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3D fatigue fracture modeling by isogeometric boundary element methods

Xuan Peng, Elena Atroshchenko, Pierre Kerfriden, Stephane Bordas

Cardiff University, UK

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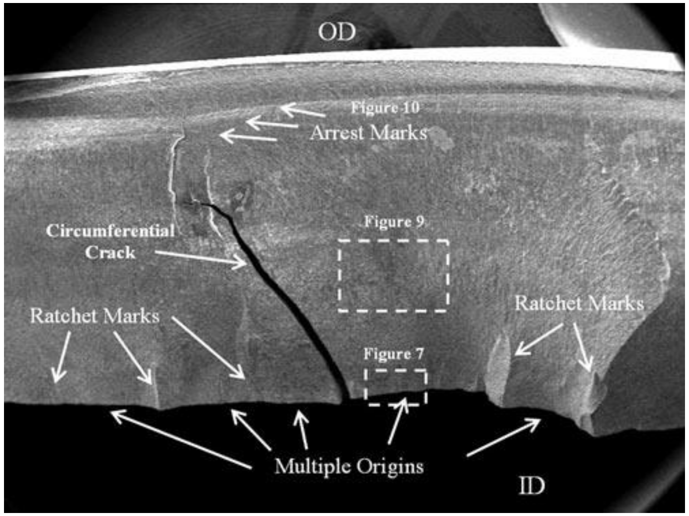
IMAM
Institute of Mechanics & Advanced Materials

Outline

- Motivation
- IGABEM formulation for crack modeling
- Embedded cracks
- Surface breaking cracks and trimmed NURBS
- Conclusion

Motivation

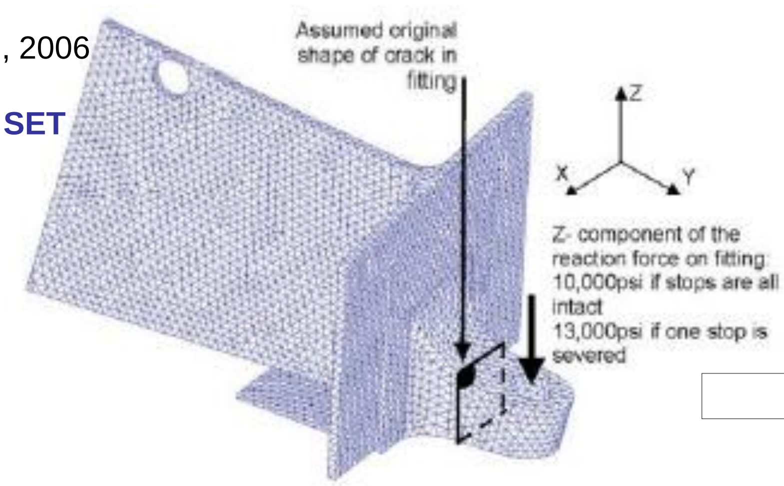
- **Fatigue Fracture Failure of Structure**
  - •Initiation: micro defects
  - •Loading : cyclic stress state (temperature, corrosion)
- **Numerical methods for crack growth**
  - •Volume methods: FEM, XFEM/GFEM, Meshfree
  - •Boundary methods: BEM

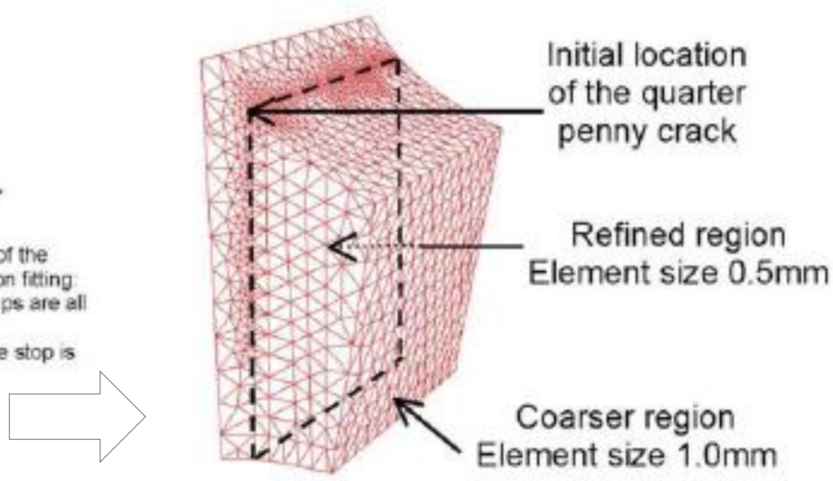


fatigue cracking of nozzle sleeve
http://met-tech.com/

Bordas & Moran, 2006

XFEM+LEVEL SET



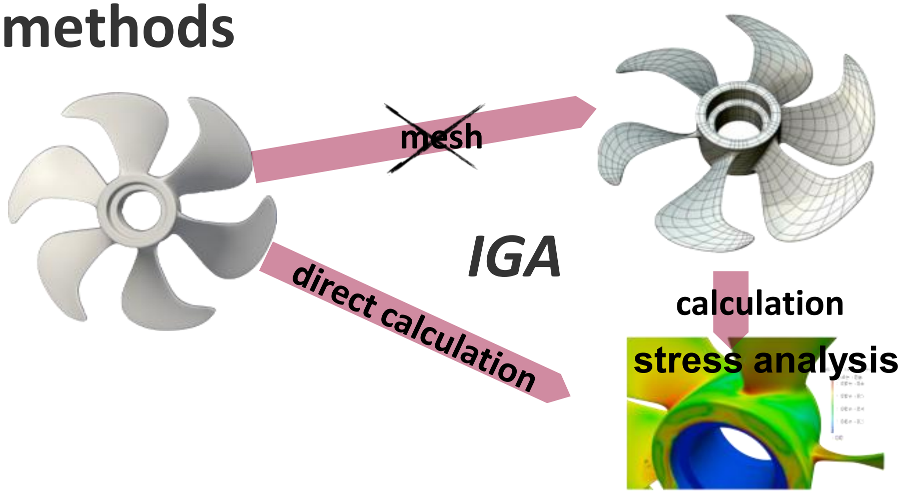


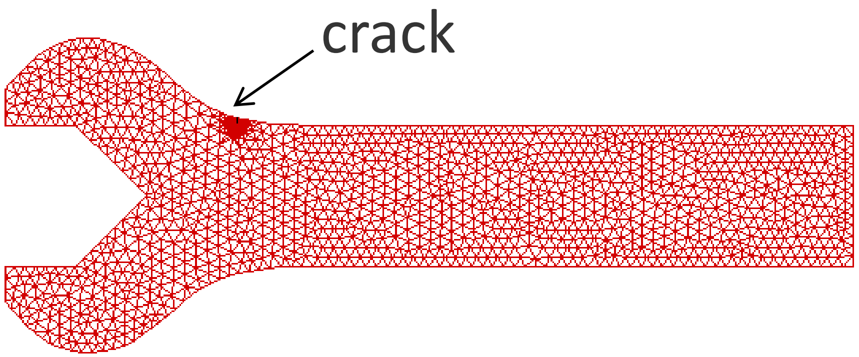
Motivation

➢ **Challenges in volume-based methods**

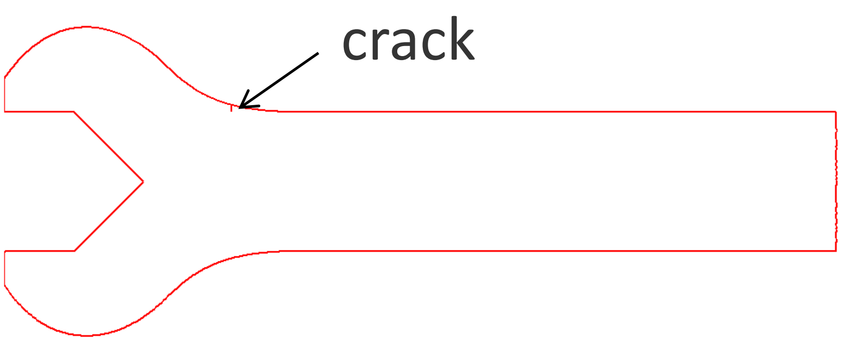
- Remeshing (FEM)
- Local mesh refinement

Efficiency & Accuracy





XFEM



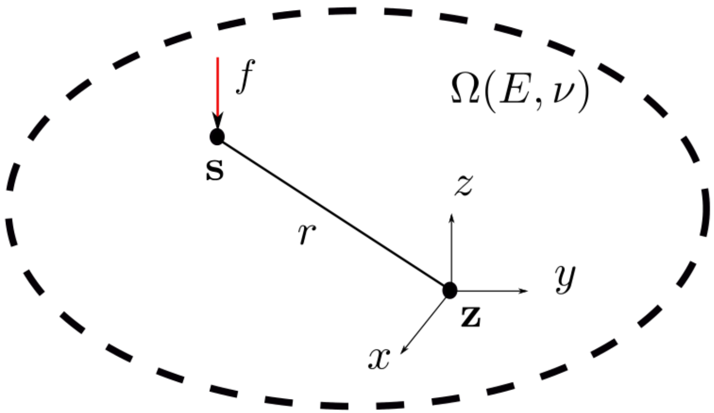
IGABEM

Kelvin fundamental solution

Navier equation: $\mu\nabla^2\mathbf{u} + (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f}$

Kelvin solution: assuming a unit concentrated force applied on a point $\mathbf{s}$ in the infinite domain $\mathbf{f}(\mathbf{z}) = \mathbf{e}\delta(\mathbf{s},\mathbf{z})$, we seek $\mathbf{u}(\mathbf{z})$ and $\mathbf{t}(\mathbf{z})$ for any point $\mathbf{z}$

$$u_i^* = U_{ij}e_j \qquad t_i^* = T_{ij}e_j$$



for 3D problem, the expressions are:

$$U_{ij}(\mathbf{s},\mathbf{z}) = \frac{1}{16\pi\mu(1-\nu)r}[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}]$$

$$T_{ij}(\mathbf{s},\mathbf{z}) = -\frac{1}{8\pi(1-\nu)}\left[\frac{\partial r}{\partial n}((1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}) - (1-2\nu)(n_j r_{,i} - n_i r_{,j})\right]$$

