ converge uniformément vers $\langle \pi(\cdot)\xi, \xi \rangle$ sur tout espace compact.

La démonstration de ce théorème se trouve dans [9], Théorème 13.5.2.

Lemme 6.9.
Soient G un groupe de Lie et $\mathfrak{g}$ l'algèbre de Lie correspondante. On définit la projection canonique p_G de $\mathfrak{g}^*$ à valeurs dans l'espace des orbites coadjointes $\mathfrak{g}^*/G$ et on munit l'espace $\mathfrak{g}^*/G$ de la topologie quotient, c.à.d. un sous-ensemble U dans $\mathfrak{g}^*/G$ est ouvert si et seulement si $p_G^{-1}(U)$ est ouvert dans $\mathfrak{g}^*$. Alors, une suite $(\mathcal{O}_k)_{k\in\mathbb{N}}$ d'éléments dans $\mathfrak{g}^*/G$ converge vers l'orbite $\mathcal{O} \in \mathfrak{g}^*/G$ si et seulement si pour $\ell \in \mathcal{O}$, il existe $\ell_k \in \mathcal{O}_k$ pour tout $k \in \mathbb{N}$ tel que $\ell = \lim\limits_{k\to\infty} \ell_k$.

Pour la preuve de ce lemme, voir [23], Chapitre 3.1.

6.3 Les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux

Dans cette sous-section, les groupes de Lie connexes réels nilpotents de pas deux seront examinés. Dans la première partie, quelques préliminaires sur les groupes de Lie nilpotents de pas deux seront donnés lesquels seront nécessaires pour comprendre la situation et en particulier la preuve des conditions énumérées dans la Définition 6.1. Dans la Section 6.3.3, les Conditions 1, 2 et 3(a) de la Définition 6.1 seront discutées et dans les Sections 6.3.4 à 6.3.8, la Condition 3(b) sera traitée. Ensuite, un résultat qui décrit les C^*-algèbres des groupes de Lie connexes réels nilpotents de pas deux sera donné.

La démonstration de la Condition 1 de la Définition 6.1 consiste en la construction des différentes couches du spectre de G et repose sur une méthode de construction d'une polarisation qui est développée dans [27]. Elle est plutôt technique mais pas très longue. La Condition 2 est la conséquence directe d'un résultat dans [8] et la Condition 3(a) dérive de la construction de la polarisation mentionnée plus haut. La majeure partie du travail pour la détermination des C^*-algèbres des groupes de Lie nilpotents de pas deux consiste en la preuve de la Propriété 3(b) de la Définition 6.1, et en particulier en la construction des applications $(\tilde{\nu}_k)_{k\in\mathbb{N}}$.

6.3.1 Préliminaires – Les groupes de Lie nilpotents de pas deux

Soit $\mathfrak{g}$ une algèbre de Lie réelle nilpotente de pas deux. Cela signifie que

$$[\mathfrak{g},\mathfrak{g}] := \operatorname{vect}\big\{[X,Y] \,|\, X,Y\in\mathfrak{g}\big\}$$

est contenu dans le centre de $\mathfrak{g}$.
On fixe un produit scalaire $\langle\cdot,\cdot\rangle$ sur $\mathfrak{g}$ comme pour les algèbres de Lie nilpotentes générales. La multiplication de Campbell-Baker-Hausdorff est dans ce cas de la forme

$$u\cdot v = u+v+\frac{1}{2}[u,v] \quad \forall u,v\in\mathfrak{g}$$

et on obtient de nouveau le groupe de Lie connexe simplement connexe $G=(\mathfrak{g},\cdot)$ avec l'algèbre de Lie $\mathfrak{g}$.

Comme $\mathfrak{g}$ est nilpotente de pas deux, $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}(\ell)$ et alors $\mathfrak{g}(\ell)$ est un idéal de $\mathfrak{g}$.
De plus, pour $\ell\in\mathfrak{g}^*$, tout sous-espace maximal isotrope $\mathfrak{p}$ de $\mathfrak{g}$ pour B_ℓ qui contient $[\mathfrak{g},\mathfrak{g}]$ est une polarisation de ℓ.

6.3.2 Préliminaires – La méthode des orbites

Dans le cas des groupes de Lie nilpotents de pas deux, pour tout élément $\ell\in\mathfrak{g}^*$ et pour tout $g\in G=(\mathfrak{g},\cdot)$ l'élément $\operatorname{Ad}^*(g)\ell$ est donné par

$$\operatorname{Ad}^*(g)\ell = \big(\mathbb{I}_{\mathfrak{g}^*} + \operatorname{ad}^*(g)\big)\ell \in \ell+\mathfrak{g}(\ell)^\perp.$$

Par conséquent, comme $\operatorname{ad}^*(\mathfrak{g})\ell=\mathfrak{g}(\ell)^\perp$,

$$\mathcal{O}_\ell := \operatorname{Ad}^*(G)\ell = \{\operatorname{Ad}^*(g)\ell \,|\, g\in G\} = \ell+\mathfrak{g}(\ell)^\perp \quad \forall\ell\in\mathfrak{g}^*, \tag{50}$$

ce qui veut dire que uniquement des orbites plates apparaissent. Ainsi,

$$\mathfrak{g}^*/G = \{\, \mathrm{Ad}^*(G)\ell |\ \ell \in \mathfrak{g}^*\} = \{\ell + \mathfrak{g}(\ell)^\perp |\ \ell \in \mathfrak{g}^*\}.$$

Alors, par la théorie de Kirillov, on obtient

$$\widehat{G} \cong \mathfrak{g}^*/G = \{\ell + \mathfrak{g}(\ell)^\perp |\ \ell \in \mathfrak{g}^*\}$$

en tant qu'espaces topologiques.

Maintenant, on considère une suite proprement convergente $\big([\pi_k]\big)_{k\in\mathbb{N}}$ dans $\widehat{G}$ avec pour ensemble limite $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$. Soient $\mathcal{O} \in \mathfrak{g}^*/G$ l'orbite coadjointe d'un élément $[\pi] \in L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$, $\mathcal{O}_k$ l'orbite coadjointe de $[\pi_k]$ pour tout $k \in \mathbb{N}$ et soit $\ell \in \mathcal{O}$. Alors, selon le Lemme 6.9, il existe pour tout $k \in \mathbb{N}$ un élément $\ell_k \in \mathcal{O}_k$ tel que $\lim\limits_{k\to\infty} \ell_k = \ell$ dans $\mathfrak{g}^*$. En considérant une sous-suite si nécessaire, on peut supposer que la suite $(\mathfrak{g}(\ell_k))_{k\in\mathbb{N}}$ converge par rapport à la topologie de la grassmannienne vers une sous-algèbre $\mathfrak{u}$ de $\mathfrak{g}(\ell)$ et qu'il existe un nombre $d \in \mathbb{N}$ tel que $\dim(\mathcal{O}_k) = 2d$ pour tout $k \in \mathbb{N}$. D'après (50), il résulte que

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} \ell_k + \mathfrak{g}(\ell_k)^\perp = \ell + \mathfrak{u}^\perp = \mathcal{O}_\ell + \mathfrak{u}^\perp \subset \mathfrak{g}^*/G.$$

Puisque dans le cas des groupes de Lie nilpotents de pas deux $\mathfrak{g}(\ell_k)$ contient $[\mathfrak{g},\mathfrak{g}]$ pour tout $k \in \mathbb{N}$, le sous-espace $\mathfrak{u}$ contient également $[\mathfrak{g},\mathfrak{g}]$. Ainsi, étant donné que l'application de Kirillov est un homéomorphisme et vu que $\mathfrak{u}^\perp$ n'est constitué que de caractères de $\mathfrak{g}$, l'ensemble limite $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$ dans $\widehat{G}$ de la suite $\big([\pi_k]\big)_{k\in\mathbb{N}}$ est le sous-ensemble "affine"

$$L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big) = \big\{\big[\chi_q \otimes \mathrm{ind}_P^G \chi_\ell\big] |\ q \in \mathfrak{u}^\perp\big\}$$

pour une polarisation $\mathfrak{p}$ de ℓ et $P := \exp(\mathfrak{p})$.

Ces observations conduisent à la proposition suivante:

Proposition 6.10.
Il y a trois types différents d'ensembles limites possibles de la suite $(\mathcal{O}_k)_{k\in\mathbb{N}}$ d'orbites coadjointes:

1. *dim$(\mathcal{O}_\ell) = 2d$: dans ce cas, l'ensemble limite $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ est le singleton $\mathcal{O}_\ell = \ell + \mathfrak{g}(\ell)^\perp$, c.à.d. $\mathfrak{u} = \mathfrak{g}(\ell)$.*

2. *dim$(\mathcal{O}_\ell) = 0$: ici, $L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \ell + \mathfrak{u}^\perp$ et l'ensemble limite $L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big)$ n'est constitué que de caractères, c.à.d. $q\big([\mathfrak{g},\mathfrak{g}]\big) = \{0\}$ pour tout $\mathcal{O}_q \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$.*

3. *La dimension de l'orbite $\mathcal{O}_\ell$ est strictement supérieure à 0 et strictement inférieure à $2d$. Dans ce cas $0 <$ dim$(\mathcal{O}) < 2d$ pour tout $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$ et*

$$L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \bigcup_{\mathfrak{q}\in\mathfrak{u}^\perp} q + \mathcal{O}_\ell, \quad \textit{c.à.d.} \quad L\Big(\big([\pi_k]\big)_{k\in\mathbb{N}}\Big) = \bigcup_{q\in\mathfrak{u}^\perp} \big[\chi_q \otimes \mathrm{ind}_P^G \chi_\ell\big]$$

pour une polarisation $\mathfrak{p}$ de ℓ et $P :=$ exp$(\mathfrak{p})$.

6.3.3 Les Conditions 1, 2 et 3(a)

A présent, on commence la vérification des conditions de la Définition 6.1.

A l'aide d'une procédure de construction de la polarisation de Vergne qui est élaborée dans [27], Section 1, on peut définir les familles de sous-ensembles $(S_i)_{i\in\{0,\dots,r\}}$ et $(\Gamma_i)_{i\in\{0,\dots,r\}}$ du spectre $\widehat{G}$ de G mentionnés dans la Condition 1 de la Définition 6.1 et on peut prouver qu'ils satisfont les propriétés souhaitées dans la Condition 1.
Dans cette construction, pour chaque $i \in \{0,...,r\}$, on obtient un nombre $d_i \in \mathbb{N}$ tel que toute orbite $\mathcal{O}$ appartenant à une représentation $\pi \in \Gamma_i$ a la même dimension $2d_i$, comme vu précédemment. On peut aussi démontrer que l'espace de Hilbert commun $\mathcal{H}_i$ qui apparaît dans la Condition 1(b) peut être choisi comme étant $\mathcal{H}_i = L^2(\mathbb{R}^{d_i})$.
Pour la preuve des Propriétés 2 et 3(a) de la Définition 6.1, on a la proposition suivante:

Proposition 6.11.
Pour tout $a \in C^*(G)$ et pour tout $i \in \{0,...,r\}$, l'application

$$\Gamma_i \to L^2(\mathbb{R}^{d_i}),\ \gamma \mapsto \mathcal{F}(a)(\gamma)$$

est continue par rapport à la norme et l'opérateur $\mathcal{F}(a)(\gamma)$ est compact pour tout $\gamma \in \Gamma_i$.

La compacité est la conséquence directe d'un théorème général qui peut être trouvé dans [8], Chapitre 4.2 ou [29], Partie II, Chapitre II.5 et qui dit que la C^*-algèbre $C^*(G)$ de tout groupe de Lie connexe nilpotent G satisfait la Condition CCR, c.à.d. l'image de toute représentation irréductible de $C^*(G)$ est un opérateur compact.
La continuité résulte de la construction des ensembles Γ_i pour $i \in \{0,...,r\}$ et du fait que les polarisations de Vergne $\mathfrak{p}_\ell^V$ de $\ell \in \mathfrak{g}^*$ qui sont utilisées pour cette construction sont continues en ℓ par rapport à la topologie de la grassmannienne.
Comme $C^*(G)$ est séparable, cela démontre les Propriétés 2 et 3(a) de la Définition 6.1. Il reste à présent à démontrer la Condition 3(b).

6.3.4 Condition 3(b) – Introduction de la situation

Pour simplifier, à partir de maintenant, les représentations seront identifiées avec leurs classes d'équivalence.

On fixe $i \in \{0,...,r\}$.
Pour $\ell' \in \mathfrak{g}^*$, soient $\mathfrak{p}_{\ell'}^V$ la polarisation de Vergne de ℓ' mentionnée en haut, $P_{\ell'}^V := \exp(\mathfrak{p}_{\ell'}^V)$ et $\pi_{\ell'}^V := \operatorname{ind}_{P_{\ell'}^V}^G \chi_{\ell'}$. De plus, pour une orbite $\mathcal{O}'$ soit $\ell_{\mathcal{O}'} \in \mathcal{O}'$ un élément fixe dont on a besoin pour la construction de la polaristation de Vergne.

Soit $\left(\pi_k^V\right)_{k\in\mathbb{N}} = \left(\pi_{\ell_{\mathcal{O}_k}}^V\right)_{k\in\mathbb{N}}$ une suite dans Γ_i dont l'ensemble limite se trouve hors de Γ_i. Comme toute suite convergente a une sous-suite proprement convergente, on supposera que $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ est proprement convergente et la transition à une sous-suite sera omise.
Alors la suite $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ correspondra à la sous-suite $(\gamma_k)_{k\in\mathbb{N}}$ de la Condition 3(b) de la Définition 6.1. C'est pourquoi pour tout $k \in \mathbb{N}$, on doit construire une application linéaire involutive $\tilde{\nu}_k : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ bornée par $C\|\cdot\|_{S_{i-1}}$ qui satisfait

$$\lim_{k\to\infty} \left\|\mathcal{F}(a)\left(\pi_k^V\right) - \tilde{\nu}_k\left(\mathcal{F}(a)_{|S_{i-1}}\right)\right\|_{op} = 0,$$

avec une constante $C > 0$ indépendante de k.
Pour réaliser ceci, on regarde la suite d'orbites coadjointes $(\mathcal{O}_k)_{k\in\mathbb{N}}$ qui correspond avec la suite $\left(\pi_k^V\right)_{k\in\mathbb{N}}$. Comme vu en haut, toute orbite $\mathcal{O}_k$ a la même dimension $2d := 2d_i$. De plus, la suite $(\mathcal{O}_k)_{k\in\mathbb{N}}$ converge proprement vers un ensemble d'orbites $L((\mathcal{O}_k)_{k\in\mathbb{N}})$.

Maintenant, l'algèbre de Lie $\mathfrak{g}$ sera examinée et décomposée en différentes parties. Ainsi, une nouvelle suite de représentations $(\pi_k)_{k\in\mathbb{N}}$ lesquelles seront équivalentes aux représentations $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ sera définie. Puis, dans les deuxième et troisième cas mentionnés dans la Proposition 6.10, on peut étudier $(\pi_k)_{k\in\mathbb{N}}$ et construire pour cette nouvelle suite des applications $(\nu_k)_{k\in\mathbb{N}}$ avec certaines propriétés similaires à celles dont on a besoin dans la Condition 3(b). A la fin, la convergence demandée peut être déduite de la convergence de $(\pi_k)_{k\in\mathbb{N}}$ et de l'équivalence des représentations π_k et π_k^V.

