CLASSIFICATION OF UNIVERSAL FORMALITY MAPS FOR QUANTIZATIONS OF LIE BIALGEBRAS

SERGEI MERKULOV AND THOMAS WILLWACHER

Abstract. We introduce an endofunctor $\mathcal{D}$ in the category of augmented props with the property that for any representation of a prop $\mathcal{P}$ in a vector space V the associated prop $\mathcal{DP}$ admits an induced representation on the graded commutative algebra $\odot^\bullet V$ given in terms of polydifferential operators.

Applying this functor to the prop $\mathcal{L}ieb$ of Lie bialgebras we show that universal formality maps for quantizations of Lie bialgebras are in in 1-1 correspondence with morphisms of dg props

$$F : \mathcal{A}ssb_\infty \longrightarrow \mathcal{DL}ieb_\infty$$

satisfying certain boundary conditions, where $\mathcal{A}ssb_\infty$ is a minimal resolution of the prop of associative bialgebras. We prove that the set of such formality morphisms (having an extra property of being *Lie connected*) is non-empty. The latter result is used in turn to give a short proof of the formality theorem for universal quantizations of arbitrary Lie bialgebras which says that for any Drinfeld associator $\mathfrak{A}$ there is an associated $\mathcal{L}ie_\infty$ quasi-isomorphism between the $\mathcal{L}ie_\infty$ algebras $\mathsf{Def}(\mathcal{A}ss\mathcal{B}_\infty \to \mathcal{E}nd_{\odot^\bullet V})$ and $\mathsf{Def}(\mathcal{L}ie\mathcal{B} \to \mathcal{E}nd_V)$ controlling, respectively, deformations of the standard bialgebra structure in $\odot V$ and deformations of any given Lie bialgebra structure in V.

We study the deformation complex of any universal formality morphism $\mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{F}{\to} \mathcal{DL}ieb_\infty)$ and show that it is quasi-isomorphic (up to one class corresponding to the standard rescaling automorphism of the properad $\mathcal{L}ieb$) to the oriented graph complex GC_3^{or} studied earlier in [W2]. This result gives a complete classification of the set $\{F_\mathfrak{A}\}$ of gauge equivalence classes of universal Lie connected formality maps — it is a torsor over the Grothendieck-Teichmüller group $GRT = GRT_1 \rtimes \mathbb{K}^*$ and can hence can be identified with the set $\{\mathfrak{A}\}$ of Drinfeld associators.

Key words. Hopf algebras, Lie bialgebras, deformation quantization, Grothendieck-Teichmüller group, graph complexes.

Contents

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1. Introduction

1.1. Polydifferential functor. We introduce an endofunctor $\mathcal{D}$ in the category of (augmented) props with the property that for any representation of a prop $\mathcal{P}$ in a vector space V the associated *polydifferential* prop $\mathcal{DP}$ admits an induced representation on the graded commutative algebra $\mathcal{O}_V := \odot^\bullet V$ given in terms of polydifferential operators. For any $\mathcal{P}$ the polydifferential prop $\mathcal{DP}$ comes equipped with a canonical injection $\mathcal{C}omb \to \mathcal{DP}$ from the prop $\mathcal{C}omb$ of commutative and cocommutative bialgebras.

We apply this functor to the minimal resolution of the prop of Lie bialgebras $\mathcal{L}ieb$ introduced by Drinfeld and obtain the following results.

1.2. Main Theorem. (i) *For any choice of a Drinfeld associator $\mathfrak{A}$ there is an associated highly non-trivial*[1] *Lie connected*[2] *morphism of dg props,*

$$F_{\mathfrak{A}} : \mathcal{A}ssb_\infty \longrightarrow \mathcal{D}\mathcal{L}ieb_\infty. \tag{1}$$

where $\mathcal{A}ssb_\infty$ stands for a minimal resolution of the prop of associative bialgebras.

(ii) *For any graded vector space V, each morphism $F_{\mathfrak{A}}$ induces a $\mathcal{L}ie_\infty$ quasi-isomorphism (called a* formality map*) between the dg $\mathcal{L}ie_\infty$ algebra*[3]

$$\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{\rho_0} \mathcal{E}nd_{\mathcal{O}_V})$$

controlling deformations of the standard graded commutative and co-commutative bialgebra structure ρ_0 in $\mathcal{O}_V$, and the Lie algebra[4]

$$\mathsf{Def}(\mathcal{L}ieb \xrightarrow{0} \mathcal{E}nd_V)$$

controlling deformations of the trivial morphism $0 : \mathcal{L}ieb \to \mathcal{E}nd_V$.

(iii) *For any Lie connected formality morphism $F_{\mathfrak{A}}$ there is a canonical morphism of complexes*

$$\mathsf{GC}_3^{or} \longrightarrow \mathsf{Def}\left(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{D}\mathcal{L}ieb_\infty\right)_{\text{Lie connected}}$$

which is a quasi-isomorphism up to one class corresponding to the standard rescaling automorphism of the prop of Lie bialgebras $\mathcal{L}ieb$. Here GC_3^{or} is the oriented version of the Kontsevich graph complex studied earlier in [W2].

(iv) The set of homotopy classes of Lie connected universal formality maps as in (1) can be identified with the set of Drinfeld associators. In particular, the Grothendieck-Teichmüller group $GRT = GRT_1 \rtimes \mathbb{K}^*$ acts faithfully and transitively on such universal formality maps.

[1] The precise meaning of what we mean by *highy non-trivial* is given in §**5.3** below. It says essentially that the morphism $F_{\mathfrak{A}}$ is non-zero on *each* member of the infinite family of generators of $\mathcal{A}ssb_\infty$ (see formula (37)).

[2] The definition of a *Lie connected* formality morphism $F_{\mathfrak{A}}$ is given §**5.4**.

[3] This is essentially the Gerstenhaber-Schack complex of $\mathcal{O}_V$.

[4] Maurer-Cartan elements of this Lie algebra are precisely strongly homotopy Lie bialgebra structures in V.

In the proof of item (i) we used the Etingof-Kazhdan theorem [EK] which says that any Lie bialgebra can deformation quantized in the sense introduced by Drinfeld in [D1]. Later different proofs of this theorem have been given by D. Tamarkin [Ta], P. Severa [Se] and the authors [MW3]. The latest proof [MW3] gives an explicit formula for a universal quantization of Lie bialgebras which is *Lie connected* in the sense used the above Theorem (see §**5.4** for a detailed definition of this notion). Item (ii) in the above Theorem is a Lie bialgebra analogue of the famous Kontsevich formality theorem for deformation quantizations of Poisson manifolds first proved in [Me2] with the help of a more complicated method. In the proof of item (iii) we used an earlier result [MW2] which established a quasi-isomorphism between the deformation complex of the properad of Lie bialgebras and the oriented graph complex. In the proof of item (iv) we used a computation [W2] of the zero-th cohomology group of the graph complex GC_3^{or},

$$H^0(\mathsf{GC}_3^{or}) = \mathfrak{grt}_1$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-Teichmüller group GRT_1 introduced by Drinfeld in [D2].

Given any non-trivial morphism of dg props

$$i : \mathcal{L}ieb_\infty \longrightarrow \mathcal{P}$$

there is, by the Main Theorem, an associated *quantized* morphism given as a composition

$$i_{\mathfrak{A}} : \mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{D}\mathcal{L}ieb_\infty \xrightarrow{\mathcal{D}(i)} \mathcal{D}\mathcal{P}$$

which depends in general on the choice of an associator $\mathfrak{A}$. An important example of such a non-trivial morphism i was found recently in [MW1],

$$i : \mathcal{L}ieb_\infty \longrightarrow \mathcal{RG}ra, \tag{2}$$

where $\mathcal{RG}ra$ stands for the prop of *ribbon* graphs; this morphism i provides with many new algebraic structures on the totality

$$\prod_{g,n} H^\bullet(\mathcal{M}_{g,n}) \otimes_{\mathbb{S}_n} sgn_n$$

of cohomology groups of moduli spaces of algebraic curves of genus g and with n (skewsymmetrized) punctures, and establishes a surprisingly strong link between the latter and the cohomology theory of ordinary (in the sense, non-ribbon) graph complexes [MW1]. The quantized version of the morphism (2) will be studied elsewhere.

1.3. Some notation. The set $\{1, 2, \ldots, n\}$ is abbreviated to $[n]$; its group of automorphisms is denoted by $\mathbb{S}_n$; the trivial one-dimensional representation of $\mathbb{S}_n$ is denoted by $\mathbb{1}_n$, while its one dimensional sign representation is denoted by sgn_n; we often abbreviate $sgn_n^d := sgn_n^{\otimes |d|}$. The cardinality of a finite set A is denoted by $\#A$.

We work throughout in the category of $\mathbb{Z}$-graded vector spaces over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ stands for the graded vector space with $V[k]^i := V^{i+k}$ and and s^k for the associated isomorphism $V \to V[k]$; for $v \in V^i$ we set $|v| := i$. For a pair of graded vector spaces V_1 and V_2, the symbol $\mathrm{Hom}_i(V_1, V_2)$ stands for the space of homogeneous linear maps of degree i, and $\mathrm{Hom}(V_1, V_2) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(V_1, V_2)$; for example, $s^k \in \mathrm{Hom}_{-k}(V, V[k])$. Furthermore, we use the notation $\odot^k V$ for the k-fold symmetric product of the vector space V.

For a prop(erad) $\mathcal{P}$ we denote by $\mathcal{P}\{k\}$ a prop(erad) which is uniquely defined by the following property: for any graded vector space V a representation of $\mathcal{P}\{k\}$ in V is identical to a representation of $\mathcal{P}$ in $V[k]$. The degree shifted operad of Lie algebras $\mathcal{L}ie\{d\}$ is denoted by $\mathcal{L}ie_{d+1}$ while its minimal resolution by $\mathcal{H}olie_{d+1}$; representations of $\mathcal{L}ie_{d+1}$ are vector spaces equipped with Lie brackets of degree $-d$.

For a right (resp., left) module V over a group G we denote by V_G (resp. ${}_GV$) the $\mathbb{K}$-vector space of coinvariants: $V/\{g(v) - v \mid v \in V, g \in G\}$ and by V^G (resp. ${}^GV$) the subspace of invariants: $\{\forall g \in G \ : g(v) = v,\ v \in V\}$. If G is finite, then these spaces are canonically isomorphic as $char(\mathbb{K}) = 0$.

2. A polydifferential endofunctor

2.1. Polydifferential operators. For an arbitrary vector space V the associated graded commutative tensor algebra $\mathcal{O}_V := \odot^\bullet V$ has a canonical structure of a commutative and co-commutative bialgebra with the multiplication denoted by central dot (or just by juxtaposition)

$$\cdot : \mathcal{O}_V \otimes \mathcal{O}_V \to \mathcal{O}_V$$

and the comultiplication denoted by

$$\Delta : \mathcal{O}_V \longrightarrow \mathcal{O}_V \otimes \mathcal{O}_V$$

This comultiplication is uniquely determined by its value on an arbitrary element $x \in V$ which is given by

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

The bialgebra $\mathcal{O}_V$ is (co)augmented, with the (co)augmentation (co-)ideal denoted by $\bar{\mathcal{O}}_V := \odot^{\bullet \geq 1} V$. The reduced diagonal $\bar{\mathcal{O}}_V \to \bar{\mathcal{O}}_V \otimes \bar{\mathcal{O}}_V$ is denoted by $\bar{\Delta}$. The multiplication in $\mathcal{O}_V$ induces naturally a multiplication in $\mathcal{O}_V^{\otimes m}$ for any $m \geq 2$,

$$\cdot : \mathcal{O}_V^{\otimes m} \otimes \mathcal{O}_V^{\otimes m} \otimes \mathcal{O}_V^{\otimes m}$$

which is also denoted by the central dot.

As $\mathcal{O}_V$ is a bialgebra, one can consider an associated Gerstenhaber-Schack complex

$$C_{GS}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V^{\otimes \bullet}) = \bigoplus_{m,n \geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes n}, \mathcal{O}_V^{\otimes m})[m+n-2]$$

equipped with the differential [GS],

$$d_{\mathfrak{gs}} = d_1 \oplus d_2 : \mathrm{Hom}(\mathcal{O}_V^{\otimes n}, \mathcal{O}_V^{\otimes m}) \longrightarrow \mathrm{Hom}(\mathcal{O}_V^{\otimes n+1}, \mathcal{O}_V^{\otimes m}) \oplus \mathrm{Hom}(\mathcal{O}_V^{\otimes n}, \mathcal{O}_V^{\otimes m+1}),$$

with d_1 given on an arbitrary $f \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ by

$$\begin{aligned}(d_1 f)(v_0, v_1, \ldots, v_n) \quad := \quad & \Delta^{m-1}(v_0) \cdot f(v_1, v_2, \ldots, v_n) - \sum_{i=0}^{n-1} (-1)^i f(v_1, \ldots, v_i v_{i+1}, \ldots, v_n) \\ & + (-1)^{n+1} f(v_1, v_2, \ldots, v_n) \cdot \Delta^{m-1}(v_n) \forall \ \ v_0, v_1, \ldots, v_n \in V,\end{aligned}$$

where

$$\Delta^{m-1} := (\Delta \otimes \mathrm{Id}^{\otimes m-2}) \circ (\Delta \otimes \mathrm{Id}^{\otimes m-3}) \circ \ldots \circ \Delta : V \to V^{\otimes m},$$

for $m \geq 2$, we set $\Delta^0 := \mathrm{Id}$. The expression for d_2 is an obvious "dual" analogue of the one for d_1.

One can consider a subspace,

$$C_{poly}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V^{\otimes \bullet}) \subset C_{GS}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V^{\otimes \bullet}), \tag{3}$$

spanned by so called *polydifferential* operators,

$$\begin{array}{cccc} \Phi : & \mathcal{O}_V^{\otimes m} & \longrightarrow & \mathcal{O}_V^{\otimes n} \\ & f_1 \otimes \ldots \otimes f_m & \longrightarrow & \Phi(f_1, \ldots, f_m), \end{array}$$

which are linear combinations of operators of the form,

$$\Phi(f_1, \ldots, f_m) = x^{\beta_{J_1}} \otimes \ldots \otimes x^{\beta_{J_n}} \cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I_1}|} f_1}{\partial x^{\alpha_{I_1}}}\right) \cdot \ldots \cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I_m}|} f_m}{\partial x^{\alpha_{I_m}}}\right). \tag{4}$$

where $\{x^\alpha\}_{\alpha \in S}$ is some basis in V, α_I and β_J stand for multi-indices (say, for $\alpha_1\alpha_2\ldots\alpha_p$, and $\beta_1\beta_2\ldots\beta_q$) from the set S,

$$x^{\beta_J} := x^{\beta_1} x^{\beta_2} \ldots x^{\beta_q}, \qquad \frac{\partial^{|\alpha_I|}}{\partial x^{\alpha_I}} := \frac{\partial^p}{\partial x^{\alpha_1} \partial x^{\alpha_2} \ldots \partial x^{\alpha_p}}.$$

In fact this subspace is a subcomplex of the Gerstenhaber-Shack complex with the differential given explicitly by

$$
\begin{aligned}
d_{\mathfrak{gs}}\Phi \quad = \quad & \sum_{i=1}^{n}(-1)^{i+1}x^{\beta_{J_1}}\otimes ...\otimes \bar{\Delta}(x^{\beta_{J_i}})\otimes ...\otimes x^{\beta_{J_n}}\cdot \Delta^{n}\left(\frac{\partial^{|\alpha_{I_1}|}}{\partial x^{\alpha_{I_1}}}\right)\cdot ...\cdot \Delta^{n}\left(\frac{\partial^{|\alpha_{I_m}|}}{\partial x^{\alpha_{I_m}}}\right)\\
& + \sum_{i=1}^{m}(-1)^{i+1}\sum_{\alpha_{I_i}=\alpha_{I'_i}\sqcup\alpha_{I''_i}}\\
& \quad x^{\beta_{J_1}}\otimes ...\otimes x^{\beta_{J_m}}\cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I_1}|}}{\partial x^{\alpha_{I_1}}}\right)\cdot ...\cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I'_i}|}}{\partial x^{\alpha_{I'_i}}}\right)\cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I''_i}|}}{\partial x^{\alpha_{I''_i}}}\right)\cdot ...\cdot \Delta^{n-1}\left(\frac{\partial^{|\alpha_{I_m}|}}{\partial x^{\alpha_{I_m}}}\right),
\end{aligned}
$$

where the second summation goes over all possible splittings, $I_i = I'_i \sqcup I''_i$, into disjoint subsets. Moreover, the inclusion (3) is a quasi-isomorphism [Me2]. Note that polydifferential operators (4) are allowed to have sets of multi-indices I_i and J_i with zero cardinalities $|I_i| = 0$ and/or $|J_i| = 0$ (so that the standard multiplication and comultiplication in $\mathcal{O}_V$ belong to the class of polydifferential operators.