Boundary integral equation: direct method

Betti's theorem

State (1) : auxiliary state, using Kelvin solution

State (2): real state, neglecting body force

$$\int_\Omega f_j^{(1)}(\mathbf{z})u_j^{(2)}(\mathbf{z})\mathrm{d}\Omega(\mathbf{z}) + \int_\Gamma t_j^{(1)}(\mathbf{x})u_j^{(2)}(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma t_j^{(2)}(\mathbf{x})u_j^{(1)}(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) + \int_\Omega f_j^{(2)}(\mathbf{z})u_j^{(1)}(\mathbf{z})\mathrm{d}\Omega(\mathbf{z})$$

$$f_j^{(1)}(\mathbf{z}) = e_i\delta_{ij}\delta(\mathbf{s},\mathbf{z}) \qquad t_j^{(1)}(\mathbf{x}) = T_{ij}(\mathbf{s},\mathbf{x})e_i \qquad u_j^{(1)}(\mathbf{x}) = U_{ij}(\mathbf{s},\mathbf{x})e_i$$

$$u_i(\mathbf{s}) + \int_\Gamma T_{ij}(\mathbf{s},\mathbf{x})u_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{s},\mathbf{x})t_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})$$

Let source point "s" approach to the boundary

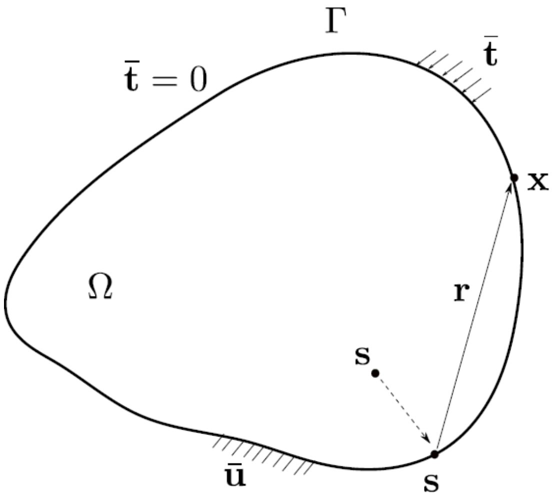
$$c_{ij}(\mathbf{s})u_j(\mathbf{s}) + ⨍_\Gamma T_{ij}(\mathbf{s},\mathbf{x})u_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{s},\mathbf{x})t_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})$$

$$\text{CPV: } T_{ij}(\mathbf{s},\mathbf{x}) \sim O(1/r^2) \qquad U_{ij}(\mathbf{s},\mathbf{x}) \sim O(1/r)$$

Take derivative w.r.t. "s"

$$c_{ij}(\mathbf{s})t_j(\mathbf{s}) + ⨎_\Gamma S_{ij}(\mathbf{s},\mathbf{x})u_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = ⨍_\Gamma K_{ij}(\mathbf{s},\mathbf{x})t_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})$$

$$\text{HFP: } S_{ij}(\mathbf{s},\mathbf{x}) \sim O(1/r^3)$$



(a)

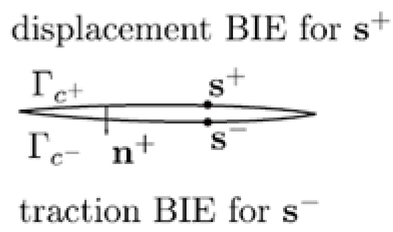
$$r = |\mathbf{r}| = |\mathbf{x} - \mathbf{s}|$$

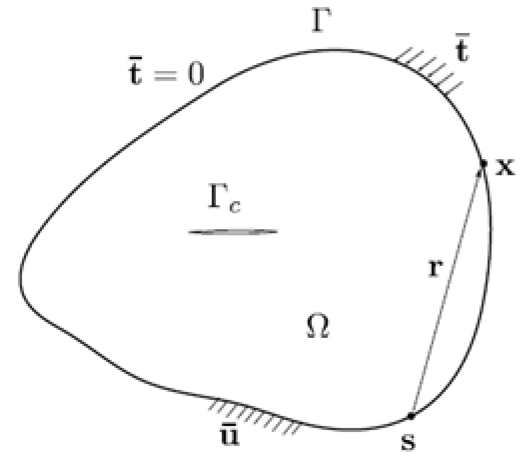
Boundary integral equations (BIEs) and crack modeling

- **Displacement BIE** $c_{ij}(\mathbf{s})u_j(\mathbf{s}) + \displaystyle -\!\!\!\!\!\!\int_\Gamma T_{ij}(\mathbf{s},\mathbf{x})u_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{s},\mathbf{x})t_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})$
- **Traction BIE** $c_{ij}(\mathbf{s})t_j(\mathbf{s}) + \displaystyle =\!\!\!\!\!\!\int_\Gamma S_{ij}(\mathbf{s},\mathbf{x})u_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x}) = -\!\!\!\!\!\!\int_\Gamma K_{ij}(\mathbf{s},\mathbf{x})t_j(\mathbf{x})\mathrm{d}\Gamma(\mathbf{x})$
- **NURBS approximation**

$$u_i(\xi) = \sum_{A=1}^{n} R_{A,p}(\xi)d_i^A$$

$$t_i(\xi) = \sum_{A=1}^{n} R_{A,p}(\xi)q_i^A$$

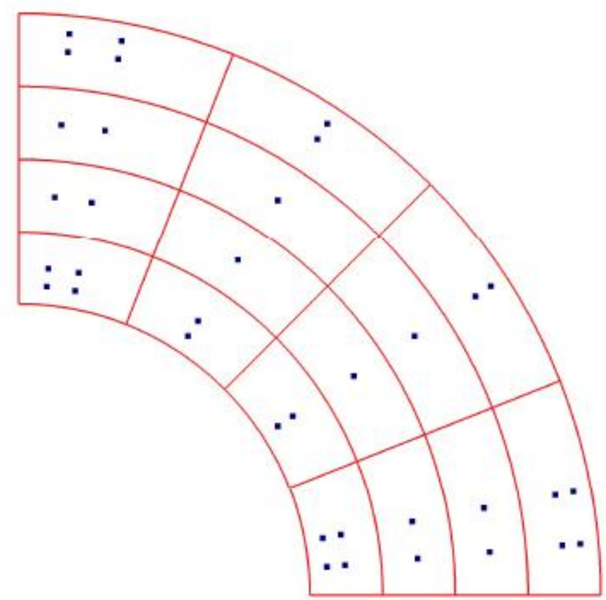




- **Collocation: Greville Abscissae**

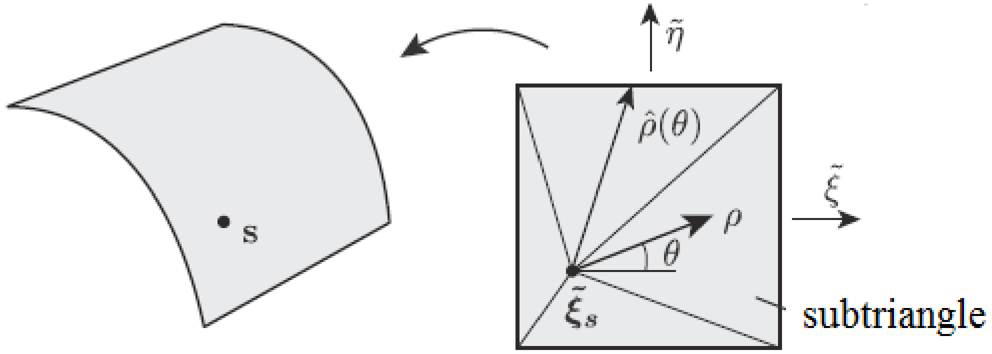
$$\xi_i^s = \frac{\xi_{i+1} + \ldots + \xi_{i+p}}{p}$$

$$\begin{cases} \xi_1^s = \xi_1^s + \alpha(\xi_2^s - \xi_1^s) \\ \xi_n^s = \xi_n^s - \alpha(\xi_n^s - \xi_{n-1}^s) \end{cases} \qquad \alpha \in [0,1)$$



Singular integration

- **Singularity subtraction technique (SST)**



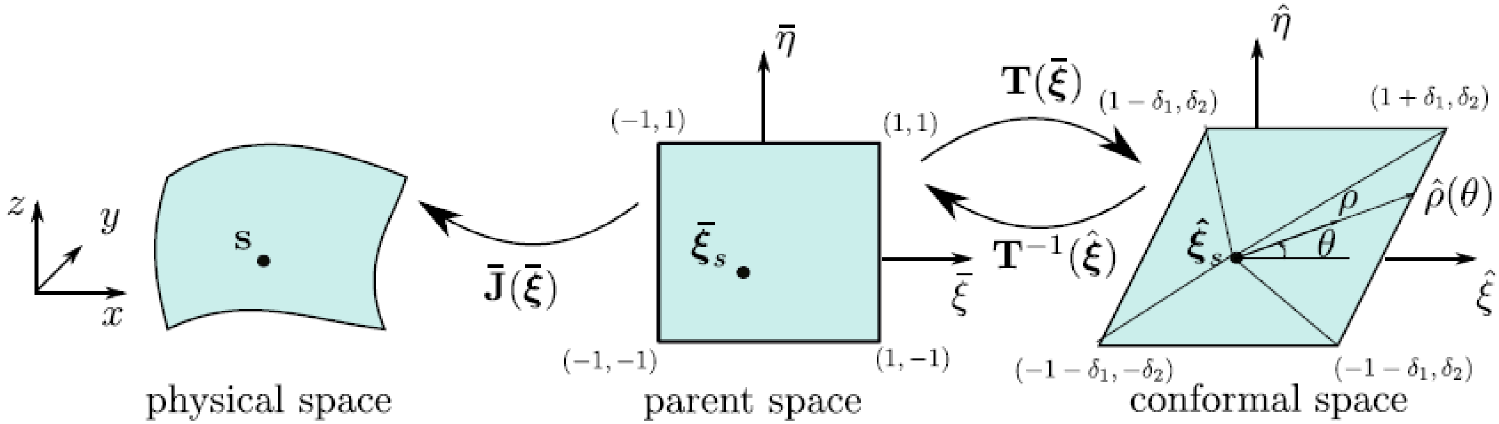
$$I = \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\hat{\rho}(\theta)} H(\rho,\theta) R(\rho,\theta) \bar{J}(\rho,\theta) \rho \mathrm{d}\rho \mathrm{d}\theta = \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\hat{\rho}(\theta)} F(\rho,\theta) \mathrm{d}\theta$$