6.3.5 Condition 3(b) – Changement de la base de Jordan-Hölder

Soit $\tilde{\ell} \in \tilde{\mathcal{O}} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$. Alors il existe une suite $\left(\tilde{\ell}_k\right)_{k\in\mathbb{N}}$ dans $(\mathcal{O}_k)_{k\in\mathbb{N}}$ telle que $\tilde{\ell} = \lim\limits_{k\to\infty} \tilde{\ell}_k$.
Comme on s'intéresse aux orbites $\mathcal{O}_k = \tilde{\ell}_k + \mathfrak{g}\big(\tilde{\ell}_k\big)^{\perp}$, on peut remplacer la suite $\left(\tilde{\ell}_k\right)_{k\in\mathbb{N}}$ par une suite $(\ell_k)_{k\in\mathbb{N}}$ avec $\ell_k(A) = 0$ pour tout $A \in \mathfrak{g}\big(\tilde{\ell}_k\big)^{\perp} = \mathfrak{g}(\ell_k)^{\perp}$.
Ainsi, on obtient une autre suite convergente $(\ell_k)_{k\in\mathbb{N}}$ dans $(\mathcal{O}_k)_{k\in\mathbb{N}}$ dont la limite ℓ appartient à une orbite $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$.
Dans ce qui suit, cette suite $(\ell_k)_{k\in\mathbb{N}}$ sera étudiée et permettra d'effectuer la décomposition de $\mathfrak{g}$ mentionnée en haut.

On peut supposer que les sous-algèbres $(\mathfrak{g}(\ell_k))_{k\in\mathbb{N}}$ convergent vers une sous-algèbre $\mathfrak{u}$ dont le groupe de Lie correspondant $\exp(\mathfrak{u})$ est dénommé U. Ces sous-algèbres $\mathfrak{g}(\ell_k)$ pour $k \in \mathbb{N}$ peuvent être écrites comme

$$\mathfrak{g}(\ell_k) \;=\; [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{s}_k,$$

où $\mathfrak{s}_k \subset [\mathfrak{g}, \mathfrak{g}]^{\perp}$. En outre, soient $\mathfrak{n}_{k,0}$ le noyau de $\ell_{k|[\mathfrak{g},\mathfrak{g}]}$ et $\mathfrak{s}_{k,0}$ le noyau de $\ell_{k|\mathfrak{s}_k}$ pour tout $k \in \mathbb{N}$. On peut supposer que $\mathfrak{s}_{k,0} \neq \mathfrak{s}_k$ et on choisit $T_k \in \mathfrak{s}_k$ orthogonal à $\mathfrak{s}_{k,0}$ et de longueur 1. Etant plus facile, le cas $\mathfrak{s}_{k,0} = \mathfrak{s}_k$ pour $k \in \mathbb{N}$ sera omis.
De la même manière, on choisit $Z_k \in [\mathfrak{g}, \mathfrak{g}]$ orthogonal à $\mathfrak{n}_{k,0}$ et de longueur 1. De plus, on note $\mathfrak{r}_k = \mathfrak{g}(\ell_k)^{\perp} \subset \mathfrak{g}$.
En prenant une sous-suite si nécessaire, on peut supposer que les limites $\lim\limits_{k\to\infty} Z_k =: Z$, $\lim\limits_{k\to\infty} T_k =: T$ et $\lim\limits_{k\to\infty} \mathfrak{r}_k =: \mathfrak{r}$ existent.

Maintenant, des nouvelles polarisations $\mathfrak{p}_k$ de ℓ_k seront construites pour ensuite définir les représentations $(\pi_k)_{k\in\mathbb{N}}$.
La restriction à $\mathfrak{r}_k$ de la forme antisymétrique B_k définie par

$$B_k(V, W) := \langle \ell_k, [V, W] \rangle \quad \forall V, W \in \mathfrak{g}$$

est non-dégénérée sur $\mathfrak{r}_k$ et il existe un endomorphisme inversible S_k de l'espace $\mathfrak{r}_k$ tel que

$$\langle x, S_k(x') \rangle = B_k(x, x') \quad \forall x, x' \in \mathfrak{r}_k.$$

Alors S_k est antisymétrique, c.à.d. $S_k^t = -S_k$, et on peut décomposer $\mathfrak{r}_k$ en une somme directe orthogonale

$$\mathfrak{r}_k = \sum_{j=1}^{d} V_k^j$$

de sous-espaces de deux dimensions S_k-invariants. On choisit une base de Hilbert $\{X_j^k, Y_j^k\}$ de V_k^j. Alors,

$$\begin{aligned}
&[X_i^k, X_j^k] \in \mathfrak{n}_{k,0} \quad \forall i,j \in \{1,...,d\},\\
&[Y_i^k, Y_j^k] \in \mathfrak{n}_{k,0} \quad \forall i,j \in \{1,...,d\} \quad \text{et}\\
&[X_i^k, Y_j^k] = \delta_{i,j} c_j^k Z_k \bmod \mathfrak{n}_{k,0} \quad \forall i,j \in \{1,...,d\},
\end{aligned}$$

où $0 \neq c_j^k \in \mathbb{R}$ et $\sup\limits_{k\in\mathbb{N}} c_j^k < \infty$ pour tout $j \in \{1,...,d\}$.
En prenant encore une sous-suite si nécessaire, la suite $(c_j^k)_{k\in\mathbb{N}}$ converge pour tout $j \in \{1,...,d\}$ vers un c_j.
Puisque $X_j^k, Y_j^k \in \mathfrak{r}_k$ et $\ell_k(A) = 0$ pour tout $A \in \mathfrak{r}_k$, on a $\ell_k(X_j^k) = \ell_k(Y_j^k) = 0$ pour tout $j \in \{1,...,d\}$. En outre, on peut supposer que les suites $(X_j^k)_{k\in\mathbb{N}}, (Y_j^k)_{k\in\mathbb{N}}$ convergent dans $\mathfrak{g}$ vers les vecteurs X_j, Y_j qui forment une base modulo $\mathfrak{u}$ dans $\mathfrak{g}$.
Il résulte que

$$\begin{aligned}
\langle \ell_k, [X_j^k, Y_j^k] \rangle &= c_j^k \lambda_k, \text{ où}\\
\lambda_k &= \langle \ell_k, Z_k \rangle \overset{k\to\infty}{\longrightarrow} \langle \ell, Z \rangle =: \lambda.
\end{aligned}$$

Comme Z_k a été choisi orthogonal à $\mathfrak{n}_{k,0}$, $\lambda_k \neq 0$ pour tout $k \in \mathbb{N}$.

Maintenant, soient

$$\mathfrak{p}_k := \text{vect}\{Y_1^k, \cdots, Y_d^k, \mathfrak{g}(\ell_k)\}$$

et $P_k := \exp(\mathfrak{p}_k)$. Alors, $\mathfrak{p}_k$ est une polarisation de ℓ_k. De plus, on définit la représentation π_k par

$$\pi_k := \text{ind}_{P_k}^G \chi_{\ell_k}.$$

Puisque π_k et π_k^V sont des représentations induites par des polarisations et des caractères χ_{ℓ_k} et $\chi_{\ell_{\mathcal{O}_k}}$, où ℓ_k et $\ell_{\mathcal{O}_k}$ se trouvent dans la même orbite coadjointe $\mathcal{O}_k$, les deux représentations sont équivalentes, comme on a vu dans la Section 6.2.2.

Soit $\mathfrak{a}_k := \mathfrak{n}_{k,0} + \mathfrak{s}_{k,0}$. Alors $\mathfrak{a}_k$ est un idéal de $\mathfrak{g}$ sur lequel ℓ_k est 0. C'est pourquoi le sous-groupe normal $\exp(\mathfrak{a}_k)$ est contenu dans le noyau de la représentation π_k. En outre, soit $\mathfrak{a} := \lim\limits_{k\to\infty} \mathfrak{a}_k$.
De plus, soient $p \in \mathbb{N}$ et $\tilde{p} \in \{1,...,p\}$. On note $\mathfrak{n}_{k,0}$ la partie de $\mathfrak{a}_k$ incluse dans $[\mathfrak{g},\mathfrak{g}]$ et on se donne une base de Hilbert $\{A_1^k, \cdots, A_{\tilde{p}}^k\}$ de $\mathfrak{n}_{k,0}$. Puis on note $\mathfrak{s}_{k,0}$ la partie de $\mathfrak{a}_k$ qui n'appartient pas à $[\mathfrak{g},\mathfrak{g}]$ et on se donne une base de Hilbert $\{A_{\tilde{p}+1}^k, \cdots, A_p^k\}$ de $\mathfrak{s}_{k,0}$. Alors $\{A_1^k, \cdots, A_p^k\}$ est une base de Hilbert de $\mathfrak{a}_k$ et, comme avant, on peut supposer que la limite $\lim\limits_{k\to\infty} A_j^k = A_j$ existe pour tout $j \in \{1,...,p\}$.

Maintenant, pour tout $k \in \mathbb{N}$, on peut prendre comme base de Hilbert de $\mathfrak{g}$ aussi bien l'ensemble de vecteurs

$$\{X_1^k, \cdots, X_d^k, Y_1^k, \cdots, Y_d^k, T_k, Z_k, A_1^k, ..., A_p^k\}$$

que l'ensemble

$$\{X_1, \cdots, X_d, Y_1, \cdots, Y_d, T, Z, A_1, ..., A_p\}.$$

Cela donne les crochets de Lie suivants:

$$\begin{aligned} [X_i^k, Y_j^k] &= \delta_{i,j} c_j^k Z_k \bmod \mathfrak{a}_k, \\ [X_i^k, X_j^k] &= 0 \bmod \mathfrak{a}_k \quad \text{et} \\ [Y_i^k, Y_j^k] &= 0 \bmod \mathfrak{a}_k. \end{aligned}$$

Les vecteurs Z_k et T_k sont centrals modulo $\mathfrak{a}_k$.

6.3.6 Condition 3(b) – Premier cas

Premièrement, on considérera le cas où $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ consiste en un seul point limite $\mathcal{O}$, le premier cas mentionné dans la Proposition 6.10.
Dans ce cas, pour tout $k \in \mathbb{N}$,

$$2d = \dim(\mathcal{O}_k) = \dim(\mathcal{O}).$$

Par conséquent, la situation regardée apparaît si et seulement si $\lambda \neq 0$ et $c_j \neq 0$ pour tout $j \in \{1, ..., d\}$.

Le premier cas est le plus facile. Après quelques observations, les opérateurs $(\tilde{\nu}_k)_{k\in\mathbb{N}}$ peuvent immédiatement être définis. En outre, les assertions de la Condition 3(b) de la Définition 6.1 peuvent facilement être démontrées. Dans ce premier cas, la transition aux représentations $(\pi_k)_{k\in\mathbb{N}}$ n'est pas nécessaire.
On considère encore la suite $(\ell_k)_{k\in\mathbb{N}}$ qui a été choisie en haut et qui converge vers $\ell \in \mathcal{O}$. Comme les dimensions des orbites $\mathcal{O}_k$ et $\mathcal{O}$ sont les mêmes, il existe une sous-suite de $(\ell_k)_{k\in\mathbb{N}}$ (qui sera aussi dénommée $(\ell_k)_{k\in\mathbb{N}}$ pour simplifier) telle que $\mathfrak{p} := \lim_{k\to\infty} \mathfrak{p}_{\ell_k}^V$ soit une polarisation de ℓ, mais pas nécessairement la polarisation de Vergne. De plus, on définit $P := \exp(\mathfrak{p}) = \lim_{k\to\infty} P_{\ell_k}^V$ et

$$\pi := \operatorname{ind}_P^G \chi_\ell.$$

Si on identifie les espaces de Hilbert $\mathcal{H}_{\pi_{\ell_k}^V}$ et $\mathcal{H}_\pi$ des représentations $\pi_{\ell_k}^V = \operatorname{ind}_{P_{\ell_k}^V}^G \chi_{\ell_k}$ et π avec $L^2(\mathbb{R}^d)$, on peut déduire de [31], Théorème 2.3, que

$$\left\|\pi_{\ell_k}^V(a) - \pi(a)\right\|_{op} = \left\|\operatorname{ind}_{P_{\ell_k}^V}^G \chi_{\ell_k}(a) - \operatorname{ind}_P^G \chi_\ell(a)\right\|_{op} \overset{k\to\infty}{\longrightarrow} 0$$

pour tout $a \in C^*(G)$.
Comme π et $\pi_\ell^V = \operatorname{ind}_{P_\ell^V}^G \chi_\ell$ sont toutes les deux des représentations induites de polarisations et du même caractère χ_ℓ, elles sont équivalentes. Par conséquent, il existe un opérateur d'entrelacement unitaire

$$F : \mathcal{H}_{\pi_\ell^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_\pi \cong L^2(\mathbb{R}^d) \quad \text{tel que} \quad F \circ \pi_\ell^V(a) = \pi(a) \circ F \quad \forall a \in C^*(G).$$

De plus, les deux représentations $\pi_k^V = \pi_{\ell_{\mathcal{O}_k}}^V = \mathrm{ind}_{P^V_{\ell_{\mathcal{O}_k}}}^G \chi_{\ell_{\mathcal{O}_k}}$ et $\pi_{\ell_k}^V = \mathrm{ind}_{P^V_{\ell_k}}^G \chi_{\ell_k}$ sont équivalentes pour tout $k \in \mathbb{N}$ parce que ℓ_{O_k} et ℓ_k sont inclus dans la même orbite coadjointe $\mathcal{O}_k$ et $\mathfrak{p}^V_{\ell_{\mathcal{O}_k}}$ et $\mathfrak{p}^V_{\ell_k}$ sont des polarisations. Alors il existe d'autres opérateurs d'entrelacement unitaires

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi^V_{\ell_k}} \cong L^2(\mathbb{R}^d) \quad \text{tels que} \quad F_k \circ \pi_k^V(a) = \pi_{\ell_k}^V(a) \circ F_k \quad \forall a \in C^*(G).$$

On définit les opérateurs souhaités $\tilde{\nu}_k$ pour tout $k \in \mathbb{N}$ par

$$\tilde{\nu}_k(\varphi) := F_k^* \circ F \circ \varphi\big(\pi_\ell^V\big) \circ F^* \circ F_k \quad \forall \varphi \in CB(S_{i-1}),$$

ce qui est raisonnable puisque π_ℓ^V est un point limite de la suite $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ et donc contenu dans S_{i-1} d'après la définition de S_{i-1}.
Comme $\varphi\big(\pi_\ell^V\big) \in \mathcal{B}(L^2(\mathbb{R}^d))$ et F et F_k sont des opérateurs d'entrelacement, ces opérateurs sont bornés et l'image de $\tilde{\nu}_k$ est contenue dans $\mathcal{B}(L^2(\mathbb{R}^d))$, comme demandé.
Il est facile de démontrer que $\tilde{\nu}_k$ est borné et involutif. La dernière chose à vérifier est la convergence dont on a besoin pour la Condition 3(b): Pour tout $a \in C^*(G)$

$$\begin{aligned}
\big\|\pi_k^V(a) - \tilde{\nu}_k\big(\mathcal{F}(a)_{|S_{i-1}}\big)\big\|_{op} &= \big\|\pi_k^V(a) - F_k^* \circ F \circ \mathcal{F}(a)_{|S_{i-1}}\big(\pi_\ell^V\big) \circ F^* \circ F_k\big\|_{op} \\
&= \big\|\pi_k^V(a) - F_k^* \circ F \circ \pi_\ell^V(a) \circ F^* \circ F_k\big\|_{op} \\
&= \big\|F_k^* \circ \pi_{\ell_k}^V(a) \circ F_k - F_k^* \circ \pi(a) \circ F_k\big\|_{op} \\
&\leq \big\|\pi_{\ell_k}^V(a) - \pi(a)\big\|_{op} \overset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Ainsi, les représentations $\big(\pi_k^V\big)_{k\in\mathbb{N}}$ et les $(\tilde{\nu}_k)_{k\in\mathbb{N}}$ construits satisfont la Condition 3(b) de la Définition 6.1. Alors l'assertion est bien démontrée pour le premier cas.

