A characterization of associative idempotent nondecreasing functions with neutral elements

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Notation: If $F : I^2 \to I$ is associative, then we also say that the pair (I, F) is a (2-ary) semigroup.

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Lemma

If F is associative, idempotent and monotone (in each variable) then it is monotone increasing (in each variable).

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If g satisfies (2) then

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2. *The 'extended' graph*

$$\Gamma_g = \{(x, y) : g(x - 0) \geq y \geq g(x + 0)\}$$

is symmetric with respect to the line $x = y$.

Characterization of associative, idempotent, monotone increasing functions with neutral element

Theorem (Martín-Mayor-Torrens,'03; K-Marichal-Teheux,'16)

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Moreover, in this case F must be commutative except perhaps on the set of points (x,y) such that $y = g(x)$ and $x = g(y)$.

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An important construction:

Let (X, F_2) be a binary semigroup and $F_n := \underbrace{F_2 \circ F_2 \circ \ldots \circ F_2}_{n-1}$.

Then F_n is n-associative.

Dudek-Mukhin's results

Theorem (Dudek-Mukhin, 2006)

If an n-associative F_n has a neutral element e, then F_n is derived from an associative function $F_2 : I^2 \to I$ where $F_2(a, b) = F_n(a, e, \ldots, e, b)$. (i.e: $F_n = \underbrace{F_2 \circ \cdots \circ F_2}_{n-1}$.)

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By the definition of F_2, the element e is also a neutral element of F_2.

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By a previous lemma, if F_2 is monotone, idempotent, associative, then F_2 is monotone increasing in each variable. Easily, F_n is also monotone increasing in each variable.

Generalization of Czogala-Drewniak theorem

We denote $\min(a_1, \ldots, a_n)$ and $\max(a_1, \ldots, a_n)$ by $\min(a_{1,\ldots,n})$ and $\max(a_{1,\ldots,n})$, respectively.

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Lemma

Let a and b be as above. Further let $x_1 = \ldots = x_l = a$ *and* $x_{l+1} = \ldots = x_k = b$*. Then for every* $\pi \in Sym(k)$ *we have*

$$F_k(x_1, \ldots, x_k) = F_k(x_{\pi(1)}, \ldots, x_{\pi(k)}).$$

Lemma

Let l *and* m *be fixed and* $l + m = k$*. Then for any* $1 \leq m \leq k - 2$

$$F_k(\underbrace{a, \ldots, a}_{l}, \underbrace{b, \ldots, b}_{m}) = F_l(\underbrace{a, \ldots, a}_{l}),$$

and $F_k(a, \underbrace{b, \ldots, b}_{k-1}) = a$.

Lemma

Let a and b be as above. Further let $x_1 = \ldots = x_l = a$ *and* $x_{l+1} = \ldots = x_k = b$*. Then for every* $\pi \in Sym(k)$ *we have*

$$F_k(x_1, \ldots, x_k) = F_k(x_{\pi(1)}, \ldots, x_{\pi(k)}).$$

Lemma

Let l *and* m *be fixed and* $l + m = k$*. Then for any* $1 \leq m \leq k - 2$

$$F_k(\underbrace{a, \ldots, a}_{l}, \underbrace{b, \ldots, b}_{m}) = F_l(\underbrace{a, \ldots, a}_{l}),$$

and $F_k(a, \underbrace{b, \ldots, b}_{k-1}) = a$.

$b = F_{k-1}(a, \ldots, a)$	$F_{k-2}(a, \ldots, a)$	$\ldots$	a
a	$b = F_{k-1}(a, \ldots, a)$	$\ldots$	$F_2(a, a)$

Thank you for your kind attention!