$$F(\rho,\theta) = \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} + F_0(\theta) + F_1(\theta)\rho + F_2(\theta)\rho^2 + \cdots$$

$$I_1 = \int_0^{2\pi} \int_0^{\hat{\rho}(\theta)} \left(F(\rho,\theta) - \frac{F_{-2}(\theta)}{\rho^2} - \frac{F_{-1}(\theta)}{\rho}\right) \mathrm{d}\rho \mathrm{d}\theta$$

$$I = I_1 + \text{semi-analytical}$$

•Conformal mapping for SST



$$\delta_1 = \cos\psi/\lambda$$
$$\delta_2 = \sin\psi/\lambda$$

curve-linear basis vectors at $\mathbf{s}$ $\quad \hat{\mathbf{m}}_1^s \cdot \hat{\mathbf{m}}_2^s = 0, |\hat{\mathbf{m}}_1^s| = |\hat{\mathbf{m}}_2^s|$

$$\mathbf{T} = \hat{\boldsymbol{\xi}} = \mathbf{T}\bar{\boldsymbol{\xi}}, \text{ and } \begin{bmatrix} 1 & \cos\psi/\lambda \\ 0 & \sin\psi/\lambda \end{bmatrix}$$

$$\begin{bmatrix} \hat{\mathbf{m}}_1^s & \hat{\mathbf{m}}_2^s \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1^s & \mathbf{m}_2^s \end{bmatrix} \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{m}_1^s & -(1/\tan\psi)\mathbf{m}_1^s + (\lambda/\sin\psi)\mathbf{m}_2^s \end{bmatrix}$$

$$\lambda = |\mathbf{m}_1^s|/|\mathbf{m}_2^s|,$$
$$\cos\psi = \mathbf{m}_1^s \cdot \mathbf{m}_2^s/|\mathbf{m}_1^s||\mathbf{m}_2^s|,$$

•Rong et al 2014, EAWBE

Evaluation of stress intensity factors (SIFs)

•Virtual crack closure integrals (VCCI)

$$G_I = \frac{1}{2R}\int_0^R \sigma_{yy}(x)[\![u_y(R-x)]\!]\mathrm{d}x$$

$$G_{II} = \frac{1}{2R}\int_0^R \sigma_{xy}(x)[\![u_x(R-x)]\!]\mathrm{d}x$$

$$G_{III} = \frac{1}{2R}\int_0^R \sigma_{yz}(x)[\![u_z(R-x)]\!]\mathrm{d}x$$

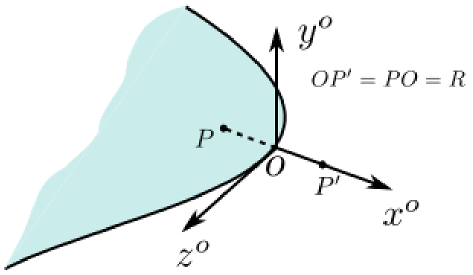


Figure: Crack tip coordinate system

•M integral

$$J_k := \lim_{\Gamma_\epsilon \to 0}\int_{\Gamma_\epsilon}(W\delta_{jk} - \sigma_{ij}u_{i,k})n_j \mathrm{d}\Gamma$$

$$M^{(1,2)} = \int_{\Gamma_\epsilon}\left[W^{(1,2)}\delta_{1j} - \sigma_{ij}^{(1)}\frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)}\frac{\partial u_i^{(1)}}{\partial x_1}\right]n_j \mathrm{d}\Gamma$$

$$M^{(1,2)} = \frac{2(1-\nu^2)}{E}(K_I^{(1)}K_I^{(2)} + K_{II}^{(1)}K_{II}^{(2)}) + \frac{1}{\mu}K_{III}^{(1)}K_{III}^{(2)}$$

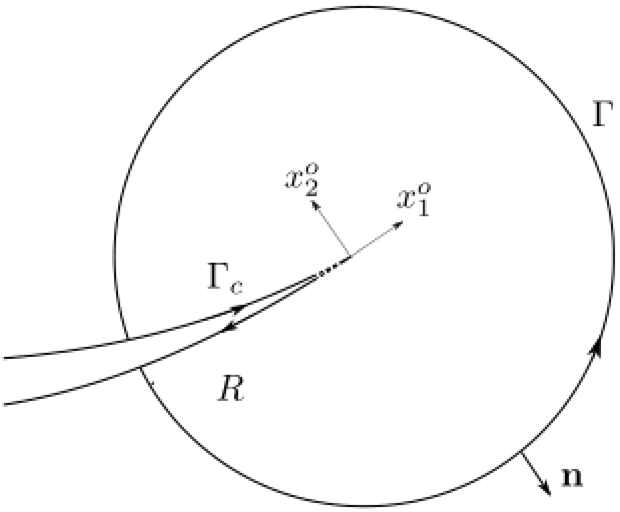
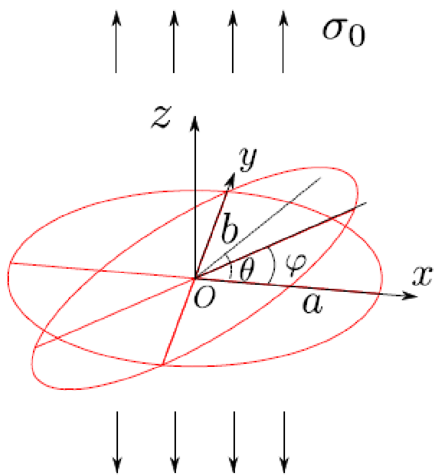


Figure: Path definition for J integral

Penny-shaped crack under remote tension (embedded crack)



$$u_z(r,\theta,0) = \frac{2(1-\nu)\sigma_0}{\pi\mu}\sqrt{a^2-r^2}, \quad r \leqslant a$$

$$K_I = \frac{2}{\pi}\sigma_0\sqrt{a\pi}\cos^2\varphi,$$

$$K_{II} = \frac{4}{\pi(2-\nu)}\sigma_0\sqrt{a\pi}\cos\varphi\sin\varphi\cos\theta,$$

$$K_{III} = \frac{4(1-\nu)}{\pi(2-\nu)}\sigma_0\sqrt{a\pi}\cos\varphi\sin\varphi\sin\theta.$$

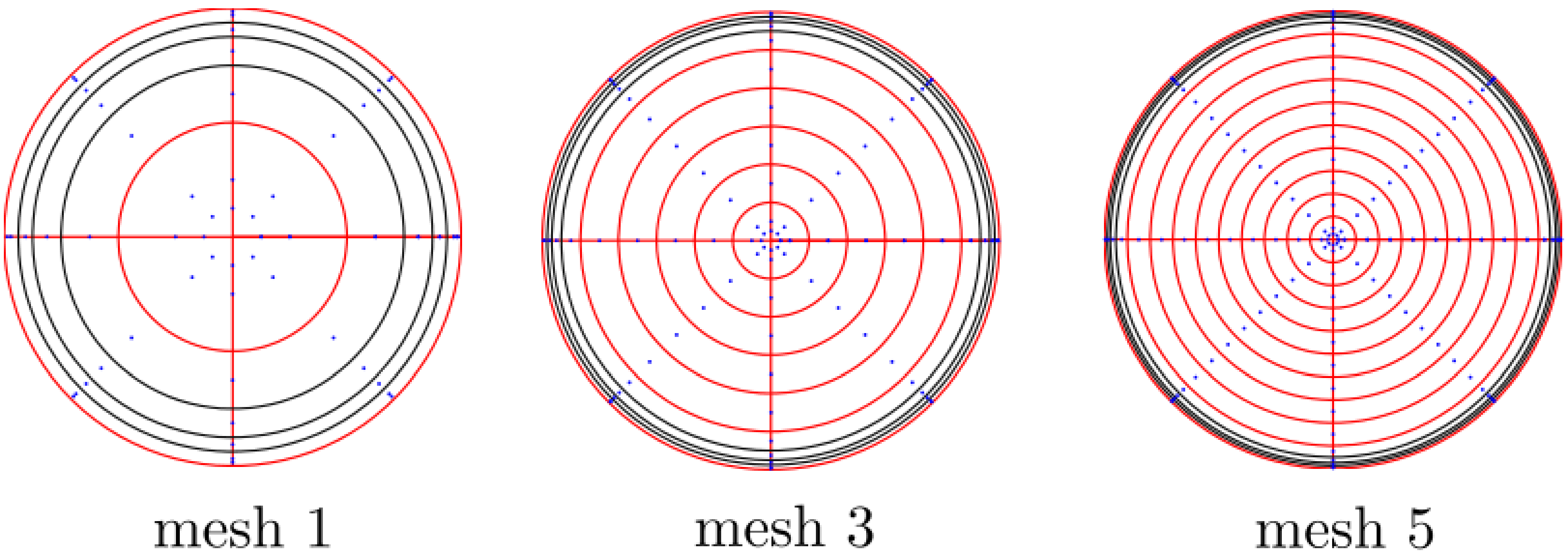
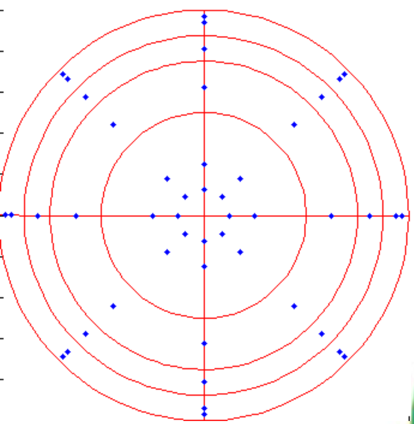
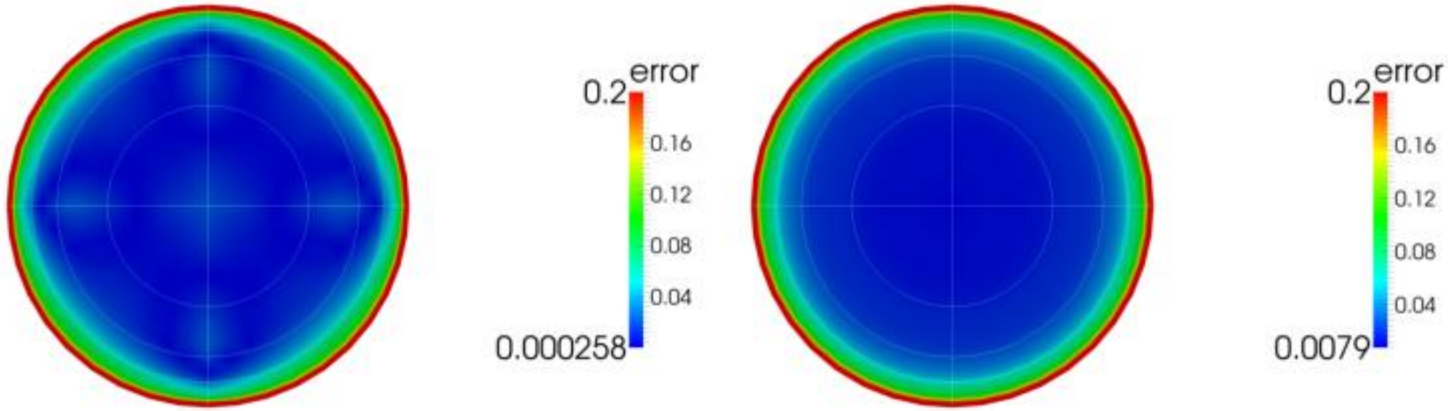