6.3.7 Condition 3(b) – Deuxième cas

Dans le deuxième cas mentionné dans la Proposition 6.10, la situation $\lambda = 0$ ou $c_j = 0$ pour tout $j \in \{1, ..., d\}$ sera considérée.
Dans ce cas,

$$\langle \ell_k, [X_j^k, Y_j^k]\rangle = c_j^k \lambda_k \overset{k\to\infty}{\longrightarrow} c_j \lambda = 0 \quad \forall j \in \{1, ..., d\},$$

tandis que $c_j^k \lambda_k \neq 0$ pour tout k et tout $j \in \{1, ..., d\}$.
Alors $\ell_{|[\mathfrak{g},\mathfrak{g}]} = 0$, et comme $\tilde{\ell}_{|[\mathfrak{g},\mathfrak{g}]}$ s'annule évidemment aussi pour tout $\tilde{\ell} \in \mathfrak{u}^\perp$, chaque $\mathcal{O} \in L((\mathcal{O}_k)_{k\in\mathbb{N}})$ s'annule sur $[\mathfrak{g}, \mathfrak{g}]$. Ceci veut dire que la représentation associée est un caractère. C'est pourquoi toute orbite limite $\mathcal{O}$ dans l'ensemble $L((\mathcal{O}_k)_{k\in\mathbb{N}})$ est de dimension 0.

Maintenant, on doit adapter les méthodes développées dans [26] à la situation donnée ici.
On choisit pour $j \in \{1, ..., d\}$ les fonctions de Schwartz $\eta_j \in \mathcal{S}(\mathbb{R})$ telles que $\|\eta_j\|_{L^2(\mathbb{R})} = 1$ et $\|\eta_j\|_{L^\infty(\mathbb{R})} \leq 1$. Soient $s = (s_1, ..., s_d)$, $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{R}^d$. Alors on définit

$$\eta_{k,\alpha,\beta}(s) = \eta_{k,\alpha,\beta}(s_1, \cdots, s_d) := e^{2\pi i \alpha s} \prod_{j=1}^{d} \big|\lambda_k c_j^k\big|^{\frac{1}{4}} \eta_j\left(|\lambda_k c_j^k|^{\frac{1}{2}}\left(s_j + \frac{\beta_j}{\lambda_k c_j^k}\right)\right).$$

Cette définition est raisonnable parce que $\lambda_k c_j^k \neq 0$ pour chaque k et chaque $j \in \{1, ..., d\}$.
Pour $0 \neq \tilde{\eta} \in L^2(G/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^d)$ soit

$$P_{\tilde{\eta}} : L^2(\mathbb{R}^d) \to \mathbb{C}\tilde{\eta},\ \xi \mapsto \tilde{\eta}\langle \xi, \tilde{\eta} \rangle.$$

Alors $P_{\tilde{\eta}}$ est la projection orthogonale sur l'espace $\mathbb{C}\tilde{\eta}$.
Soit $h \in C^*(G/U, \chi_\ell)$. On identifie G/U avec $\mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$ et l'utilisation de la base limite introduite dans la Section 6.3.5 est exprimée par un index ∞:

$$h_\infty(x, y) = h_\infty(x_1, ..., x_d, y_1, ..., y_d) := \sum_{j=1}^{d} x_j X_j \ + \ \sum_{j=1}^{d} y_j Y_j.$$

Maintenant, $\hat{h}_\infty$ peut être vu comme une fonction dans $C_\infty(\ell + \mathfrak{u}^\perp) \cong C_\infty(\mathbb{R}^{2d})$. En utilisant cette identification, on peut définir l'opérateur linéaire

$$\nu_k(h) := \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) \ P_{\eta_{k,\tilde{x},\tilde{y}}} \ \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^{d} \left|\lambda_k c_j^k\right|}.$$

Alors on a la proposition suivante:

Proposition 6.12.

1. *Pour tout $k \in \mathbb{N}$ et tout $h \in \mathcal{S}(G/U, \chi_\ell)$, l'intégrale qui définit $\nu_k(h)$ converge par rapport à la norme d'opérateurs.*
2. *L'opérateur $\nu_k(h)$ est compact et $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U, \chi_\ell)}$.*
3. *ν_k est involutif, c.à.d. $\nu_k(h)^* = \nu_k(h^*)$ pour tout $h \in C^*(G/U, \chi_\ell)$.*

La preuve de cette proposition est plutôt longue et technique.
Avec cette proposition sur la suite de représentations $(\pi_k)_{k\in\mathbb{N}}$, on peut maintenant déduire l'assertion pour la suite initiale $(\pi_k^V)_{k\in\mathbb{N}}$.
Comme pour tout $k \in \mathbb{N}$ les deux représentations π_k et π_k^V sont équivalentes, il existe des opérateurs d'entrelacement unitaires

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \to \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{tels que} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall a \in C^*(G).$$

En outre, puisque l'ensemble limite $L\big((\pi_k^V)_{k\in\mathbb{N}}\big)$ de la suite $(\pi_k^V)_{k\in\mathbb{N}}$ est contenu dans S_{i-1}, en identifiant $\widehat{G}$ avec l'ensemble des orbites coadjointes $\mathfrak{g}^*/G$, on peut restreindre un champ d'opérateurs $\varphi \in CB(S_{i-1})$ à $L((\mathcal{O}_k)_{k\in\mathbb{N}}) = \ell + \mathfrak{u}^\perp$ et on obtient un élément dans $CB(\ell + \mathfrak{u}^\perp)$.
Alors, comme $\big\{\mathcal{F}(a)_{|L((\mathcal{O}_k)_{k\in\mathbb{N}})} |\ a \in C^*(G)\big\} = C_\infty(L((\mathcal{O}_k)_{k\in\mathbb{N}})) = C_\infty(\ell + \mathfrak{u}^\perp)$, on peut définir le $*$-isomorphisme

$$\tau : C_\infty(\mathbb{R}^{2d}) \cong C_\infty(\ell + \mathfrak{u}^\perp) \to C^*(G/U, \chi_\ell) \cong C^*(\mathbb{R}^{2d}),\ \mathcal{F}(a)_{|L((\mathcal{O}_k)_{k\in\mathbb{N}})} \mapsto p_{G/U}(a).$$

Maintenant, pour chaque $k \in \mathbb{N}$ on définit $\tilde{\nu}_k$ par

$$\tilde{\nu}_k(\varphi) \ := \ F_k^* \circ (\nu_k \circ \tau)\big(\varphi_{|L((\mathcal{O}_{\tilde{k}})_{\tilde{k}\in\mathbb{N}})}\big) \circ F_k \quad \forall \varphi \in CB(S_{i-1}).$$

Puisque l'image de ν_k est dans $\mathcal{B}(L^2(\mathbb{R}^d))$ et F_k est un opérateur d'entrelacement et donc borné, l'image de $\tilde{\nu}_k$ est aussi contenue dans $\mathcal{B}(L^2(\mathbb{R}^d))$.
De plus, on obtient que l'opérateur $\tilde{\nu}_k$ est borné et involutif en utilisant le fait que ν_k possède ces mêmes propriétés (voir Proposition 6.12). La convergence demandée dans la Condition 3(b) de la Définition 6.1 convient grâce à l'équivalence des représentations $(\pi_k)_{k\in\mathbb{N}}$ et $\left(\pi_k^V\right)_{k\in\mathbb{N}}$.

Il s'ensuit que les représentations $\left(\pi_k^V\right)_{k\in\mathbb{N}}$ satisfont la Propriété 3(b) et ainsi, les conditions de la Définition 6.1 sont démontrées.

6.3.8 Condition 3(b) – Troisième cas

Dans le troisième et dernier cas mentionné dans la Proposition 6.10, $\lambda \neq 0$ et il existe $1 \leq m < d$ tels que $c_j \neq 0$ pour chaque $j \in \{1, ..., m\}$ et $c_j = 0$ pour chaque $j \in \{m+1, ..., d\}$. Ceci veut dire que

$$\left\langle \ell_k, [X_j^k, Y_j^k] \right\rangle = c_j^k \lambda_k \overset{k\to\infty}{\longrightarrow} c_j\lambda = 0 \quad \Longleftrightarrow \quad j \in \{m+1, ..., d\}.$$

Dans ce cas $\mathfrak{p} := \text{vect}\{X_{m+1}, ..., X_d, Y_1, ..., Y_d, T, Z, A_1, ..., A_p\}$ est une polarisation de ℓ.
En outre, pour $\tilde{\mathfrak{p}}_k := \text{vect}\{X_{m+1}^k, ..., X_d^k, Y_1^k, ..., Y_d^k, T_k, Z_k, A_1^k, ..., A_p^k\}$, on a $\tilde{\mathfrak{p}}_k \overset{k\to\infty}{\longrightarrow} \mathfrak{p}$.

La démarche utilisée dans ce troisième cas est similaire à celle du deuxième cas. Ainsi, le troisième cas sera omis. Les formules et les calculs sont plus compliqués que dans les autres cas.

6.3.9 Résultat pour les groupes de Lie connexes réels nilpotents de pas deux

Avec les familles de sous-ensembles $(S_i)_{i\in\{0,...,r\}}$ et $(\Gamma_i)_{i\in\{0,...,r\}}$ de $\widehat{G}$ et les espaces de Hilbert $(\mathcal{H}_i)_{i\in\{0,...,r\}}$ qui satisfont les conditions citées dans la Définition 6.1, on obtient le résultat suivant:

Théorème 6.13.
La C^*-algèbre $C^*(G)$ d'un groupe de Lie G connexe réel nilpotent de pas deux est isomorphe (sous la transformation de Fourier) à l'ensemble de tous les champs d'opérateurs φ définis sur le spectre $\widehat{G}$ du groupe respectif tels que:

1. *$\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ pour tout $i \in \{1, ..., r\}$ et tout $\gamma \in \Gamma_i$.*
2. *$\varphi \in l^\infty(\widehat{G})$.*
3. *Sur chacun des différents ensembles Γ_i, l'application $\gamma \mapsto \varphi(\gamma)$ est continue par rapport à la norme.*
4. *Pour toute suite $(\gamma_k)_{k\in\mathbb{N}} \subset \widehat{G}$ tendant vers l'infini, $\lim\limits_{k\to\infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$.*
5. *Pour $i \in \{1, ..., r\}$ et toute suite proprement convergente $\overline{\gamma} = (\gamma_k)_{k\in\mathbb{N}} \subset \Gamma_i$ dont l'ensemble limite est contenu dans S_{i-1} (en prenant une sous-suite si nécessaire) et pour les applications $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma},k} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ construites dans les sections précédentes, on a*

$$\lim_{k\to\infty} \left\| \varphi(\gamma_k) - \tilde{\nu}_k\left(\varphi_{|S_{i-1}}\right) \right\|_{op} = 0.$$

6.4 La C^*-algèbre du groupe de Lie $SL(2,\mathbb{R})$

Dans cette sous-section, la C^*-algèbre de $SL(2,\mathbb{R})$ sera examinée. Au début, le groupe de Lie $SL(2,\mathbb{R})$ et certains de ses sous-groupes seront introduits, et des définitions et des résultats importants, nécessaires pour démontrer les conditions énumérées dans la Définition 6.1 et alors pour la détermination de la C^*-algèbre de $SL(2,\mathbb{R})$, seront évoqués. Les Sections 6.4.2 à 6.4.4 traitent du spectre de $SL(2,\mathbb{R})$ et de sa topologie et dans la Section 6.4.5 les conditions spécifiées en haut seront discutées pour le groupe $G = SL(2,\mathbb{R})$. Finalement, dans la Section 6.4.6, un résultat sur la C^*-algèbre de $SL(2,\mathbb{R})$ sera présenté, et la structure concrète de $C^*(G)$ sera donnée.

Pour le groupe $SL(2,\mathbb{R})$, la Condition 1 de la Définition 6.1 est rendue claire par les définitions des ensembles Γ_i et S_i pour $i \in \{0, ..., r\}$, lesquelles seront données dans la Section 6.4.5. La Condition 2 est satisfaite aussi, comme $SL(2,\mathbb{R})$ est un groupe de Lie connexe linéaire semi-simple. La démonstration de la Condition 3(a) est assez courte et ne représente pas de difficulté particulière, tandis que l'essentiel du travail de la preuve des conditions de la Définition 6.1 consiste encore une fois en la vérification de la Condition 3(b). Néanmoins, sa preuve est moins longue et moins technique que dans le cas des groupes de Lie nilpotents de pas deux.

6.4.1 Préliminaires

Dans cette section, soit

$$G := SL(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) |\ \det A = 1\}$$

et soit

$$K := SO(2) = \left\{ k_\varphi := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \,\middle|\, \varphi \in [0, 2\pi) \right\}$$

son sous-groupe compact maximal. De plus, on définit les sous-groupes unidimensionnels nilpotent N et abélien A de G comme étant

$$N := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \quad \text{et} \quad A := \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Soit

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) |\ \operatorname{tr} A = 0\}$$

l'algèbre de Lie de G.

Grâce à la décomposition de Iwasawa, on sait que $G = KAN$. Par conséquent, pour tout $g \in G$ il existe $\kappa(g) \in K$, $\mu \in N$ et $H(g) \in \mathfrak{a}$, où $\mathfrak{a} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$ est l'algèbre de Lie de A, tels que

$$g = \kappa(g) e^{H(g)} \mu.$$

En outre, on définit sur $\mathfrak{a}$ les applications ρ et ν_u pour $u \in \mathbb{C}$ via

$$\rho \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := t \quad \text{et} \quad \nu_u \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} := ut \quad \forall t \in \mathbb{R}.$$

Soient

$$\begin{aligned} L^2(K)_+ &:= \big\{f \in L^2(K) |\ f(k) = f(-k)\ \forall k \in K\big\} \quad \text{et} \\ L^2(K)_- &:= \big\{f \in L^2(K) |\ f(k) = -f(-k)\ \forall k \in K\big\}. \end{aligned}$$

Alors on définit pour tout $u \in \mathbb{C}$ les représentations $\mathcal{P}^{+,u}$ sur $\mathcal{H}_{\mathcal{P}^{+,u}} := L^2(K)_+$ et $\mathcal{P}^{-,u}$ sur $\mathcal{H}_{\mathcal{P}^{-,u}} := L^2(K)_-$ comme suit:

$$\mathcal{P}^{\pm,u}(g)f(k) := e^{-(\nu_u+\rho)H(g^{-1}k)} f\Big(\kappa\big(g^{-1}k\big)\Big) \quad \forall g \in G\ \forall f \in L^2(K)_\pm\ \forall k \in K.$$

Pour des raisons de simplicité, dans cette section, les représentations seront identifiées avec leurs classes d'équivalence.

