Optimal output consensus for linear systems: A topology free approach [*]

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Abstract

In this paper, for any homogeneous system of agents with linear continuous time dynamics, we formulate an optimal control problem. In this problem a convex cost functional of the control signals of the agents shall be minimized, while the outputs of the agents shall coincide at some given finite time. This is an instance of the rendezvous or finite time consensus problem. We solve this problem without any constraints on the communication topology and provide a solution as an explicit feedback control law for the case when the dynamics of the agents is output controllable. It turns out that the communication graph topology induced by the solution is complete. Based on this solution for the finite time consensus problem, we provide a solution to the case of infinite time horizon. Furthermore, we investigate under what circumstances it is possible to express the controller as a feedback control law of the output instead of the states.

Key words: Consensus control, multi-agent systems, time-invariant, optimal control, network topologies.

1 Introduction

Here we study the output consensus problem Kim et al. (2011), Xi et al. (2012a,b) for a homogeneous system of agents with linear dynamics, both in finite time and in the asymptotic case (as time tends to infinity). In the finite time case (rendezvous) Wang & Xiao (2010), Sundram & Hadjicostis (2007), the outputs for all the agents shall be the same at some predefined finite time. It is easy to show that for homogeneous systems of agents with linear dynamics, it is not possible to construct a linear, time-invariant feedback control law based on relative information such that the agents reach consensus in their states in finite time. With relative information in this context, we are referring to pairwise differences between the states of the agents.

Regarding the output consensus problem, using a decomposition of the state space, we show that if the dynamics for the agents is output controllable and the nullspace of the matrix which maps the state to the output satisfies a certain invariance condition, there cannot exist a linear continuous in state, time-invariant feed-

back control law that solves the problem while using only relative output information in the form of pairwise differences between the outputs of the agents. If only relative information is used, the control laws need to be either time-varying or non-Lipschitz in order to solve the finite time consensus problem. Furthermore, the output controllability is a standing assumption in order to guarantee consensus for arbitrary initial conditions.

In Cao & Ren (2009, 2010), an optimal linear consensus problem is addressed for systems of mobile agents with single-integrator dynamics. In this setting, the authors constrain the agents to use only relative information in their controllers, *i.e.*, the controller of each agent consists of a weighted sum of the differences between its state and the states of its neighbors. In this setting the authors show that the graph Laplacian matrix used in the optimal controller for the system corresponds to a complete directed graph. There is another line of research on the optimal consensus problem Semsar-Kazerooni & Khorasani (2008), in which the consensus requirement is reflected in the cost function. However, with such a formulation the optimal controller in general can not be implemented with relative state information only.

We formulate the consensus problem as an optimal control problem, where there are no restrictions on the controllers besides that the agents shall reach consensus at some predefined time. Note that we do not impose any

communication topology on the system beforehand, instead we are interested in the topology generated by the solution to the optimal control problem. By solving the problem, we show that the optimal controller uses only relative information. Moreover, the connectivity graph needs to be completely connected. Thus, for any other topology between the agents than the complete graph, any controller constructed will be suboptimal. The provided control laws are given in closed form and are bounded and continuous. The input and output dimensions are arbitrary.

Not surprisingly, the optimal controller requires the measurement of state errors in general. We identify cases where the optimal controller is only based on the output errors. We also show that in the asymptotic case, there is a corresponding observer based controller, that is only based on the output errors.

Regarding the theoretical contribution of this work, we use linear vector space optimization methods in order to solve the consensus problems. We show that the problem can be posed as a certain minimum norm problem in a Hilbert space Luenberger (1997). In this framework the finite time consensus problem is a solution of an optimization problem in this Hilbert space.

2 Preliminaries

We consider a system of N agents, where each agent i in the system has the dynamics

$$\dot{x}_i = Ax_i(t) + Bu_i(t),$$
$$y_i = Cx_i.$$

The variable $x_i(t_0) = x_0$, $x_i(t) : \mathbb{R} \to \mathbb{R}^n$, $u_i(t) : \mathbb{R} \to \mathbb{R}^m$ and $y_i(t) : \mathbb{R} \to \mathbb{R}^p$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. It is assumed that B and C are full rank matrices and that the system is output controllable. Let us define

$$x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T \in \mathbb{R}^{nN},$$
$$u(t) = [u_1^T(t), u_2^T(t), \ldots, u_N^T(t)]^T \in \mathbb{R}^{mN},$$
$$y(t) = [y_1^T(t), y_2^T(t), \ldots, y_N^T(t)]^T \in \mathbb{R}^{pN},$$

and the vector

$$a = [a_1, a_2, ..., a_N]^T,$$

where $a_i \in \mathbb{R}^+$ for all i. For each agent i, the positive scalar a_i determines how much the control signal of agent i shall be penalized in the objective function of the optimal control problem; see (4) further down.