Figure 9: NURBS represented crack surface meshes with 1, 5, and 9 uniformed refinement in radial direction, followed by graded refined elements (with black edges) close to crack front. The blue dots are collocation points.

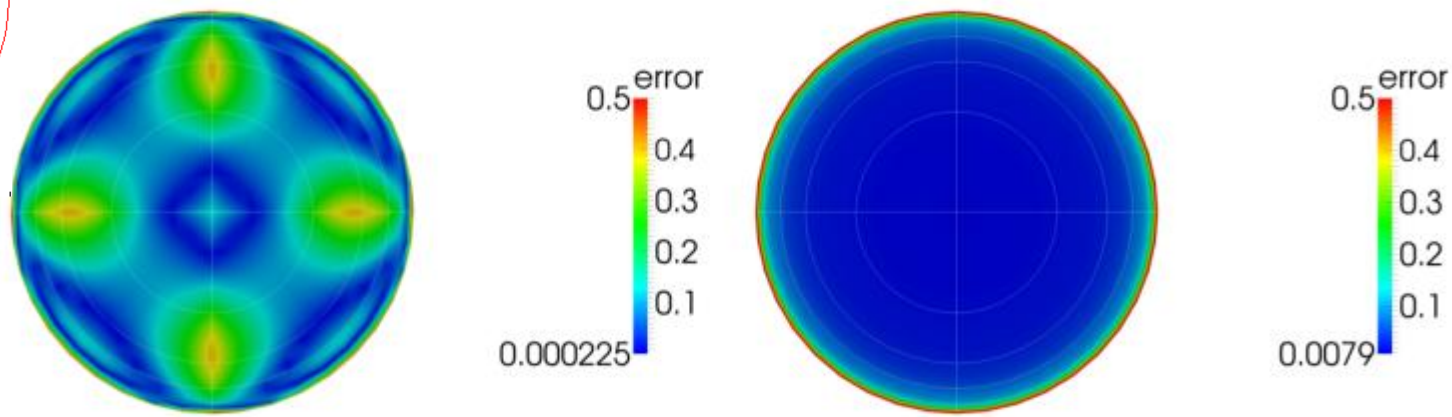
Penny crack under remote tension (embedded crack)





(a) original SST, ngp_s=30, e_{L_2} =3.344418e-2

(b) improved SST, ngp_s=30, e_{L_2} =3.844282e-2

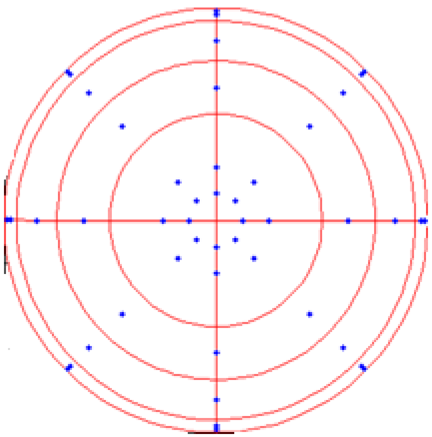


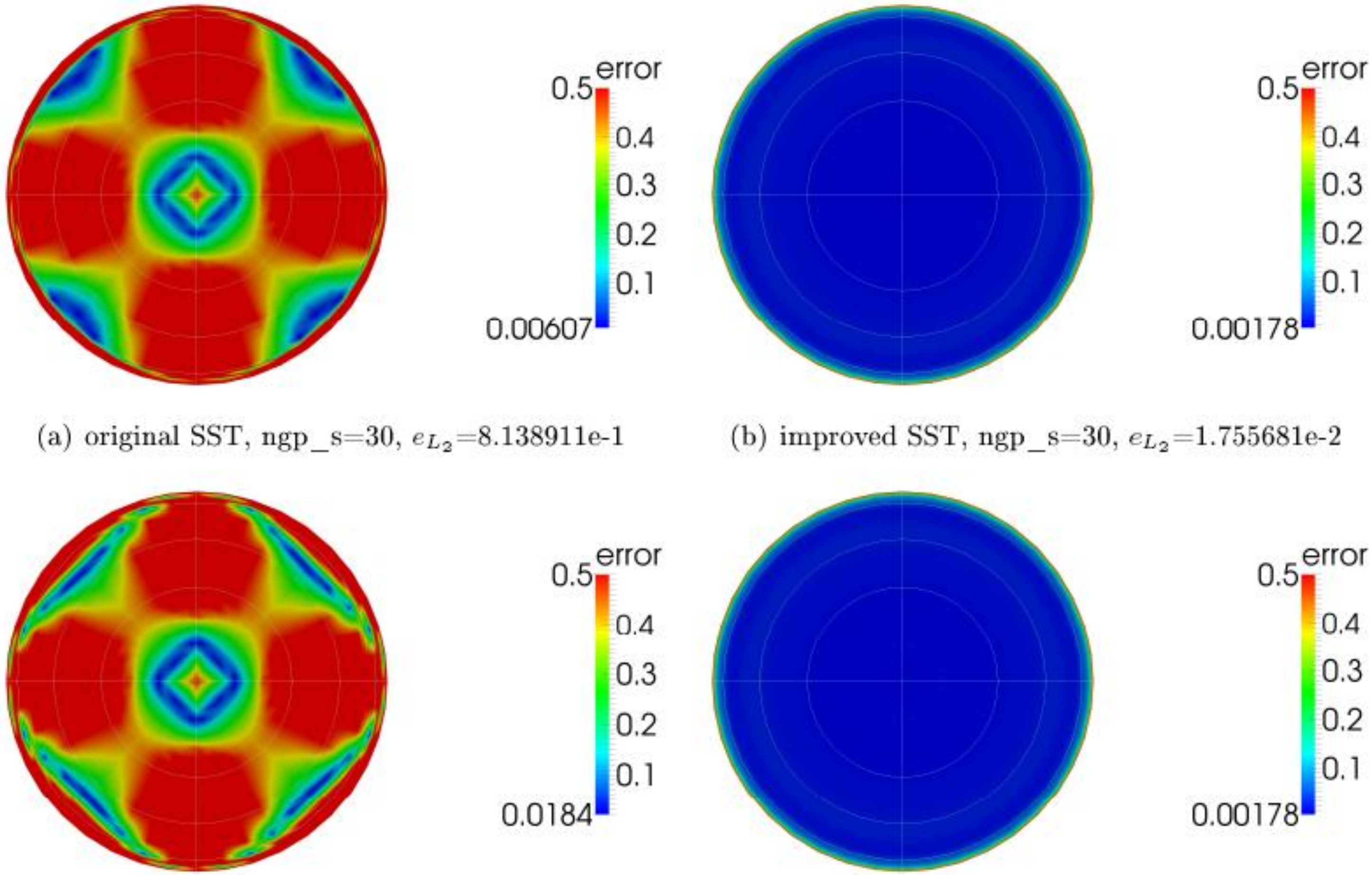
(c) original SST, ngp_s=18, e_{L_2} =1.467716e-1

(d) improved SST, ngp_s=18, e_{L_2}=3.844282e-2

Figure 7: Error in crack opening displacement for penny crack. 'ngp_s' denotes the number of Gauss points in angular direction in each sub-triangle. Knot vectors: angular direction $\boldsymbol{\xi}$=[0,0,0,0.25,0.25,0.5,0.5,0.75,0.75,1,1,1], radial direction $\boldsymbol{\eta}$=[0,0,0,0.5,0.75,0.875,1,1,1]

Penny crack under remote tension (embedded crack)