Définition 6.14 (n-ième composante isotypique)**.**
Pour une représentation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ de K on définit pour tout $n \in \mathbb{Z}$ la n-ième composante isotypique de $\tilde{\pi}$ comme suit:

$$\mathcal{H}_{\tilde{\pi}}(n) := \big\{v \in \mathcal{H}_{\tilde{\pi}} |\ \tilde{\pi}(k_\varphi)v = e^{in\varphi}v\ \forall \varphi \in [0, 2\pi)\big\}.$$

Une représentation $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$ de G est dite paire (respectivement impaire), si $\mathcal{H}_{\tilde{\pi}_{|K}}(n) = \{0\}$ pour tout n impair (respectivement n pair).

Toute représentation irréductible unitaire de G est paire ou impaire.
En outre, la somme directe algébrique

$$\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_{\tilde{\pi}}(n)$$

est dense dans $\mathcal{H}_{\tilde{\pi}}$.

Remarque 6.15.
D'après la définition des espaces de Hilbert $L^2(K)_\pm$ de $\mathcal{P}^{\pm,u}$ pour $u \in \mathbb{C}$, il est facile de vérifier que $\mathcal{P}^{+,u}$ est pair pour tout $u \in \mathbb{C}$ et que $\mathcal{P}^{-,u}$ est impair pour tout $u \in \mathbb{C}$.

Définition 6.16 (p_n)**.**
Pour $n \in \mathbb{Z}$, on dénote $b_n(f)$ le n-ième coefficient de Fourier de $f \in L^2(K)_\pm$, défini par

$$b_n(f) := \frac{1}{|K|} \int\limits_K f(k_\varphi) e^{-in\varphi} dk_\varphi,$$

et on note

$$p_n(f) := b_{-n}(f) e^{-in\cdot}.$$

On peut facilement démontrer que pour tout $u \in \mathbb{C}$ et pour tout $n \in \mathbb{Z}$ l'opérateur p_n est la projection de $\mathcal{H}_{\mathcal{P}^{\pm,u}} = L^2(K)_\pm$ sur la n-ième composante isotypique de la représentation $\mathcal{P}^{\pm,u}$.

6.4.2 Le spectre de $SL(2,\mathbb{R})$ – Introduction de l'opérateur K_u

En utilisant l'opérateur de Knapp-Stein, maintenant un opérateur K_u, dont on a besoin pour décrire le spectre de G sera introduit.

On définit

$$\begin{aligned} C^\infty(K)_+ &:= \left\{f \in C^\infty(K) | \; f(k) = f(-k) \; \forall k \in K\right\} \quad \text{et} \\ C^\infty(K)_- &:= \left\{f \in C^\infty(K) | \; f(k) = -f(-k) \; \forall k \in K\right\} \end{aligned}$$

et soit

$$J_u : C^\infty(K)_+ \to C^\infty(K)_+ \quad \text{pour } u \in \mathbb{C} \text{ avec } \operatorname{Re} u > 0$$

l'opérateur d'entrelacement de Knapp-Stein de la représentation $\mathcal{P}^{+,u}$ vers la représentation $\mathcal{P}^{+,-u}$. En outre, soit $w := k_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. On étend $f \in L^2(K)$ à G en utilisant la décomposition de Iwasawa $G \ni g = \kappa(g) e^{H(g)} \mu$ pour $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ et $\mu \in N$. Ensuite, en définissant $\tilde{f}_u\big(\kappa(g)e^{H(g)}\mu\big) := e^{-(\nu_u+\rho)H(g)} f\big(\kappa(g)\big)$, cet opérateur peut être écrit comme

$$J_u f(k) = \int_N \tilde{f}_u(k\mu w) d\mu \quad \forall f \in C^\infty(K)_+ \; \forall k \in K. \tag{51}$$

Cette intégrale converge pour $\operatorname{Re} u > 0$ (voir [20], Chapitre VII ou [33], Chapitre 10.1).
L'application $f \mapsto J_u f$ est continue, et la famille d'opérateurs $\{J_u | \; u \in \mathbb{C}\}$ est holomorphe en u pour $\operatorname{Re} u > 0$ par rapport à des topologies appropriées (voir [20], Chapitre VII.7 ou [33], Chapitre 10.1).
Pour $u \in \mathbb{R}_{>0}$ l'opérateur J_u est autoadjoint par rapport au produit scalaire de $L^2(K)$ habituel.
De plus, on peut étendre la fonction $u \mapsto J_u$ méromorphiquement à $\mathbb{C}$ (voir [33], Chapitre 10.1). Puis, pour tout $u \in \mathbb{C}$ pour lequel l'opérateur J_u est régulier, J_u est un opérateur d'entrelacement de $\mathcal{P}^{+,u}$ vers $\mathcal{P}^{+,-u}$.

Grâce à la propriété d'entrelacement de J_u, on obtient la remarque suivante:

Remarque 6.17.
L'opérateur J_u commute avec les projections p_n pour tout $n \in \mathbb{Z}$ et pour tout $u \in \mathbb{C}$ pour lequel J_u est régulier.

Maintenant, on peut déduire que les opérateurs J_u ont la propriété

$$J_{u|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} = c_n(u) \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{pour tout } n \in \mathbb{Z} \text{ pair},$$

comme une équation de fonctions méromorphes, où $c_n : \mathbb{C} \to \mathbb{C}$ est une fonction méromorphe pour tout $n \in \mathbb{Z}$ pair. Ceci est une conséquence de la remarque ci-dessus considérée avec le fait que $\mathcal{H}_{\mathcal{P}^{+,u}}(n)$ est unidimensionnel.

En utilisant des formules d'intégrales standard (voir [20], Chapitre V.6), on obtient avec (51) que pour $u = 1$

$$J_1(f) = c \int_K f(k) dk \quad \forall f \in C^\infty(K)_+$$

pour une constante $c > 0$. De plus, pour tout $n \in \mathbb{Z}$ pair on a $\mathcal{H}_{\mathcal{P}^{+,u}}(n) = \mathbb{C} \cdot e^{-in\cdot}$. Ainsi, on obtient

$$c_n(1) \qquad \begin{cases} \neq 0 & \text{pour } n = 0 \\ = 0 & \text{pour } n \in \mathbb{Z} \setminus \{0\} \text{ pair.} \end{cases}$$

On peut démontrer que

$$c_0(u) \neq 0 \quad \text{pour } u \in (0,1)$$

et par conséquent, on peut définir

$$\tilde{J}_u := \frac{1}{c_0(u)}\ J_u$$

pour $u \in \mathbb{C}$ comme une fonction méromorphe. De surcroît, on peut prouver que

$$\tilde{J}_u \text{ est régulier pour } u = 0 \text{ et } \tilde{J}_0 = id.$$

Maintenant, on définit un produit scalaire sur $C^\infty(K)_+$ de la manière suivante:

$$\langle f_1, f_2 \rangle_u := \left\langle \tilde{J}_u f_1, f_2 \right\rangle_{L^2(K)}.$$

Ce produit scalaire est invariant et défini positif pour $u \in (0,1)$.
Le complété de $C^\infty(K)_+$ par rapport à ce produit scalaire $\langle \cdot, \cdot \rangle_u$ donne un espace de Hilbert $\mathcal{H}_u$. En considérant la restriction de la représentation $\mathcal{P}^{+,u}$ à $C^\infty(K)_+$ et en la prolongeant de façon continue sur l'espace $\mathcal{H}_u$, on obtient une représentation unitaire que l'on nomme $\mathcal{P}^{+,u}$ aussi. G agit sur $\mathcal{H}_u$ via cette représentation unitaire $\mathcal{P}^{+,u}$.

On pose $d_n(u) := \sqrt{\frac{c_n(u)}{c_0(u)}} > 0$ pour $u \in (0,1)$. On définit une bijection unitaire

$$K_u : \mathcal{H}_u \to L^2(K)_+ \quad \forall u \in (0,1)$$

comme suit: sur la n-ième composante isotypique de $\mathcal{H}_u$, on définit K_u par

$$K_{u|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} := d_n(u) \cdot id_{|\mathcal{H}_{\mathcal{P}^{+,u}}(n)} \quad \text{pour tout } n \in \mathbb{Z} \text{ pair.}$$

Puis, on peut étendre cette définition à des sommes finies de composantes isotypiques. Cet opérateur est aussi autoadjoint par rapport au produit scalaire habituel de $L^2(K)$ et pour des sommes finies de composantes isotypiques f_1 et f_2

$$\left\langle K_u f_1, K_u f_2 \right\rangle_{L^2(K)} = \left\langle K_u^2 f_1, f_2 \right\rangle_{L^2(K)} = \left\langle \tilde{J}_u f_1, f_2 \right\rangle_{L^2(K)} = \langle f_1, f_2 \rangle_u.$$

Par conséquent, l'opérateur est unitaire (si on regarde l'espace $\mathcal{H}_u$ muni de $\langle \cdot, \cdot \rangle_u$ et $L^2(K)_+$ muni de $\langle \cdot, \cdot \rangle_{L^2(K)}$) et ainsi, en raison de la densité de $\bigoplus_{n \in \mathbb{Z}} \mathcal{H}_{\mathcal{P}^{+,u}}(n)$ dans $\mathcal{H}_u$, on peut prolonger K_u de façon continue sur tout l'espace $\mathcal{H}_u$.
En outre, K_u est continu en u et on a aussi la propriété $\lim_{u \to 0} K_u = id$.
D'après sa définition, l'opérateur K_u commute également avec les projections p_n pour tout $n \in \mathbb{Z}$.

Ensuite, un autre produit scalaire sur $C^\infty(K)_{++} := \big\{f \in C^\infty(K)_+ | \; p_n(f) = 0 \; \forall n \leq 0\big\}$ est nécessaire.
Pour cela, on définit

$$\tilde{J}_{(u)} := \frac{1}{c_2(u)} \; J_u$$

pour $u \in \mathbb{C}$ comme une famille méromorphe d'opérateurs.
De surcroît, on définit l'espace $C^\infty(K)_{+-} := \big\{f \in C^\infty(K)_+ | \; p_n(f) = 0 \; \forall n \geq 0\big\}$.

Lemme 6.18.

(a) $\tilde{J}_{(-u)} \circ \tilde{J}_{(u)} = \tilde{J}_{(u)} \circ \tilde{J}_{(-u)} = id$ *en tant qu'opérateurs méromorphes.*

(b) $\tilde{J}_{(u)}$ *est régulier pour* $u = -1$ *et*

$$ker\big(\tilde{J}_{(-1)}\big) \cap C^\infty(K)_{++} = ker\big(\tilde{J}_{(-1)}\big) \cap C^\infty(K)_{+-} = \{0\}.$$

De plus, $\tilde{J}_{(-1)}$ *est un opérateur d'entrelacement de* $\mathcal{P}^{+,-1}$ *vers* $\mathcal{P}^{+,1}$.

(c) $\tilde{J}_{(u)|C^\infty(K)_{++}\oplus C^\infty(K)_{+-}}$ *est régulier pour* $u = 1$.

L'opérateur $\tilde{J}_{(1)}$ n'est pas identiquement zéro sur l'espace $C^\infty(K)_{++}$, étant donné que sa restriction à $\mathcal{H}_{\mathcal{P}^{+,1}}(2)$ est l'opérateur identité. Alors, on peut définir pour toutes fonctions $f_1, f_2 \in C^\infty(K)_{++}$

$$\langle f_1, f_2\rangle_{(1)} := \big\langle \tilde{J}_{(1)} f_1, f_2 \big\rangle_{L^2(K)}.$$

Ceci est un produit scalaire invariant et défini positif.
Maintenant, le complété de l'espace $C^\infty(K)_{++}$ par rapport à ce produit scalaire donne un espace de Hilbert que l'on nommera $\mathcal{H}_{(1)}$.

La même procédure peut être accomplie pour $C^\infty(K)_{+-}$.
Dans ce cas, on définit pour $u \in (0,1)$ l'opérateur $\tilde{J}_{[u]}$ de la manière suivante:

$$\tilde{J}_{[u]} := \frac{1}{c_{-2}(u)} \; J_u.$$

Comme précédemment, on peut démontrer que $\tilde{J}_{[u]}$ est régulier pour $u = 1$ sur $C^\infty(K)_{+-}$ et n'est pas identiquement nul sur cet espace.
Par conséquent, on définit de nouveau pour tous $f_1, f_2 \in C^\infty(K)_{+-}$

$$\langle f_1, f_2\rangle_{[1]} := \big\langle \tilde{J}_{[1]} f_1, f_2 \big\rangle_{L^2(K)}.$$

Ceci est un autre produit scalaire invariant et défini positif. Le complété de l'espace $C^\infty(K)_{+-}$ par rapport à ce produit scalaire donne un espace de Hilbert noté $\mathcal{H}_{[1]}$.

6.4.3 Le spectre de $SL(2,\mathbb{R})$ – Description des représentations irréductibles unitaires

Maintenant, des réalisations appropriées pour le spectre de $SL(2,\mathbb{R})$ seront fournies.

Le spectre $\widehat{G}$ de $G = SL(2,\mathbb{R})$ se compose des représentations suivantes:

1. Les représentations de la **série principale**:
   (a) $\mathcal{P}^{+,iv}$ pour $v \in [0,\infty)$.
   (b) $\mathcal{P}^{-,iv}$ pour $v \in (0,\infty)$.

   Voir Section 6.4.1 en haut pour les définitions.

2. Les représentations de la **série complémentaire** $\mathcal{C}^u$ pour $u \in (0,1)$:
   L'espace de Hilbert $\mathcal{H}_{\mathcal{C}^u}$ est défini comme

   $$\mathcal{H}_{\mathcal{C}^u} := L^2(K)_+$$

   et l'action est donnée par

   $$\mathcal{C}^u(g) := K_u \circ \mathcal{P}^{+,u}(g) \circ K_u^{-1}$$

   pour tout $g \in G$, où ici encore on considère la restriction de $\mathcal{P}^{+,u}(g)$ à $C^\infty(K)_+$ et puis on le prolonge de façon continue sur l'espace $\mathcal{H}_u$ (voir la définition de $\mathcal{H}_u$ dans la Section 6.4.2).