2.2. The polydifferential prop associated to a prop. In this section we construct an endofunctor in the category of (augmented) props which has the property that for any prop $\mathcal{P} = \{\mathcal{P}(m,n)\}_{m,n\geq 1}$ and its representation

$$\rho : \mathcal{P} \longrightarrow \mathcal{E}nd_V$$

in a vector space V, the associated prop $\mathcal{DP} = \{\mathcal{DP}(k)\}_{k\geq 1}$ admits an associated representation,

$$\rho^{poly} : \mathcal{DP} \longrightarrow \mathcal{E}nd_{\odot^\bullet V}$$

in the graded commutative algebra $\odot^\bullet V$ on which elements $p\in\mathcal{P}$ act as polydifferential operators.

The idea is simple, and is best expressed in some basis $\{x^\alpha\}_{\alpha\in S}$ in V (so that $\odot^\bullet V \simeq \mathbb{K}[x^\alpha]$). Any element $p \in \mathcal{P}(m,n)$ gets transformed by ρ into a linear map

$$
\begin{array}{rccc}
\rho(p): & \otimes^n V & \longrightarrow & \otimes^m V\\
& x^{\alpha_1}\otimes x^{\alpha_2}\otimes\ldots\otimes x^{\alpha_n} & \longrightarrow & \displaystyle\sum_{\beta_1,\beta_2,\ldots,\beta_m} A^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots,\beta_m} x^{\beta_1}\otimes x^{\beta_2}\otimes\ldots\otimes x^{\beta_n}
\end{array}
$$

for some $A^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots,\beta_m}\in\mathbb{K}$.

Then, for any partitions

$$[m] = J_1\sqcup\ldots\sqcup J_l \qquad \text{and} \qquad [n] = I_1\sqcup\ldots\sqcup I_k$$

of the sets $[m]$ and $[n]$ into a disjoint union of (possibly, not all non-empty) subsets,

$$J_j = \{s_{a_1}, s_{a_2},\ldots, s_{a_{\# J_j}}\},\ 1\leq j\leq l, \qquad \text{and} \qquad I_i = \{s_{b_1}, s_{b_2},\ldots, s_{b_{\# I_i}}\},\ 1\leq i\leq k,$$

we can associate to $p\in\mathcal{P}(m,n)$ a polydifferential operator

$$
\begin{array}{rccc}
p^{poly}: & \otimes^k(\odot^\bullet V) & \longrightarrow & \otimes^l(\odot^\bullet V)\\
& f_1(x)\otimes f_2(x)\otimes\ldots\otimes f_n(x) & \longrightarrow & p^{poly}(f_1,\ldots,f_k)
\end{array}
$$

where

$$p^{poly}(f_1,\ldots,f_k) := \sum_{\substack{\alpha_1,\ldots,\alpha_n\\ \beta_1,\ldots,\beta_m}} A^{\alpha_{I_1}\ldots\alpha_{I_k}}_{\beta_{J_1}\ldots,\beta_{J_l}} x^{\beta_{J_1}}\otimes\ldots\otimes x^{\beta_{J_l}}\cdot\Delta^{l-1}\left(\frac{\partial^{|\alpha_{I_1}|}f_1}{\partial x^{\alpha_{I_1}}}\right)\cdot\ldots\cdot\Delta^{l-1}\left(\frac{\partial^{|\alpha_{I_k}|}f_k}{\partial x^{\alpha_{I_k}}}\right).$$

This association $p\to p^{poly}$ (for any fixed partition of $[m]$ and $[n]$) is independent of the choice of a basis used in the construction. Our purpose is to construct a prop $\mathcal{DP}$ out of the prop $\mathcal{P}$ together with a linear map

$$F^{[m]=J_1\sqcup\ldots\sqcup J_l}_{[n]=I_1\sqcup\ldots\sqcup I_k} : \mathcal{P}(m,n)\longrightarrow\mathcal{OP}(l,k)$$

such that, for any representation $\rho:\mathcal{P}\to\mathcal{E}nd_V$, the prop $\mathcal{DP}$ admits a natural representation ρ^{poly} in $\odot^\bullet V$ and

$$p^{poly} = \rho^{poly}\left(F^{[m]=J_1\sqcup\ldots\sqcup J_l}_{[n]=I_1\sqcup\ldots\sqcup I_k}(p)\right).$$

Note that $\odot^\bullet V$ carries a natural representation of the prop $\mathcal{C}omb$ controlling commutative and cocommutative bialgebras so that the latter can also be incorporated into $\mathcal{DP}$ in the form of operators corresponding, in the above notation, to the case when all the sets I_i and J_j are empty.

2.2.1. Remark. The functor $\mathcal{D}$ (see below) can be defined for an arbitrary unital prop. However, some of our constructions in this paper become nicer if we assume that the props we consider are augmented,

$$\mathcal{P} = \mathbb{K} \oplus \bar{\mathcal{P}},$$

and apply the functor $\mathcal{D}$ to the augmentation ideal $\bar{\mathcal{P}}$ only[5].

2.2.2. Definition. Let $\mathcal{P}$ be an augmented prop. Define a collection of (completed with respect to the filtration by the number of outputs and inputs) $\mathbb{S}$-modules,

$$\mathcal{DP}(l,k) := \mathcal{C}omb(l,k) \ \oplus \prod_{m,n\geq 1} \bigoplus_{\substack{[m]= \ J_1\sqcup\ldots\sqcup J_l \\ [n]= \ I_1\sqcup\ldots\sqcup I_k \\ \#I_i,\ldots,\#J_j\geq 0}} \mathcal{DP}^{J_1,\ldots,J_l}_{I_1,\ldots,I_k}$$

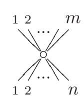
where

$$\begin{aligned} \mathcal{C}omb(l,k) \ &:= \ \mathbb{1}_l \otimes \mathbb{1}_k \\ \mathcal{DP}^{J_1,\ldots,J_l}_{I_1,\ldots,I_k} \ &:= \ (\mathbb{1}_{S_{J_1}} \otimes \ldots \otimes \mathbb{1}_{S_{J_l}}) \otimes_{S_{J_1}\times\ldots\times S_{J_l}} \otimes \bar{\mathcal{P}}(m,n) \otimes_{S_{I_1}\times\ldots\times S_{I_k}} (\mathbb{1}_{S_{I_1}} \otimes \ldots \otimes \mathbb{1}_{S_{I_k}}) \end{aligned}$$

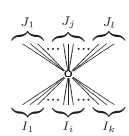
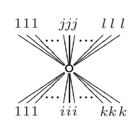
where $\mathbb{1}_I$ stands for the trivial one-dimensional representation of the permutation group S_I. Thus an element of the summand $\mathcal{DP}^{J_1,\ldots,J_l}_{I_1,\ldots,I_k}$ is an element of $\mathcal{P}(\#I_1+\ldots+\#I_l, \#J_1+\ldots+\#J_k)$ with inputs (resp., outputs) belonging to each bunch I_i (resp., J_j) symmetrized. We assume from now on that all legs in each bunch I_i (resp., J_j) are labelled by the same integer i (resp., j); this defines an action of the group $\mathbb{S}_l^{op} \times \mathbb{S}_k$ on $\mathcal{DP}(l,k)$. There is a canonical linear projection,

$$\pi^{J_1,\ldots,J_l}_{I_1,\ldots,I_k} : \bar{\mathcal{P}}(m,n) \longrightarrow \mathcal{DP}^{J_1,\ldots,J_l}_{I_1,\ldots,I_k}. \tag{5}$$

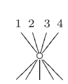
One can represent elements p of the (non-unital) prop $\bar{\mathcal{P}}$ as (decorated) corollas,

$p \ \sim$  $\in \bar{\mathcal{P}}(m,n)$

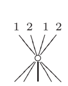
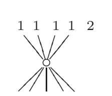
The image of p under the projection (5) is represented pictorially as the same corolla but with output and input legs decorated by *not necessarily distinct* numbers,

$\pi^{J_1,\ldots,J_l}_{I_1,\ldots,I_k}(p) \sim$  $\sim$  $\qquad 1 \leq i \leq k, \ 1 \leq j \leq l.$

Note that some of the sets I_i and/or J_j can be empty so that some of the numbers decorating inputs and/or outputs can have no legs attached at all. For example, one and the same element

$q =$  $\in \bar{\mathcal{P}}(4,5)$

can generate, for different partitions, several different elements in $\mathcal{DP}$,

(6)  $\in \mathcal{DP}(2,2)$,  $\in \mathcal{OP}(2,4)$, *etc.*

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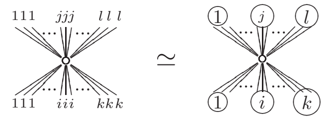
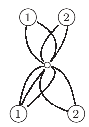
[5]The restriction of the functor $\mathcal{D}$ to the augmentation ideal comes, however, at the price of loosing some *rescaling* operators in $\mathcal{DP}$ which can be useful in applications.

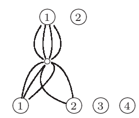
We shall often represent elements of $\mathcal{DP}$ as graphs having two types of vertices: the small ones which are decorated by elements of $\bar{\mathcal{P}}$, and new big ones corresponding to those input, respectively, output legs of $\mathcal{DP}$ which having the same numerical labels,



In this notation elements (6) gets represented, respectively, as



and



while the unique generator of $\mathcal{C}omb(k,l) \subset \mathcal{DP}(k,l)$ as a two leveled collection of solely white vertices

(7)
$$\begin{array}{c} \textcircled{1}\ \textcircled{2}\ \cdots \textcircled{l} \\ \\ \textcircled{1}\ \textcircled{2}\ \cdots \textcircled{k} \end{array}.$$

In particular the multiplication generator in $\mathcal{C}omb$ is given by the graph $\begin{array}{c}\textcircled{1}\\ \textcircled{1}\ \textcircled{2}\end{array}$ while the comultiplication generator by $\begin{array}{c}\textcircled{1}\ \textcircled{2}\\ \textcircled{1}\end{array}$. These new big white vertices come in two types — the incoming ones and outgoing ones; we call them from now on *in-vertices* and, respectively, *out-vertices* of al element $p \in \mathcal{DP}(l,k)$.
The $\mathbb{S}$-bimodule $\mathcal{DP} = \{\, \mathcal{DP}(k,l)\}$ has a natural structure of a prop with respect to the following composition operations:

(I) The *horizontal composition*

$$\begin{array}{rccc} \Box_h: & \mathcal{DP}(k,l) \otimes \mathcal{DP}(k',l') & \longrightarrow & \mathcal{DP}(k+k, l+l') \\ & p \otimes q & \longrightarrow & p\Box_h q \end{array}$$

is given as follows:

(i) if $p \in \mathcal{DP}^{J_1,\dots,J_l}_{I_1,\dots,I_k}$ and $q \in \mathcal{DP}^{J'_1,\dots,J'_{l'}}_{I'_1,\dots,I'_{k'}}$ the $\mathbb{S}$-equivariance of the horizontal composition $\circ_h$ in $\mathcal{P}$ assures that
$$p \circ_h q \in \mathcal{DP}^{J_1,\dots,J_l,J'_1,\dots,J'_{l'}}_{I_1,\dots,I_k,I'_1,\dots,I'_{k'}}$$
and we set $p\Box_h q := p \circ_h q$.

(ii) if $p \in \mathcal{DP}^{J_1,\dots,J_l}_{I_1,\dots,I_k} \subset \bar{P}(m,n)$ and q is the generator $\begin{array}{c} \textcircled{1}\ \textcircled{2}\ \cdots \textcircled{l'} \\ \textcircled{1}\ \textcircled{2}\ \cdots \textcircled{k'} \end{array}$ of $\mathcal{C}omb(l',k')$, then we define
$$p\Box_h q := \pi^{J_1,\dots,J_l,J'_1,\dots,J'_{l'}}_{I_1,\dots,I_k,I'_1,\dots,I'_{k'}}(p) \in \mathcal{DP}^{J_1,\dots,J_l,J'_1,\dots,J'_{l'}}_{I_1,\dots,I_k,I'_1,\dots,I'_{k'}}$$
where $J'_1 = \dots = J'_{l'} = I'_1 = \dots = I'_{k'} = \emptyset$. Similarly one defines $q\Box_h p$.

(iii) if $p \in \mathcal{C}omb(l,k)$ and $q \in \mathcal{C}omb(l',k')$ then $p\Box_h q$ is defined to be the horizontal composition of these elements in the prop $\mathcal{C}omb$.

(II) The *vertical composition*

$$\begin{array}{rccc} \Box_v: & \mathcal{DP}(l,n) \otimes \mathcal{DP}(n,k) & \longrightarrow & \mathcal{DP}(l,k) \\ & \Gamma \otimes \Gamma' & \longrightarrow & \Gamma\Box_v\Gamma', \end{array}$$

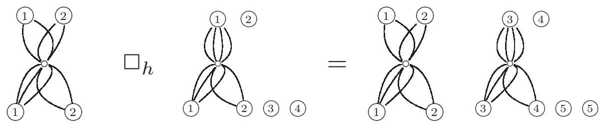
by the following three step procedure:

(1) erase all n out-vertices of Γ' and all n in-vertices of Γ; this procedure leave output legs of Γ' (which are labelled by elements of the set $[n]$) and input legs of Γ (which are also labelled by elements of the same set $[n]$) "hanging in the air";

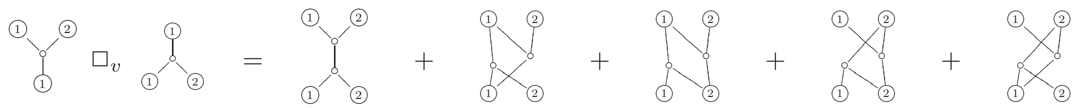
(2) take the sum over all possible ways of
   - (2i) attaching a number of the hanging out-legs of Γ' to the same number of hanging in-legs of Γ *with the same numerical label*,
   - (2ii) attaching the remaining hanging edges of Γ' to the out-vertices of Γ,
   - (2iii) attaching the remaining (after step (2i)) hanging in-legs of Γ to the in-vertices of Γ';

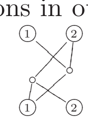
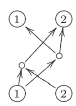
(3) perform the vertical composition (along new internal edges in step (2i)) and horizontal composition (of all remaining disjoint internal vertices) a in $\mathcal{P}$ to get a unique internal vertex in $\Gamma\Box_v\Gamma'$.

Often props $\mathcal{P}$ are defined as quotients of free props $\mathcal{F}ree\langle E\rangle$ generated by some $\mathbb{S}$-bimodules E modulo ideals $\mathcal{R}$ of relations; in this representation elements of $\mathcal{P}$ (and hence internal vertices of elements from $\mathcal{DP}$) are themselves graphs (modulo some equivalence relation). Therefore it is enough to understand the above compositions $\Box_h$ and $\Box_v$ for props of the form $\mathcal{P} = \mathcal{F}ree\langle E\rangle$ in which case one has, for example,



and



It is worth emphasizing that all graphs above are *oriented* in the sense that every edge is directed (but there are now directed paths of directed edge forming a closed path). We do not show directions in our pictures assuming that the flow goes by default from bottom to the top. For example, the graph  stands in fact for the directed graph  .

Let us summarize the above discussion in the following proposition-definition.

2.2.3. Proposition-definition. *(i) There is an endofunctor $\mathcal{D}$ in the category of augmented props. The value $\mathcal{DP}$ of the endofunctor $\mathcal{D}$ on an augmented prop $\mathcal{P}$ is called the* polydifferential prop associated to $\mathcal{P}$.

(ii) Any representation of the augmented prop $\mathcal{P}$ in a graded vector space V induces canonically an associated representation of the polydifferential $\mathcal{DP}$ in $\odot^\bullet V$ given by polydifferential operators.

(iii) For any augmented prop $\mathcal{P}$ there is a canonical injection of props,

$$i : \mathcal{C}omb \longrightarrow \mathcal{DP} \tag{8}$$

sending the generator 1 *of* $\mathcal{C}omb(l,k) = \mathbb{1}_l \otimes \mathbb{1}_k = \mathbb{K}$ *to the graph (7).*

2.2.4. Remark. The suboperad $\{\mathcal{DP}(1,n)\}_{n\geq 1}$ of the prop $\mathcal{DP}$ has been studied earlier in [MW1] where it was denoted by $\mathcal{OP}$.