(a) original SST, ngp_s=30, e_{L_2}=8.138911e-1

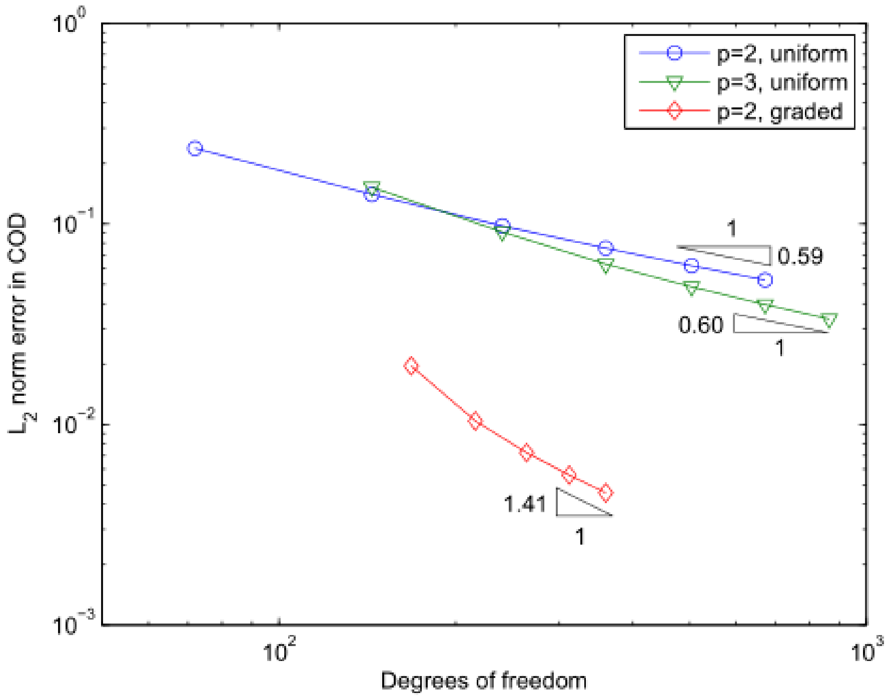
(b) improved SST, ngp_s=30, e_{L_2}=1.755681e-2

(c) original SST, ngp_s=18, e_{L_2}=7.110011e-1

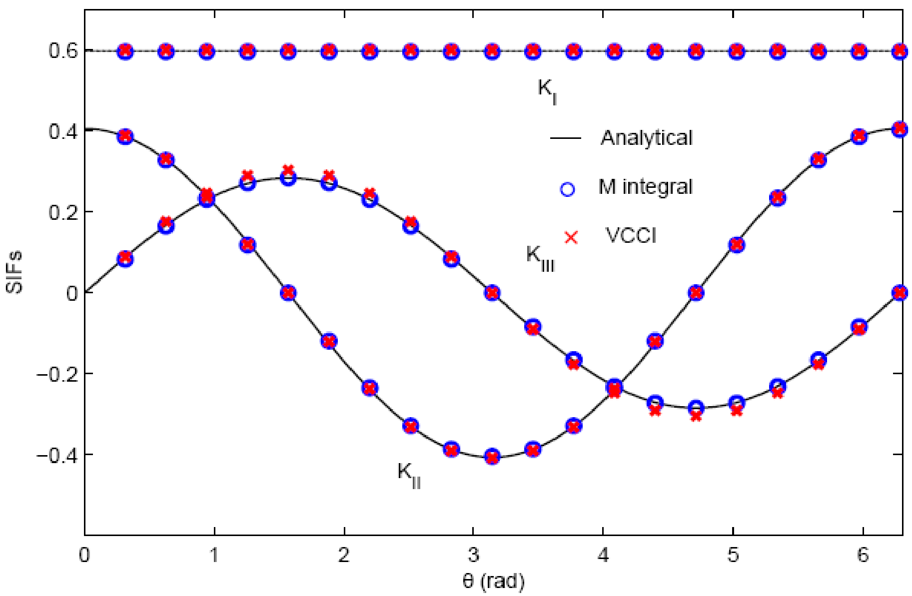
(d) improved SST, ngp_s=18, e_{L_2}=1.755679e-002

Figure 8: Error in crack opening displacement for penny crack. Knot vectors: angular direction $\boldsymbol{\xi}$=[0,0,0,0.25,0.25,0.5,0.5,0.75,0.75,1,1,1], radial direction $\boldsymbol{\eta}$=[0,0,0,0.5,0.75,0.94,1,1,1]