The matrix

$$L(a) = -\left(\left(\sum_{i=1}^{N} a_i\right)^{-1} 1_N a^T - \mathrm{diag}\left([1, \ldots, 1]^T\right)\right),$$
(1)

plays an important role as one of the building blocks of the proposed control laws. The vector 1_N is a vector of dimension N with all entries equal to one. The matrix $L(a) \in \mathbb{R}^{N \times N}$ has one eigenvalue 0 and has $N-1$ eigenvalues equal, positive and real.

We now define the matrices

$$V_1(a) = \left[\frac{1}{a_1} 1_{N-1}, -\mathrm{diag}\left(\left[\frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_N}\right]^T\right)\right],$$
$$V_2(a) = \mathrm{diag}\left(\left[\frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_N}\right]^T\right) + \frac{1}{a_1} 1_{N-1} 1_{N-1}^T,$$
$$V_3 = [-1_{N-1}, I_{N-1}],$$

and formulate the following lemma.

Lemma 1 $L(a) = -V_1(a)^T V_2(a)^{-1} V_3$.

Proof: A proof can be found in Thunberg (2014). ∎

Lemma 2 *Assume that $C \in \mathbb{R}^{p \times n}$ has full row rank, $P \in \mathbb{R}^{n \times n}$ is nonsingular and $\ker(C)$ is P-invariant, then*

$$P^T C^T (CPWP^T C^T)^{-1} CP = C^T (CWC^T)^{-1} C.$$

Proof: A proof can be found in Thunberg (2014).

Let us define

$$W(t, T) = \int_t^T Ce^{A(T-s)} BB^T e^{A^T(T-s)} C^T ds. \quad (2)$$

The matrix $W(t, T)$ is the output controllability Gramian, and since the system is assumed to be output controllable, this matrix is nonsingular (for $t < T$). Let us also define the related matrix

$$G(t, T) = \int_0^{T-t} Ce^{-Ar} BB^T e^{-A^T r} C^T dr. \quad (3)$$

Beware of the difference between the transpose operator $(\cdot)^T$ and the time T.

The approach we use in this work relies to a large extent on the projection theorem in Hilbert spaces. We recall the following version of the projection theorem where inner product constraints are present.

Theorem 3 (Luenberger (1997)) *Let H be a Hilbert space and $\{z_1, z_2, \ldots, z_N\}$ a set of linearly independent vectors in H. Among all vectors $w \in H$ satisfying*

$$\langle w, z_1 \rangle = c_1,$$
$$\langle w, z_2 \rangle = c_2,$$
$$\vdots$$
$$\langle w, z_M \rangle = c_M,$$

let z_0 have minimum norm. Then

$$z_0 = \sum_{i=1}^{N} \beta_i z_i,$$

where the coefficients β_i satisfy the equations

$$\langle z_1, z_1 \rangle \beta_1 + \langle z_2, z_1 \rangle \beta_2 + \cdots + \langle z_N, z_1 \rangle \beta_N = c_1,$$
$$\langle z_1, z_2 \rangle \beta_1 + \langle z_2, z_2 \rangle \beta_2 + \cdots + \langle z_N, z_2 \rangle \beta_N = c_2,$$
$$\vdots$$
$$\langle z_1, z_M \rangle \beta_1 + \langle z_2, z_M \rangle \beta_2 + \cdots + \langle z_N, z_M \rangle \beta_N = c_M.$$

In Theorem 3 $\langle \cdot, \cdot \rangle$ denotes the inner product.

3 Finite time consensus

In this section we consider the following problem.

Problem 4 *For any finite $T > t_0$, construct a control law $u(t)$ for the system of agents such that the agents reach consensus in their outputs at time T, i.e.,*

$$y_i(T) = y_j(T) \ \forall i \neq j,$$

while minimizing the following cost functional

$$\int_{t_0}^{T} \sum_{i=1}^{N} a_i u_i^T u_i dt, \tag{4}$$

where $a_i \in \mathbb{R}^+$, $i = 1, 2, ..., N$.