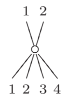
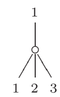
2.2.5. Proposition. *The endofunctor $\mathcal{D}$ is exact.*

Proof. Due to the classical Maschke's theorem, the functor of taking invariants and/or coinvariants in the category of representations of finite groups in dg vector spaces over a field of characteristic zero is exact. Hence the required claim. □

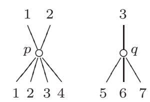
2.3. Polydifferential props and hypergraphs. In the definition of the polydifferential prop $\mathcal{DP}$ horizontal compositions in a prop $\mathcal{P}$ play as important role as vertical ones. Therefore, to apply apply the polydifferential functor to a *properad* (or operad) $\mathcal{P}$ one has to take first its enveloping prop $\mathcal{UP}$ and then apply the endofunctor $\mathcal{D}$ to the latter; for an augmented properad $\mathcal{P}$ we understand the prop $\mathcal{UP}$ as $\mathbb{K}\oplus\mathcal{U}\bar{\mathcal{P}}$ and define

$$\mathcal{DP} := \mathcal{D}(\mathcal{UP}).$$

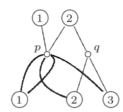
The case of properads is of special interest as elements of $\mathcal{DP}$ can be understood now as *hypergraphs*, that is, generalizations of graphs in which edges can connect more than two vertices (cf. [MW1]). For example, elements of a properad $\mathcal{P}$

$p =$  $\in \mathcal{P}(2,4)$ and $q =$  $\in \mathcal{P}(1,3)$

generate an element

 $\in \mathcal{UP}(3,7)$

which in turn generates an element

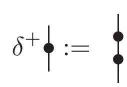
 $\in \mathcal{DP}(2,3)$

which looks like a real graph with many vertices of two types — the small ones which are decorated by elements of the properad $\mathcal{P}$ and big ones corresponding to the inputs and outputs of the properad $\mathcal{DP}$. Sometimes it is useful to understand such a graph as a *hypergraph* with small vertices playing the role of *hyperedges* (cf. [MW1]).

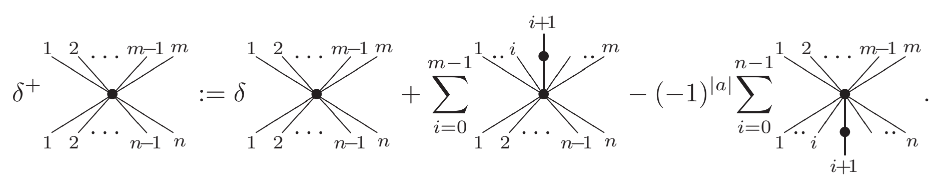
We consider below several concrete examples of polydifferential props and their applications.

2.4. A note on deformation complexes and Lie algebras of derivations of properads. Our definitions of these notions are slightly non-standard, but only with these non-standard definitions the $\mathcal{L}ie_\infty$ claim (ii) in the Main theorem **1.2** holds true.

Let (cP,δ) be an arbitrary dg properad, and $\mathcal{P}^+$ be the properad freely generated by $\mathcal{P}$ and one extra operation of arity $(1,1)$ and of cohomological degree $+1$. We make $\mathcal{P}^+$ into a *differential* properad by setting the value of the differential δ^+ on the new generator by



and on any other element $a \in \mathcal{P}(m,n)$ (which we identify pictorially with the (m,n)-corolla whose vertex is decorated with a) by the formula



The dg properad $(\mathcal{P}^+,\delta^+)$ is uniquely characterized by the fact that there is a 1-1 correspondence between representations

$$\rho : \mathcal{P}^+ \longrightarrow \mathcal{E}nd_V$$

of $(\mathcal{P}^+,\delta^+)$ in a dg vector space (V,d), and representations of $\mathcal{P}$ in the same space V but equipped with a deformed differential $d+d'$, where $d' := \rho(\,)$. Clearly any $\mathcal{P}$-algebra is a $\mathcal{P}^+$-algebra (with $d'=0$) so that there a canonical epimorphism $\pi : \mathcal{P}^+ \to \mathcal{P}$ of dg properad; we also have a canonical monomorphism $\mathcal{P} \to \mathcal{P}^+$ of *non-differential* properads. We define the dg Lie algebra, $\mathrm{Der}(\mathcal{P})$, of derivations of $(\mathcal{P},\delta)$ as the *complex of derivations of* $(\mathcal{P}^+,\delta^+)$ *with values in* $\mathcal{P}$, that is, as the Lie algebra generated by linear maps

$$D : \mathcal{P}^+ \longrightarrow \mathcal{P}$$

which satisfy the condition

$$D(a \boxtimes_{1,1} b) = D(a) \boxtimes_{1,1} \pi(b) + (-1)^{|a||D|}\pi(a) \boxtimes_{1,1} D(b), \quad \forall\ a,b \in \mathcal{P}^+,$$

where $\boxtimes_{1,1}$ stands [V] for the properadic composition along any connected oriented 2-vertex graph. The differential d in $\mathrm{Der}(\mathcal{P})$ is given by the formula

$$dD := \delta \circ D - (-1)^{|D|} D \circ \delta^{+}.$$

Let next $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of dg properads. Its deformation complex $Def(\mathcal{P} \stackrel{f}{\to} \mathcal{Q})$ is a $\mathcal{L}ie_\infty$ algebra which is defined in full details (and in two different ways) in [MeVa] with the help of a cofibrant resolution $\tilde{\mathcal{P}}$ of the prop $\mathcal{P}$. Most often $\tilde{\mathcal{P}}$ is equal to the cobar construction $\Omega(\mathcal{C}) = \mathcal{F}ree\langle \overline{\mathcal{C}}[-1]\rangle$ of some coaugmented homotopy co(pr)operad $\mathcal{C}$ (with the cokernel denoted by $\overline{\mathcal{C}}$), in which case one has an isomorphism of $\mathbb{S}$-modules,

$$Def(\mathcal{P} \stackrel{f}{\to} \mathcal{Q}) \equiv Def(\mathcal{P} \stackrel{f}{\to} \mathcal{Q}) \simeq \prod_{m,n\geq 1} \mathrm{Hom}_{\mathbb{S}_m^{op}\times \mathbb{S}_n}\left(\overline{C}(m,n), Q(m,n)\right).$$

Again we need a slightly extended version of the deformation functor,

$$\mathsf{Def}(\mathcal{P} \stackrel{f}{\to} \mathcal{Q}) := Def(\mathcal{P} \stackrel{f}{\to} \mathcal{Q}) \oplus Q(1,1)[1] \simeq \prod_{m,n\geq 1} \mathrm{Hom}_{\mathbb{S}_m^{op}\times \mathbb{S}_n}\left(C(m,n), Q(m,n)\right).$$

which includes an extra degree of freedom for deformations of the differentials. With this definition we have an isomorphism of complexes

$$\mathrm{Der}(P) = \mathsf{Def}(\mathcal{P} \stackrel{\mathrm{Id}}{\to} \mathcal{P})[1]$$

relating our extended version of the complex derivations with the extended version of the deformations of the identity map. Note however that this isomorphism does *not* respect Lie brackets — they are quite different (with the r.h.s. being in general a $\mathcal{L}ie_\infty$ algebra, while the l.h.s. being in a general a dg Lie algebra).

Finally we warn the reader that the formality theorem for quantization of Lie bialgebras as formulated in item (ii) of Theorem **1.2** holds true only for our extended version Def of the deformation functor Def.

3. Associative bialgebras, Lie bialgebras and their deformation complexes

3.1. A properad of Lie bialgebras. A Lie bialgebra is a graded vector space V equipped with compatible Lie brackets

$$[\ ,\ ] : \wedge^2 V \to V, \qquad \triangle : V \to \wedge^2 V$$

The compatibility relations are best described in terms of the properad $\mathcal{L}ieb$ governing Lie bialgebras via its representations $\rho_0 : \mathcal{L}ieb \to \mathcal{E}nd_V$ which send one generator to the Lie bracket and another generator to the Lie cobracket,

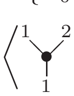
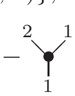
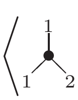
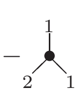
$\rho_0 \Big($  $\Big) = [\ ,\ ]$, $\rho_0 \Big($  $\Big) = \triangle$.

More precisely, the properad $\mathcal{L}ieb$ is defined [D1] as a quotient,

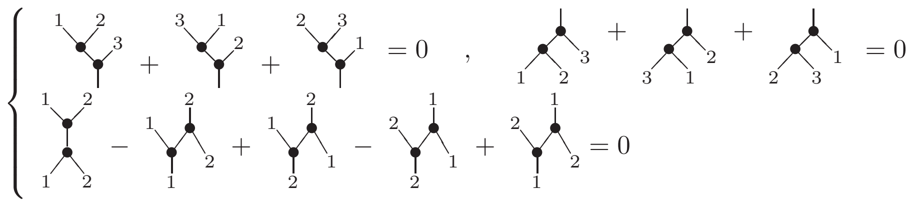
$$\mathcal{L}ieb := \mathcal{F}ree\langle E_0\rangle/(R)$$

of the free properad generated by an $\mathbb{S}$-bimodule $E_0 = \{E_0(m,n)\}$,

$E_0(m,n) :=$

- $sgn_2 \otimes 1\!\!1_1 \equiv \mathrm{span}\Big\langle$  $= -$  $\Big\rangle$ if $m=2, n=1$,
- $1\!\!1_1 \otimes sgn_2 \equiv \mathrm{span}\Big\langle$  $= -$  $\Big\rangle$ if $m=1, n=2$,
- 0 otherwise

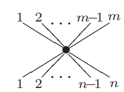
modulo the ideal generated by the following relations[6]

(9) $R:$ 

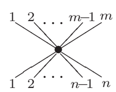
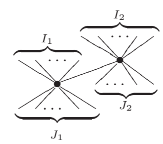
Its minimal resolution, $\mathcal{L}ieb_\infty$, is a dg free prop,

$$\mathcal{L}ieb_\infty = \mathcal{F}ree\langle E\rangle,$$

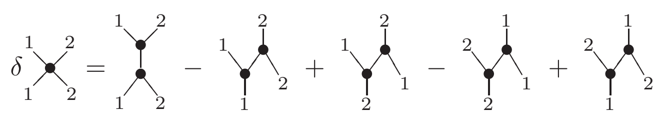
generated by the $\mathbb{S}$–bimodule $E = \{E(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

$$E(m,n) := sgn_m \otimes sgn_n[m+n-3] = \text{span}\Big\langle$$  $$\Big\rangle,$$

and with the differential given on generating corollas by [MaVo, V]

(10) δ  $$= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2\\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2\\ |J_1|\geq 1, |J_2|\geq 1}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1||I_2|+|J_1||J_2|}$$ 

where $\sigma(I_1 \sqcup I_2)$ and $\sigma(J_1 \sqcup J_2)$ are the signs of the shuffles $[1,\ldots,m] \to I_1 \sqcup I_2$ and, respectively, $[1,\ldots,n] \to J_1 \sqcup J_2$. For example,

(11) 

Strongly homotopy Lie bialgebra structures, $\rho : \mathcal{L}ieb_\infty \longrightarrow \mathcal{E}nd_V$, in a dg vector space V can be identified with Maurer-Cartan (MC, for short) elements of a dg Lie algebra $\mathfrak{g}_V$ defined next.

3.1.1. Strongly homotopy Lie bialgebra structures as Maurer-Cartan elements. Let V be a dg vector space (with differential denoted by d). According to the general theory [MeVa], there is a one-to-one correspondence between the set of representations, $\{\rho : \mathcal{L}ieb_\infty \to \mathcal{E}nd_V\}$, and the set of Maurer-Cartan elements in the dg Lie algebra

(12) $$\mathsf{Def}(\mathcal{L}ieb_\infty \xrightarrow{0} \mathcal{E}nd_V) \simeq \prod_{m,n\geq 1} \wedge^m V^* \otimes \wedge^n V[2-m-n] = \prod_{m,n\geq 1} \odot^m(V^*[-1]) \otimes \odot^n(V[-1])[2] =: \mathfrak{g}_V[2]$$

controlling deformations of the zero map $\mathcal{L}ieb_\infty \xrightarrow{0} \mathcal{E}nd_V$. The differential in $\mathfrak{g}_V$ is induced by the differential in V while the Lie bracket can be described explicitly as follows. First one notices that the completed graded vector space

$$\mathfrak{g}_V = \prod_{m,n\geq 1} \odot^m(V^*[-1]) \otimes \odot^n(V[-1]) = \widehat{\odot^{\bullet\geq 1}}(V^*[-1]) \oplus V[-1])$$

is naturally a 3-algebra with degree -2 Lie brackets, $\{\ ,\ \}$, given on generators by

$$\{sv, sw\} = 0, \quad \{s\alpha, s\beta\} = 0, \quad \{s\alpha, sv\} = <\alpha, v>, \quad \forall v, w \in V, \alpha, \beta \in V^*.$$

where $s : V \to V[-1]$ and $s : V^* \to V^*[-1]$ are natural isomorphisms. Maurer-Cartan elements in $\mathfrak{g}_V$, that is degree 3 elements ν satisfying the equation

$$\{\nu, \nu\} = 0,$$

are in 1-1 correspondence with representations $\nu : \mathcal{L}ieb_\infty \to \mathcal{E}nd_V$. Such elements satisfying the condition

$$\nu \in \odot^2(V^*[-1] \otimes V[-1] \ \oplus V^*[-1] \otimes \odot^2(V[-1])$$

[6]Here and everywhere all internal edges and legs in the graphical representations of elements of all the props considered are implicitly (but sometimes explicitly) oriented by the flow which runs from the bottom of a picture to the top.

are precisely Lie bialgebra structures in V.

Sometimes it is useful to use a coordinate representation of the above structure. Let $\{x^i\}_{i\in I}$ be a basis in V and $\{x_i\}_{i\in I}$ the dual basis in V^*. Set $\{\psi := sx_i\}_{i\in I}$ and $\{\psi^i := sx^i\}$ for the associated bases of $V[-1]$ and $V^*[-1]$ respectively. Note that $|\psi_i| + |\psi^i| = 2$ and that $\mathfrak{g}_V \simeq \mathbb{K}[[\psi_i, \psi^i]]$. Then the degree -2 Lie brackets in $\mathfrak{g}_V$ are given explicitly by

$$\{f, g\} = \sum_{i\in I} (-1)^{|f||\psi^i|} \frac{\partial f}{\partial \psi_i} \frac{\partial g}{\partial \psi^i} - (-1)^{|f||g|} \frac{\partial g}{\partial \psi_i} \frac{\partial f}{\partial \psi^i}.$$

It is an easy exercise to check that elements of the form

$$\nu := \sum_{i,j,k\in I} C_{ij}^k \psi_k \psi^i \psi^j + \Phi_k^{ij} \psi^k \psi_i \psi_j.$$

satisfy $\{\nu, \nu\} = 0$ if and only if the maps

$$\big(\triangle : V \to \wedge^2 V, [\ ,\ ] : \wedge^2 V \to V\big)$$

with the structure constants C_{ij}^k and Φ_k^{ij} respectively,

$$[x_i, x_j] =: \sum_{k\in I} C_{ij}^k x_k, \qquad \triangle(x_k) =: \sum_{i,j\in I} \Phi_k^{ij} x_i \wedge x_j.$$

define a Lie bialgebra structure in V. (Here we assumed for simplicity that V is concentrated in homological degree zero to avoid standard Koszul signs in the formulae.)

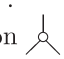
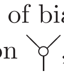
3.2. Associative bialgebras. An *associative bialgebra* is, by definition, a graded vector space V equipped with two degree zero linear maps,

$$\begin{array}{rccc} \mu: & V\otimes V & \longrightarrow & V \\ & a\otimes b & \longrightarrow & ab \end{array} \quad , \quad \begin{array}{rccc} \Delta: & V & \longrightarrow & V\otimes V \\ & a & \longrightarrow & \sum a_1 \otimes a_2 \end{array}$$

satisfying,

(i) the associativity identity: $(ab)c = a(bc)$;
(ii) the coassociativity identity: $(\Delta \otimes \mathrm{Id})\Delta a = (\mathrm{Id} \otimes \Delta)\Delta a$;
(iii) the compatibility identity: Δ is a morphism of algebras, i.e. $\Delta(ab) = \sum (-1)^{a_2 b_1} a_1 b_1 \otimes a_2 b_2$,

for any $a, b, c \in V$. We often abbreviate "associative bialgebra" to simply "bialgebra".