3. Les représentations de la **série discrète**:
   (a) $\mathcal{D}_m^+$ pour $m \in \mathbb{N}^*$ impair:
      (i) $\mathcal{D}_1^+$:
      L'espace de Hilbert $\mathcal{H}_{\mathcal{D}_1^+}$ est donné par

      $$\mathcal{H}_{\mathcal{D}_1^+} := \mathcal{H}_{(1)}$$

      défini dans la Section 6.4.2. L'action est donnée par

      $$\mathcal{D}_1^+ := \mathcal{P}^{+,1}.$$

      Ici encore, comme décrit précédemment et comme dans toutes les définitions dans cette sous-section, on restreint la représentation $\mathcal{P}^{+,u}$ pour les différentes valeurs $u \in \mathbb{C}$ au sous-espace correspondant de $L^2(K)$ et puis on la prolonge de façon continue sur l'espace de Hilbert correspondant.
      (ii) $\mathcal{D}_m^+$ pour $m \in \mathbb{N}_{\geq 3}$ impair:
      Comme espace de Hilbert $\mathcal{H}_{\mathcal{D}_m^+}$ pour $\mathcal{D}_m^+$ et $m \in \mathbb{N}_{\geq 3}$ impair, on peut prendre le complété de l'espace

      $$\left\{f \in C^\infty(K)_+ \mid p_n(f) = 0 \ \forall n \leq m-1\right\}$$

      par rapport à un produit scalaire approprié, et comme action on peut considérer

      $$\mathcal{D}_m^+ := \mathcal{P}^{+,m}.$$

Avec cette réalisation, les espaces de Hilbert $\mathcal{H}_{\mathcal{D}_m^+}$ pour $m \in \mathbb{N}_{\geq 3}$ impair dépendent de m. Mais puisqu'ils sont tous de dimension infinie et séparable, on peut les identifier si on conjugue l'action de G correspondante. Alors, on fixe un espace de Hilbert $\mathcal{H}_{\mathcal{D}}$ séparable de dimension infinie. En outre, l'action de G n'est pas nécessaire pour la détermination de $C^*(G)$.

(b) $\mathcal{D}_m^-$ pour $m \in \mathbb{N}^*$ impair:

(i) $\mathcal{D}_1^-$:
L'espace de Hilbert $\mathcal{H}_{\mathcal{D}_1^-}$ est donné par

$$\mathcal{H}_{\mathcal{D}_1^-} := \mathcal{H}_{[1]}$$

défini précédemment dans la Section 6.4.2 et l'action est donnée par

$$\mathcal{D}_1^- := \mathcal{P}^{+,1}.$$

(ii) $\mathcal{D}_m^-$ pour $m \in \mathbb{N}_{\geq 3}$ impair:
De la même manière que pour $\mathcal{D}_m^+$, comme espace de Hilbert $\mathcal{H}_{\mathcal{D}_m^-}$ pour $\mathcal{D}_m^-$ et $m \in \mathbb{N}_{\geq 3}$ impair, on peut prendre le complété de l'espace

$$\left\{f \in C^\infty(K)_+ \,|\, p_n(f) = 0 \ \forall n \geq -m+1\right\}$$

par rapport à un produit scalaire approprié, et comme action on peut considérer

$$\mathcal{D}_m^- := \mathcal{P}^{+,m}.$$

Encore une fois, les espaces de Hilbert dépendent de m. On les identifie et on prend l'espace de Hilbert $\mathcal{H}_{\mathcal{D}}$ commun séparable de dimension infinie fixé dans (a)(ii). Une fois de plus, l'action de G n'est pas nécessaire pour la détermination de $C^*(G)$.

(c) $\mathcal{D}_m^+$ pour $m \in \mathbb{N}^*$ pair:
Comme espace de Hilbert $\mathcal{H}_{\mathcal{D}_m^+}$ pour $\mathcal{D}_m^+$ et $m \in \mathbb{N}^*$ pair, on peut prendre le complété de l'espace

$$\left\{f \in C^\infty(K)_- \,|\, p_n(f) = 0 \ \forall n \leq m-1\right\}$$

par rapport à un produit scalaire approprié, et comme action on peut considérer

$$\mathcal{D}_m^+ := \mathcal{P}^{-,m}.$$

Encore une fois, les espaces de Hilbert sont identifiés et on prend l'espace de Hilbert $\mathcal{H}_{\mathcal{D}}$ commun séparable de dimension infinie, comme dans (a)(ii).

(d) $\mathcal{D}_m^-$ pour $m \in \mathbb{N}^*$ pair:
Comme espace de Hilbert $\mathcal{H}_{\mathcal{D}_m^-}$ pour $\mathcal{D}_m^-$ et $m \in \mathbb{N}^*$ pair, on peut prendre le complété de l'espace

$$\left\{f \in C^\infty(K)_- \,|\, p_n(f) = 0 \ \forall n \geq -m+1\right\}$$

par rapport à un produit scalaire approprié, et comme action on peut considérer

$$\mathcal{D}_m^- := \mathcal{P}^{-,m}.$$

Une fois de plus, les espaces de Hilbert sont identifiés et on prend l'espace de Hilbert $\mathcal{H}_{\mathcal{D}}$ commun, comme dans (a)(ii).

4. Les **limites de la série discrète**:

   (a) $\mathcal{D}_+$:
   L'espace de Hilbert $\mathcal{H}_{\mathcal{D}_+}$ est défini comme étant
   $$\mathcal{H}_{\mathcal{D}_+} := \left\{ f \in L^2(K)_- \middle| \, p_n(f) = 0 \ \forall n \leq 0 \right\}$$
   et l'action est donnée par
   $$\mathcal{D}_+ := \mathcal{P}^{-,0}.$$

   (b) $\mathcal{D}_-$:
   L'espace de Hilbert $\mathcal{H}_{\mathcal{D}_-}$ est défini comme étant
   $$\mathcal{H}_{\mathcal{D}_-} := \left\{ f \in L^2(K)_- \middle| \, p_n(f) = 0 \ \forall n \geq 0 \right\}$$
   et l'action est donnée par
   $$\mathcal{D}_- := \mathcal{P}^{-,0}.$$

5. La représentation **trivial** $\mathcal{F}_1$:
   Son espace de Hilbert $\mathcal{H}_{\mathcal{F}_1} = \mathbb{C}$ sera identifié avec l'espace de fonctions constantes
   $$\left\{ f \in L^2(K)_+ \middle| \, p_n(f) = 0 \ \forall n \neq 0 \right\}.$$
   Ici, l'action est donnée par
   $$\mathcal{F}_1(g) := id$$
   pour tout $g \in G$. On a aussi
   $$\mathcal{F}_1 = \mathcal{P}^{+,-1},$$
   puisque $\nu_{-1} + \rho = 0$ et tout $f \in \mathcal{H}_{\mathcal{F}_1}$ est une fonction constante.

Remarque 6.19.
Pour tout $n \in \mathbb{Z}$ et pour toute représentation π irréductible unitaire de G, l'opérateur p_n est la projection de $\mathcal{H}_\pi$ sur la n-ième composante isotypique $\mathcal{H}_\pi(n)$.

6.4.4 Le spectre de $SL(2,\mathbb{R})$ – La topologie sur $\widehat{SL(2,\mathbb{R})}$

En calculant les valeurs que les représentations $\mathcal{P}^{\pm,u}$ prennent appliquées à l'opérateur de Casimir, il est maintenant possible de décrire la topologie sur $\widehat{G}$.

Proposition 6.20.
La topologie sur $\widehat{G}$ peut être caractérisée de la manière suivante:

1. *Pour toute suite $(v_j)_{j\in\mathbb{N}}$ et tout v dans $[0,\infty)$,*
   $$\mathcal{P}^{+,iv_j} \overset{j\to\infty}{\longrightarrow} \mathcal{P}^{+,iv} \iff v_j \overset{j\to\infty}{\longrightarrow} v.$$

2. *Pour toute suite $(v_j)_{j\in\mathbb{N}}$ et tout v dans $(0,\infty)$,*
   $$\mathcal{P}^{-,iv_j} \overset{j\to\infty}{\longrightarrow} \mathcal{P}^{-,iv} \iff v_j \overset{j\to\infty}{\longrightarrow} v.$$
   Pour toute suite $(v_j)_{j\in\mathbb{N}}$ dans $[0,\infty)$,
   $$\mathcal{P}^{-,iv_j} \overset{j\to\infty}{\longrightarrow} \left\{ \mathcal{D}_+, \mathcal{D}_- \right\} \iff v_j \overset{j\to\infty}{\longrightarrow} 0.$$

3. Pour toute suite $(u_j)_{j\in\mathbb{N}}$ et tout u dans $(0,1)$,

$$\mathcal{C}^{u_j} \overset{j\to\infty}{\longrightarrow} \mathcal{C}^u \iff u_j \overset{j\to\infty}{\longrightarrow} u.$$

Pour toute suite $(u_j)_{j\in\mathbb{N}}$ dans $(0,1)$,

$$\mathcal{C}^{u_j} \overset{j\to\infty}{\longrightarrow} \mathcal{P}^{+,0} \iff u_j \overset{j\to\infty}{\longrightarrow} 0.$$

Pour toute suite $(u_j)_{j\in\mathbb{N}}$ dans $(0,1)$,

$$\mathcal{C}^{u_j} \overset{j\to\infty}{\longrightarrow} \{\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{F}_1\} \iff u_j \overset{j\to\infty}{\longrightarrow} 1.$$

Toutes les autres suites convergentes doivent d'une manière ou d'une autre se composer des suites regardées en haut.

6.4.5 Les conditions limites duales sous contrôle normique

Dans cette sous-section, le spectre sera divisé en différents sous-ensembles. Après, il doit être vérifié que ceux-ci satisfont les exigences des conditions "limites duales sous contrôle normique".
On définit

$$\begin{aligned}
\Gamma_0 &:= \{\mathcal{F}_1\},\\
\Gamma_1 &:= \{\mathcal{D}_1^+\},\\
\Gamma_2 &:= \{\mathcal{D}_1^-\},\\
\Gamma_3 &:= \{\mathcal{D}_+\},\\
\Gamma_4 &:= \{\mathcal{D}_-\},\\
\Gamma_5 &:= \{\mathcal{D}_m^\pm |\ m \in \mathbb{N}_{>1}\},\\
\Gamma_6 &:= \{\mathcal{P}^{+,iv} |\ v \in [0,\infty)\},\\
\Gamma_7 &:= \{\mathcal{P}^{-,iv} |\ v \in (0,\infty)\} \quad \text{et}\\
\Gamma_8 &:= \{\mathcal{C}^u |\ u \in (0,1)\}.
\end{aligned}$$

Clairement, tous les ensembles Γ_i pour $i \in \{0, ..., 8\}$ sont Hausdorff. De plus, les ensembles

$$S_i := \bigcup_{j\in\{0,\dots,i\}} \Gamma_j$$

sont fermés et $S_0 = \Gamma_0$ constitue l'ensemble de tous les caractères de $G = SL(2,\mathbb{R})$. En outre, comme défini dans la Section 6.4.3, pour tout $i \in \{0, ..., 8\}$, il existe un espace de Hilbert $\mathcal{H}_i$ commun sur lequel toutes les représentations dans Γ_i agissent. C'est pourquoi la Condition 1 de la Définition 6.1 est satisfaite.
Comme tout groupe de Lie réel connexe linéaire semi-simple satisfait la Condition CCR (voir [9], Théorème 15.5.6), la Condition 2 de la Définition 6.1 est remplie aussi. Il reste donc la Condition 3.
La Condition 3(a) est claire pour les ensembles Γ_i pour $i \in \{0, ..., 5\}$, comme ceux-ci sont des ensembles discrets. En revanche, des calculs doivent être effectués pour Γ_6, Γ_7 et Γ_8.

La Condition 3(b) est la partie la plus compliquée de la preuve des différentes conditions énumérées dans la Définition 6.1. La situation dans laquelle une suite $(\gamma_j)_{j\in\mathbb{N}}$ dans Γ_i converge vers un ensemble limite contenu dans $S_{i-1} = \bigcup_{l<i} \Gamma_l$ qui est regardée dans la Condition 3(b) ne peut apparaître que dans les cas suivants:

(i) $(\gamma_j)_{j\in\mathbb{N}} = \left(\mathcal{P}^{-,iv_j}\right)_{j\in\mathbb{N}}$ est une suite dans Γ_7 avec l'ensemble limite $\Gamma_3 \cup \Gamma_4 = \left\{\mathcal{D}_+, \mathcal{D}_-\right\}$.

(ii) $(\gamma_j)_{j\in\mathbb{N}} = \left(\mathcal{C}^{u_j}\right)_{j\in\mathbb{N}}$ est une suite dans Γ_8 avec l'ensemble limite $\left\{\mathcal{P}^{+,0}\right\} \subset \Gamma_7$.

(iii) $(\gamma_j)_{j\in\mathbb{N}} = \left(\mathcal{C}^{u_j}\right)_{j\in\mathbb{N}}$ est une suite contenue dans Γ_8 qui converge vers l'ensemble limite $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 = \left\{\mathcal{F}_1, \mathcal{D}_1^+, \mathcal{D}_1^-\right\}$.

Pour $i \in \{0, ..., 6\}$, les ensembles Γ_i sont fermés et, par conséquent, la situation regardée ne peut pas apparaître pour des suites $(\gamma_j)_{j\in\mathbb{N}}$ dans Γ_i pour $i \in \{0, ..., 6\}$.

6.4.6 Résultat pour le groupe de Lie $SL(2,\mathbb{R})$

Quand on a vérifié toutes les conditions énumérées dans la Section 6.3.4, le Théorème 6.2 est démontré pour le groupe de Lie $SL(2,\mathbb{R})$. C'est pourquoi, avec les ensembles Γ_i et S_i et les espaces de Hilbert $\mathcal{H}_i$ pour $i \in \{0, ..., 8\}$ définis dans les Sections 6.4.5 et 6.4.3 et les applications $\tilde{\nu}_j$ qui sont nécessaires pour la Condition 3(b) de la Définition 6.1, la C^*-algèbre de $G = SL(2,\mathbb{R})$ peut être caractérisée par le théorème suivant:

Théorème 6.21.
La C^*-algèbre $C^*(G)$ de $G = SL(2,\mathbb{R})$ est isomorphe (sous la transformation de Fourier) à l'ensemble de tous les champs d'opérateurs φ définis sur $\widehat{G}$ tels que:

1. *$\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$ pour tout $i \in \{1, ..., 8\}$ et tout $\gamma \in \Gamma_i$.*

2. *$\varphi \in l^\infty(\widehat{G})$.*

3. *Les applications $\gamma \mapsto \varphi(\gamma)$ sont continues par rapport à la norme sur les différents ensembles Γ_i.*

4. *Pour toute suite $(\gamma_j)_{j\in\mathbb{N}} \subset \widehat{G}$ tendant vers l'infini, $\lim_{j\to\infty} \|\varphi(\gamma_j)\|_{\mathrm{op}} = 0$.*

5. *Pour tout $i \in \{1, ..., 8\}$ et toute suite $\overline{\gamma} = (\gamma_j)_{j\in\mathbb{N}} \subset \Gamma_i$ proprement convergente dont l'ensemble limite se trouve dans S_{i-1} (en prenant une sous-suite si nécessaire) et pour les applications $\tilde{\nu}_j = \tilde{\nu}_{\overline{\gamma},j} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ nécessaires pour la Condition 3(b) de la Définition 6.1, on a*

$$\lim_{j\to\infty} \left\|\varphi(\gamma_j) - \tilde{\nu}\left(\varphi_{|S_{i-1}}\right)\right\|_{op} = 0.$$

Pour un espace topologique de Hausdorff V et une C^*-algèbre B, on note $C_\infty(V, B)$ la C^*-algèbre de toutes les fonctions continues définies sur V à valeurs dans B qui s'annulent à l'infini. Alors, de ce théorème on peut déduire plus concrètement le résultat suivant pour $G = SL(2,\mathbb{R})$:

Théorème 6.22.
Soient p_+ l'opérateur de projection de $L^2(K)_\pm$ sur $\big\{f \in L^2(K)_\pm \mid p_n(f) = 0\ \forall n \leq 0\big\}$, p_- la projection de $L^2(K)_\pm$ sur $\big\{f \in L^2(K)_\pm \mid p_n(f) = 0\ \forall n \geq 0\big\}$ et p_0 la projection de $L^2(K)_+$ sur $\big\{f \in L^2(K)_+ \mid p_n(f) = 0\ \forall n \neq 0\big\} = \mathbb{C}$.
Alors la C^*-algèbre $C^*(G)$ de $G = SL(2, \mathbb{R})$ est isomorphe à la somme directe de C^*-algèbres

$$\begin{aligned} & \Big\{F \in C_\infty\Big(i[0,\infty) \cup [0,1], \mathcal{K}\big(L^2(K)_+\big)\Big) \,\Big|\, F(1) \text{ commute avec } p_+,\ p_- \text{ et } p_0\Big\} \\ \oplus\ & \Big\{F \in C_\infty\Big(i[0,\infty), \mathcal{K}\big(L^2(K)_-\big)\Big) \,\Big|\, F(0) \text{ commute avec } p_+ \text{ et } p_-\Big\} \\ \oplus\ & C_\infty\Big(\mathbb{Z} \setminus \{-1,0,1\}, \mathcal{K}\big(\mathcal{H}_\mathcal{D}\big)\Big) \end{aligned}$$

pour l'espace de Hilbert $\mathcal{H}_\mathcal{D}$ séparable de dimension infinie fixé dans la Section 6.4.3.