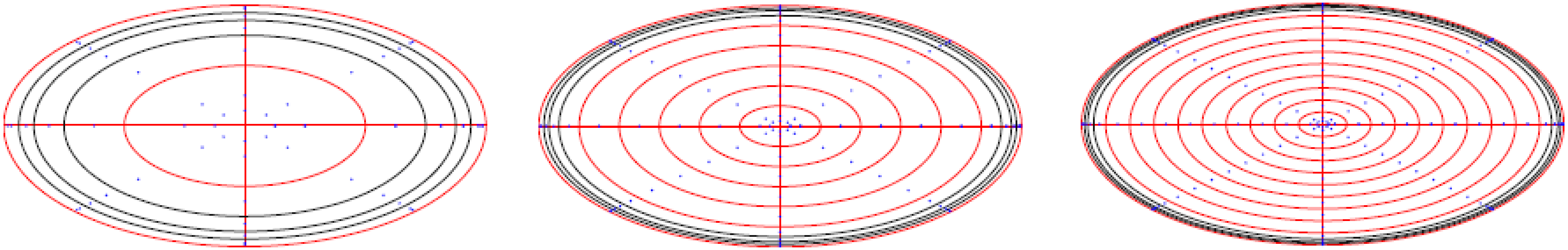
Penny-shaped crack under remote tension

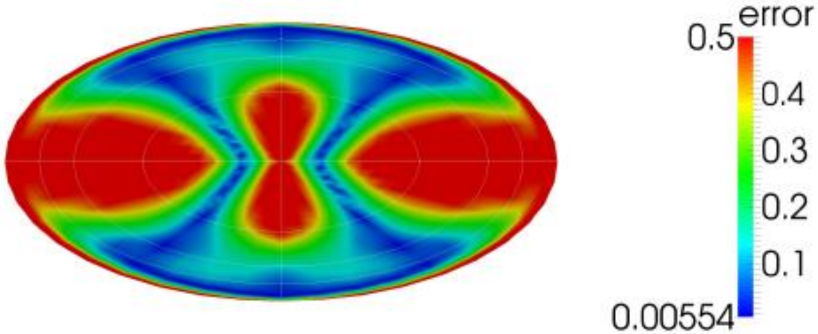


L_2 norm error of COD for penny-shaped crack

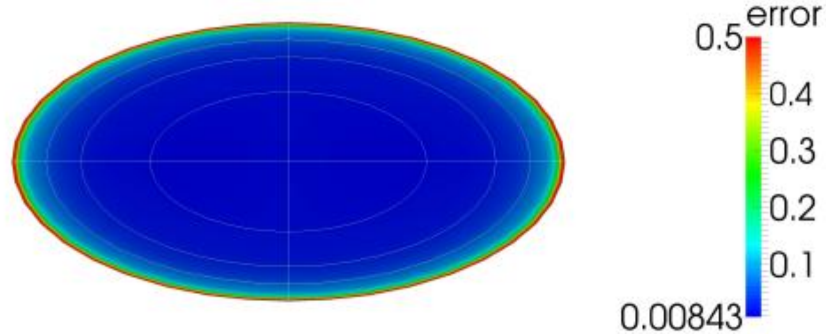


stress intensity factors for penny crack with $\varphi = \pi/6$

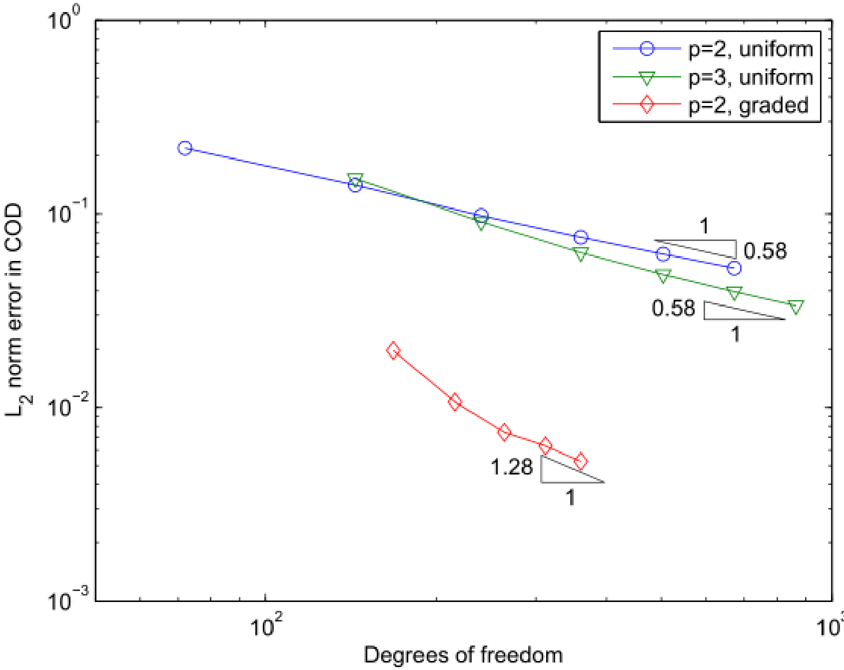


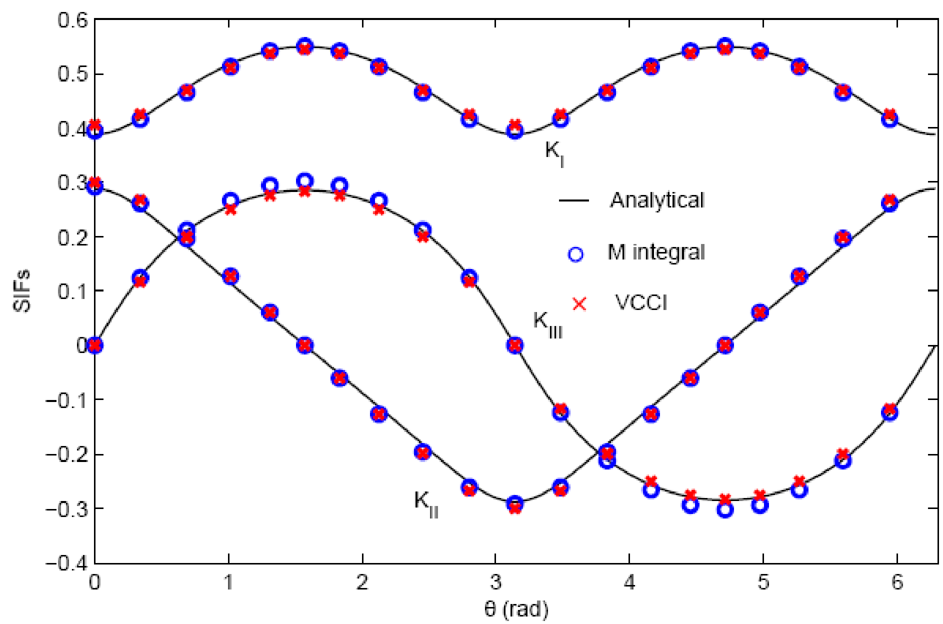


(a) original SST, ngp_s=18, e_{L_2}=4.603473e-1



(b) improved SST, ngp_s=18, e_{L_2}=3.798002e-2





NURBS-represented crack growth algorithm

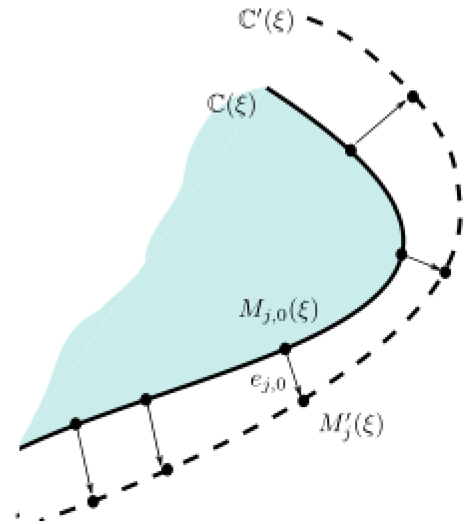
- **Fatigue fracture: Paris law**

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C(\Delta K)^m$$

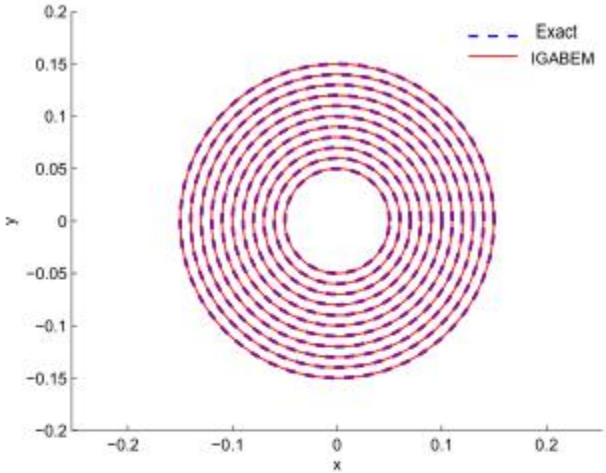
$$\Delta a^i = C(\Delta K^i_{\mathrm{eq}})^m \frac{\Delta a^{\max}}{C(\Delta K^{\max}_{\mathrm{eq}})} = \Delta a^{\max}\left(\frac{\Delta K^i_{\mathrm{eq}}}{\Delta^{\max}_{\mathrm{eq}}}\right)^m$$

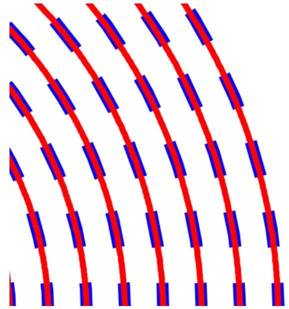
Algorithm 1 Crack front updating algorithm

Data: old crack front curve $\mathbb{C}(\xi)$; sample points M_j; new positions of sample points M'_j
Result: new crack front curve that passes through all M'_j
$t = 0$;
$tol = 1.e-4$;
$e_{j,0} = \overrightarrow{M_{j,0}M'_j}$;
while $\|\mathbf{e}_t\| > tol$ **do**
    $t = t + 1$;
    $m_{i,t} = \frac{1}{N}\sum_{j=0}^{N-1} f_{ij} e_{j,t-1}$;
    $P_{i,t} = P_{i,t-1} + m_{i,t}$;
    $e_{j,t} = e_{j,t-1} - \frac{1}{N}\sum_{k=0}^{N-1}\langle \mathbf{R}_j, \mathbf{f}_k\rangle e_{k,t-1}$;
end

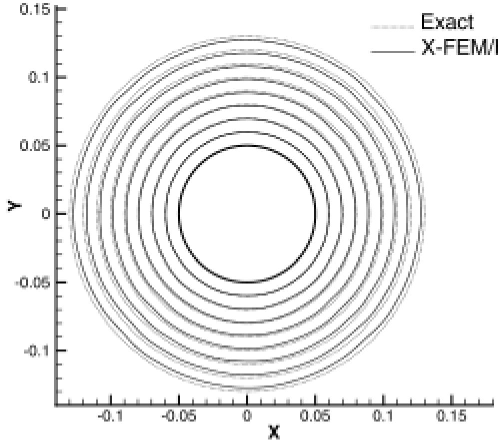


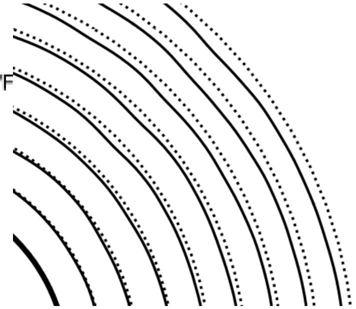
Numerical example of Mode-I penny crack growth (first 10 steps)



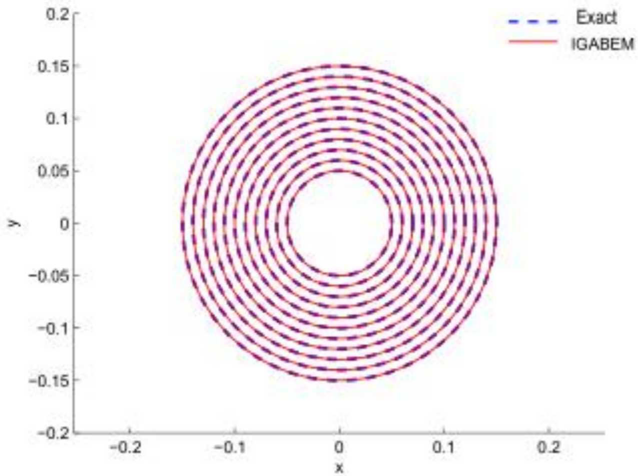


(a) IGABEM, $m = 2.1$

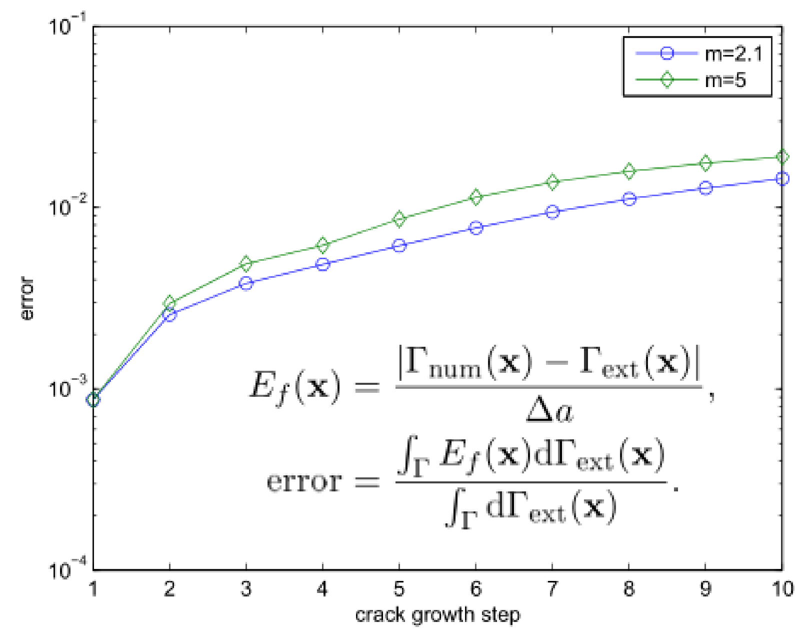




(b) XFEM/FMM, $m = 2.1$, Sukumar *et al* 2003

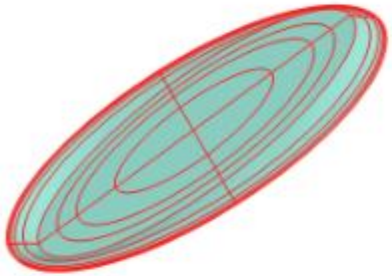


(c) IGABEM, $m = 5$

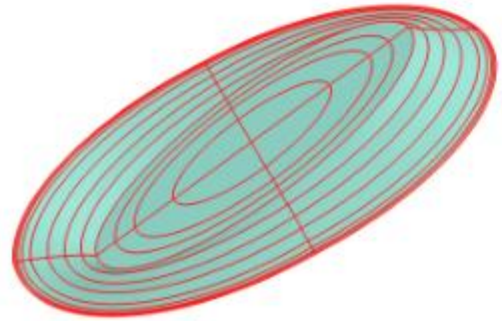


Relative error of the crack front for in each crack growth step by IGABEM

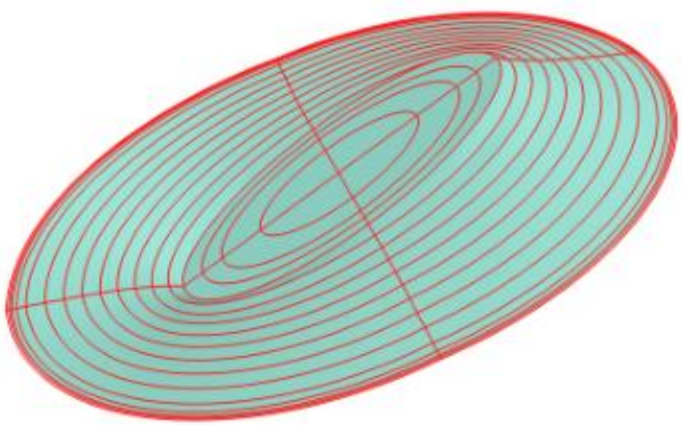
Numerical example of inclined elliptical crack growth (first 10 steps)