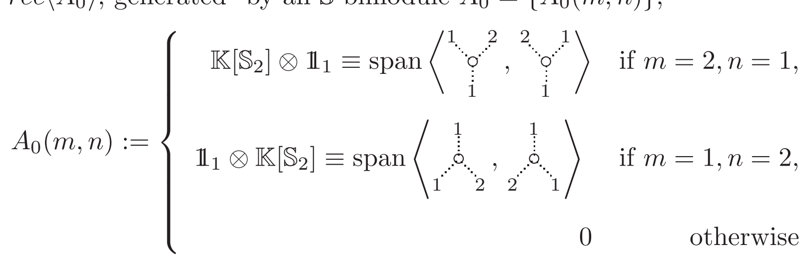
A prop of bialgebras, $\mathcal{A}ssb$, is generated by two kinds of operations, the multiplication  and the comultiplication , subject to relations which assure that its representations,

$$\rho : \mathcal{A}ssb \longrightarrow \mathcal{E}nd_A,$$

in one-to-one correspondence with associative bialgebra structures on a graded vector space A. More precisely, $\mathcal{A}ss\mathcal{B}$ is the quotient,

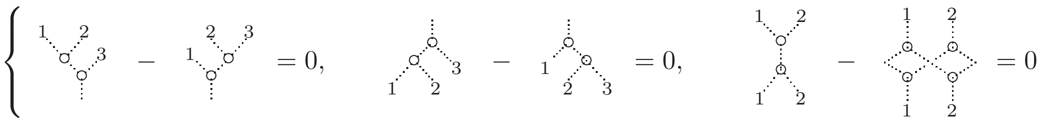
$$\mathcal{A}ssb := \mathcal{F}ree\langle A_0\rangle/(R)$$

of the free prop, $\mathcal{F}ree\langle A_0\rangle$, generated[7] by an $\mathbb{S}$-bimodule $A_0 = \{A_0(m,n)\}$,



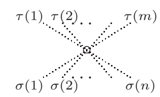
[7]Later we shall work with 2-coloured props so we reserve from now on the "dashed colour" to $\mathcal{A}ssb_\infty$ operations.

modulo the ideal generated by relations

(13) $R:$ 

which are not quadratic in the properadic sense (it is proven, however, in [MeVa] that $\mathcal{A}ssb$ is *homotopy Koszul*). A minimal resolution, $(\mathcal{A}ssb_\infty, \delta)$ of $\mathcal{A}ssb$ exists and is generated by a relatively "small" $\mathbb{S}$-bimodule $A = \{A(m,n)\}_{m,n\geq 1, m+n\geq 3}$,

$$A(m,n) := \mathbb{K}[\mathbb{S}_m] \otimes \mathbb{K}[\mathbb{S}_n][m+n-3] = \mathrm{span}\left\langle \quad \right\rangle_{\substack{\tau\in\mathbb{S}_n\\ \sigma\in\mathbb{S}_m}},$$



The differential δ in $\mathcal{A}ssb_\infty$ is not quadratic, and its explicit value on generic (m,n)-corolla is not known at present, but we can (and will) assume from now on that δ preserves the *path grading* of $\mathcal{A}ssb_\infty$ (which associates to any graph, G, from $\mathcal{A}ssb_\infty$ the total number of directed paths connecting input legs of G to the output ones, see [MaVo] for more details).

Let V be a $\mathbb{Z}$-graded vector space over a field $\mathbb{K}$ of characteristic zero. The associated symmetric tensor algebra $\mathcal{O}_V := \odot^\bullet V = \oplus_{n\geq 0} \odot^n V$ comes equipped with the standard graded commutative and co-commutative bialgebra structure, i.e. there is a non-trivial representation,

$$\rho_0 : \mathcal{A}ssb \longrightarrow \mathcal{E}nd_{\mathcal{O}_V}. \tag{14}$$

According to [MeVa], the deformation complex

$$C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V) = \mathsf{Def}\left(\mathcal{A}ssb \xrightarrow{\rho_0} \mathcal{E}nd_{\mathcal{O}_V}\right) \simeq \prod_{m,n\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2-m-n]$$

and its polydifferential subcomplex $C^\bullet_{poly}(\mathcal{O}_V, \mathcal{O}_V)$ come equipped with a $\mathcal{L}ie_\infty$ algebra structure, $\{\mu_n : \wedge^n C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V) \longrightarrow C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V)[2-n]\}_{n\geq 1}$, such that μ_1 coincides precisely with the Gerstenhaber-Shack differential. According to [GS], the cohomology of the complex $(C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V), \mu_1)$ is precisely the vector space $\mathfrak{g}_V$; moreover, it is not hard to see that the operation μ_2 induces the Lie brackets $\{\ ,\ \}$ in $\mathfrak{g}_V$. As we show in this paper, the set of $\mathcal{L}ie_\infty$ quasi-isomorphisms,

$$\{F : (\mathfrak{g}_V[2], \{\ ,\ \}) \longrightarrow (C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V), \mu_\bullet)\}, \tag{15}$$

is always non-empty.

4. A prop governing formality maps for Lie bialgebras

4.1. An operad of directed graphs. Let $G_{n,l}$ be a set of directed graphs Γ with n vertices and l edges such that some bijections $V(\Gamma) \to [n]$ and $E(\Gamma) \to [l]$ are fixed, i.e. every edges and every vertex of Γ has a fixed numerical label. There is a natural right action of the group $\mathbb{S}_n \times \mathbb{S}_l$ on the set $G_{n,l}$ with $\mathbb{S}_n$ acting by relabeling the vertices and $\mathbb{S}_l$ by relabeling the edges.

For each fixed integer d, a collection of $\mathbb{S}_n$-modules,

$$d\mathcal{G}ra_d = \left\{ d\mathcal{G}ra_d(n) = \prod_{l\geq 0} \mathbb{K}\langle G_{n,l}\rangle \otimes_{\mathbb{S}_l} sgn_l^{\otimes |d-1|}[l(d-1)] \right\}_{n\geq 1}$$

is an operad with respect to the following operadic composition,

$$\begin{array}{rccl} \circ_i : & d\mathcal{G}ra_d(n) \times d\mathcal{G}ra_d(m) & \longrightarrow & d\mathcal{G}ra_d(m+n-1), \quad \forall\ i \in [n] \\ & (\Gamma_1, \Gamma_2) & \longrightarrow & \Gamma_1 \circ_i \Gamma_2, \end{array}$$

where $\Gamma_1 \circ_i \Gamma_2$ is defined by substituting the graph Γ_2 into the i-labeled vertex v_i of Γ_1 and taking a sum over re-attachments of dangling edges (attached before to v_i) to vertices of Γ_2 in all possible ways.

The operad of directed graphs $d\mathcal{G}ra_d$ contains a suboperad $\mathcal{G}ra^{or}$ spanned by graphs with no *closed* paths of directed edges (wheels); we call such graphs *oriented*.

For any operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ in the category of graded vector spaces, the linear the map

$$\begin{array}{rccc} [\ ,\ ]: & \mathsf{P}\otimes\mathsf{P} & \longrightarrow & \mathsf{P} \\ & (a\in\mathcal{P}(n), b\in\mathcal{P}(m)) & \longrightarrow & [a,b] := \sum_{i=1}^{n} a\circ_i b - (-1)^{|a||b|}\sum_{i=1}^{m} b\circ_i a \ \in \mathcal{P}(m+n-1) \end{array}$$

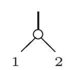
makes a graded vector space $\mathsf{P} := \prod_{n\geq 1}\mathcal{P}(n)$ into a Lie algebra [KM]; moreover, these brackets induce a Lie algebra structure on the subspace of invariants $\mathsf{P}^{\mathbb{S}} := \prod_{n\geq 1}\mathcal{P}(n)^{\mathbb{S}_n}$. In particular, the graded vector spaces

$$\mathsf{dfGC}_d := \prod_{n\geq 1} d\mathcal{G}ra_d(n)^{\mathbb{S}_n}[d(1-n)], \qquad \mathsf{fGC}_d^{or} := \prod_{n\geq 1}\mathcal{G}ra_d^{or}(n)^{\mathbb{S}_n}[d(1-n)]$$

are Lie algebra with respect to the above Lie brackets, and as such they can be identified with the deformation complexes $\mathsf{Def}(\mathcal{L}ie_d \stackrel{0}{\to} d\mathcal{G}ra_d)$ and, respectively, $\mathsf{Def}(\mathcal{L}ie_d \stackrel{0}{\to} \mathcal{G}ra_d^{or})$ of athe zero morphism. Hence non-trivial Maurer-Cartan elements of $(\mathsf{dfGC}_d/\mathsf{fGC}_d^{or}, [\ ,\ ])$ give us non-trivial morphisms of operads

$$f: \mathcal{L}ie_d \longrightarrow d\mathcal{G}ra_d \quad \text{and, respectively} \quad f: \mathcal{L}ie_d \longrightarrow \mathcal{G}ra_d^{or}$$

One such non-trivial morphism f is given explicitly on the generator of $\mathcal{L}ie_d$ by [W1]

(16) $$f\left(\ \right) = \textcircled{1}\!\!\to\!\!\textcircled{2} \ -(-1)^d\ \textcircled{2}\!\!\to\!\!\textcircled{1} \ =: \bullet\!\!\to\!\!\bullet$$


Note that elements of dfGC_d can be identified with graphs from $\mathcal{DG}ra_d$ whose vertices' labels are symmetrized (for d even) or skew-symmetrized (for d odd) so that in pictures we can forget about labels of vertices and denote them by unlabelled black bullets as in the formula above. Note also that graphs from dfGC_d come equipped with an orientation, or, which is a choice of ordering of edges (for d even) or a choice of ordering of vertices (for d odd) up to an even permutation on both cases. Thus every graph $\Gamma\in\mathsf{dfGC}_d$ has at most two different orientations, pr and or^{opp}, and one has the standard relation, $(\Gamma, or) = -(\Gamma, or^{opp})$; as usual, the data (Γ, or) is abbreviate by Γ (with some choice of orientation implicitly assumed). Note that the homological degree of graph Γ from dfGC_d is given by $|\Gamma| = d(\#V(\Gamma)-1) + (1-d)\#E(\Gamma)$.

The above morphism (16) makes $(\mathsf{dfGC}_d, [\ ,\ ])$ into a *differential* Lie algebra with the differential

$$\delta := [\bullet\!\!\to\!\!\bullet,\ ].$$

This dg Lie algebra contains a dg subalgebra dGC_d spanned by connected graphs with at least bivalent vertices.It was proven in [W2] that

$$H^\bullet(\mathsf{dfGC}_d) = \mathsf{dGC}_d$$

so that there is no loss of generality of working with dGC_d instead of dfGC_d. Moreover, one has an isomorphism of Lie algebras [W2],

$$H^0(\mathsf{dGC}_d) = \mathfrak{grt}_1,$$

where $\mathfrak{grt}_1$ is the Lie algebra of the Grothendieck-teichmü ller group GRT_1 introduced by Drinfeld [D2] in the context of deformation quantization of Lie bialgebras.

Let GC_d^{or} be the subcomplex of fGC^{or} spanned by connected graphs with at least bivalent vertices and with no bivalent vertices of the form $\to\!\bullet\!\to$. This subcomplex determines the cohomology of the full oriented graph complex, $H^\bullet(\mathsf{fGC}_d^{or}) = \odot^\bullet(H^\bullet(\mathsf{GC}_d^{or})$. It was proven in [W2] that

$$H^\bullet(\mathsf{GC}_{d+1}^{or}) = H^\bullet(\mathsf{dGC}_d).$$

In particular, one has a remarkable isomorphism of Lie algebras, $H^0(\mathsf{GC}_3^{or}) = \mathfrak{grt}$. Moreover [W2] $H^i(\mathsf{GC}_3^{or}) = 0$ for $i\leq 2$ and $H^{-1}(\mathsf{GC}_3^{or})$ is a 1-dimensional space generated by the graph  .

4.2. A canonical representation of $\mathcal{G}ra_3^{or}$ in $\mathfrak{g}_V$. For *any* graded vector space the operad $\mathcal{G}ra_3^{or}$ has a canonical representation in the associated vector space $\mathfrak{g}_V$,

$$\begin{array}{rccc} \rho: & \mathcal{G}ra_3^{or}(n) & \longrightarrow & \mathcal{E}nd_{\mathfrak{g}_V}(n) = \mathrm{Hom}(\mathfrak{g}_V^{\otimes n}, \mathfrak{g}_V) \\ & \Gamma & \longrightarrow & \Phi_\Gamma \end{array} \tag{17}$$

given by the formula,

$$\Phi_\Gamma(\gamma_1, \ldots, \gamma_n) := \mu\left(\prod_{e\in E(\Gamma)} \Delta_e\left(\gamma_1(\psi)\otimes\gamma_2(\psi)\otimes\ldots\otimes\gamma_n(\psi)\right)\right)$$

where, for an edge $e = \overset{a}{\bullet}\!\longrightarrow\!\overset{b}{\bullet}$ connecting a vertex labeled by $a \in [n]$ and to a vertex labelled by $b \in [n]$, we set

$$\Delta_e\left(\gamma_1\otimes\gamma_2\otimes\ldots\otimes\gamma_n\right) = \left\{\begin{array}{ll} \sum\limits_{i\in I}(-1)^{|\psi^i|(|\gamma_a|+|\gamma_{a+1}|+\ldots+|\gamma_{b-1}|)}\gamma_1\otimes\ldots\otimes\frac{\partial\gamma_a}{\partial\psi_i}\otimes\ldots\otimes\frac{\partial\gamma_b}{\partial\psi^i}\otimes\ldots\otimes\gamma_n & \text{for } a<b \\ \sum\limits_{i\in I}(-1)^{|\psi_i|(|\gamma_b|+|\gamma_{b+1}|+\ldots+|\gamma_{a-1}|+1)}\gamma_1\otimes\ldots\otimes\frac{\partial\gamma_b}{\partial\psi^i}\otimes\ldots\otimes\frac{\partial\gamma_a}{\partial\psi_i}\otimes\ldots\otimes\gamma_n & \text{for } b<a \end{array}\right.$$

and where μ is the multiplication map,

$$\begin{array}{rccc} \mu: & \mathfrak{g}_V^{\otimes n} & \longrightarrow & \mathfrak{g}_V \\ & \gamma_1\otimes\gamma_2\otimes\ldots\otimes\gamma_n & \longrightarrow & \gamma_1\gamma_2\cdots\gamma_n. \end{array}$$

If V is *finite* dimensional, then the above formulae can be used to define a representation of the operad $d\mathcal{G}ra_3$ in $\mathfrak{g}_V$.

4.3. A prop of directed graphs. Let $k \geq 0$, $m \geq 1$ and $n \geq 1$ be integers. Let $\mathsf{G}_{k;m,n}$ be a set of directed graphs consists, by definition, of directed oriented graphs Γ with k vertices called *internal*, m vertices called *in*-vertices and n vertices called *out*-vertices (i.e. there is a partition) satisfying the following conditions

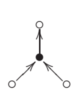
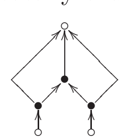
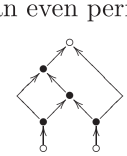
(i) the set of vertices is partitioned, $V(\Gamma) := V_{int}(\Gamma) \sqcup V_{in}(\Gamma) \sqcup V_{out}(\Gamma)$, into three subsets of the cardinalities,
$$\#V_{int}(\Gamma) = k, \quad \#V_{in}(\Gamma) = m, \quad \#V_{out}(\Gamma) = n;$$
elements of $V_{int}(\Gamma)$ are called *internal* vertices, elements of $V_{in}(\Gamma)$ are called *in*-vertices, and elements of $V_{out}(\Gamma)$ are called *out*-vertices;

(iii) the vertices are labelled via isomorphisms,
$$i_{int}: V_{int}(\Gamma) \to [k], \quad i_{in}: V_{in}(\Gamma) \to [m], \quad i_{out}: V_{int}(\Gamma) \to [n];$$

(iv) every in-vertex can have only outgoing edges (called *in-legs*), while every out-vertex can have only ingoing edges (called *in-legs*);

(v) there are no edges connecting in-vertices to out-vertices, i.e. the set of all edges is partitioned into a disjoint union, $E(\Gamma) := E_{int}(\Gamma) \sqcup E_{in}(\Gamma) \sqcup E_{out}(\Gamma)$, where $E_{in}(\Gamma)$ is the set of in-legs, $E_{out}(\Gamma)$ the set of out-legs, and $E_{int}(\Gamma)$ the set of *internal* edges which connect internal vertices to internal ones;

(vi) the set of legs $E_{in}(\Gamma) \sqcup E_{out}(\Gamma)$ is totally ordered up to an even permutation.

