Asset Pricing Models with Underlying Time-varying Lévy Processes

Bachelier World Congress 2016, New York

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July 15th, 2016

*Research funded by Fonds National de la Recherche Luxembourg (FNR)

Literature

Time-varying Jump Diffusion Framework

- Time-varying volatility: Empirical studies on the statistical properties of realized and/or implied volatilities have given rise to various stochastic volatility models in the literature, such as the Heston model, CEV models and also stochastic volatility models with jumps etc.

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- Existence of jumps is empirically supported: Carr and Wu (2003), Pan (2002).
- Jump intensity is time-varying: Christoffersen et al (2012).

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- Theoretical part: We assume a time-varying Lévy process, with time-varying drift, volatility and jump intensity parameters, to model the jump diffusion economy. We study an equilibrium asset and option pricing model in this economy.

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- Theoretical part: We assume a time-varying Lévy process, with time-varying drift, volatility and jump intensity parameters, to model the jump diffusion economy. We study an equilibrium asset and option pricing model in this economy.
- Empirical part: Under this general framework, we decompose S&P500 index into time-varying processes of drift, volatility and jump, using the Hodrick-Prescott filter and a particle filter.

Model Build-up

Stock Market

- An investment of S_t in the *stock market* is governed by:

$$\frac{dS_t}{S_{t-}} = \mu(t)dt + \sigma(t)dB_t + (e^x - 1)dN_t - \lambda(t)E(e^x - 1)dt, \quad (1)$$

where S_{t-} is the value of S_t before a possible jump occurs; $\mu(t)$ and $\sigma(t)$ are the *rate of return* and the *volatility* of the investment.

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The jump part is assumed to be a Poisson process, with jump intensity $\lambda(t)$ and jump size x which follows an arbitrary distribution.

Money Market Account

- We further assume that there is a market for instantaneous borrowing and lending at a *risk-free rate* $r(t)$. The *money market account*, M_t, follows

$$\frac{dM_t}{M_t} = r(t)dt. \tag{2}$$

The risk-free rate, $r(t)$, will be derived from the general equilibrium later, as a part of the solution.

Representative Investor

- Maximize the expected utility function of life time consumption

$$\max_{c_t} E_t \int_t^T p(t)U(c_t)dt,$$

where c_t is the rate of consumption at time t,
$U(c)$ the utility function with $U' > 0$, $U'' < 0$, and
$p(t) \geq 0$, $0 \leq t \leq T$ the *time preference function*.

- Assume *constant relative risk aversion (CRRA) utility* function.

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$$U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1, \\ \ln c, & \gamma = 1, \end{cases}$$

where the constant γ is the *relative risk aversion coefficient*,
$\gamma = -cU''/U'$.

Total Wealth

- The total wealth of the representative investor at time t:

$$W_t = W_{1t} + W_{2t}$$

where $W_{1t} = \omega W_t$ is invested in the stock market, and $W_{2t} = (1 - \omega) W_t$ is invested in the money market.

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ω is called the *wealth ratio*.

- Representative Investor's Optimal Control Problem:

$$\max_{c_t,\omega} E_t \int_t^T p(t)U(c_t)dt, \tag{3}$$

subject to

$$\begin{aligned}\frac{dW_t}{W_t} &= \omega\frac{dS_t}{S_{t-}} + (1-\omega)\frac{dM_t}{M_t} - \frac{c_t}{W_t}dt \\ &= [r(t) + \omega\mu(t) - \omega r(t) - \omega\lambda(t)E(e^x - 1) - \frac{c_t}{W_t}]dt + \omega\sigma(t)dB_t \\ &\quad + \omega(e^x - 1)dN_t,\end{aligned}$$

where $\phi(t) = \mu(t) - r(t)$ is the *equity premium*.

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where $\phi(t) = \mu(t) - r(t)$ is the *equity premium*.

- Market Clearing: Because there is only one investor in the economy, he has to put all the wealth into the stock market. The general equilibrium occurs at $\omega = 1$, under which the market is cleared.

Equity Premium

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Proposition

- *In the production economy with jump diffusion and one representative investor with CRRA utility function, the equilibrium equity premium is given by*

$$\phi(t) = \phi_\sigma(t) + \phi_J(t),$$

where $\phi_\sigma(t) = \gamma\sigma(t)^2$ *-diffusion risk premium*

$$\phi_J(t) = \lambda(t)E[(1 - e^{-\gamma x})(e^x - 1)] \text{ -jump risk premium}$$

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- *The risk-free rate is a time-varying function:*

$$r(t) = \mu(t) - \phi(t) = \mu(t) - \phi_\sigma(t) - \phi_J(t).$$

General Pricing Kernel

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- *The pricing kernel is given by*

$$\frac{d\pi_t}{\pi_t} = -r(t)dt - \gamma\sigma(t)dB_t + (e^y - 1)dN_t - \lambda(t)E(e^y - 1)dt,$$