6.5 Sur la topologie duale des groupes $\mathbf{U(n)} \ltimes \mathbb{H}_\mathbf{n}$

Dans cette section, le produit semi-direct $G_n = U(n) \ltimes \mathbb{H}_n$ pour $n \in \mathbb{N}^*$ sera analysé et la topologie de son spectre sera décrit.

Après avoir donné quelques préliminaires, le spectre et l'espace des orbites coadjointes admissibles de G_n seront décrits. Puis, la topologie du spectre de G_n sera comparée avec celle de l'espace des orbites coadjointes admissibles et, pour finir, des résultats seront déduits.

Comme déjà mentionné dans l'introduction, cette section est basée sur une prépublication par M.Elloumi et J.Ludwig qui peut être trouvée dans la thèse de doctorat de M.Elloumi (voir [11], Chapitre 3). Dans la présente thèse, elle a été élaborée et complétée et plusieurs changements importants ont été effectués. Pour plus de détails voir Section 5.

6.5.1 Préliminaires

Soit $\mathbb{C}^n$ l'espace vectoriel complexe n-dimensionnel muni du produit scalaire standard $\langle.,.\rangle_{\mathbb{C}^n}$ et soient respectivement $(.,.)_{\mathbb{C}^n}$ et $\omega(.,.)_{\mathbb{C}^n}$ les parties réelle et imaginaire de $\langle.,.\rangle_{\mathbb{C}^n}$. Le groupe de Heisenberg associé $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ de dimension $2n+1$ sur $\mathbb{R}$ est donné par la multiplication

$$(z,t)(z',t') := \Big(z + z', t + t' - \frac{1}{2}\omega(z,z')_{\mathbb{C}^n}\Big) \quad \forall (z,t),(z',t') \in \mathbb{H}_n.$$

En outre, on considère le groupe unitaire $U(n)$ d'automorphismes de $\mathbb{H}_n$ qui préserve $\langle.,.\rangle_{\mathbb{C}^n}$ sur $\mathbb{C}^n$ et qui s'immerge injectivement dans $\mathrm{Aut}(\mathbb{H}_n)$ via

$$A.(z,t) := (Az, t) \quad \forall A \in U(n)\ \forall (z,t) \in \mathbb{H}_n.$$

Par cette immersion, $U(n)$ est un sous-groupe compact connexe maximal de $\mathrm{Aut}(\mathbb{H}_n)$ (voir [14], Théorème 1.22 et [20], Chapitre I.1). On note $G_n = U(n) \ltimes \mathbb{H}_n$ le produit semi-direct de $U(n)$ avec le groupe de Heisenberg $\mathbb{H}_n$ muni de la multiplication

$$(A,z,t)(B,z',t') := \Big(AB, z + Az', t + t' - \frac{1}{2}\omega(z,Az')_{\mathbb{C}^n}\Big) \quad \forall (A,z,t),(B,z',t') \in G_n.$$

L'algèbre de Lie $\mathfrak{h}_n$ de $\mathbb{H}_n$ sera identifiée avec $\mathbb{H}_n$ lui-même via l'application exponentielle. Le crochet de Lie de $\mathfrak{h}_n$ est défini par

$$\big[(z,t),(w,s)\big] := \big(0, -\omega(z,w)_{\mathbb{C}^n}\big) \quad \forall (z,t),(w,s) \in \mathfrak{h}_n$$

et l'action dérivée de l'algèbre de Lie $\mathfrak{u}(n)$ de $U(n)$ sur $\mathfrak{h}_n$ est

$$A.(z,t) := (Az, 0) \quad \forall A \in \mathfrak{u}(n)\ \forall (z,t) \in \mathfrak{h}_n.$$

Notant $\mathfrak{g}_n = \mathfrak{u}(n) \ltimes \mathfrak{h}_n$ l'algèbre de Lie de G_n, on obtient le crochet de Lie

$$\big[(A,z,t),(B,w,s)\big] = \big(AB - BA, Aw - Bz, -\omega(z,w)_{\mathbb{C}^n}\big)$$

pour tous $(A,z,t),(B,w,s) \in \mathfrak{g}_n$.

L'algèbre de Lie $\mathfrak{u}(n)$ sera identifiée avec son espace vectoriel dual $\mathfrak{u}^*(n)$ à l'aide du produit scalaire $U(n)$-invariant

$$\langle A, B\rangle_{\mathfrak{u}(n)} := \operatorname{tr}(AB) \quad \forall A, B \in \mathfrak{u}(n)$$

et le dual $\mathfrak{g}_n^* = \big(\mathfrak{u}(n) \ltimes \mathfrak{h}_n\big)^*$ sera identifié avec $\mathfrak{u}(n) \oplus \mathfrak{h}_n$ à l'aide du produit scalaire

$$\big\langle (U,u,x),(B,w,s)\big\rangle_{\mathfrak{g}_n} = \langle U, B\rangle_{\mathfrak{u}(n)} + \omega(u,w)_{\mathbb{C}^n} + xs \quad \forall (U,u,x),(B,w,s) \in \mathfrak{g}_n.$$

Puis, en définissant $\times : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathfrak{u}^*(n)$ par $z \times w(B) := \omega(w, Bz)_{\mathbb{C}^n}$ pour tout $B \in \mathfrak{u}(n)$, on peut calculer que les orbites coadjointes de G_n ont la forme

$$\mathcal{O}_{(U,u,x)} = \left\{ \Big(AUA^* + z \times (Au) + \frac{x}{2} z \times z, Au + xz, x\Big) \Big| \ A \in U(n), z \in \mathbb{C}^n \right\}$$

pour chaque $(U,u,x) \in \mathfrak{g}_n^* = \mathfrak{u}(n) \times \mathbb{C}^n \times \mathbb{R}$, où A^* est la matrice adjointe de A.

6.5.2 Le spectre et les orbites coadjointes admissibles de G_n

La description du spectre de G_n est basée sur une méthode de Mackey (voir [28], Chapitre 10) qui dit qu'on doit déterminer les représentations irréductibles unitaires du sous-groupe $\mathbb{H}_n$ afin d'en construire des représentations de G_n.
Soit P_n l'ensemble des poids entiers maximaux λ pour $U(n)$ qui peuvent être écrits de la façon $\sum_{j=1}^{n} i\lambda_j e_j$ ou simplement comme $\lambda = (\lambda_1, \cdots, \lambda_n)$, où λ_j sont des entiers pour tout $j \in \{1, ..., n\}$ tels que $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Alors $P_n \cong \mathbb{Z}^n$.

Le spectre $\widehat{G}_n$ se compose des familles de représentations suivantes

(i) $\pi_{(\lambda,\alpha)}$ pour $\lambda \in P_n$ et $\alpha \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$,

(ii) $\pi_{(\mu,r)}$ pour $\mu \in P_{n-1}$ et $r \in \mathbb{R}_{>0}$, et

(iii) τ_λ pour $\lambda \in P_n$.

Alors $\widehat{G_n}$ est en bijection avec l'ensemble

$$(P_n \times \mathbb{R}^*) \cup \big(P_{n-1} \times \mathbb{R}_{>0}\big) \cup P_n.$$

Une fonctionnelle linéaire ℓ dans $\mathfrak{g}_n^*$ est dite admissible s'il existe un caractère unitaire χ de la composante connexe de l'identité de $G_n[\ell]$, le stabilisateur de ℓ dans G_n, tel que $d\chi = i\ell_{|\mathfrak{g}_n[\ell]}$. Maintenant, on peut associer des fonctionnelles linéaires $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ et ℓ_λ aux représentations $\pi_{(\lambda,\alpha)}$, $\pi_{(\mu,r)}$ et τ_λ et trouver qu'elles sont tous admissibles. Alors d'après [25], les représentations $\pi_{(\lambda,\alpha)}$, $\pi_{(\mu,r)}$ et τ_λ sont équivalentes aux représentations de G_n obtenues par induction holomorphe à partir des fonctionnelles linéaires associées respectives.
On note $\mathcal{O}_{(\lambda,\alpha)}$, $\mathcal{O}_{(\mu,r)}$ et $\mathcal{O}_\lambda$ les orbites coadjointes associées respectivement aux formes linéaires $\ell_{\lambda,\alpha}$, $\ell_{\mu,r}$ et ℓ_λ. En outre, soient $\mathfrak{g}_n^\ddagger \subset \mathfrak{g}_n^*$ l'union de tous les éléments dans $\mathcal{O}_{(\lambda,\alpha)}$, $\mathcal{O}_{(\mu,r)}$ et $\mathcal{O}_\lambda$ et on note $\mathfrak{g}_n^\ddagger/G_n$ l'ensemble correspondant dans l'espace des orbites. On peut déduire de [25] que $\mathfrak{g}_n^\ddagger$ est l'ensemble des fonctionnelles linéaires admissibles de $\mathfrak{g}_n$.
Alors l'espace quotient $\mathfrak{g}_n^\ddagger/G_n$ des orbites coadjointes admissibles est en bijection avec le spectre $\widehat{G_n}$.

On note v_r la forme linéaire identifiée avec $(0,...,0,r)^T$ dans $\mathbb{C}^n$ et $\mathcal{W}$ le sous-espace de $\mathfrak{u}(n)$ généré par les matrices $z \times v_r$ pour $z \in \mathbb{C}^n$. L'espace $\mathfrak{g}_n^\ddagger/G_n$ peut être écrit comme l'ensemble des orbites

$$\begin{aligned} \mathcal{O}_{(\lambda,\alpha)} &= \left\{ \left(AJ_\lambda A^* + \frac{i\alpha}{2} zz^*, \alpha z, \alpha \right) \Big|\ z \in \mathbb{C}^n,\ A \in U(n) \right\}, \\ \mathcal{O}_{(\mu,r)} &= \left\{ \big(A(J_\mu + \mathcal{W})A^*, Av_r, 0 \big) \big|\ A \in U(n) \right\} \quad \text{et} \\ \mathcal{O}_\lambda &= \left\{ (AJ_\lambda A^*, 0, 0) \big|\ A \in U(n) \right\} \end{aligned}$$

pour $\alpha \in \mathbb{R}^*$, $r > 0$, $\mu \in P_{n-1}$ et $\lambda \in P_n$ et où J_λ et J_μ sont des matrices dans $U(n)$ associées à λ et à μ respectivement.

6.5.3 Convergence dans l'espace quotient $\mathfrak{g}_n^\ddagger/G_n$ et la topologie du spectre de G_n

Dans le théorème suivant, les topologies de l'espace des orbites coadjointes admissibles et du spectre de G_n sont caractérisées.

Théorème 6.23.
Soient $\alpha \in \mathbb{R}^*$, $r > 0$, $\mu \in P_{n-1}$ et $\lambda \in P_n$.

1. *Une suite d'orbites coadjointes $\big(\mathcal{O}_{\lambda^k}\big)_{k\in\mathbb{N}}$ converge vers l'orbite $\mathcal{O}_\lambda$ dans $\mathfrak{g}_n^\ddagger/G_n$ si et seulement si $\lambda^k = \lambda$ pour k suffisamment grand. Ceci est le cas si et seulement si la suite $(\tau_{\lambda^k})_{k\in\mathbb{N}}$ de représentations irréductibles unitaires converge vers la représentation τ_λ dans $\widehat{G_n}$.*
2. *Une suite d'orbites coadjointes $\big(\mathcal{O}_{(\mu^k,r_k)}\big)_{k\in\mathbb{N}}$ converge vers $\mathcal{O}_{(\mu,r)}$ dans $\mathfrak{g}_n^\ddagger/G_n$ si et seulement si $\lim\limits_{k\to\infty} r_k = r$ et $\mu^k = \mu$ pour k suffisamment grand. Ceci est le cas si et seulement si la suite $\big(\pi_{(\mu^k,r_k)}\big)_{k\in\mathbb{N}}$ de représentations irréductibles unitaires converge vers la représentation $\pi_{(\mu,r)}$ dans $\widehat{G_n}$.*

3. *Une suite d'orbites coadjointes* $\left(\mathcal{O}_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ *converge vers* $\mathcal{O}_\lambda$ *dans* $\mathfrak{g}_n^\ddagger/G_n$ *si et seulement si* $(r_k)_{k\in\mathbb{N}}$ *converge vers* 0 *et* $\lambda_1 \geq \mu_1^k \geq \lambda_2 \geq \mu_2^k \geq ... \geq \lambda_{n-1} \geq \mu_{n-1}^k \geq \lambda_n$ *pour* k *assez grand. Ceci est le cas si et seulement si la suite* $\left(\pi_{(\mu^k,r_k)}\right)_{k\in\mathbb{N}}$ *de représentations irréductibles unitaires converge vers la représentation* τ_λ *dans* $\widehat{G_n}$.

4. *Une suite d'orbites coadjointes* $\left(\mathcal{O}_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *converge vers* $\mathcal{O}_{(\lambda,\alpha)}$ *dans* $\mathfrak{g}_n^\ddagger/G_n$ *si et seulement si* $\lim\limits_{k\to\infty}\alpha_k = \alpha$ *et* $\lambda^k = \lambda$ *pour* k *suffisamment grand. Ceci est le cas si et seulement si la suite* $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *de représentations irréductibles unitaires converge vers la représentation* $\pi_{(\lambda,\alpha)}$ *dans* $\widehat{G_n}$.

5. *Une suite d'orbites coadjointes* $\left(\mathcal{O}_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *converge vers l'orbite* $\mathcal{O}_{(\mu,r)}$ *dans* $\mathfrak{g}_n^\ddagger/G_n$ *si et seulement si* $\lim\limits_{k\to\infty}\alpha_k = 0$ *et la suite* $\left(\mathcal{O}_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *satisfait une des conditions suivantes:*

   (i) *Pour* k *suffisamment grand,* $\alpha_k > 0$, $\lambda_j^k = \mu_j$ *pour chaque* $j \in \{1,...,n-1\}$ *et* $\lim\limits_{k\to\infty}\alpha_k\lambda_n^k = -\frac{r^2}{2}$.