(a) Step 2



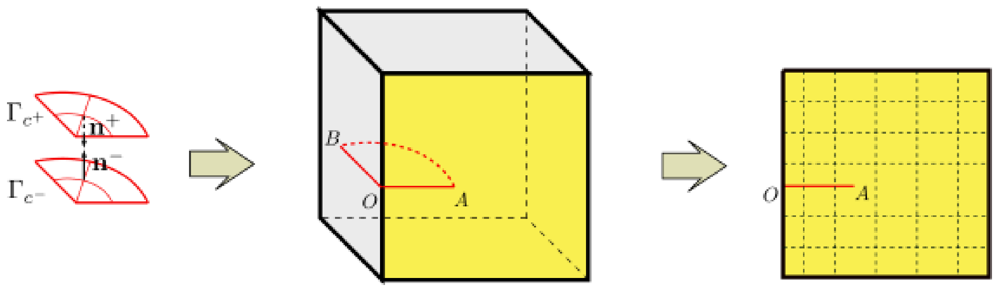
(b) Step 5



(c) Step 10

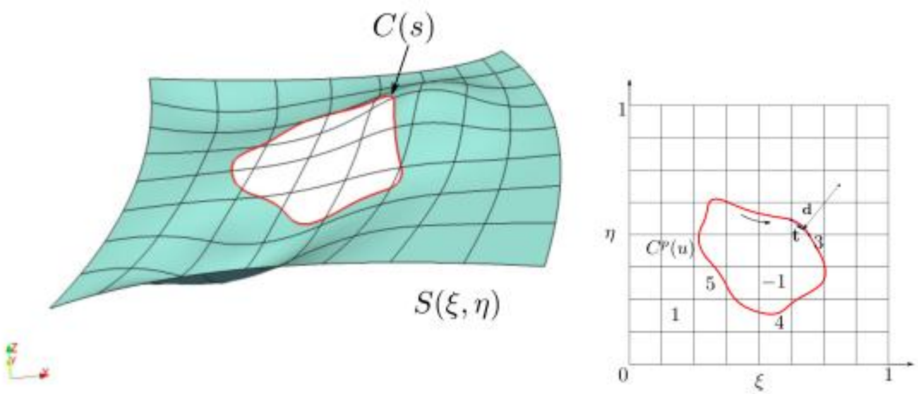
Fatigue crack growth simulation of an elliptical crack

Modeling techniques for surface breaking cracks



(a) (b) (c)

- Surface discontinuity is introduced



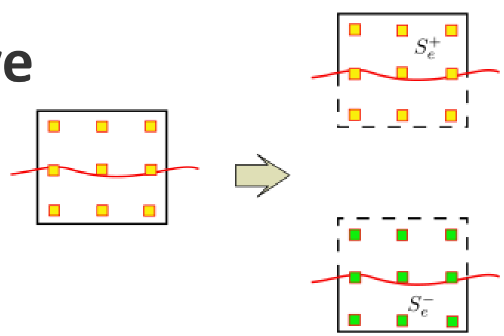
(a) (b)

- Trimmed NURBS technique

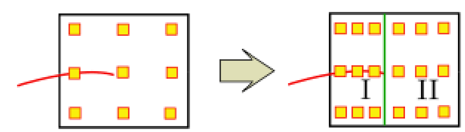
- Crack → trimming curve
- Phantom node method

$$\mathbf{u}^+(\mathbf{x}) = \sum_{j}^{N^e} \mathbf{R}_j(\mathbf{x})\mathbf{d}_j, \quad \mathbf{x} \in S_e^+$$

$$\mathbf{u}^-(\mathbf{x}) = \sum_{k}^{N^e} \mathbf{R}_k(\mathbf{x})\mathbf{d}_k, \quad \mathbf{x} \in S_e^-$$



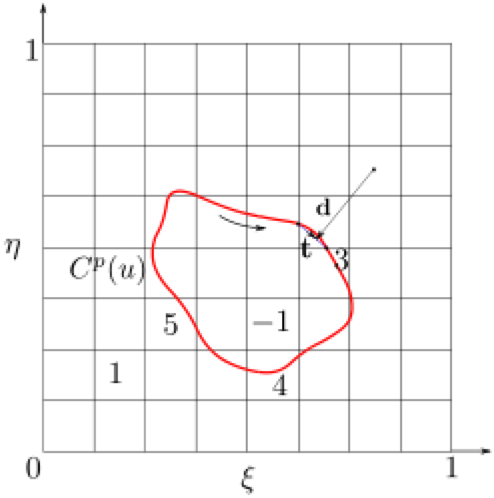
(a)

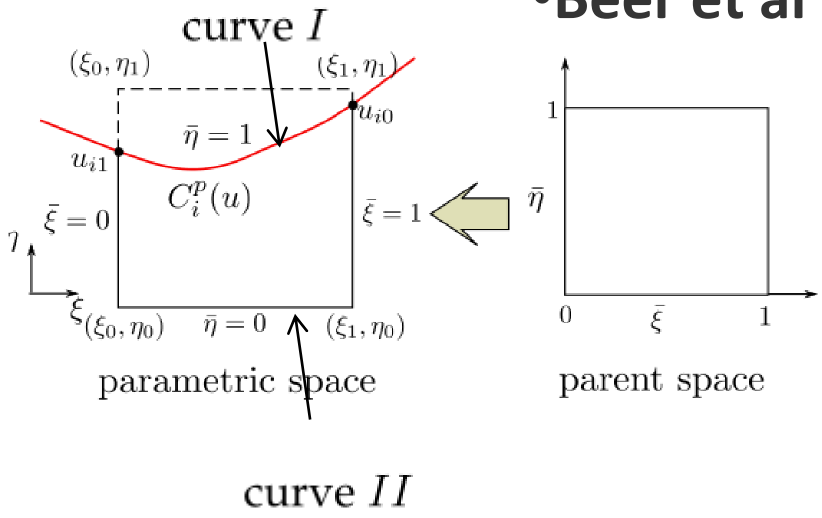


(b)

Integration and collocation for trimmed NURBS

- **Beer et al 2015, CMAME**

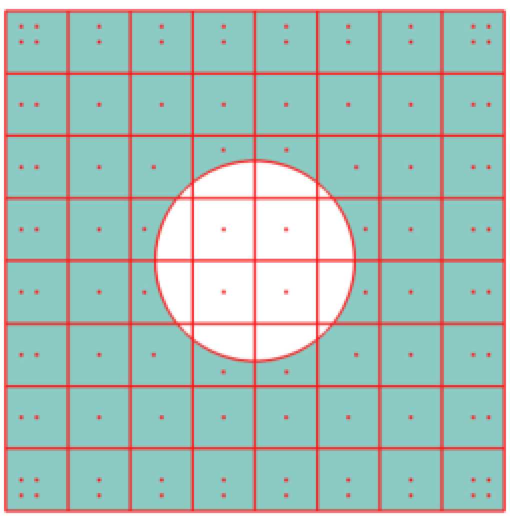




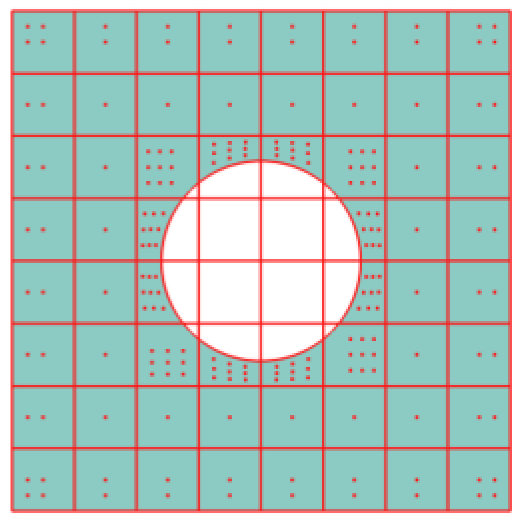
taking $u = \bar{\xi}$

$$\xi^I = \sum_{j=1}^{n_I} R_j^I(\bar{\xi}) \xi_j^I$$

$$\xi = (1 - \bar{\eta})\xi^I + \bar{\eta}\xi^{II},$$
$$\eta = (1 - \bar{\eta})\eta^I + \bar{\eta}\eta^{II}.$$



The Greville Abscissae (GA) collocation
(a)



The mixed collocation approach
(b)

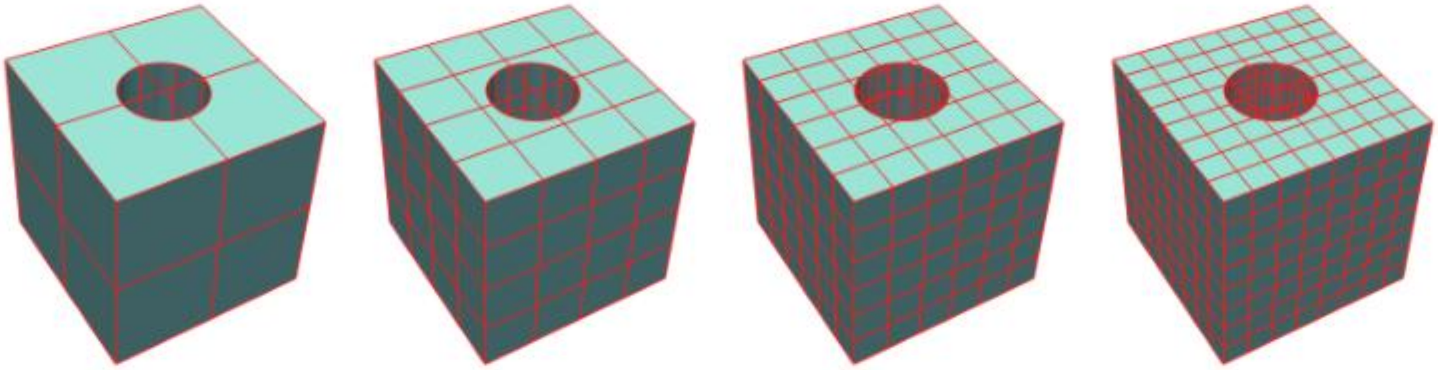
Convergence study for a cube with cylindrical cutout