Note that the criterion in Problem 4 only regards the time T and does not impose any constraints on $y(t)$ when $t > T$. When we say that a control law u for the system is based on only relative information, we mean that

$$u_i = g(y_1 - y_i, \ldots, y_N - y_i), \quad \forall i, t \geq t_0$$

for some function g. An interesting question to answer, is under what circumstances it is possible to construct a control law that solves Problem 4 using only relative information. The following lemma provides a first step on the path to the answer of this question.

Lemma 5 *Suppose $\ker(C)$ is A-invariant and u is based on only relative information, then there is no locally Lipschitz continuous in state, time-invariant feedback control law u that solves Problem 4 and for which $g(0, \ldots, 0) = 0$.*

Proof of Lemma 5: Let us introduce the invertible matrix

$$P = \begin{bmatrix} C^T & C_{\ker}^T \end{bmatrix},$$

where $C_{\ker}$ has full row rank and the columns of $C_{\ker}^T$ span $\ker(C)$. Let us now define $\tilde{x}_i = (x_{i,1}, x_{i,2})^T$ through the following relation

$$x_i = P\tilde{x}_i,$$

for all i. The dynamics for $\tilde{x}$ is given by

$$\dot{\tilde{x}}_i = \tilde{A}\tilde{x}_i + \tilde{B}u_i, \quad y_i = \tilde{C}\tilde{x}_i,$$

where

$$\tilde{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{ and } \tilde{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix}.$$

The structure of $\tilde{A}$ is a consequence of the fact that $\ker(C)$ is A-invariant.

Suppose there is a linear time-invariant feedback control law u that solves the Problem 4 with only relative information. We note that

$$y_i = CC^T x_{i,1}.$$

We define $y_{1j} = y_1 - y_j$ for all $j > 1$. The control law u has the following form

$$u = f(y_{12}, \ldots, y_{1N}),$$

but $y_{1j} = CC^T x_{1j,1}$, where $x_{1j,1} = x_{1,1} - x_{j,1}$, so u can be written as

$$u = f(x_{12,1}, \ldots, x_{1N,1}).$$

Since CC^T is invertible it follows that $y_{1j} = 0$ for all $j > 1$ if and only if $x_{1j,1} = 0$ for all $j > 1$. But, by the structure of f and $\tilde{A}$ it follows that $(x_{12,1}, \ldots, x_{1N,1})^T = 0$ is an equilibrium for the dynamics of $(x_{12,1}, \ldots, x_{1N,1})^T$. Since the right-hand side of this dynamics is locally Lipschitz continuous, $(x_{12,1}, \ldots, x_{1N,1})^T$ cannot have reached the point 0 in finite time. This is a contradiction. ∎

We now provide the solution to Problem 4.

Theorem 6 *For $T < \infty$ the solution to Problem 4 is*

$$u(t) = \\ - L(a) \otimes \left(B^T e^{A^T(T-t)} C^T W(t_0,T)^{-1} Ce^{A(T-t_0)} \right) x_0 \tag{5}$$

$$u(x,t) = \\ - L(a) \otimes \left(B^T e^{A^T(T-t)} C^T W(t,T)^{-1} Ce^{A(T-t)} \right) x. \tag{6}$$

Furthermore, if $ker(C)$ is A-invariant, the solution to Problem 4 is

$$u(y,t) = -L(a) \otimes B^T C^T G(t,T)^{-1} y. \tag{7}$$

or equivalently

$$u(y,t) = B^T C^T G(t,T)^{-1} \sum_{j=1}^{N} (\alpha a_j y_j - y_i),$$

where $\alpha = \left(\sum_{i=1}^{N} a_i \right)^{-1}$.

All the control laws (5-7) are equivalent (provided $ker(C)$ is A-invariant), but expressed in different ways. The control law (5) is the open loop controller and (6) is the closed loop version of (5). The matrix $W(t,T)$ is invertible due to the assumption of output controllability. We take the liberty of denoting all the controllers (5-7) by u. Provided u is used during $[t_0,T)$, at the time T we have that

$$\lim_{t \uparrow T} u(x(t),t) = \lim_{t \uparrow T} u(y(t),t) = u(T).$$

Even though the feedback controllers in (6) and (7) are bounded and continuous for $t \in [t_0,T)$ (see the open loop version of u in (5)), computational difficulties arise as $t \to T$ when (5) and (6) are used, since $W(T)$ is not invertible.