   (ii) *Pour* k *suffisamment grand,* $\alpha_k < 0$, $\lambda_j^k = \mu_{j-1}$ *pour chaque* $j \in \{2,...,n\}$ *et* $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = -\frac{r^2}{2}$.

   Ceci est le cas si et seulement si la suite $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *de représentations irréductibles unitaires converge vers la représentation* $\pi_{(\mu,r)}$ *dans* $\widehat{G_n}$.

6. *Une suite d'orbites coadjointes* $\left(\mathcal{O}_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *converge vers l'orbite* $\mathcal{O}_\lambda$ *dans* $\mathfrak{g}_n^\ddagger/G_n$ *si et seulement si* $\lim\limits_{k\to\infty}\alpha_k = 0$ *et la suite* $\left(\mathcal{O}_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *satisfait une des conditions suivantes:*

   (i) *Pour* k *assez grand,* $\alpha_k > 0$, $\lambda_1 \geq \lambda_1^k \geq ... \geq \lambda_{n-1} \geq \lambda_{n-1}^k \geq \lambda_n \geq \lambda_n^k$ *et* $\lim\limits_{k\to\infty}\alpha_k\lambda_n^k = 0$.

   (ii) *Pour* k *assez grand,* $\alpha_k < 0$, $\lambda_1^k \geq \lambda_1 \geq \lambda_2^k \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n^k \geq \lambda_n$ *et* $\lim\limits_{k\to\infty}\alpha_k\lambda_1^k = 0$.

   Ceci est le cas **si** *la suite* $\left(\pi_{(\lambda^k,\alpha_k)}\right)_{k\in\mathbb{N}}$ *de représentations irréductibles unitaires converge vers la représentation* τ_λ *dans* $\widehat{G_n}$.

Pour la description entière de la topologie de $\widehat{G_n}$, il manque encore la seconde implication de 6), c.à.d. que la convergence de la suite de représentations s'ensuit des conditions indiquées dans 6). On suppose que cette implication est correcte aussi mais la preuve est encore en construction. En revanche, pour $n = 1$, cette implication a déjà pu être démontrée.

6.5.4 Résultats

On peut déduire les résultats suivants:

Théorème 6.24.
L'application

$$\widehat{G}_n \longrightarrow \mathfrak{g}_n^{\ddagger}/G_n,\ \pi \mapsto \mathcal{O}_\pi$$

est continue.

Si on réussit à décrire entièrement la topologie de $\widehat{G_n}$ de la façon présentée précédemment, on obtient le résultat ci-dessous:

Conjecture 6.25.
Pour $n \in \mathbb{N}^*$ quelconque, le spectre du groupe $G_n = U(n) \ltimes \mathbb{H}_n$ est homéomorphe à l'espace $\mathfrak{g}_n^{\ddagger}/G_n$ des orbites coadjointes admissibles associé.

Pour $n = 1$ la situation est beaucoup plus facile que dans le cas général et donc, comme mentionné précédemment, cette conjecture a pu être prouvée pour ce cas-là.

Théorème 6.26.
Le spectre du produit semi-direct $U(1) \ltimes \mathbb{H}_1$ est homéomorphe à l'espace des orbites coadjointes admissibles associé.

7 Appendix

Lemma 7.1.
Using the notation from Section 3.2, let $d \in \{0, ..., [\frac{n}{2}]\}$ and $(J, K) \in \mathcal{M}$ with $|J| = |K| = d$. Then, the set

$$S_{i_{JK}} = \left\{ \left[\pi_{\ell_O}\right] \middle| \ \exists\ (J', K') \leq (J, K) : \ O \in (\mathfrak{g}^*/G)_{(J',K')} \right\}$$

is closed.

Proof:
Let $O_k \in (\mathfrak{g}^*/G)_{(J^k,K^k)}$ for $(J^k, K^k) \leq (J, K)$ for every $k \in \mathbb{N}$, let $O \in \mathfrak{g}^*/G$ and let $\left[\pi_{\ell_{O_k}}\right] \overset{k\to\infty}{\longrightarrow} \left[\pi_{\ell_O}\right]$. It needs to be shown that $O \in (\mathfrak{g}^*/G)_{(J',K')}$ for $(J', K') \leq (J, K)$.
Assume that this is not the case, i.e. that $O \in (\mathfrak{g}^*/G)_{(J',K')}$ for $(J', K') > (J, K)$.
Since $|J'| = |K'| > |J| = |K|$ is obviously not possible because of the convergence of the sequence $\left(\left[\pi_{\ell_{O_k}}\right]\right)_{k\in\mathbb{N}}$ to $\left[\pi_{\ell_O}\right]$, one has $|J'| = |K'| = |J| = |K| = d$.
So, let $J = \{j_1, ..., j_d\}$, $K = \{k_1, ..., k_d\}$, $J' = \{j'_1, ..., j'_d\}$ and $K' = \{k'_1, ..., k'_d\}$. Then, because $(J', K') > (J, K)$, there are two cases to regard:

1. There is an index $m \in \{1, ..., d\}$ such that $j_i = j'_i$ and $k_i = k'_i$ for all $i < m$ and $j_m < j'_m$.
2. There is an index $m \in \{1, ..., d\}$ such that $j_i = j'_i$ and $k_i = k'_i$ for all $i < m$, $j_m = j'_m$ and $k_m < k'_m$.

Regard the first case:
The case $m = 1$ being easier than the other cases, for simplicity, let $m = 2$. Hence, $j_1 = j'_1$, $k_1 = k'_1$ and $j_2 < j'_2$.
Now, take the largest index $\tilde{m} \in \mathbb{N}$ such that $j^k_i = j_i$ for all $i < \tilde{m}$ and $j^k_{\tilde{m}} < j_{\tilde{m}}$ for infinitely many $k \in \mathbb{N}$. Then, take a subsequence of $\left((J^k, K^k)\right)_{k\in\mathbb{N}}$ whose members all have this property. Since $(J^k, K^k) \leq (J, K)$, $\tilde{m}$ has to be smaller or equal to m.
So, without restriction, one can assume that $\tilde{m} = m = 2$, i.e. one can assume that $j^k_1 = j'_1$ and $j^k_2 < j'_2$ for all $k \in \mathbb{N}$.
By its definition, j'_2 is the largest index fulfilling $H^{1,\ell_O}_{j'_2} \notin \mathfrak{g}^{1,\ell_O}\left(\ell_{O_{|\mathfrak{g}^{1,\ell_O}}}\right)$. Furthermore, j^k_2 is the largest index fulfilling $H^{1,\ell_{O_k}}_{j^k_2} \notin \mathfrak{g}^{1,\ell_{O_k}}\left(\ell_{{O_k}_{|\mathfrak{g}^{1,\ell_{O_k}}}}\right)$. As $j^k_2 < j'_2$ for all $k \in \mathbb{N}$, one gets

$$\begin{aligned} \left\langle \ell_O, \left[H^{1,\ell_O}_{j'_2}, \mathfrak{g}^{1,\ell_O}\right] \right\rangle &\neq 0 \quad \text{and} \\ \left\langle \ell_{O_k}, \left[H^{1,\ell_{O_k}}_{j'_2}, \mathfrak{g}^{1,\ell_{O_k}}\right] \right\rangle &= 0 \quad \forall k \in \mathbb{N}. \end{aligned} \tag{52}$$

Moreover, since $j^k_1 = j'_1$ for all $k \in \mathbb{N}$,

$$\begin{aligned} \mathfrak{g}^{1,\ell_O} &= \left\{ U \in \mathfrak{g} \middle| \ \left\langle \ell_O, [U, H_{j'_1}] \right\rangle = 0 \right\} \quad \text{and} \\ \mathfrak{g}^{1,\ell_{O_k}} &= \left\{ U \in \mathfrak{g} \middle| \ \left\langle \ell_{O_k}, [U, H_{j^k_1}] \right\rangle = 0 \right\} \\ &= \left\{ U \in \mathfrak{g} \middle| \ \left\langle \ell_{O_k}, [U, H_{j'_1}] \right\rangle = 0 \right\} \quad \forall k \in \mathbb{N}. \end{aligned}$$

Because $\ell_{O_k} \overset{k\to\infty}{\longrightarrow} \ell_O$, one gets $\mathfrak{g}^{1,\ell_{O_k}} \overset{k\to\infty}{\longrightarrow} \mathfrak{g}^{1,\ell_O}$. Furthermore, by their construction, the basis elements $H_i^{1,\ell}$ depend continuously on ℓ for all $i \in \{1, ..., n\}$. Hence,

$$\left\langle \ell_{O_k}, \left[H_{j_2'}^{1,\ell_{O_k}}, \mathfrak{g}^{1,\ell_{O_k}}\right]\right\rangle \overset{k\to\infty}{\longrightarrow} \left\langle \ell_O, \left[H_{j_2'}^{1,\ell_O}, \mathfrak{g}^{1,\ell_O}\right]\right\rangle,$$

which is a contradiction to (52).
The second case is similar to the first case and will thus be skipped.
Therefore, $(J', K') > (J, K)$ is not possible and the closure is shown.

□

Lemma 7.2.
Let V be a finite-dimensional euclidean vector space and S an invertible skew-symmetric endomorphism. Then, V can be decomposed into an orthogonal direct sum of two-dimensional S-invariant subspaces.

Proof:
S extends to a complex endomorphism $S_{\mathbb{C}}$ on the complexification $V_{\mathbb{C}}$ of V, which has purely imaginary eigenvalues.
If $i\lambda \in i\mathbb{R}$ is an eigenvalue, then also $-i\lambda$ is a spectral element. Denote by $E_{i\lambda}$ the corresponding eigenspace. These eigenspaces are orthogonal to each other with respect to the Hilbert space structure of $V_{\mathbb{C}}$ coming from the euclidean scalar product $\langle \cdot, \cdot \rangle$ on V.
Let for $i\lambda$ in the spectrum of $S_{\mathbb{C}}$

$$V^\lambda := (E_{i\lambda} + E_{-i\lambda}) \cap V.$$

If $\lambda \neq 0$, $\dim\big(V^\lambda\big)$ is even and V^λ is S-invariant and orthogonal to $V^{\lambda'}$, whenever $|\lambda| \neq |\lambda'|$: Indeed, one then has for $x \in V^\lambda, x' \in V^{\lambda'}$,

$$x + iy \in E_{i\lambda} \quad \text{and} \quad x - iy \in E_{-i\lambda} \text{ for some } y \in V, \quad \text{as well as}$$
$$x' + iy' \in E_{i\lambda'} \quad \text{and} \quad x' - iy' \in E_{-i\lambda'} \text{ for some } y' \in V.$$

Therefore,
$$\langle x + iy, x' + iy'\rangle = 0 \quad \text{and} \quad \langle x - iy, x' + iy'\rangle = 0.$$

Thus, one has
$$\langle x, x' + iy'\rangle = 0 \quad \text{and hence,} \quad \langle x, x'\rangle = 0.$$

Suppose that $\dim\big(V^\lambda\big) > 2$, choose a vector $x \in V^\lambda$ of length 1 and let $y = S(x)$. Since $S_{\mathbb{C}}^2 = -\lambda^2\mathbb{I}$, both, on $E_{i\lambda}$ and on $E_{-i\lambda}$,

$$S(y) = S^2(x) = -\lambda^2 x.$$

This shows that $W_1^\lambda := \operatorname{span}\{x, y\}$ is an S-invariant subspace of V^λ. If V_1^λ denotes the orthogonal complement of W_1^λ in V^λ, then V_1^λ is S-invariant, since $S^t = -S$.
In this way, one can find a decomposition of V^λ into an orthogonal direct sum of two-dimensional S-invariant subspaces W_j^λ and by summing up over the eigenvalues, one obtains the required decomposition of V.

□

Lemma 7.3.
Let G be a Lie group, M a dense subset of $C^*(G)$ and $\mathcal{H}$ a Hilbert space. Furthermore, let for every $j \in \mathbb{N}$ the mapping $\vartheta_j : C^*(G) \to \mathcal{B}(\mathcal{H})$ be linear and bounded by $c\|\cdot\|_{C^*(G)}$ for a constant $c > 0$ (independent of j) and let $(\pi_j)_{j\in\mathbb{N}}$ be a sequence of representations of G on the Hilbert space $\mathcal{H}$ such that

$$\|\pi_j(h) - \vartheta_j(h)\|_{op} \overset{j\to\infty}{\longrightarrow} 0 \quad \forall h \in M.$$

Then,

$$\|\pi_j(a) - \vartheta_j(a)\|_{op} \overset{j\to\infty}{\longrightarrow} 0 \quad \forall a \in C^*(G).$$

Proof:
Let $a \in C^*(G)$ and $\varepsilon > 0$.
Since M is dense in $C^*(G)$, there exists $h \in M$ such that $\|a-h\|_{C^*(G)} < \frac{\varepsilon}{3\tilde{c}}$ for $\tilde{c} := \max\{1, c\}$.
Furthermore, there exists $J(\varepsilon) > 0$ in such a way that for all integers $j \geq J(\varepsilon)$, one has $\|\pi_j(h) - \vartheta_j(h)\|_{op} < \frac{\varepsilon}{3}$.
For all $j \geq J(\varepsilon)$,

$$\|\pi_j(a) - \vartheta_j(a)\|_{op} \leq \|\pi_j(a) - \pi_j(h)\|_{op} + \|\pi_j(h) - \vartheta_j(h)\|_{op} + \|\vartheta_j(h) - \vartheta_j(a)\|_{op}.$$

By assumption, $\|\pi_j(h) - \vartheta_j(h)\|_{op} < \frac{\varepsilon}{3}$. In addition, as π_j is a homomorphism,

$$\|\pi_j(a) - \pi_j(h)\|_{op} = \|\pi_j(a-h)\|_{op} \leq \sup_{\tilde{\pi}\in\widehat{G}} \left\|\tilde{\pi}(a-h)\right\|_{op} = \|a-h\|_{C^*(G)} < \frac{\varepsilon}{3\tilde{c}}.$$

Moreover,

$$\|\vartheta_j(h) - \vartheta_j(a)\|_{op} = \|\vartheta_j(h-a)\|_{op} \leq c\|h-a\|_{C^*(G)} < \frac{c\varepsilon}{3\tilde{c}} \leq \frac{\varepsilon}{3}.$$

Thus,

$$\|\pi_j(a) - \vartheta_j(a)\|_{op} < \varepsilon$$

and the claim is shown.

□

Corollary 7.4.
Let M be a dense subset of $C^*(G)$ and let π and $(\pi_j)_{j\in\mathbb{N}}$ be irreducible representations of G such that

$$\|\pi_j(h) - \pi(h)\|_{op} \overset{j\to\infty}{\longrightarrow} 0 \quad \forall h \in M.$$

Then,

$$\|\pi_j(a) - \pi(a)\|_{op} \overset{j\to\infty}{\longrightarrow} 0 \quad \forall a \in C^*(G).$$

Letting $\vartheta_j := \pi$ for all $j \in \mathbb{N}$, Corollary 7.4 follows directly with Lemma 7.3.