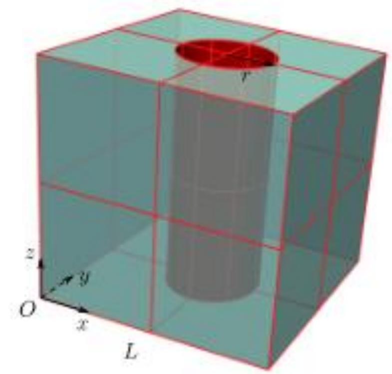
$$u_i(\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)r}[(3-4\nu)\delta_{i3} + r_{,i}r_{,3}],$$

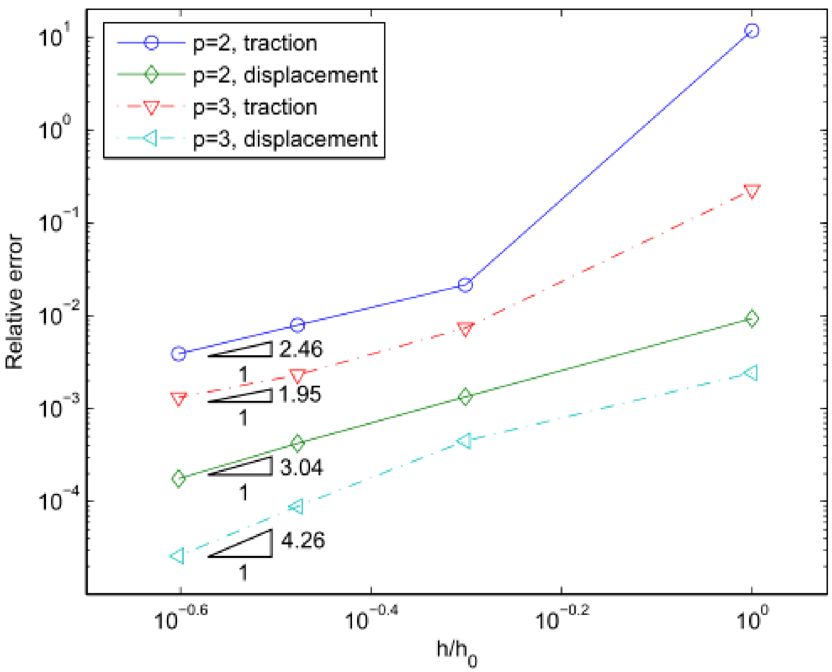
$$r = |\mathbf{s}_P - \mathbf{x}|$$

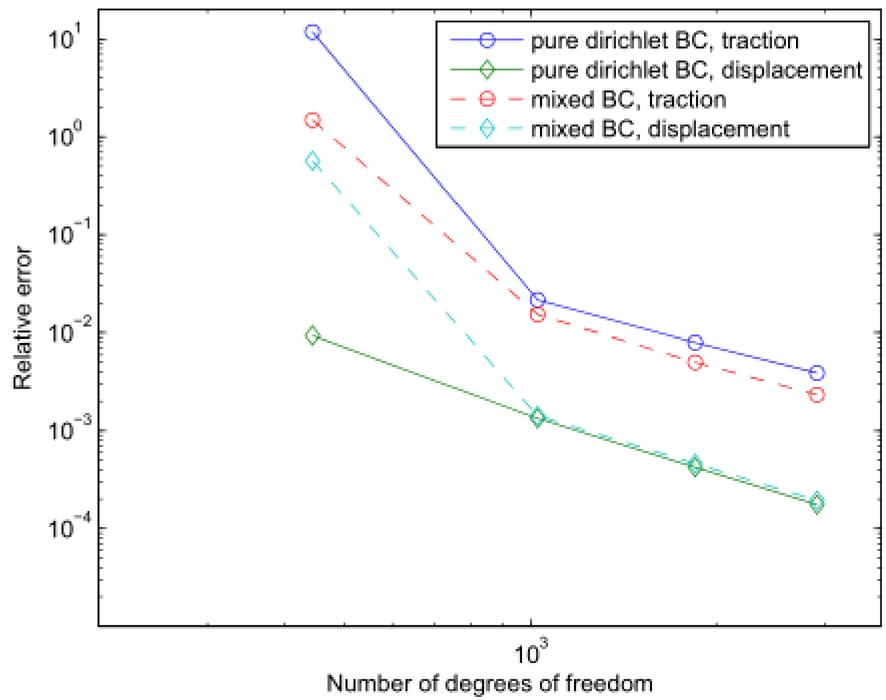
$$L = 1, \mathbf{s}_P(0,0,1.5)$$

$$t_i(\mathbf{x}) = -\frac{1}{8\pi(1-\nu)r^2}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{i3} + 3r_{,i}r_{,3}] - (1-2\nu)(r_{,i}n_3 - r_{,3}n_i)\right\}, \quad \forall \mathbf{x} \in \bar{\Omega}$$

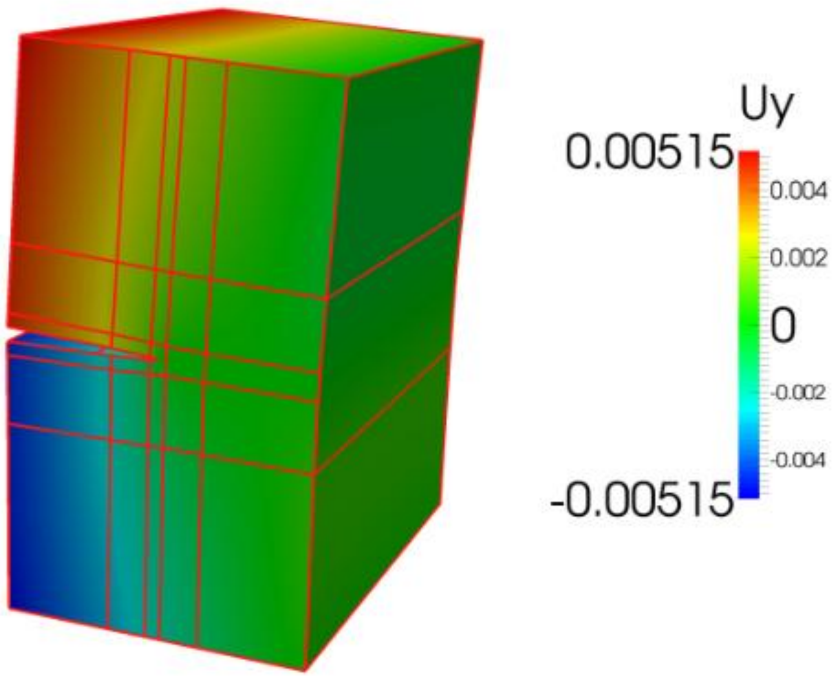


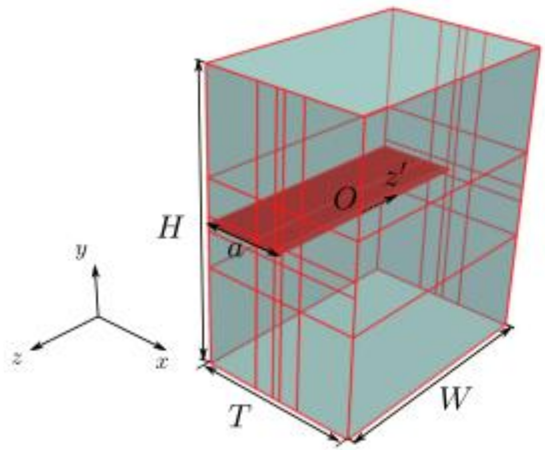






Example of surface breaking cracks: edge crack under uniform tension



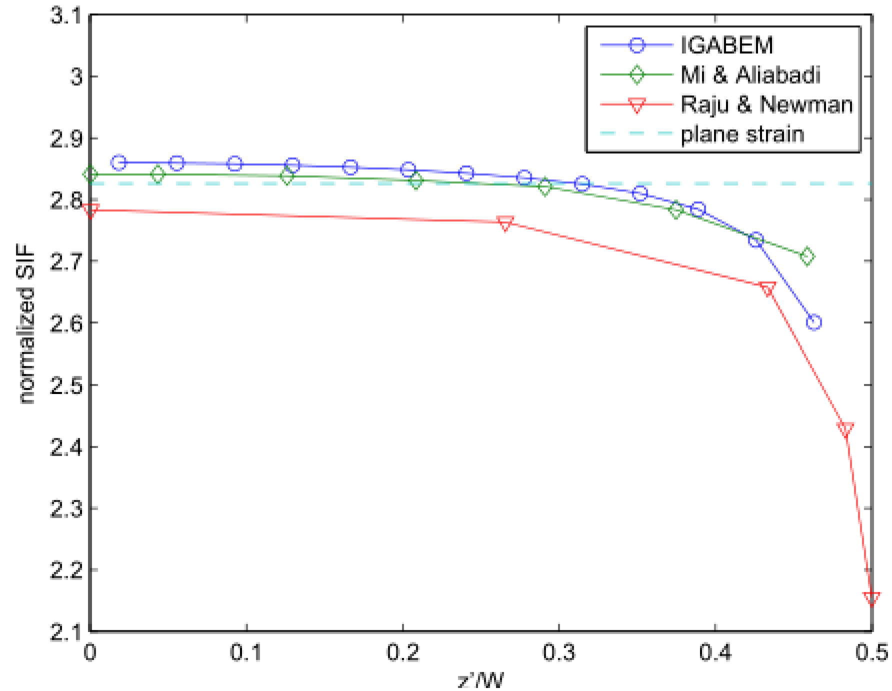


$E = 1.0e3$, $\nu = 0.3$
top face:
$t_x = t_z = 0$, $t_y = 1$
bottom face:
$t_x = t_z = 0, t_y = -1$

$T/a = 2$

$W/a = 3$

$H/a = 3.5$



Relative error in $K_I/\sqrt{\pi a}$

Conclusions & Future work

- Dual BIEs are used for NURBS-represented fracture modeling
- Improved numerical singular integration scheme
- Approaches for SIFs evaluation
- Fatigue crack growth algorithm
- IGABEM for trimmed NURBS and surface crack modeling

- Improve the integration and collocation schemes for trimmed NURBS
- Acceleration algorithm
- T-spline for local refinement

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Many thanks for YOUR attention