Proof of Theorem 6: Problem 1 is formally stated as follows

$$\text{minimize} \int_{t_0}^{T} \sum_{i=1}^{N} a_i u_i^T u_i dt \quad a_i \in \mathbb{R}^+ \ i = 1,2,...,N,$$

when

$$y_i(t) = Ce^{A(t-t_0)} x_i(t_0) + \int_{t_0}^{t} Ce^{A(t-s)} Bu_i ds, \text{ for all } i,$$

and

$$\int_{t_0}^{T} Ce^{A(T-s)} B (u_1 - u_i) ds = \\ - Ce^{A(T-t_0)} (x_1(t_0) - x_i(t_0)), \tag{8}$$

for $i \in \{2,...,N\}$. Here we have without loss of generality assumed that the outputs of the agents at time T, $y_i(T)$ shall be equal to the output of agent 1 at time T, *i.e.*, $y_1(T)$.

We notice that this problem is a minimum norm problem in the Hilbert space of all functions

$$f = (f_1(t), f_2(t), ..., f_N(t))^T : \mathbb{R} \to \mathbb{R}^{mN},$$

such that the Lebesgue integral

$$\int_{t_0}^{T} \sum_{i=1}^{N} a_i f_i^T(t) f_i(t) dt \tag{9}$$

converges. Here $f_i : \mathbb{R} \to \mathbb{R}^m$. We denote this space H, and the norm is given by the square root of (9).

Now we continue along the lines of Theorem 3 and reformulate the constraints (8) into inner product constraints in H.

$$\int_{t_0}^{T} Ce^{A(T-s)} B(u_1(s) - u_i(s)) ds = \\ \left\langle \left[\frac{Ce^{A(T-t)}B}{a_1}, 0,...,0, -\frac{Ce^{A(T-t)}B}{a_i}, 0,...,0 \right]^T, \right. \\ \left. \left[u_1^T, 0,...,0, u_i^T, 0,...,0 \right]^T \right\rangle = \\ -Ce^{A(T-t_0)} (x_1(t_0) - x_i(t_0)).$$

Depending on context the symbol $\langle \cdot, \cdot \rangle$ shall be interpreted as follows. If f and g belongs to H, $\langle f, g \rangle$ denotes the inner product between these two elements. If $f(t)$ and $g(t)$ are matrices of proper dimensions, then $\langle f, g \rangle$ is a matrix inner product where each element in the matrix is an inner product between a column in f and a column in g.

To simplify the notation we define

$$p_i = \left[\frac{Ce^{A(T-t)}B}{a_1}, 0,...,0, -\frac{Ce^{A(T-t)}B}{a_i}, 0,...,0 \right]^T.$$

Since we have a minimum norm problem and all the columns of all the p_i are independent, by Theorem 3 we get that the optimal controller $u(t)$ is given by

$$u(t) = [p_2, ..., p_N]\beta, \tag{10}$$

where β is the solution to

$$Q\beta = V_3 \otimes Ce^{A(T-t_0)}x_0, \qquad (11)$$

where

$$Q = \begin{bmatrix} \langle p_2, p_2 \rangle & \langle p_3, p_2 \rangle & \cdots & \langle p_N, p_2 \rangle \\ \langle p_2, p_3 \rangle & \langle p_3, p_3 \rangle & \cdots & \langle p_N, p_3 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle p_2, p_N \rangle & \langle p_3, p_N \rangle & \cdots & \langle p_N, p_N \rangle \end{bmatrix}. \qquad (12)$$

From (10-12) we get that $\beta = Q^{-1}V_3 \otimes Ce^{A(T-t_0)}x_0$ and $u = [p_2, ..., p_N]Q^{-1}V_3 \otimes Ce^{A(T-t_0)}x_0$. Now we have that $[p_2, ..., p_N] = V_1(a)^T \otimes (B^T e^{A^T(T-t)}C^T)$.