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- *or equivalently, after integration*

$$\frac{\pi_T}{\pi_t} = \exp\{-\int_t^T \gamma\sigma(s)dB_s - \int_t^T [r(s) + \frac{1}{2}\gamma^2\sigma^2(s)]ds - E(e^y - 1)\int_t^T \lambda(s)ds + \sum_{i=1}^{N_{t,T}} y_i\}.$$

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- *The random variable y modeling the jump size in the logarithm of the pricing kernel, satisfies $E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$.*

European Call

Proposition

The price of a European call, $c(S_t, t)$, in the jump diffusion economy satisfies

$$\frac{\partial c(S_t,t)}{\partial t} + \frac{1}{2}\sigma^2(t)S_t^2\frac{\partial^2 c(S_t,t)}{\partial S^2} + [r(t) - \lambda^Q(t)E^Q(e^x - 1)]S_t\frac{\partial c(S_t,t)}{\partial S} - r(t)c(S_t,t) + \lambda^Q(t)\{E^Q[c(S_te^x,t)] - c(S_t,t)\} = 0,$$

with final condition

$$c(S_T, T) = max(S_T - K, 0),$$

where $\lambda^Q(t) \equiv \lambda(t)E(e^y)$: jump intensity in the risk-neutral measure Q,defined by $E^Q[f(x)] := \frac{E[e^y f(x)]}{E(e^y)}$, for any function $f(x)$.

European Call

Proposition

Pricing formula of a European call option:

$$c(S_t, t) = \sum_{n=0}^{+\infty} e^{-\int_t^T \lambda^Q(s)ds} \frac{(\int_t^T \lambda^Q(s)ds)^n}{n!} E_n^Q[c^{BS}(Se^X e^{-E^Q(e^x-1)\int_t^T \lambda^Q(s)ds}, t)],$$

where $c^{BS}(S,t)$ *is the Black-Scholes formula price for the European call option and*

$X = \sum_{i=1}^n x_i.$

Empirical Part

Decompose the S&P500 Index into time-varying components, using

- the Hodrick-Prescott Filter
- a particle filter (Sequential Monte Carlo Method)

Hodrick-Prescott Filter

- The Hodrick-Prescott filters was first proposed in Whittaker (1923), then popularized in economics by Hodrick and Prescott (1997).

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- The method serves to decompose the time series $y_t = ln(S_t)$ into a trend component τ_t, and a cyclical component c_t:

$$y_t = \tau_t + c_t, \text{ for } t = 1, \ldots, T.$$

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$$y_t = \tau_t + c_t, \text{ for } t = 1, \ldots, T.$$

- Condition: For a given a, τ_t satisfies

$$\min_{\tau}\left(\sum_{t=1}^{T}(y_t - \tau_t)^2 + a\sum_{t=2}^{T-1}[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2\right),$$

where $a = 129600$ for monthly data[1].

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Extract Drift

Data:

- S&P500 index, daily, 1985 - 2014.
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- S&P500 index, daily, 1985 - 2014.
- In each month, we use the 5% to 95% quantile of $ln(S_t)$, compute the mean as a monthly data input for HP filter.

As a result, we decompose the stock index into a time-varying trend component T and a component C:

$$ln(S_t) = T + C.$$

T is a monthly drift, C the remaining process of volatility plus jumps.

Time-varying Drift

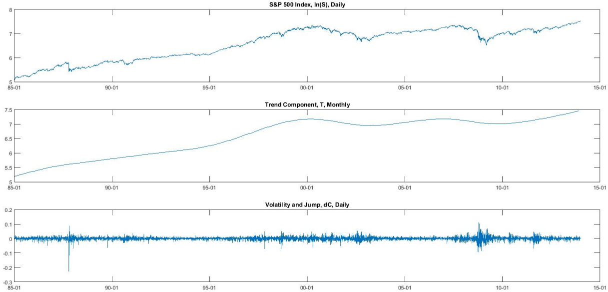

Figure 1.

	mean $(\times 10^{-5})$	volatility	skewness	kurtosis
Δ ln(S)	31.9	0.0115	-1.3044	31.8
ΔC	1.91	0.0117	-1.2229	30.1

Table 1.

By taking the difference of the time-varying trend (ΔT), we can observe that:

- Regime Switching: Before 2000, stock return was positive. However, after 2000 we can observe that it fluctuates around zero.
- Volatility/Jump Clustering: In negative return periods, there exists jumps and volatility clustering. By contrast, in positive return period, volatility/jump process is much less volatile.

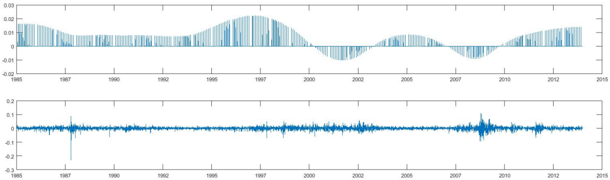

Figure 2.