 $\in \mathsf{G}_{1;1,2}$,  $\in \mathsf{G}_{3;1,2}$,  $\in \mathsf{G}_{4;1,2}$

where in- and out-vertices are depicted by small white circles, while internal vertices by small black ones. (and we do not show numerical labels of the vertices).

Define a graded vector space
$$\mathcal{BG}ra^{or}(k;m,n) := \mathbb{R}\langle \mathsf{G}_{k;m,n}\rangle$$
by assigning to each graph $\Gamma \in \mathsf{G}_{k;m,n}$ homological degree,
$$|\Gamma| = -2\#E_{int}(\Gamma) - \#E_{in}(\Gamma) - \#E_{out}(\Gamma),$$

i.e. by assigning to each internal edge degree -2 and to each leg degree -1. Consider next an $\mathbb{S}$-bimodule,

$$\mathcal{BG}ra^{or} = \left\{ \mathcal{BG}ra^{or}(m,n) := \sum_{k\geq 0} \mathcal{BG}ra^{or}(k;m,n) \right\}.$$

We claim that it has a natural structure of a prop with

- horizontal composition

$$\begin{array}{rccc} \boxtimes: & \mathcal{BG}ra^{or}(k;m,n)\otimes \mathcal{BG}ra^{or}(k';m',n') & \longrightarrow & \mathcal{BG}ra^{or}(k+k';m+m',n+n') \\ & \Gamma\otimes\Gamma' & \longrightarrow & \Gamma\boxtimes\Gamma' \end{array}$$

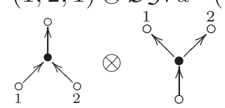
  by taking the disjoint union of the graphs Γ and Γ' and relabelling in-, out- and internal vertices of Γ' accordingly;
- vertical composition,

$$\begin{array}{rccc} \circ: & \mathcal{BG}ra^{or}(k;m,n)\otimes \mathcal{BG}ra^{or}(k';n,l) & \longrightarrow & \mathcal{BG}ra^{or}(k+k';m,l) \\ & \Gamma\otimes\Gamma' & \longrightarrow & \Gamma\circ\Gamma', \end{array}$$

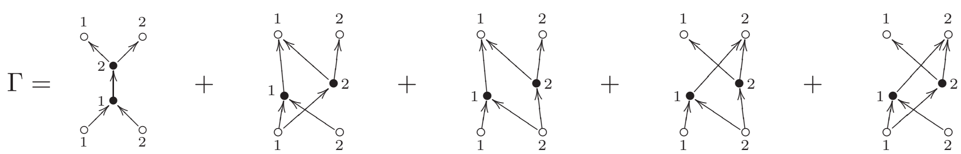
  by the following three step procedure: (a) erase all m out-vertices of Γ and all m in-vertices of Γ', (b) take a sum over all possible ways of attaching the hanging out-legs of Γ to hanging in-legs of Γ' as well as to out-vertices of Γ', and also attaching the remaining in-legs of Γ' to in-vertices of Γ, (c) relabel internal vertices of Γ' by adding the value k to each label.

For example, a vertical composition of the following two graphs,

$$\circ: \quad \mathcal{BG}ra^{or}(1;2,1)\otimes \mathcal{BG}ra^{or}(1;1,2) \quad\longrightarrow\quad \mathcal{BG}ra^{\uparrow}(2;2,2)$$


$\longrightarrow \quad \Gamma$

is given by the following sum (cf. §**2.2.2**)



Note that the $\mathbb{S}$-bimodule $\{\mathcal{BG}ra^{or}(0;m,n)\}$ is a subprop of $\mathcal{BG}ra^{\uparrow}$ isomorphic to $\mathcal{C}omb$.

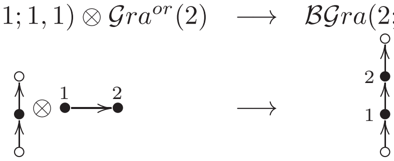
Note also that we can also define compositions

$$\circ: \mathcal{G}ra_3^{or}(p_1)\otimes \mathcal{G}ra_3^{or}(p_2)\otimes\ldots\otimes \mathcal{G}ra_3^{or}(p_k)\otimes \mathcal{BG}ra^{or}(k,m,n) \longrightarrow \mathcal{BG}ra^{or}(p_1+p_2+\ldots+p_k,m,n)$$

by substituting k graphs from $\mathcal{G}ra_3^{or}$ into internal vertices of a graph from $\mathcal{BG}ra^{or}(k,m,n)$ and redistributing edges in exactly the same way as in the definition of the operadic composition in $\mathcal{G}ra_3^{or}$ (and setting to zero all graphs which do not satisfy the condition that every black vertex has at least one incoming edge and at least one outgoing edge). For example,

$$\bullet: \quad \mathcal{BG}ra(1;1,1)\otimes \mathcal{G}ra^{or}(2) \quad\longrightarrow\quad \mathcal{BG}ra(2;1,1)$$

(18)



If we denote by the same symbol $\mathcal{G}ra_3^{or}$ the *prop* generated by the operad $\mathcal{G}ra_3^{or}$ then we conclude that the data

$$\mathcal{B}ra^{or} := \{\mathcal{BG}ra^{or}, \mathcal{G}ra_3^{or}\}$$

generates a *2-coloured* prop in the category of graded vector spaces.

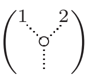
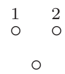
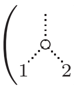
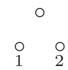
Abusing notations, let us denote by $\mathcal{L}ie_3$ the prop generated by the operad $\mathcal{L}ie\{2\}$. Then formula (16) says that there is a canonical morphism of props,

$$i_1 : \mathcal{L}ie_3 \longrightarrow \mathcal{B}ra^{or}. \tag{19}$$

There is also a morphism of props

$$i_2 : \mathcal{A}ssb \longrightarrow \mathcal{B}ra^{or} \tag{20}$$

given on the generators as follows,

$i_2\Bigg($  $\Bigg) =$  , $\qquad i_2\Bigg($  $\Bigg) =$ 

4.3.1. A canonical representation of the prop $\mathcal{B}ra^{or}$. Let V be an arbitrary graded vector space with a basis $\{x^i\}$ so that $\mathcal{O}_V = \mathbb{K}[x^i]$ and $\mathfrak{g}_V = \mathbb{K}[[\psi^i, \psi_i]]$ with $\psi^i = sx^i$. Let

$$\mathcal{E}nd_{\mathcal{O}_V,\ \mathfrak{g}_V} = \{\mathrm{Hom}(\mathfrak{g}_V^{\otimes p} \otimes \mathcal{O}_V^{\otimes m}, \mathfrak{g}_V^{\otimes q} \otimes \mathcal{O}_V^{\otimes n}\}$$

be the two coloured endomorphism prop of vector spaces $\mathcal{O}_V$ (in white colour) and $\mathfrak{g}_V$ (in black colour). A representation,

$$\rho : \mathcal{B}ra^{or} \longrightarrow \mathcal{E}nd_{\mathcal{O}_V,\ \mathfrak{g}_V}$$

is uniquely determined by its values on the generators. The values of ρ on the generators of $\mathcal{G}ra_3^{gr}$ are given by (17). The value,

$$\rho_\Gamma \in \mathrm{Hom}\left(\mathfrak{g}_V^{\otimes k} \otimes \mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n}\right),$$

of ρ on a graph $\Gamma \in \mathsf{G}_{k;m,n}$ of $\mathcal{B}ra^{or}$ is defined as a composition of maps,

(21)

$$\begin{array}{rccccl} \rho_\Gamma : & \mathfrak{g}_V^{\otimes k} \otimes \mathcal{O}_V^{\otimes m} & \longrightarrow & \mathcal{O}_V^{\otimes n} \otimes \mathfrak{g}_V^{\otimes k} \otimes \mathcal{O}_V^{\otimes m} & \xrightarrow{\Psi_\Gamma} & \mathcal{O}_V^{\otimes n} \otimes \mathcal{O}_V^{\otimes m} \xrightarrow{\mu} \mathcal{O}_V^{\otimes n} \\ & \gamma_1 \otimes ... \otimes \gamma_k \otimes f_1 \otimes ... \otimes f_m & \longrightarrow & 1^{\otimes n} \otimes \gamma_1 \otimes ... \otimes \gamma_k \otimes f_1 \otimes ... \otimes f_m & & \end{array}$$

where

$$\Psi_\Gamma(\underbrace{1, ..., 1}_{n}, \gamma_1, ..., \gamma_k, f_1, ..., f_m) = \left(\prod_{\overline{e} \in E_{out}(\Gamma)} \Delta_{\overline{e}} \prod_{e \in E_{int}(\Gamma)} \Delta_e \prod_{\underline{e} \in E_{in}(\Gamma)} \Delta_{\underline{e}}(\underbrace{1 \otimes ... \otimes 1}_{n} \otimes \gamma_1 \otimes ... \otimes \gamma_k \otimes f_1 \otimes ... \otimes f_m)\right)_{\substack{\psi_i = 0 \\ \psi^i = 0}}$$

with

- for an in-leg $e = \overset{\alpha}{\circ}\!\longrightarrow\!\overset{a}{\bullet}$ connecting α-th in-vertex to a-th internal vertex, $\alpha \in [m]$, $a \in [k]$,

$$\Delta_{\underline{e}} := \sum_{i \in I} \underbrace{\frac{\partial}{\partial \psi^i}}_{\substack{\text{acts on } a-\text{th} \\ \text{tensor factor of } \mathfrak{g}_V^{\otimes k}}} \otimes \underbrace{\frac{\partial}{\partial x_i}}_{\substack{\text{acts on } \alpha-\text{th} \\ \text{tensor factor of } \mathcal{O}_V^{\otimes m}}}$$

- for an internal edge $e = \overset{a}{\bullet}\!\longrightarrow\!\overset{b}{\bullet}$, $a, b \in [k]$,

$$\Delta_e := \sum_{i \in I} \underbrace{\frac{\partial}{\partial \psi^i}}_{\substack{\text{acts on} a-\text{th} \\ \text{tensor factor of } \mathfrak{g}_V^{\otimes k}}} \otimes \underbrace{\frac{\partial}{\partial x^i}}_{\substack{\text{acts on } b-\text{th} \\ \text{tensor factor of } \mathfrak{g}_V^{\otimes k}}}$$

- for an out-leg $\overline{e} = \overset{a}{\bullet}\!\longrightarrow\!\overset{\beta}{\circ}$, $a \in [k]$, $\beta \in [n]$,

$$\Delta_{\overline{e}} := \sum_{i \in I} \underbrace{x_i}_{\substack{\text{acts as multiplication on } \beta-\text{th} \\ \text{tensor factor of } 1^{\otimes n}}} \otimes \underbrace{\frac{\partial}{\partial \psi_i}}_{\substack{\text{acts on } a-\text{th} \\ \text{tensor factor of } \mathfrak{g}_V^{\otimes k}}}$$

and where

$$\begin{array}{cccc} \mu: & \mathcal{O}_V^{\otimes n}\otimes\mathcal{O}_V^{\otimes m} & \longrightarrow & \mathcal{O}_V^{\otimes n} \\ & g_1\otimes ...\otimes g_n\otimes h_1\otimes ...\otimes h_m & \longrightarrow & (g_1\otimes ...\otimes g_n)\cdot(\Delta^{n-1}(h_1\cdots h_m)) \end{array}$$

with $\cdot$ being the standard multiplication in the algebra $\mathcal{O}_V^{\otimes n}$.

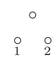
The proof of the claim that the above formula for $\rho(\Gamma)$ gives a morphism of props (i.e. respects prop compositions) is a straightforward untwisting of the definitions; in fact, the prop compositions in $\mathcal{BG}ra^{or}$ have been defined just precisely for the purpose to make this claim true, i.e. they have been read out from the compositions of the operators $\rho(\Gamma)$.

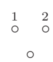
4.3.2. Remarks. (i) Note that operators $\Delta_{\underline{e}}$ and $\Delta_{\overline{e}}$ have degree -1 (in contrast to operators Δ_e which have degree -2) so that the ordering of legs of a graph Γ from $\mathcal{BG}ra^{or}$ is required to make the definition of $\rho(\Gamma)$ unambiguous. However, we shall see below that this choice is irrelevant for the application of the morphism ρ to deformation quantization.

(ii) As our formula for $\rho(\Gamma)$ involves evaluation at $\psi_\bullet=\psi^\bullet=0$, we can say that $\rho(\Gamma)(\gamma_1,...,\gamma_k,f_1,...,f_m)$ is non-zero only in the case when for any $a\in[k]$ one has

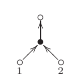
$$\gamma_a\in\odot^{\#In(v_a)}V^*[1]\otimes\odot^{\#Out(v_a)}V[1]$$

where v_a is the a-labelled internal vertex of Γ and $In(v_a)$ (resp., $Out(v_a)$) is the set of input (resp., output) half-edges.

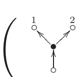
4.3.3. Examples (cf. [Sh]). 1) $\rho\Big($  $\Big):\mathcal{O}_V^{\otimes 2}\to\mathcal{O}_V$ is just the multiplication in $\mathcal{O}_V$.

2) $\rho\Big($  $\Big):\mathcal{O}_V\to\mathcal{O}_V^{\otimes 2}$ is the comultiplication Δ in $\mathcal{O}_V$.

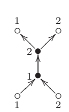
3) If $\gamma=\sum_{i,j,k\in I}C_{ij}^k\psi_k\psi^i\psi^j$, $C_{ij}^k\in\mathbb{R}$, then, for any $f_1(x),f_2(x)\in\mathcal{O}_V$, one has (modulo a sign coming from a choice of ordering of legs)

$\rho\Big($  $\Big)$ $$(\gamma,f_1,f_2)=\sum_{i,j,k\in I}\pm x_kC_{ij}^k\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j}$$

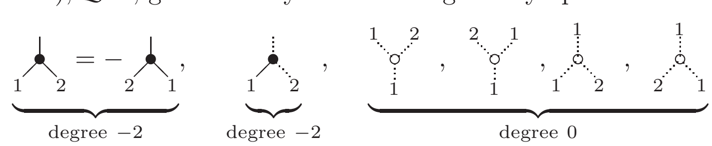
4) If $\gamma=\sum_{i,j,k\in I}\Phi_k^{ij}\psi_i\psi_j\psi^k$, $\Phi_k^{ij}\in\mathbb{R}$, then, for any $f(x)\in\mathcal{O}_V$, one has (modulo a sign coming from a choice of ordering of legs)

$\rho\Big($  $\Big)$ $$(\gamma,f)=\sum_{i,j,k\in I}\pm(x_i\otimes x_j)\cdot\Phi_k^{ij}\Delta(\frac{\partial f}{\partial x_k})$$

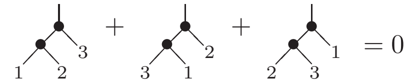
5) If f $\gamma_1=\sum_{i,j,k\in I}C_{ij}^k\psi_k\psi^i\psi^j$,$\gamma_2=\sum_{i,j,k\in I}\Phi_k^{ij}\psi_i\psi_j\psi^k$, $C_{ij}^k,\Phi_k^{ij}\in\mathbb{R}$, then, for any $f_1,f_2\in\mathcal{O}_V$, one has (modulo a sign coming from a choice of ordering of legs)

$\rho\Big($  $\Big)$ $$(\gamma,f)=\sum_{i,j,k,m,n\in I}\pm(x_m\otimes x_n)\cdot\Phi_k^{mn}C_{ij}^k\Delta(\frac{\partial f_1}{\partial x_i}\frac{\partial f_2}{\partial x_j})$$

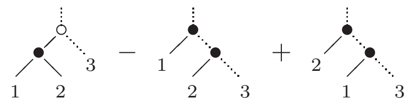
4.4. A 2-coloured prop of $\mathcal{L}ie_3$ actions on bialgebras. Consider an auxiliary 2-coloured prop (in straight and dashed colours), $\mathcal{Q}ua$, generated by the following binary operations

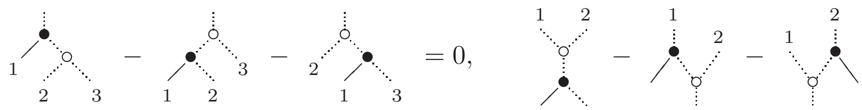
(22) 

modulo Jacobi relations

(23) 

the associative bialgebra relations (13), and also the following ones

(24)  $= 0,$

(25)  $= 0.$

Rather surprisingly, this prop arises naturally in the context of formality maps for universal quantizations of Lie bialgebras.