Since

$$\langle p_i, p_j \rangle = \begin{cases} \frac{1}{a_1}W(t_0, T) & \text{if } i \neq j, \\ \left(\frac{1}{a_1} + \frac{1}{a_{i+1}}\right)W(t_0, T) & \text{if } i = j, \end{cases}$$

where $W(t_0, T) = \int_{t_0}^{T} Ce^{A(T-s)}BB^T e^{A^T(T-s)}C^T ds$, we have that

$$Q = V_2(a) \otimes W(t_0, T).$$

$$\begin{aligned} u(t) &= \left(V_1(a)^T \otimes (B^T e^{A^T(T-t)}C^T)\right) \cdot \\ &\quad \left(V_2(a)^{-1} \otimes W(t_0, T)^{-1}\right)\left(V_3 \otimes Ce^{A(T-t_0)}\right)x_0 \\ &= (V_1(a)^T V_2(a)^{-1}V_3) \otimes \\ &\quad \left(B^T e^{A^T(T-t)}C^T W(t_0, T)^{-1}Ce^{A(T-t_0)}\right)x_0 \\ &= -L(a) \otimes \\ &\quad \left(B^T e^{A^T(T-t)}C^T W(t_0, T)^{-1}Ce^{A(T-t_0)}\right)x_0. \end{aligned}$$

By Bellman's Principle we get that

$$\begin{aligned} u(x, t) &= \\ &- L(a) \otimes \left(B^T e^{A^T(T-t)}C^T W(t, T)^{-1}Ce^{A(T-t)}\right)x. \end{aligned}$$

Now, provided $\ker(C)$ is A-invariant, we can use Lemma 2 to obtain

$$u(y, t) = -L(a) \otimes B^T C^T G(t, T)^{-1}y \qquad (13)$$

(see Thunberg (2014) for details). ∎

Corollary 7 *The controllers (6-7) use only relative information, i.e., differences of the states (outputs) of the agents.*

Proof: Straight forward by using the structure of the matrix $L(a)$. ∎

Let us define $y_c = \frac{1}{\sum_{i=1}^{N} a_i} \sum_{i=1}^{N} a_i y_i$, and $\bar{y}_c = (y_c, ..., y_c)^T \in \mathbb{R}^{pN}$.

Lemma 8 *Suppose that A has not full rank and $x_i(0) = x_{i0} \in \ker(A)$ for all $i = 1, ..., N$, then the consensus point for the system of agents using the controller (6) or (7) is $y_c(0)$.*

Proof: We have that

$$\begin{aligned} y(T) &= I_N \otimes Ce^{A(T-t_0)}x_0 + \int_{t_0}^{T}\left(I_N \otimes Ce^{A(T-t)}B\right) \cdot \\ &\quad \left(-L(a) \otimes B^T e^{A^T(T-t)}C^T W(t_0, T)^{-1}Ce^{A(T-t_0)}\right)x_0 dt \\ &= y_0 + \int_{t_0}^{T}(-L(a) \otimes \\ &\quad Ce^{A(T-t)}BB^T e^{A^T(T-t)}C^T W(t_0, T)^{-1})y_0 dt = \bar{y}_c(t_0). \end{aligned}$$

∎

Theorem 9 *For non-homogeneous output controllable (A_i, B_i, C_i), $i = 1, \cdots, N$, the optimal controller can be derived in a similar way and has the following form:*

$$u_i(t) = B_i^T e^{A_i^T t}C_i^T W_i(0, T)^{-1}(\alpha^* - C_i e^{A_i t}x_i(0)), \quad i \leq N,$$

where

$$W_i(0, T) = \int_0^T C_i e^{A_i(T-s)}B_i B_i^T e^{A_i(T-s)}C_i^T ds,$$

$$\alpha^* = \left(\sum_{j=1}^{N} a_j W_j(0, T)^{-1}\right)^{-1} \cdot$$

$$\sum_{i=1}^{N} a_i W_i(0, T)^{-1}C_i e^{A_i t}x_i(0).$$

Proof:
Step 1:
Solve the problem

$$\begin{cases} \min_{u_i} \int_0^T \frac{1}{2}u_i(t)u_i(t)dt \\ \text{s.t. } \dot{x} = A_i x + B_i u, \\ \qquad y_i(T) = \alpha. \end{cases}$$

Using any standard approach, one obtains that

$$u_i^*(t,\alpha) = B_i^T e^{A_i^T t} C_i^T W_i(0,T)^{-1}(\alpha - C_i e^{A_i t} x_i(0))$$

is the solution to this problem provided (A_i, B_i, C_i) is output controllable.

