On strategically equivalent contests

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joint work with

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Outline

1. Basic Framework - Deterministic rent

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2. General Framework - Risky rent
   - Strategically equivalent contests
   - Situations in which the proportional contest dominates

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The vector of efforts (x_i, x_j) determines what each player will finally get.

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Assumption 2. Players are expected utility maximizer.

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Strategically equivalent contests

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Definition (Chowdhury and Sheremeta (2014))

Contests are effort equivalent if they result in the same Nash equilibrium level of effort.

The case of a deterministic rent

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- **Lottery contest (L)**

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Payoff

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Expected utility

$$EU\pi_i^L(x) = \frac{x_i}{x_i + x_j} U_i(V - x_i) + \frac{x_j}{x_i + x_j} U_i(-x_i).$$

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$$\pi_i^P = \left(p_i V - x_i; 1\right).$$

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$$EU\pi_i^P(x) = U(\frac{x_i}{x_i + x_j} V - x_i).$$

The case of risk-neutral players

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If players are **risk-neutral** expected utility maximizer ($U(z) = z$)

$$EU\pi_i^L(x) = \frac{x_i}{x_i + x_j}V - x_i \equiv UE\pi_i^L(x) \equiv U\pi_i^P(x).$$

$$D_{x_i}(\frac{x_i}{x_i + x_j}V - x_i) = V\frac{x_j}{(x_i + x_j)^2} - 1.$$

Theorem

The two contests have Nash equilibrium $x^* = V/4$ and are thus strategically equivalent.

The case of risk averse players

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Assumption 3. The players have constant absolute risk-averse (CARA) preferences represented by the negative exponential utility function

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We can then show that

- $x_P^* = \frac{V}{4}$.
- $x_L^* = \frac{e^{Vr}-1}{2r(e^{Vr}+1)} = \frac{1}{2r}\tanh(\frac{Vr}{2})$.

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$$2r(x_P^* - x_L^*) = 2r(\frac{V}{4} - \frac{1}{2r}\tanh(\frac{Vr}{2}))$$

$$= \frac{Vr}{2} - \tanh(\frac{Vr}{2}) > 0,$$

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It's Laplace transform f^* is defined by

$$f^*(t) = E[exp(tV)].$$

The proportional contest

The proportional contest

Expected utility:

$$Eu(\pi_i) = E(-e^{-r\pi_i}) = \int_{-\infty}^{+\infty} (-e^{-r(p_i u - x_i)}) f(u) du$$
$$= -e^{rx_i} \int_{-\infty}^{+\infty} e^{-rp_i u} f(u) du$$
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This gives us an equilibrium effort

$$x_P^* = \frac{1}{4} \left(\frac{f^* \left(-\frac{r}{2}\right)'}{f^* \left(-\frac{r}{2}\right)} \right).$$

The lottery contest

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Expected utility:

$$
\begin{aligned}
Eu(\pi_1) &= \frac{x_1}{x_1+x_2}\left(\int_{-\infty}^{+\infty}(-\exp(-r(u-x_1))f(u)du\right) + \frac{x_2}{x_1+x_2}(-e^{rx_1}) \\
&= -\frac{x_1\exp(rx_1)}{x_1+x_2}\left(\int_{-\infty}^{+\infty}\exp(-ru)f(u)du\right) - \frac{x_2}{x_1+x_2}(e^{rx_1}) \\
&= -\frac{x_1\exp(rx_1)}{x_1+x_2}f^*(-r) - \frac{x_2\exp(rx_1)}{x_1+x_2}.
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This gives us an equilibrium effort

$$x_L^* = \frac{1-f^*(-r)}{2r(1+f^*(-r))}.$$

Differential equation

Strategically equivalent contests thus exist if

$$\frac{1}{4}\frac{(f^*)'\left(-\frac{r}{2}\right)}{f^*\left(-\frac{r}{2}\right)} = \frac{1-f^*(-r)}{2r(1+f^*(-r))},$$

where f^* is the Laplace transform of the density of the rent.

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This is a nonlinear, non local differential equation.

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Then, $f(x) = \lambda e^{-\lambda x}$, $f^*(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$ and $(f^*(t))' = \frac{\lambda}{(t-\lambda)^2}$.

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Hence,

$$x_P^* = \frac{1}{4}\left(\frac{f^*\left(-\frac{r}{2}\right)'}{f^*\left(-\frac{r}{2}\right)}\right) = \frac{1}{4}\left(\frac{\frac{\lambda}{\left(-\frac{r}{2}-\lambda\right)^2}}{\frac{\lambda}{\lambda - -\frac{r}{2}}}\right) = \frac{1}{2r + 4\lambda}$$

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$\Rightarrow$

$$x_P^* = x_L^* = \frac{1}{2r + 4\lambda}.$$

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$x_L^* = \frac{1-a+br-cr^2}{2r(1+a-br+cr^2)}$.

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Both contests are strategically equivalent and f is a density function of a random variable iff $a = 1$, $c = 0$ and $0 < b < 2/r$.

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Moreover,

$$f(t) = \text{dirac}(t) + b\,\text{dirac}(1, t),$$

where $\text{dirac}(1, t)$ denotes the derivative of $\text{dirac}(t)$.

Open question 1

Are these all possible solution?

The normal distribution

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If the rent is normally distributed with mean V and variance σ^2 then $f^*(r) = \exp(rV + \frac{1}{2}r^2\sigma^2)$ and $(f^*(r))' = \left(r\sigma^2 + V\right) e^{\frac{1}{2}r^2\sigma^2 + Vr}$.

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The contest equilibrium levels are then given by

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Hence,

$$x_P^* - x_L^* = 2r(\frac{1}{4}V - \frac{1}{8}r\sigma^2) - \tanh(2r(\frac{1}{4}V - \frac{1}{8}r\sigma^2)) > 0$$

since $x \geq \tanh(x)\ \forall x \geq 0$.

The uniform distribution

The uniform distribution

If the rent is uniformally distributed on an invertavl $[a, b]$ then $f^*(r) = \frac{e^{rb} - e^{ra}}{r(b-a)}$ and $(f^*)'(r) = -\frac{1}{r^2(a-b)}\left(e^{ar} - e^{br} - are^{ar} + bre^{br}\right)$.

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Again, one can show that $X_P^* > X_L^*$.

Open question 2

Does the proportional contest always dominate the lottery contest?