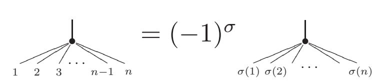
4.5. A prop governing formality maps. Sometimes it is useful to define a prop in terms of generators and relations, sometimes it is easier to define it by specifying its generic representation. We shall define a useful for our purposes prop $\mathcal{Q}ua_\infty$ using both approaches (but start with the second one which explains its relation to formality maps)) and then prove that $\mathcal{Q}ua_\infty$ is a minimal resolution of $\mathcal{Q}ua$.

Let $\mathcal{Q}ua_\infty$ be a 2-coloured prop whose arbitrary representation, $\rho : \mathcal{Q}ua \to \mathcal{E}nd_{\mathfrak{g},A}$, in a pair of graded vector spaces $\mathfrak{g}$ and A provides these spaces with the following list of algebraic structures,

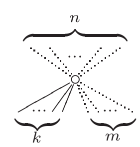
(i) the structure of a $\mathcal{H}olie_3$-algebra in $\mathfrak{g}$, i.e. a representation $\rho_1 : \mathcal{H}olie_3 \to \mathcal{E}nd_{\mathfrak{g}}$ (note that $\mathfrak{g}[2]$ is then an ordinary $\mathcal{L}ie_\infty$ algebra);
(ii) the structure of an $\mathcal{A}ssb_\infty$-algebra in A, i.e. a representation $\rho_2 : \mathcal{A}ssb_\infty \to \mathcal{E}nd_A$
(iii) a morphism of $\mathcal{L}ie_\infty$-algebras,

$$F : \mathfrak{g}[2] \longrightarrow \mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{\rho} \mathcal{E}nd_A)$$

The prop $\mathcal{Q}ua_\infty$ is free with the space of generators in the colour responsible for datum (i) given by

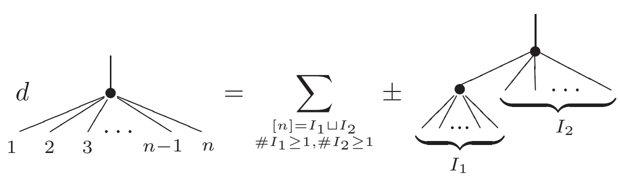
(26) $E(n) := sgn_n[3n-4] = \Big\langle$  $\Big\rangle_{\sigma\in\mathbb{S}_n}$

and the space of generators (of mixed colours) responsible for data (ii) and (iii) given by,

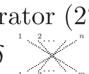
(27) $E(n, k+m) := \mathbb{R}[\mathbb{S}_n] \otimes sgn_k \otimes \mathbb{R}[\mathbb{S}_m][m+n+3k-3] = \mathrm{span}\Big\langle$  $\Big\rangle_{\substack{n\geq 1, m\geq 1, k\geq 0\\ n+k+m\geq 3}}$

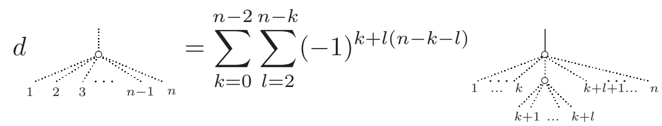
Corollas with $k = 0$ take care about datum (ii), i.e. they are the standard generators of $\mathcal{A}ssb_\infty$, while corollas with $k \geq 1$ take care about datum (ii). The prop $\mathcal{Q}ua_\infty$ is differential; the values of its differential d on

- the generators (26) of $\mathcal{H}olie_3$ is given by the standard formula,

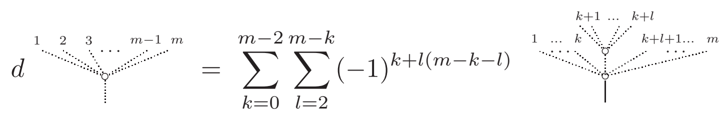


- the generators (27) with $k = 0$ are determined by a choice of a minimal resolution, $\mathcal{A}ss\mathcal{B}_\infty$, i.e. d restricted to these generators equals precisely the differential δ in $\mathcal{A}ssb_\infty$;
- generators (27) with $k \geq 1, n \geq 1$ are determined uniquely by the choice of a minimal resolution $\mathcal{A}ssb_\infty$ and the standard formulae for a $\mathcal{L}ie_\infty$ morphism.

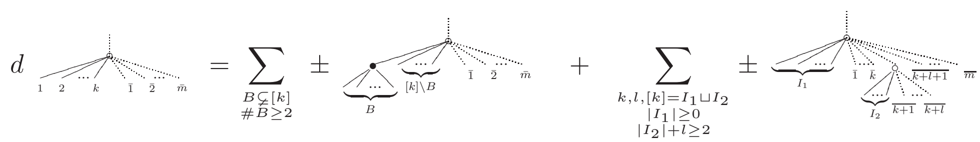
Roughly speaking, the value of the differential on a generator (27) with $k \geq 1, n \geq 1$ a sum over all possible attachments of k solid legs to the vertices in the graph δ  plus standard terms of "$\mathcal{L}ie_\infty$" type. For example, one has in the simplest cases $m=1$, $k=0$, $n\geq 2$,

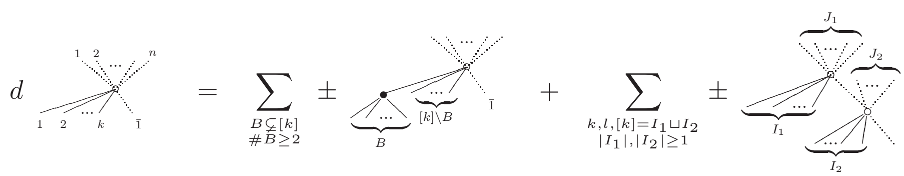


and $n=1$, $k=0$, $m\geq 2$,

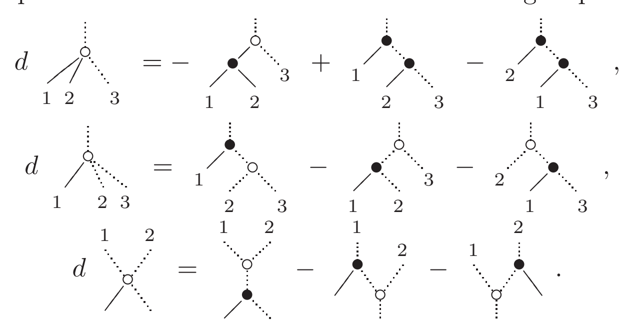


so that the formulae for the values of d on corollas with $m=1$, $k\geq 1$, $n\geq 1$ and, respectively, with $m\geq 1$, $k\geq 1$, $n=1$, are given (modulo signs) by

(28) 

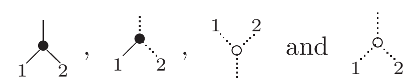


We are interested in a few special cases of these formulae for which we give precise expressions,



These equations say that there exists a canonical morphism of dg props,

$$p : \mathcal{Q}ua_\infty \longrightarrow \mathcal{Q}ua \tag{29}$$

which sends to zero all generators of $\mathcal{Q}ua_\infty$ except 

4.5.1. Theorem. *The morphism p is a quasi-isomorphism.*

Proof. The differential in the complex $(\mathcal{Q}ua_\infty, d)$ is connected (that is, its value on any generator is given by a sum of connected graphs) and preserves the path grading[8]. Hence by Theorem 27 in [MaVo], the genus filtration

$$F_p^{genus} := \{\text{span}\langle \Gamma\in \mathcal{Q}ua_\infty(m,n)\rangle \ : \ \text{the genus of } \Gamma > -p\}$$

induces a *converging* spectral sequence whose first term we denote by $Gr^{gen}(\mathcal{Q}ua_\infty)$. The projection map p respects genus filtrations of both sides, and hence induces a morphism of complexes

$$p' : Gr^{gen}(\mathcal{Q}ua_\infty) \longrightarrow Gr^{gen}(\mathcal{Q}ua).$$

[8]To any graph $\Gamma\in\mathcal{P}(m,n)$ of a free prop generated by by an $\mathbb{S}$-bimodule $E=\{E(m,n)\}_{m+n\geq 3, m,n\geq 1}$ we can associate the number $|\Gamma|_{path}\leq mn$ of directed paths connecting input legs of Γ with output ones.

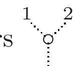
The complex $Gr^{gen}(\mathcal{Q}ua_\infty)$ admits a path filtration

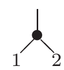
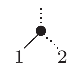
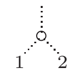
$$F_p^{path} := \{\text{span}\langle \Gamma \in Gr^{gen}(\mathcal{Q}ua_\infty)(m,n)\rangle \ : \ |\Gamma|_{path} < p\}$$

with the converging spectral sequence whose first term we denote by $Gr^{gen,path}(\mathcal{Q}ua_\infty)$. The map p' above preserves path filtrations of both sides and hence induces a morphism of the associated graded complexes,

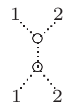
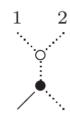
$$p'' : Gr^{gen,path}(\mathcal{Q}ua_\infty) \longrightarrow Gr^{gen,path}(\mathcal{Q}ua)$$

The r.h.s. equals the prop enveloping of a quadratic $\frac{1}{2}$-prop $\mathcal{P}_0$ generated by

(i) the *Koszul* operad $\mathcal{A}ss$ with generators ,

(ii) the 2-coloured *Koszul* [A, HL] operad of non-commutative Gerstenhaber algebras (also known as operad of Leibniz pairs) with generators  ,  , and 

modulo $\frac{1}{2}$-prop relations

 $= 0$ and  $= 0$.

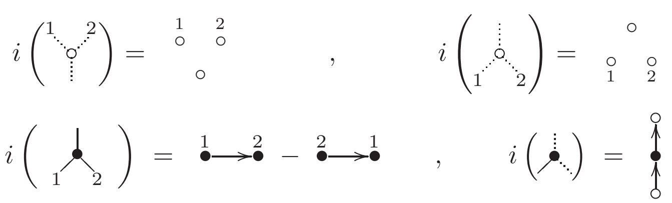
It is easy to compute the minimal resolution of the $\frac{1}{2}$-prop $\mathcal{P}_0$ and notice that its prop enveloping equals precisely $Gr^{gen,path}(\mathcal{Q}ua_\infty)$. Hence the morphism p'' is a quasi-isomorphism which implies that the morphism p' is a quasi-isomorphism which in tern implies that the morphism p is a quasi-isomorphism. $\square$

4.6. Formality morphisms as Maurer-Cartan elements. The morphism (19) factors through the inclusion $\mathcal{G}ra^{or} \to \mathcal{BG}ra^{or}$, and extends to the following one.

4.6.1. Proposition. *There is a morphism of props*

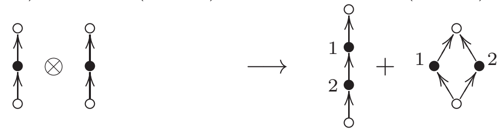
$$i : \mathcal{Q}ua \longrightarrow \mathcal{B}ra^{or} \tag{30}$$

given on the generators as follows,



Proof. Relation (24) follows from example (18) and the fact that

$$\circ : \quad \mathcal{PG}ra(1;1,1) \otimes \mathcal{PG}ra(1;1,1) \quad \longrightarrow \quad \mathcal{PG}ra(2;1,1)$$



Analogously one checks that all other relations (13) and (25) are also respected by the map i. $\square$

The composition of maps $p : \mathcal{Q}ua_\infty \to \mathcal{Q}ua$ and i from (30) gives us a canonical morphism of props,

$$q_0 : \mathcal{Q}ua_\infty \longrightarrow \mathcal{B}ra^{or},$$

so that we can consider a graph complex (in fact, a $\mathcal{L}ie_\infty$ algebra)[9]

$$\mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{B}ra) =: \mathsf{fGC}_3^{or} \oplus \mathsf{BGC}^{or}$$

[9] The symbol $\oplus$ means here a direct sum of graded vector spaces, *not* a direct sum of $\mathcal{L}ie_\infty$-algebras. It is worth noting, however, that each summand is a $\mathcal{L}ie_\infty$ subalgebra (with fGC_3^{or} being an ordinary dg Lie algebra). Moreover, we can (and will) assume without loss of generality that BGC^{or} is spanned by graphs with at least trivalent vertices (cf. [W1]).

controlling deformations of the morphism q_0. Elements of BGC can be understood as (linear combinations of) oriented directed graphs Γ from $\mathsf{G}_{k;m,n}$ which have labels of *black* vertices skew-symmetrized (so that labelling of *black* vertices can be omitted in the pictures), and which are assigned the homological degree

$$\begin{aligned} |\Gamma| &= 3|V_{int}(\Gamma)| + |V_{in}(\Gamma)| + |V_{out}(\Gamma)| - 2|E_{int}(\Gamma)| - |E_{in}(\Gamma)| - |E_{out}(\Gamma)| - 2 \\ &= \sum_{i=1}^{|V_{int}(\Gamma)|} (3 - |v_i|) + |V_{in}(\Gamma)| + |V_{out}(\Gamma)| - 2. \end{aligned} \tag{31}$$

where $|v_i|$ stands for the valency (the total number of input and output half-edges) of the i-th internal vertex.

4.7. Universal formality maps. A generic Maurer-Cartan element in the $\mathcal{L}ie_\infty$ algebra $\mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{B}ra^{or})$ (or its oriented version) is a direct sum

$$(q', q) \in \mathsf{fGC_3}^{or} \oplus \mathsf{BGC}^{or},$$

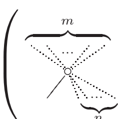
where q' is responsible for the deformation of the map i_1 in (19). In this paper we are interested in deformations which keeps i_0 fixed (speaking plainly, this means that we want to preserve the Poisson brackets $\{\ ,\ \}$ in $\mathfrak{g}_V$ which define strongly homotopy bialgebras). This leads us to the following notion.

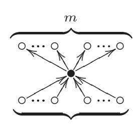
4.7.1. Definitions. A Maurer-Cartan element of the $\mathcal{L}ie_\infty$ algebra $\mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{B}ra^{or})$ of the form $(0, q)$, that is, a generic Maurer-Cartan element q of its $\mathcal{L}ie_\infty$ subalgebra BGC^{or} is called a *universal quantization* of arbitrary (possibly, infinite-dimensional) strongly homotopy Lie bialgebras. A universal quantization $q \in \mathcal{MC}(\mathsf{BGC}^{or})$ is a called a *universal formality map* if the associated morphism of dg props,

$$F : \mathcal{Q}ua_\infty \stackrel{q_0+q}{\longrightarrow} \mathcal{B}ra^{or},$$

satisfies the following condition,

$$F\left(\right) = \frac{1}{m!n!} \tag{32}$$





where the graph in the r.h.s. is assumed to be skew-symmetrized over the labels of in- and out-vertices.

The set of universal formality maps is denoted by $\mathcal{MC}_{\text{form}}$.