Step 2:
We want to find the α^* that minimizes

$$\int_0^T \sum_{i=1}^N a_i u_i^*(t,\alpha)^T u_i^*(t,\alpha) dt.$$

Now,

$$\min_\alpha \int_0^T \sum_{i=1}^N a_i u_i^*(t,\alpha)^T u_i^*(t,\alpha) dt$$

$$= \min_\alpha \sum_{i=1}^N a_i(\alpha - C_i e^{A_i t} x_i(0))^T W_i(0,T)^{-1} \cdot$$

$$(\alpha - C_i e^{A_i t} x_i(0)).$$

This gives that

$$\sum_{i=1}^N a_i W_i(0,T)^{-1}(\alpha^* - Ce^{A_i t} x_i(0)) = 0$$

or

$$\alpha^* = \left(\sum_{j=1}^N a_j W_j(0,T)^{-1} \right)^{-1} \cdot$$

$$\sum_{i=1}^N \alpha_i W_i(0,T)^{-1} C_i e^{A_i t} x_i(0).$$

Thus the optimal controller $u = [u_1^T, u_2^T, \ldots, u_N^T]$ is given by

$$u_i(t) = B_i^T e^{A_i^T t} C_i^T W_i(0,T)^{-1} \cdot$$

$$(\frac{1}{\sum_{j=1}^N a_j} \sum_{i=1}^N a_i C_i e^{A_i t} x_i(0) - Ce^{A_i t} x_i(0))$$

for all i. ∎

4 Extension to the asymptotic consensus problem

We now examine the asymptotic case, *i.e.*, we want the system to asymptotically reach consensus while minimizing the cost functional. The problem is formally stated as follows.

Problem 10 *Construct a control law $u(t)$ for the system of agents such that the agents asymptotically reach consensus in the outputs,* i.e.,

$$\lim_{t\to\infty}(y_i(t) - y_j(t)) = 0 \quad \text{for all } i, j \text{ such that } i \neq j,$$

while minimizing the following cost functional

$$\int_{t_0}^\infty \sum_{i=1}^N a_i u_i^T(t) u_i(t) dt \qquad (14)$$

where $a_i \in \mathbb{R}^+$ for $i = 1, 2, ..., N$.

In order to solve Problem 10, we start by defining the matrix

$$P(t,T) = e^{A^T(T-t)} C^T W(t,T)^{-1} C e^{A(T-t)}$$

which satisfies the following differential Riccati equation

$$\frac{dP}{dt} = -A^T P - PA + PBB^T P. \qquad (15)$$

The matrix $P(t,T)$ is an essential part of the control laws that were presented in the last section, and here we see that this matrix is provided as the solution to a differential matrix Riccati equation. It is well known that (15) has a positive semidefinite limit P_0 as $T - t \to \infty$ if (A, B) is stabilizable and A does not have any eigenvalue on the imaginary axis. In order to see this we consider the following problem

$$\min \int_0^\infty \|u\|^2 dt \qquad (16)$$
$$\text{s.t. } \dot{x} = Ax + Bu.$$

If (A, B) is stabilizable and A does not have any eigenvalue on the imaginary axis,

$$u = -B^T P_0 x$$

is the optimal control law that solves (16), where P_0 is the positive semi-definite solution to

$$-A^T P_0 - P_0 A + P_0 BB^T P_0 = 0.$$

This Algebraic Riccati equation is obtained by letting the left-hand side of (15) be equal to zero.

The problem (16) is not a consensus problem, and the question is, besides the fact the same matrix P_0 is used in the optimal control law, how it is related to our consensus problem. It turns out that the control law, besides being the solution of the consensus problem, is also the solution

of N problems on the form (16). In order to show this we introduce

$$x_c = \frac{1}{\sum_{i=1}^{N} a_i} \sum_{i=1}^{N} a_i x_i, \text{ and } \delta_i = x_i - x_c.$$

The dynamics of x_c and δ_i are given by

$$\dot{x}_c = A x_c \quad \text{and} \quad \dot{\delta}_i = A \delta_i + B u_i.$$

Now each control law $u_i(t)$ contained in the vector

$$u(t) = -L(a) \otimes (B^T P_0) x,$$

can be written as

$$u_i = B^T P_0 \delta_i$$

where u_i solves the problem

$$\min \int_0^\infty \|u\|^2 dt \tag{17}$$
$$\text{s.t. } \dot{\delta}_i = A \delta_i + B u_i.$$