Corollary 7.5.
Let the sets Γ_i and S_i and the Hilbert spaces $\mathcal{H}_i$ for $i \in \{0, ..., 8\}$ be defined as in Section 4.2.5 and Section 4.2.2.
Let $i \in \{0, ..., 8\}$, let M be a dense subset of $C^*(G)$, let $\tilde{\nu} : CB(S_{i-1}) \to \mathcal{B}(\mathcal{H}_i)$ be a linear

map bounded by $c\|\cdot\|_{S_{i-1}}$ for a constant $c > 0$ and let $(\pi_j)_{j\in\mathbb{N}}$ be a sequence of representations in Γ_i such that

$$\Big\|\pi_j(h) - \tilde{\nu}\big(\mathcal{F}(h)_{|S_{i-1}}\big)\Big\|_{op} \stackrel{j\to\infty}{\longrightarrow} 0 \quad \forall h \in M.$$

Then,

$$\Big\|\pi_j(a) - \tilde{\nu}\big(\mathcal{F}(a)_{|S_{i-1}}\big)\Big\|_{op} \stackrel{j\to\infty}{\longrightarrow} 0 \quad \forall a \in C^*(G).$$

Proof:
Let $\mathcal{H} := \mathcal{H}_i$ and $\vartheta_j := \tilde{\nu}\big(\mathcal{F}(\cdot)_{|S_{i-1}}\big)$ for all $j \in \mathbb{N}$. Then, for all $a \in C^*(G)$,

$$\begin{aligned} \|\vartheta_j(a)\|_{op} &= \Big\|\tilde{\nu}\big(\mathcal{F}(a)_{|S_{i-1}}\big)\Big\|_{op} \leq c\big\|\mathcal{F}(a)_{|S_{i-1}}\big\|_{S_{i-1}} = c \sup_{\tilde{\pi}\in\mathcal{S}_{i-1}} \big\|\mathcal{F}(a)(\tilde{\pi})\big\|_{op} \\ &= c \sup_{\tilde{\pi}\in\mathcal{S}_{i-1}} \big\|\tilde{\pi}(a)\big\|_{op} \leq c \sup_{\gamma\in C^*(G)} \big\|\gamma(a)\big\|_{op} = c\|a\|_{C^*(G)}. \end{aligned}$$

Therefore, $\vartheta_j : C^*(G) \to \mathcal{B}(\mathcal{H})$ is bounded by $c\|\cdot\|_{C^*(G)}$ and Corollary 7.5 follows with Lemma 7.3.

□

Corollary 7.6.
Use the notations from Section 3.5 and Section 3.6 and let M be a dense subset of $C^*(G)$ such that

$$\Big\|\pi_j(h) - \nu_j\big(p_{G/U}(h)\big)\Big\|_{op} \stackrel{j\to\infty}{\longrightarrow} 0 \quad \forall h \in M.$$

Then,

$$\Big\|\pi_j(a) - \nu_j\big(p_{G/U}(a)\big)\Big\|_{op} \stackrel{j\to\infty}{\longrightarrow} 0 \quad \forall a \in C^*(G).$$

Proof:
Let $\mathcal{H} := L^2(\mathbb{R}^d)$ and let $\vartheta_j := \nu_j\big(p_{G/U}(\cdot)\big)$ for all $j \in \mathbb{N}$. Then, for all $a \in C^*(G)$, by Proposition 3.3 and Proposition 3.5, respectively,

$$\|\vartheta_j(a)\|_{op} = \Big\|\nu_j\big(p_{G/U}(a)\big)\Big\|_{op} \leq \big\|p_{G/U}(a)\big\|_{C^*(G/U,\chi_\ell)} \leq \|a\|_{C^*(G)}.$$

Therefore, $\vartheta_j : C^*(G) \to \mathcal{B}(\mathcal{H})$ is bounded by $c\|\cdot\|_{C^*(G)}$ and Corollary 7.6 follows with Lemma 7.3.

□

Lemma 7.7.
Let $n \in \mathbb{N}^*$, let $B^{2n}_{\mathbb{R}}$ be the $2n$-dimensional real unit ball and define the mapping

$$\begin{aligned} &\psi : [0,1]^{n-1} \times [0,2\pi)^n \times (0,1] \to B^{2n}_{\mathbb{R}}, \\ &\psi(s_1, ..., s_{n-1}, t_1, ..., t_n, \rho) := \\ &\rho\Big(\sqrt{s_1}\cos(t_1), \sqrt{s_1}\sin(t_1), ..., \sqrt{s_{n-1}}\cos(t_{n-1}), \sqrt{s_{n-1}}\sin(t_{n-1}), \\ &\quad \sqrt{1-s}\cos(t_n), \sqrt{1-s}\cos(t_n)\Big), \end{aligned}$$

where $s = \sum\limits_{i=1}^{n-1} s_i$.
Then, the absolute value of the determinant of the Jacobian of ψ equals $\frac{1}{2^{n-1}} \cdot \rho^{2n-1}$.

Proof:
Denote for $i \in \{1, ..., 2n\}$ by C_i the i-th column and by R_i the i-th row of the Jacobian of ψ.
For $i \in \{1, ..., n-1\}$, one has

$$C_{2i-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\rho\cos(t_i)}{2\sqrt{s_i}} \\ 0 \\ \vdots \\ 0 \\ -\rho\sqrt{s_i}\sin(t_i) \\ 0 \\ \vdots \\ 0 \\ \sqrt{s_i}\cos(t_i) \end{pmatrix}, \quad \begin{matrix} \leftarrow\ i - \text{th row}\ \rightarrow \\ \\ \\ \leftarrow\ (n-1+i) - \text{th row}\ \rightarrow \end{matrix} \quad C_{2i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\rho\sin(t_i)}{2\sqrt{s_i}} \\ 0 \\ \vdots \\ 0 \\ \rho\sqrt{s_i}\cos(t_i) \\ 0 \\ \vdots \\ 0 \\ \sqrt{s_i}\sin(t_i) \end{pmatrix}$$

and

$$C_{2n-1} = \begin{pmatrix} -\frac{\rho\cos(t_n)}{2\sqrt{1-s}} \\ \vdots \\ -\frac{\rho\cos(t_n)}{2\sqrt{1-s}} \\ 0 \\ \vdots \\ 0 \\ -\rho\sqrt{1-s}\sin(t_n) \\ \sqrt{1-s}\cos(t_n) \end{pmatrix}, \quad \leftarrow\ (n-1) - \text{th row}\ \rightarrow \quad C_{2n} = \begin{pmatrix} -\frac{\rho\sin(t_n)}{2\sqrt{1-s}} \\ \vdots \\ -\frac{\rho\sin(t_n)}{2\sqrt{1-s}} \\ 0 \\ \vdots \\ 0 \\ \rho\sqrt{1-s}\cos(t_n) \\ \sqrt{1-s}\sin(t_n) \end{pmatrix}.$$

Now, in several steps, this matrix will be transformed into a new matrix whose determinant can easily be calculated. For simplicity, the columns and rows of the matrices appearing in each step will also be denoted by C_i and R_i for $i \in \{1, ..., 2n\}$.
First, one takes out the factor ρ in the rows R_i for $i \in \{1, ..., 2n-1\}$, the factor $\frac{1}{2}$ in the rows R_i for $i \in \{1, ..., n-1\}$, the factor $\sqrt{s_i}$ in the columns C_{2i-1} and C_{2i} for every $i \in \{1, ..., n-1\}$ and the factor $\sqrt{1-s}$ in the columns C_{2n-1} and C_{2n}. Hence, one has the prefactor $\rho^{2n-1}\frac{1}{2^{n-1}}s_1 \cdots s_{n-1}(1-s)$ and the columns of the remaining matrix have the shape

$$C_{2i-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\cos(t_i)}{s_i} \\ 0 \\ \vdots \\ 0 \\ -\sin(t_i) \\ 0 \\ \vdots \\ 0 \\ \cos(t_i) \end{pmatrix}, \quad \begin{matrix} \leftarrow \ i - \text{th row} \ \rightarrow \\ \\ \\ \leftarrow \ (n-1+i) - \text{th row} \ \rightarrow \end{matrix} \quad C_{2i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\sin(t_i)}{s_i} \\ 0 \\ \vdots \\ 0 \\ \cos(t_i) \\ 0 \\ \vdots \\ 0 \\ \sin(t_i) \end{pmatrix}$$

for all $i \in \{1, ..., n-1\}$ and

$$C_{2n-1} = \begin{pmatrix} -\frac{\cos(t_n)}{1-s} \\ \vdots \\ -\frac{\cos(t_n)}{1-s} \\ 0 \\ \vdots \\ 0 \\ -\sin(t_n) \\ \cos(t_n) \end{pmatrix}, \quad \leftarrow \ (n-1) - \text{th row} \ \rightarrow \quad C_{2n} = \begin{pmatrix} -\frac{\sin(t_n)}{1-s} \\ \vdots \\ -\frac{\sin(t_n)}{1-s} \\ 0 \\ \vdots \\ 0 \\ \cos(t_n) \\ \sin(t_n) \end{pmatrix}.$$

Next, for every $i \in \{1, ..., n\}$, the column C_{2i-1} shall be replaced by $\sin(t_i)C_{2i-1} - \cos(t_i)C_{2i}$. Then, the prefactor changes to $\rho^{2n-1}\frac{1}{2^{n-1}}\frac{s_1 \cdots s_{n-1}(1-s)}{\sin(t_1)\cdots\sin(t_n)}$ and for every $i \in \{1, ..., n\}$,

$$C_{2i-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \ \leftarrow \ (n-1+i) - \text{th row}$$

The columns C_{2i} for $i \in \{1, ..., n\}$ stay the same.
Now, for all $i \in \{1, ..., n-1\}$, the rows R_i and R_{n-1+i} will be interchanged. Therefore, the prefactor is multiplied by $(-1)^{n-1}$ and for every $i \in \{1, ..., n-1\}$,

$$C_{2i-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \leftarrow i-\text{th row} \qquad C_{2n-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

and

$$C_{2i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \cos(t_i) \\ 0 \\ \vdots \\ 0 \\ \frac{\sin(t_i)}{s_i} \\ 0 \\ \vdots \\ 0 \\ \sin(t_i) \end{pmatrix}, \quad \begin{matrix} \leftarrow i-\text{th row} \\ n-\text{th row} \rightarrow \\ \leftarrow (n-1+i)-\text{th row} \end{matrix} \qquad C_{2n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{\sin(t_n)}{1-s} \\ \vdots \\ -\frac{\sin(t_n)}{1-s} \\ \cos(t_n) \\ \sin(t_n) \end{pmatrix}.$$

In the next step, for every $i \in \{1, ..., n-1\}$, the matrix will be developed with respect to the i-th row, which has only one non-zero entry, namely the entry -1 in the $(2i-1)$-th column. Furthermore, one develops with respect to the $(2n-1)$-th row which also only consists of one non-zero entry, -1, in the $(2n-1)$-th column. The prefactor is then multiplied by $(-1)^n(-1)^{2n-1+2n-1}\prod_{i=1}^{n-1}(-1)^{i+2i-1} = \prod_{i=1}^{n-1}(-1)^{n+i-1}$, i.e. the prefactor now equals

$$(-1)^{n-1}\prod_{i=1}^{n-1}(-1)^{n+i-1}\rho^{2n-1}\frac{1}{2^{n-1}}\frac{s_1\cdots s_{n-1}(1-s)}{\sin(t_1)\cdots\sin(t_n)} = \prod_{i=1}^{n-1}(-1)^{i}\rho^{2n-1}\frac{1}{2^{n-1}}\frac{s_1\cdots s_{n-1}(1-s)}{\sin(t_1)\cdots\sin(t_n)}.$$

One has a $n \times n$-matrix left, whose columns have the shape

$$C_{i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\sin(t_i)}{s_i} \\ 0 \\ \vdots \\ 0 \\ \sin(t_i) \end{pmatrix}, \quad \leftarrow i-\text{th row} \qquad C_{n} = \begin{pmatrix} -\frac{\sin(t_n)}{1-s} \\ \vdots \\ -\frac{\sin(t_n)}{1-s} \\ \sin(t_n) \end{pmatrix}$$

for all $i \in \{1,...,n-1\}$. Now, in every column C_i for $i \in \{1,...,n\}$, one can take out the factor $\sin(t_i)$. Then, the prefactor changes to $\prod_{i=1}^{n-1}(-1)^i\rho^{2n-1}\frac{1}{2^{n-1}}s_1\cdots s_{n-1}(1-s)$ and one has the following columns:

$$C_i \quad = \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{s_i} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \leftarrow \ i-\text{th row} \qquad C_n \quad = \quad \begin{pmatrix} -\frac{1}{1-s} \\ \vdots \\ -\frac{1}{1-s} \\ 1 \end{pmatrix}$$

for all $i \in \{1,...,n-1\}$. In the last step, the column C_n will be replaced by $C_n + \frac{1}{1-s}\sum_{i=1}^{n-1} s_i C_i$. Since

$$1 + \frac{1}{1-s}\sum_{i=1}^{n-1} s_i = \frac{1-s}{1-s} + \frac{1}{1-s}\ s = \frac{1}{1-s},$$

one obtains the columns

$$C_i \quad = \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{s_i} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \leftarrow \ i-\text{th row} \qquad C_n \quad = \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{1-s} \end{pmatrix}$$

for $i \in \{1,...,n-1\}$ and the prefactor stays the same, i.e. $\prod_{i=1}^{n-1}(-1)^i\rho^{2n-1}\frac{1}{2^{n-1}}s_1\cdots s_{n-1}(1-s)$. Since the remaining matrix is a triangular matrix, one can easily calculate its determinant and gets

$$\prod_{i=1}^{n-1}(-1)^i\rho^{2n-1}\frac{1}{2^{n-1}}\ s_1\cdots s_{n-1}(1-s)\ \frac{1}{s_1\cdots s_{n-1}(1-s)} = \prod_{i=1}^{n-1}(-1)^i\frac{1}{2^{n-1}}\cdot\rho^{2n-1}.$$

□

8 Acknowledgements

First, I want to express my deepest gratitude to my Ph.D. advisor Professor Jean Ludwig for the continuous support during the last years. I appreciated our regular meetings, his patience and his advice that I received despite his retirement two years ago.

Furthermore, my sincere thanks goes to my advisor Professor Martin Olbrich for his guidance during my Ph.D., for many fruitful discussions and for sharing his great expertise with me.

Besides my advisors, I want to thank Professor Ingrid Beltita and Professor Joachim Hilgert for having accepted to refer this thesis and Professor Angela Pasquale and Professor Anton Thalmaier for having joined my Dissertation Committee.

Moreover, I would like to express my thanks to the proofreaders for reading my manuscript, for their suggestions and ideas.

I also want to thank the Ph.D. students, Post-Docs and friends at the University of Luxembourg for the good working atmosphere, the nice and funny lunch and coffee breaks and the numerous joint activities that we undertook during the last four years.

Besides, I would like to give thanks to the Ph.D. students and friends at the Université de Lorraine in Metz, for the lunch breaks and the conferences we participated in together.

Last, I want to thank my parents who supported and encouraged me with everything and who have always been there for me. Without them, I would not have succeeded in my Ph.D.

This work is supported by the Fonds National de la Recherche, Luxembourg (Project Code 3964572).

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