4.8. On GRT_1 action. The $\mathcal{L}ie_\infty$ algebra structure

$$\left\{ \mu_n : \wedge^n \mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{B}ra^{or}) \to \mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{PG}ra^{or})[2-n] \right\}_{n \geq 1}$$

satisfies the condition that the maps μ_n restricted to the subspace $\mathsf{fGC_3^{or}} \otimes \wedge^{n-1} \mathsf{Def}(\mathcal{Q}ua_\infty \stackrel{q_0}{\to} \mathcal{PG}ra^{or})$ vanish for $n \geq 3$. This means that the map

$$\begin{array}{rccc} Ad: & \mathsf{fGC_3^{or}} \otimes \mathsf{BGC^{or}} & \longrightarrow & \mathsf{BGC^{or}} \\ & \gamma \otimes \Gamma & \longrightarrow & ad_\gamma \Gamma := \mu_2(\gamma, \Gamma) \end{array}$$

gives us a injective morphism of Lie algebras,

$$\begin{array}{ccc} Z\mathsf{fGC_3^{or}} & \longrightarrow & \mathrm{Der}(\mathsf{BGC^{or}}) \\ \gamma & \longrightarrow & ad_\gamma \end{array} \tag{33}$$

where $Z\mathsf{fGC_3^{or}} \subset \mathsf{fGC_3^{or}}$ is the graded Lie subalgebra of co-cycles, and $\mathrm{Der}(\mathsf{BGC^{or}})$ is the Lie algebra of derivations of the $\mathcal{L}ie_\infty$ algebra $\mathsf{BGC^{or}}$.Hence we also have a canonical map

$$\mathfrak{grt}_1 \longrightarrow H^0(\mathrm{Der}(\mathsf{BGC^{or}}))$$

which implies the following

4.8.1. Proposition. *The prounipotent Grothendieck-Teichmüller group $GRT_1 = \exp(\mathfrak{grt})$ acts on the set of gauge equivalence classes, $\mathcal{MC}(\mathsf{BGC^{or}})/\sim$, of Maurer-Cartan elements in the $\mathcal{L}ie_\infty$ algebra $\mathsf{BGC^{or}}$. Moreover, this action decsends to the action on the subset $\mathcal{MC}_{\mathrm{form}} \subset \mathcal{MC}(\mathsf{BGC^{or}})$ of formality maps. The action*

$$\begin{array}{ccc} GRT_1 \times \mathcal{MC}_{\mathrm{form}} & \longrightarrow & \mathcal{MC}_{\mathrm{form}} \\ (\exp(\gamma), F = \sum \Gamma) & \longrightarrow & \sum \exp(\gamma) \cdot \Gamma \end{array}$$

is given by (repeated) substitutions of a graph representative of an element $\gamma \in \mathfrak{grt}_1$ in $Z\mathsf{fGC}_3^{\mathsf{or}}$ into black vertices of graphs Γ representing the formality map F (cf. [W1]*).*

4.8.2. From formality maps to quantizations of strongly homotopy algebras. Any element $q \in \mathcal{MC}(\mathsf{BGC}^{or})$ is the same as a morphism of props, which deforms the standard morphism q_0 in such a way that its component i_0 stays invariant. As the prop $\mathcal{B}ra^{or}$ admits a canonical representation in $\mathcal{E}nd_{\mathcal{O}_V, \mathfrak{g}_V}$ for any (possibly, infinite-dimensional) graded vector space V, the universal quantization q gives us a representation

$$\mathcal{Q}ua_\infty \stackrel{q_0+q}{\longrightarrow} \mathcal{B}ra \longrightarrow \mathcal{E}nd_{\mathfrak{g}_V, \mathcal{O}_V},$$

that is, a $\mathcal{L}ie_\infty$ morphism,

$$F = \left\{ F_k : \wedge^k(\mathfrak{g}_V[2]) \longrightarrow C^\bullet_{GS}(\mathcal{O}_V, \mathcal{O}_V), \quad |F_k| = 1-k \right\}_{k\geq 1}$$

If q satisfies in addition the condition (32), this $\mathcal{L}ie_\infty$ morphism F must be a quasi-isomorphism.

Let ν be an arbitrary Maurer-Cartan element in $\mathfrak{g}_V$, that is, an arbitrary strongly homotopy Lie bialgebra structure on V. Then the morphism F gives us in turn an associated Maurer-Cartan element,

$$\rho^\hbar := \rho_0 + \sum_{k=1}^{\infty} \frac{\hbar^k}{k!} F_k(\gamma, \ldots, \gamma), \tag{34}$$

in $\mathfrak{gs}(\mathcal{O}_V, \mathcal{O}_V)[[\hbar]]$, that is, a continuous $\mathcal{A}ss\mathcal{B}_\infty$ structure in $\mathcal{O}_V[[\hbar]]$ which deforms the standard $\mathcal{A}ss\mathcal{B}$ structure in $\mathcal{O}_V$,

$$\rho^\hbar|_{\hbar=0} = \rho_0,$$

and also satisfies,

$$\frac{d\rho^\hbar}{d\hbar}|_{\hbar=0} = \gamma.$$

Thus any universal formality map gives a *universal quantization of strongly homotopy algebras.*

5. Existence and classification of universal formality maps

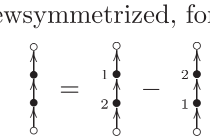
5.1. A one coloured prop of graph complexes. Consider an $\mathbb{S}$-bimodule,

$$\mathcal{B}^{or} := \left\{ \mathcal{B}^{or}(m,n) := \bigoplus_{k\geq 0} \mathcal{BG}ra^{or}(k;m,n) \otimes_{\mathbb{S}_k} sgn_k[3k] \right\}.$$

An element of $\mathcal{B}^{or}$ can be understood as a graph Γ from $\mathcal{B}ra^{or}$ whose *internal* vertices have *no* labels and are totally ordered (up to an even permutation)[10], and which is assigned the homological degree

$$|\Gamma| = 3|V_{int}(\Gamma)| - 2|E_{int}(\Gamma)| - |E_{in}(\Gamma)| - |E_{out}(\Gamma)|. \tag{35}$$

One should, however, keep in mind that in reality such a graph Γ stands for a linear combination of graphs from $\mathcal{PG}ra^{or}$ whose internal vertices are skewsymmetrized, for example

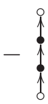


[10]A choice of such a total ordering is called sometimes an *orientation* on Γ. Every graph Γ from $\mathcal{B}^{or}$ with more than one internal vertex has precisely two possible orientations, or and or^{opp}, and one tacitly identifies $(\Gamma, or) = -(\Gamma, or^{opp})$. In our pictures we show only Γ, a choice of orientation on Γ being tacitly assumed.

We use such an identification in the following definition of a degree 1 operation on $\mathcal{B}^{or}$,

$$
\begin{array}{rccc}
\delta_{\bullet\!-\!\bullet}: & \mathcal{B}^{or} & \longrightarrow & \mathcal{B}^{or} \\
& \Gamma & \longrightarrow & \delta_{\bullet\!-\!\bullet}\Gamma = \left\{\begin{array}{cl} 0 & \text{if } V_{int}(\Gamma)=\emptyset \\ (-1)^{|\Gamma|}\Gamma\bullet_1\left(\overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet} - \overset{2}{\bullet}\!\!\longrightarrow\!\!\overset{1}{\bullet}\right) & \text{if } V_{int}(\Gamma)\neq\emptyset \end{array}\right.
\end{array}
$$

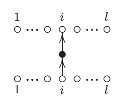
where $\bullet_1$ stands for the substitution of the graph $\overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet} - \overset{2}{\bullet}\!\!\longrightarrow\!\!\overset{1}{\bullet}$ into the internal vertex labeled by 1 in the skewsymmetrized linear combination of graphs corresponding to Γ, and then the summation over all possible ways of reconnecting the "hanging" half-edges to the two vertices of the substituted graph which respect the directed flow $\uparrow$ on edges and the condition that each internal vertex of the resulting graph has at least one incoming half-edge and at least one outgoing half-edge (cf. (18)). For example,

$\delta_{\bullet\!-\!\bullet}$ $= -$ 

Note that each element Γ from $\mathcal{B}^{or}$ comes equipped with an *orientation*, i.e. an element (of unit length) in $\det\mathbb{R}^{|V_{int}(\Gamma)|}\otimes\det\mathbb{R}^{|E_{in}(\Gamma)|}\otimes\det\mathbb{R}^{|E_{in}(\Gamma)|}$, and that Γ is identified with $-\Gamma^{opp}$, where Γ^{opp} is the unique graph which is identical to Γ in all data except orientation.

It is obvious that the prop structure in $\mathcal{B}ra^{or}$ induces a prop structure in $\mathcal{B}^{or}$. What is less obvious is that the one-coloured prop $\mathcal{B}^{or}$ has a natural differential $\delta_\bullet$.

For any $l\geq 1$ and any $i\in[l]$ define a degree one element,

$\theta_i(l,l) :=$ 

in $B^{or}(l,l)$, and then consider a linear map

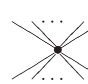
$$
\begin{array}{rccc}
\delta_{\circ\!-\!\bullet\!-\!\circ}: & \mathcal{B}^{or}(m,n) & \longrightarrow & \mathcal{B}^{or}(m,n) \\
& \Gamma & \longrightarrow & \delta_{\circ\!-\!\bullet\!-\!\circ}\Gamma := \sum_{i=1}^m \theta_i(m,m)\circ_v\Gamma - (-1)^{|\Gamma|}\sum_{i=1}^n \Gamma\circ_v\theta_i(n,n)
\end{array}
$$

where $\circ_v$ stands for the vertical composition in the prop $\mathcal{B}ra^{or}$.

5.1.1. Lemma. *The linear map $\delta_\bullet := \delta_{\bullet\!-\!\bullet} + \delta_{\circ\!-\!\bullet\!-\!\circ}$ is a differential in the prop $\mathcal{B}^{or}$ which makes it identical to the dg prop $\mathcal{DL}ieb_\infty$.*

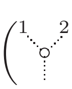
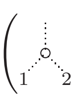
Proof. This claim can be proven in two different ways.

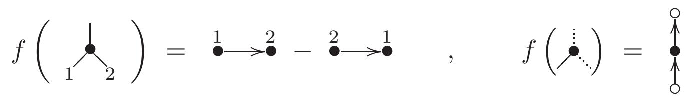
(i) By construction, every internal vertex of $\mathcal{B}^{or}$ is at least trivalent. A direct inspection of the formula for $\delta_\bullet$ shows that it acts only on black (i.e. internal) vertices,

 ,

of graphs from $\mathcal{B}^{or}$ by splitting them precisely as in (10). The only non-obvious part of this claim is the formula for signs. Recall that any graph Γ from $\mathcal{BG}ra^{or}$ comes equipped with an orientation, that is, with a choice of an element (of unit length) in $\det\mathbb{R}^{|V_{int}(\Gamma)|}\otimes\det\mathbb{R}^{|E_{in}(\Gamma)|}\otimes\mathbb{R}^{|E_{out}(\Gamma)|}$. It was proven in [Th] that the latter space can be canonically identified with $\det\mathbb{R}^{|e(\Gamma)|}\otimes\det\mathbb{R}^{\dim H_1(|Ga|)}$, where $e(\Gamma)$ is the set of all edges of Γ and $H_1(|Ga|)$ is the first homology group of the geometric realization of Γ. The map $\delta_\bullet$ leaves $H_1(|Ga|)$ invariant so that, if we want to understand induced orientations on summands of the expression $d\Gamma$ for any $\Gamma\in\mathcal{BG}ra^{or}$, it is enough to take care only about the orientation of the space $\mathbb{R}^{|e(\Gamma)|}$ generated by all edges. Then the translation of graph orientations into signs becomes more or less straightforward: it was first done in [MaVo] (in a different context), and we refer to §7.1 of that paper for the proof that the formula for signs is exactly the one given by (10).

(ii) Consider a morphism of props, $f:\mathcal{Q}ua\longrightarrow\mathcal{B}ra^{or}$, given on the generators as follows (cf. (30)),

$f\Big($  $\Big) = 0,$ $\qquad , \qquad$ $f\Big($  $\Big) = 0,$



Then, as before, the dg vector space

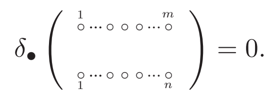
$$\mathsf{Def}(\mathcal{Q}ua_\infty \xrightarrow{f} \mathcal{B}ra^{or}) \simeq \mathsf{fGC}_3^{or} \oplus \mathsf{BGC}_3^{\mathsf{or}}$$

splits into a direct sum of complexes. Up to a degree shift graphs from BGC_3^{or} can be identified with graphs from $\mathcal{B}^{or}$. It remains to notice that the differential in BGC_3^{or} induced by the morphism f is precisely the sum $d_{\bullet\!-\!\bullet} + d_{\circ-\bullet-\circ}$. □

As the functor $\mathcal{D}$ exact, the above result computes immediately the cohomology of the dg prop $\mathcal{B}^{or} \equiv \mathcal{DL}ieb_\infty$.

5.1.1. Corollary. *The canonical surjection $\pi : \mathcal{DL}ieb_\infty \longrightarrow \mathcal{DL}ieb$ is a quasi-isomorphism.*

5.2. The graph complex $\mathsf{BGC}^{\mathsf{or}}$ reinterpreted. Note that elements in BBC_3^{or} which have no internal vertices are cycles with respect to the differential $\delta_\bullet$, that is,



Therefore there is a morphism of *dg* props,

$$i_\circ : (\mathcal{A}ss\mathcal{B}, 0) \longrightarrow (\mathcal{DL}ieb_\infty, \delta_\bullet)$$

given on the generators as follows

(36) 

and hence a morphism of dg props $\mathcal{A}ss\mathcal{B}_\infty \to \mathcal{DL}ieb_\infty$ denoted by the same letter $i_\circ$. Using the above identification of $\mathcal{DL}ieb_\infty$ with $\mathcal{B}^{or}$ and the relation of the latter with $\mathcal{B}ra^{or}$, it is now straightforward to see that the deformation complex of the morphism $i_\circ$,

$$\mathsf{Def}(\mathcal{A}ss\mathcal{B} \xrightarrow{i_\circ} \mathcal{DL}ieb_\infty)$$

can be identified as a $\mathcal{L}ie_\infty$ algebra with the earlier defined graph complex BGC^{or}. Thus the Maurer-Cartan elements γ of, for example, the $\mathcal{L}ie_\infty$ algebra BGC^{or} can be understood not only as morphisms of two-coloured dg props,

$$(\mathcal{Q}ua_\infty, \delta) \xrightarrow{\gamma} (\mathcal{B}ra^{or}, 0)$$

but also as morphisms of one-coloured dg props,

$$(\mathcal{A}ss\mathcal{B}_\infty, \delta) \xrightarrow{\gamma} (\mathcal{DL}ieb_\infty, \delta_\bullet).$$

This double meaning of one and the same element $\gamma \in \mathcal{MC}(\mathsf{BGC}^{\mathsf{or}})$ is not surprising and can be understood in terms of representations as follows.

Let $\nu : \mathcal{L}ieb_\infty \to \mathcal{E}nd_V$ be a strongly homotopy Lie bialgebra structure in a dg vector space V; we can understand it as a Maurer-Cartan element ν in the Lie algebra $\mathfrak{g}_V$. According to the general principle (see §2), there is an associated representation, $\rho_\nu : \mathcal{DL}ieb_\infty \longrightarrow \mathcal{E}nd_{\mathcal{O}_V}$ which we would like to describe explicitly.

Recall (see §**4.3.1**) that there is a canonical representation of the 2-coloured prop $\mathcal{B}ra^{or}$ in the pair of vector spaces $(\mathfrak{g}, \mathcal{O}_V)$ which associates to a graph Γ from $\mathcal{BG}ra^{or}(k; m, n)$ a linear map (21),

$$\begin{array}{rccc} \rho_\Gamma : & \mathfrak{g}_V^{\otimes k} \otimes \mathcal{O}_V^{\otimes n} & \longrightarrow & \mathcal{O}_V^{\otimes m} \\ & \gamma_1 \otimes \ldots \otimes \gamma_k \otimes f_1 \otimes \ldots \otimes f_n & \longrightarrow & \rho_\Gamma(\gamma_1 \otimes \ldots \otimes \gamma_k \otimes f_1 \otimes \ldots \otimes f_n). \end{array}$$

According to Lemma **5.1.1**, a generator of $\mathcal{DL}ieb_\infty(m, n)$ can be understood as a graph Γ^{skew} obtained by skewsymmetrization of labels of internal vertices of some graph Γ from $\mathcal{BG}ra(k; m, n)$. Then the formula

$$\begin{array}{rccc} \Phi_\Gamma^\nu := \rho(\Gamma^{skew}) : & \mathcal{O}_V^{\otimes n} & \longrightarrow & \mathcal{O}_V^{\otimes m}[[\hbar]] \\ & f_1 \otimes \ldots \otimes f_n & \longrightarrow & \frac{1}{k!}\rho_\Gamma(\nu \otimes \ldots \otimes \nu \otimes f_1 \otimes \ldots \otimes f_n) \end{array}$$

gives us the required representation of $\mathcal{DL}ieb_\infty$ in $\mathcal{O}_V$. Comparing this formula with (34), we see that any Maurer-Cartan element γ of the $\mathcal{L}ie_\infty$ algebra $\mathsf{BGC}^{\mathsf{or}}$ can indeed be understood as a morphisms of one-coloured dg props $\gamma' : \mathcal{A}ss\mathcal{B}_\infty \xrightarrow{\gamma} \mathcal{DL}ieb_\infty$ such that the $\mathcal{A}ss\mathcal{B}_\infty$ algebra structure in $\mathcal{O}_V$ induced from the representation ν in accordance with the formality map $\gamma \in \mathsf{BGC}^{\mathsf{or}}$ and formula (34) coincides precisely with the composition

$$\mathcal{A}ss\mathcal{B}_\infty \xrightarrow{\gamma'} \mathcal{B}ra^{or} \xrightarrow{\rho_\nu} \mathcal{E}nd_{\mathcal{O}_V}[[\hbar]].$$

Put another way, the identification of $\mathcal{L}ie_\infty$ algebras $\mathsf{BGC}^{\mathsf{or}}$ and $\mathsf{Def}(\mathcal{A}ss\mathcal{B} \xrightarrow{i_\circ} \mathcal{DL}ieb_\infty)$ is no more than a universal (graph) incarnation of the main observation in §**4.8.2** that formality maps give us quantizations of strongly homotopy Lie bialgebras.