Filtering Problems

- For filtering problem, the data is generated by the state space model, which consists of the observation and state evolution equations,
  - Observation equation: $y_t = f(x_t, \epsilon_t^y)$
  - State evolution: $x_{t+1} = g(x_t, \epsilon_{t+1}^x)$,

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  where ϵ_{t+1}^y is the observation error or "noise", and ϵ_{t+1}^x are state shocks.
- Particle filters belong to statistical filtering methods, which usually refer to an algorithm for extracting a latent state variable (e.g. volatility) from noisy observations (e.g. stock price/return) using a statistical model.

Particle Filters

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- They are simulation-based estimation methods, which include a set of algorithms that estimate the posterior density by directly implementing the Bayesian recursion equations.
- The state space model used in particle filters can be non-linear and the initial state and noise distributions can take any form required.

SIR Algorithm

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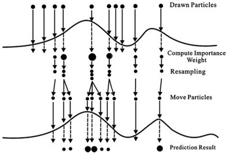
- SIR includes two steps: given samples from $p^N(x_t|y^t)$,
  - S1. Propagation: for $i = 1, ..., N$, draw $x_{t+1}^{(i)} \sim p(x_{t+1}|x_t^{(i)})$.
  - S2. Resampling: for $i = 1, ..., N$, draw $z^{(i)} \sim Mult_N(w_{t+1}^{(1)}, \ldots, w_{t+1}^{(N)})$, with importance sampling weights $w_{t+1}^{(i)} = \frac{p(y_{t+1}|x_{t+1}^{(i)})}{\sum_{l=1}^{N} p(y_{t+1}|x_{t+1}^{(l)})}$, and set $x_{t+1}^{(i)} = x_{t+1}^{z^{(i)}}$.



State Variable of Volatility

- As a state space model (for volatility) is necessary to implement particle filters, we assume that following dynamics for the stochastic variance:

$$d\nu_t = k(\theta - \nu_t)dt + \sigma_\nu \sqrt{\nu_t} dB_t^\nu,$$

where ν_t is a mean-reverting stochastic process. B_t^ν is a Brownian motion correlated with B_t, with correlation coefficient ρ.

Filter out Volatility

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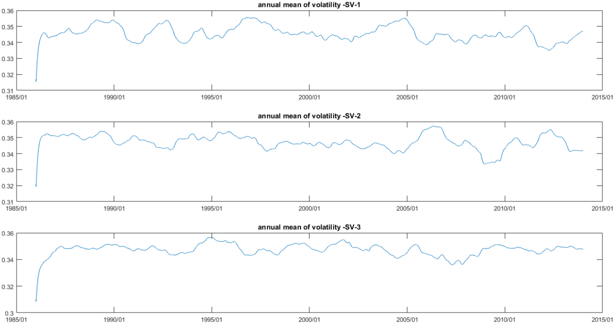
- We start with particle filters under stochastic volatility (SV) model without jumps, then we apply particle filters under stochastic volatility and jump (SVJ) model.
- The parameters used for the particle filters are taken from Eraker, Johannes and Polson (2003).

Filtered Volatility Processes I - SV model

- We run particle filters under SV model three times. Estimated volatilities stay around 0.34-0.35. However, the pattern of the volatility processes varies each time.
- Note that the hump shape on the left sides are caused by an adaptation period (around 200 initial data points) needed by the algorithm.



Filtered Volatility Processes II - SVJ model

- Following Eraker, Johannes and Polson (2003), we assume a jump intensity of $\lambda = 0.006$, meaning 1 to 2 jumps per year; the jump size follows a normal distribution.

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- With the SVJ model, filtered volatilities decrease to 0.2-0.25, as jumps account for some of the excess variance.
- We detect a high possibility of jumps around 1987-1988, and some other infrequent jumps. Overall jumps are rare in this model. We observe high level of volatilities when the probability of a jump occuring is high.

Filtered Volatility Processes II - SVJ model

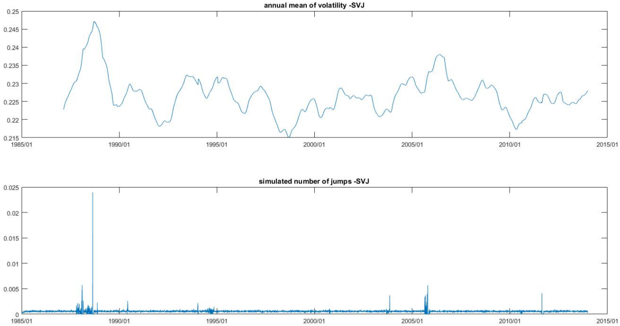

Figure 4. Filtered volatility and jump processes under SVJ model

Future Research

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- It will be interesting to use option data jointly with return data in the filtering methods.

Main References

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Thank you!