Provided $\ker(C)$ is A-invariant, it can be shown that

$$P(t,T) = e^{A^T(T-t)} C^T W(t,T)^{-1} C e^{A(T-t)}$$
$$= C^T G(t,T)^{-1} C,$$

which implies that

$$(CC^T)^{-1} C P(t,T) C^T (CC^T)^{-1} = G(t,T)^{-1}.$$

Now, as $T - t \to \infty$, it holds that $P(t,T) \to P_0$. This means that, as $T - t \to \infty$,

$$(CC^T)^{-1} C P(t,T) C^T (CC^T)^{-1} \to$$
$$(CC^T)^{-1} C P_0 C^T (CC^T)^{-1}.$$

Let $G_0 = (CC^T)^{-1} C P_0 C^T (CC^T)^{-1}$. Then for the asymptotic consensus problem, controller (13) becomes

$$u = -L(a) \otimes (B^T C^T G_0) y.$$

Proposition 11 *If A is stabilizable with no eigenvalues on the imaginary axis. Then P_0 exists, is positive semidefinite and the optimal control law that solves Problem 10 is*

$$u = -L(a) \otimes (B^T P_0) x.$$

Furthermore, if $\ker(C)$ is also A-invariant then

$$u = -L(a) \otimes (B^T C^T G_0) y.$$

When only the output $y_i = C x_i$ is available for control action and $\ker(C)$ is not necessarily A-invariant, an observer can be designed.

$$\dot{\hat{\delta}}_i = (A - BB^T P_0)\hat{\delta}_i - QC^T \left(\frac{1}{\sum_{i=1}^{N} a_i} \sum_{j=1}^{N} a_j(y_i - y_j) - C\hat{\delta}_i \right).$$

Under the assumption that (A, C) is detectable and A does not have any eigenvalue on the imaginary axis we have that

$$\dot{\hat{\delta}} = A\hat{\delta} - BB^T P_0 \hat{\delta} - QC^T((y_i - y_c) - C\hat{\delta}),$$

where $Q \le 0$ satisfies

$$AQ + QA^T = -QC^T C Q.$$

We summarize these results in the following proposition.

Proposition 12 *Suppose (A, B) is stabilizable and (A, C) is detectable, and A has no eigenvalue on the imaginary axis. Then, if the following dynamic output control law is used,*

$$u_i = -B^T P_0 \hat{\delta}_i,$$
$$\dot{\hat{\delta}}_i = (A - BB^T P_0)\hat{\delta}_i -$$
$$QC^T \left(\frac{1}{\sum_{i=1}^{N} a_i} \sum_{j=1}^{N} a_j(y_i - y_j) - C\hat{\delta}_i \right),$$

the system reaches asymptotic consensus in the outputs.

References

Cao, Y. & Ren, W. (2009), LQR-based optimal linear consensus algorithms, *in* 'American Control Conference', IEEE, pp. 5204–5209.

Cao, Y. & Ren, W. (2010), 'Optimal linear-consensus algorithms: an LQR-perspective', *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics* **40**(3), 819–830.

Kim, H., Shim, H. & Seo, J. (2011), 'Output consensus of heterogeneous uncertain linear multi-agent systems', *IEEE Transactions on Automatic Control* **56**(1), 200–206.

Luenberger, D. (1997), *Optimization by vector space methods*, John Wiley & Sons, Inc.

Semsar-Kazerooni, E. & Khorasani, K. (2008), 'Optimal consensus algorithms for cooperative team of agents subject to partial information', *Automatica* **44**(11), 2766–2777.

Sundram, S. & Hadjicostis, C. (2007), Finite-time distributed consensus in graphs with time-invariant topologies, American Control Conference.

Thunberg, J. (2014), Consensus and Pursuit-Evasion in Nonlinear Multi-Agent Systems, PhD thesis, KTH Royal Institute of Technology.

Wang, L. & Xiao, F. (2010), 'Finite-time consensus problems for networks of dynamic agents', *IEEE Transactions on Automatic Control* .

Xi, J., Shi, Z. & Zhong, Y. (2012a), 'Output consensus analysis and design for high-order linear swarm systems: partial stability method', *Automatica* **48**(9), 2335–2343.

Xi, J., Shi, Z. & Zhong, Y. (2012b), 'Output consensus for high-order linear time-invariant swarm systems', *International Journal of Control* **85**(4), 350–360.