Similarly, morphisms of (non-differential) props,

$$\mathcal{A}ss\mathcal{B} \longrightarrow \mathcal{DL}ieb$$

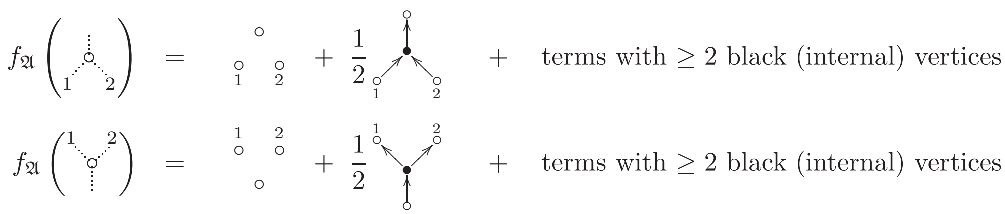
can be understood as universal quantizations of Lie bialgebras.

5.3. Existence Theorem. *For any Drinfeld associator $\mathfrak{A}$ there is an associated formality map $F_\mathfrak{A} \in \mathcal{MC}_{\rm form}$. In particular, the set of universal formality maps is non-empty.*

Proof. The Etingof-Kazhdan theorem [EK] (see also [EE]) says that for any Drinfeld associator $\mathfrak{A}$ there there is a morphism of props

$$f_\mathfrak{A} : \mathcal{A}ssb \longrightarrow \mathcal{T}ra^{or} = \mathcal{DL}ieb$$

such that



The theorem is proven once we show that there exists a morphism of dg props

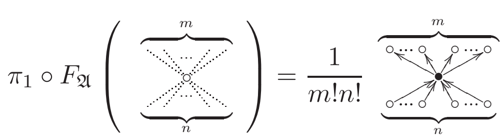
$$F_\mathfrak{A} : \mathcal{A}ssb_\infty \longrightarrow \mathcal{B}^{or} \equiv \mathcal{DL}ieb_\infty$$

which makes the following diagram commutative,

$$\begin{tikzcd} \mathcal{A}ssb_\infty \arrow[r, "F_\mathfrak{A}"] \arrow[d, "p"'] & \mathcal{DL}ieb_\infty \arrow[d, "\pi"] \\ \mathcal{A}ssb \arrow[r, "f_\mathfrak{A}"'] & \mathcal{DL}ieb \end{tikzcd}$$

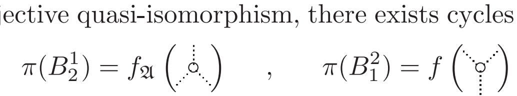
and satisfies the conditions

(37)

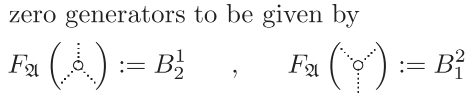


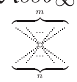
for all $m+n \geq 4$, $m,n \geq 1$. Here $\pi_1 : \mathcal{DL}ieb_\infty \to \mathcal{DL}ieb_\infty^{(1)}$ is the projection to the subspaces spanned by graphs with precisely one black (internal) vertex.

One can construct the required morphism F by a standard induction on the degree of the generators of $\mathcal{A}ssb_\infty$. As the map π is a surjective quasi-isomorphism, there exists cycles B_2^1 and B_1^2 in $\mathcal{DL}ieb_\infty$ such that



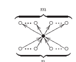
We set the values of F on degree zero generators to be given by



Assume now that values of F are defined on the generators of $\mathcal{A}ssb_\infty$ of degree $\geq -N+1$. Let us define F on generators of degree $-N$, that is, on (m,n) corollas, $c_n^m :=$  satisfying $m+n-3=N$. Note that δe_n^m is a linear combination of graphs whose vertices are decorated by corollas of degrees $\geq -N+1$ (as δ increases degree by $+1$). By induction, $F(\delta e_n^m)$ is a well-defined closed element in $\mathcal{B}ra^{or}$. As $f\circ p(e_n^m)=0$, we have $\pi(F(\delta e_n^m))=0$. Since the surjection π is a quasi-isomorphism, the element $F(\delta e_n^m)$ must be exact. Thus there exists $B_n^m \in \mathcal{B}ra^{or}$ such that

$$\delta_\bullet B_n^m = \mathcal{F}(\delta e_n^m).$$

We set $F(e_n^m) := B_n^m$.

To complete the inductive construction of F one has to show that $\pi_1 \circ F(e_n^m) = \frac{1}{m!n!}$  . This step is the same as in [Me1] and we omit the details. $\square$

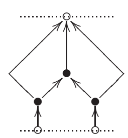
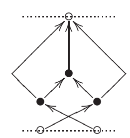
5.4. Lie connected formality maps. Let $F_{\mathfrak{A}} \in \mathcal{MC}_{\mathrm{form}}$ be any universal formality map. We want to study its deformation complex

$$\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)$$

As a graded vector space

$$\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty) \simeq \mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{0} \mathcal{DL}ieb_\infty) \simeq \prod_{m,n\geq 1} \mathcal{B}^{or}(m,n) \otimes_{\mathbb{S}_m \times \mathbb{S}_n} (\mathbb{K}[\mathbb{S}_m]] \otimes \mathbb{K}[\mathbb{S}_n])[2-m-n]$$

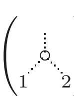
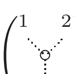
There are several different natural bases in this graded vector space. One basis is given by graphs from $\mathcal{B}^{or}$ in which the white *in*-vertices are totally ordered, and the white *out*-vertices are also totally ordered. We represent elements of this basis as graphs whose white vertices are put on a pair of dashed straight lines equipped with their canonical total order. For example, the graphs

 and 

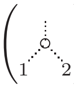
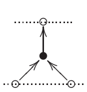
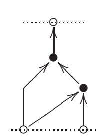
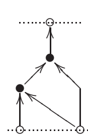
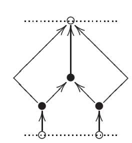
represent two different elements in $\mathsf{Def}(\mathcal{A}ssb \to \mathcal{DL}ieb_\infty)$ as total orders on input white vertices are different. The Maurer-Cartan element $\Upsilon_{\mathfrak{A}}$ in the $\mathcal{L}ie_\infty$ algebra $\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{0} \mathcal{DL}ieb_\infty)$ corresponding to the morphism $F_{\mathfrak{A}}$ can be decomposed into a sum,

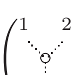
$$\Upsilon_{\mathfrak{A}} = \sum_{N\geq 3} \Upsilon_{\mathfrak{A}}^{[N]},$$

over the total number N of white vertices. The initial terms in this sum are given (up to a gauge equivalence) in this basis by [D1, MW3]

$\Upsilon_{\mathfrak{A}}^{[3]} = F_{\mathfrak{A}}\Big($  $\Big) + F_{\mathfrak{A}}\Big($  $\Big)$

with

$F_{\mathfrak{A}}\Big($  $\Big) =$ $+\ \frac{1}{2}$  $+\ \frac{1}{12}$  $+\ \frac{1}{12}$  $+\ \frac{1}{24}$  $+ \dots$

and similarly for $F_{\mathfrak{A}}\Big($  $\Big)$.

We shall need a different basis for the graded vector space $\mathsf{Def}(\mathcal{A}ssb_\infty \to \mathcal{DL}ieb_\infty)$ which we construct with the help of the following observations. The vector space $\mathbb{K}[\mathbb{S}_n][-n]$ can be identified with the subspace of the free associative algebra

$$\mathfrak{ass}_n = \mathbb{K}\langle x_1, \dots, x_n \rangle$$

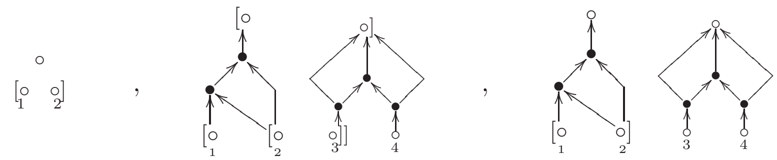
spanned by order n monomials such that each formal variable x_i (which has homological degree 1) occurs precisely once. If $\mathfrak{lie}_n$ stands for the free Lie algebra in the same generators, then

$$\mathfrak{ass}_n = \mathcal{U}(\mathfrak{lie}_n)$$

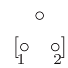
where $\mathcal{U}$ is the universal enveloping functor. By the Poincarè-Birkhoff-Witt Theorem, there is an isomorphism of graded vector spaces

$$\mathfrak{ass}_n = \odot^\bullet(\mathfrak{lie}_n)$$

so that one can identify $\mathbb{K}[\mathbb{S}_n][-n]$ with the subspace of $\odot^\bullet(\mathfrak{lie}_n)$ spanned by graded commutative products of Lie words in (odd) letters $x_1, \ldots, x_n$ such that each letter occurs precisely once. Therefore, one can introduce a new basis in the graded vector space $\mathsf{Def}(\mathcal{A}ssb_\infty \to \mathcal{DL}ieb_\infty)$ spanned by *Lie decorated* graphs of the form (cf. [W3])

(38) 

These are graphs from the prop $\mathcal{DL}ieb_\infty$ whose white *in*-vertices (resp., *out*-vertices) are grouped into graded commutative products of Lie words. The two sets of basis vectors are related to each other in the standard way, for example,

$\frac{1}{2}$  $\equiv$ 

5.4.1. Definition. We say that two vertices in a Lie decorated graph $\Gamma \in \mathsf{Def}(\mathcal{A}ssb_\infty \to \mathcal{DL}ieb_\infty)$ are *connected* if they are connected by an edge or belong to the same Lie word (the latter condition applies only to external vertices, of course). Then Γ is called *Lie connected* if it is connected in this generalized sense.

For example, the first two graphs in (38) are Lie connected while the third one is not.

5.4.2. Lemma. *Let*

$$\mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{0}{\to} \mathcal{DL}ieb_\infty)_{\rm Liec} \subset \mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{0}{\to} \mathcal{DL}ieb_\infty)$$

be the subspace spanned by Lie connected graphs. Then it is a $\mathcal{L}ie_\infty$ subalgebra.

Proof. The subspace of Lie connected graphs in $\mathcal{B}^{or} = \mathcal{DL}ieb_\infty$ is closed with respect to the vertical (properadic) composition. Hence the claim. □

5.4.3. Definition-Lemma. *A formality map $\mathcal{F} : \mathcal{A}ssb_\infty \to \mathcal{DL}ieb_\infty$ is called* Lie connected *if it belongs to the subspace $\mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{0}{\to} \mathcal{DL}ieb_\infty)_{\rm Liec}$. The subset in $\mathcal{MC}_{\rm form} \subset \mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{0}{\to} \mathcal{DL}ieb_\infty)$ consisting of Lie connected formality maps is non-empty.*

Proof. It is shown in [MW3] that there exists a universal quantization of Lie bialgebras

$$f : \mathcal{A}ssb \longrightarrow \mathcal{T}ra^{or} = \mathcal{DL}ieb$$

which is given by a sum of graphs $\sum \gamma$ with the following property: if γ contains at least one internal (black) vertex, then it is connected (as ordinary graph). The induction procedure used in the proof of Existence Theorem **5.3** preserves this property and gives us a formality map $F = \sum \Gamma$ with the same property: if Γ contains at least one internal (black) vertex, then it is connected (as ordinary graph). In particular, every graph Γ in this sum is Lie connected, i.e. belongs to the subspace $\mathsf{Def}(\mathcal{A}ssb_\infty \stackrel{0}{\to} \mathcal{DL}ieb_\infty)_{\rm Liec}$. □

5.4.4. Corollary. *For any Lie connected formality map $F_{\mathfrak{A}}$ the subspace*

$$\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{\rm Liec} \subset \mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)$$

is a $\mathcal{L}ie_\infty$ subalgebra.

5.5. Deformation complex of Lie connected formality maps. Let $F_{\mathfrak{A}}$ be any Lie connected formality map. By composing derivations of the *properad* $\mathcal{L}ieb_\infty$ with $F_{\mathfrak{A}}$ we obtain a morphism of complexes,

$$s : \mathrm{Der}(\mathcal{L}ieb_\infty)[-1] \equiv \mathsf{Def}(\mathcal{L}ieb_\infty \xrightarrow{\mathrm{Id}} \mathcal{L}ieb_\infty) \longrightarrow \mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{Liec} \tag{39}$$

5.5.1. Proposition. *The map s in (39) is a quasi-isomorphism.*

Proof. The complex $\mathrm{Der}(\mathcal{L}ieb_\infty)$ of *properadic* derivations is described very explicitly in terms of connected graphs in [MW2]. Both complexes in (39) admit filtrations by the number of edges in the graphs, and the map s preserves these filtrations, and hence induces a morphism of the associated spectral sequences,

$$s_r : (\mathcal{E}_r\mathrm{Der}(\mathcal{L}ieb_\infty)[-1], d_r) \longrightarrow \left(\mathcal{E}_r\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{Liec}, \delta_r\right), \quad r \geq 0,$$

The induced differential d_0 on the initial page of the spectral sequence of the l.h.s. is trivial, $d_0 = 0$. The induced differential on the initial page of the spectral sequence of the r.h.s. is not trivial and is determined by the following summand

in $F_{\mathfrak{A}}$. Hence the differential δ_0 acts only on white vertices of graphs from $\mathsf{Def}(\mathcal{A}ssb \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{\mathrm{Liec}}$ by splitting them in two white vertices and redistributes outgoing or incoming edges and all possible ways. The cohomology

$$E_1\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty) = H^\bullet(E_0\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty), \delta_0)$$

is spanned by graphs all of whose white vertices are precisely univalent (see Theorem 3.2.4 in [Me2] or Appendix A in [W1]) and hence is isomorphic to $\mathrm{Der}(\mathcal{L}ieb_\infty)[-1]$ as a graded vector space. By boundary condition (37), the induced differential δ_1 in $E_1\mathsf{Def}(\mathcal{A}ssb \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)$ agrees with the induced differential d_1 in $\mathcal{E}_1\mathrm{Der}(\mathcal{L}ieb_\infty)[-1]$ so that the morphism of the next pages of the spectral sequences,

$$s_1 : (\mathcal{E}_1\mathrm{Der}(\mathcal{L}ieb_\infty)[-1], d_1) \simeq \mathrm{Der}(\mathcal{L}ieb_\infty) \longrightarrow \left(\mathcal{E}_1\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{Liec}, \delta_1\right)$$

is an isomorphism. By the Comparison Theorem, the morphism s is a quasi-isomorphism. □

5.5.2. Corollary. *For any Lie connected formality morphism $F_{\mathfrak{A}}$ the associated deformation complex comes equipped with a canonical morphism of complexes*

$$\mathsf{GC}_3^{or} \longrightarrow \mathsf{Def}\left(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty\right)_{\mathrm{Liec}}[1]$$

which is a quasi-isomorphism up to one class corresponding to the standard rescaling automorphism of the prop of Lie bialgebras $\mathcal{L}ieb$. In particular,

$$H^{\bullet+1}(\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)_{conn}) = H^\bullet(\mathsf{GC}_3^{or}) \oplus \mathbb{K}[0].$$

Proof. In [MW1] the authors constructed an explicit morphism of dg Lie algebras

$$\mathsf{GC}_3^{or} \longrightarrow \mathrm{Der}(\mathcal{L}ieb_\infty)$$

which was proven to be a quasi-isomorphism up to one rescaling class. Hence the claim follows from Proposition **5.5.1**. □

5.6. Proof of the Main Theorem 1.2. Claim (i) follows from Theorem **5.3** and Lemma **5.4.3**.

Claim (ii) follows from the identification of the deformation complex $\mathsf{Def}(\mathcal{A}ssb_\infty \xrightarrow{F_{\mathfrak{A}}} \mathcal{DL}ieb_\infty)$ with the subcomplex BGC_3^{or} of the complex $\mathsf{Def}(\mathcal{Q}ua_\infty \xrightarrow{f} \mathcal{B}ra^{or})$.

Claim (iii) is the Corollary **5.5.2**.

Claim (iv) follows from Corollary **5.5.2** and the fact [W2] that $H^0(\mathsf{GC}_3^{or}) = \mathfrak{grt}_1$.

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