A GLOBAL DESCRIPTION OF THE FINE SIMPSON MODULI SPACES OF 1-DIMENSIONAL SHEAVES SUPPORTED ON PLANE QUARTICS

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Abstract. A global description of the fine Simpson moduli spaces of 1-dimensional sheaves supported on plane quartics is given: we describe the gluing of the Brill-Noether loci described by Drézet and Maican and show that the Simpson moduli space $M = M_{4m\pm1}(\mathbb{P}_2)$ is a blow-down of a blow-up of a projective bundle over a smooth moduli space of Kronecker modules. An easy computation of the Poincaré polynomial of M is presented.

Introduction

Fix an algebraically closed field $\Bbbk$, $\operatorname{char}\Bbbk = 0$. Let V be a 3-dimensional vector space over $\Bbbk$ and let $\mathbb{P}_2 = \mathbb{P}V$ be the corresponding projective plane. Let $P(m) = dm + c$, $d \in \mathbb{Z}_{>0}$, $c \in \mathbb{Z}$ be a linear Hilbert polynomial. Let $M = M_{dm+c} = M_{dm+c}(\mathbb{P}_2)$ be the Simpson moduli space (cf. [14]) of semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm + c$.

In [12] and [15] it has been shown that $M_{dm+c} \cong M_{d'm+c'}$ iff $d = d'$ and $c = \pm c' \mod d$. Therefore, in order to understand, for fixed d, the Simpson moduli spaces M_{dm+c} it is enough to understand at most $d/2 + 1$ different moduli spaces.

If $\gcd(d, c) = 1$, every semi-stable sheaf is stable and M_{dm+c} is a fine moduli space whose points correspond to the isomorphism classes of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm + c$. As shown in [10, Proposition 3.6], M_{dm+c} is a smooth projective variety of dimension $d^2 + 1$ in this case.

As shown in [10, Théorème 5.1] $M_{dm+c} \cong \mathbb{P}(S^dV^*)$ for $d = 1$, and $d = 2$. For $d = 3$, $M_{3m\pm1}$ is isomorphic to the universal cubic plane curve

$$\{(C, p) \in \mathbb{P}(S^3V^*) \times \mathbb{P}_2 \mid p \in C\}.$$

By [12] one has the isomorphisms $M_{4m+b} \cong M_{4m-1}$ for $d = 4$ and $\gcd(4, b) = 1$. In [4] a description of the moduli space M_{4m-1} is given in terms of two strata: an open stratum M_0 and its closed complement M_1 in codimension 2. The open stratum is naturally described as an open subvariety if a projective bundle $\mathbb{B} \to N$ associated to a vector bundle of rank 12 over a smooth 6-dimensional projective variety N.

The main result of the paper. The aim of this paper is to give a global description of $M = M_{4m-1}$ and hence of all fine Simpson moduli spaces of 1-dimensional sheaves supported on plane quartics. We describe how the strata from [4] are "glued together". We notice that every sheaf from M can be given as the cokernel of a morphism

$$2\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2}(-2) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-1),$$

which gives a simple way to deform the sheaves from M_0 to the ones from M_1. We show that M is a blow-down to M_1 of the exceptional divisor D of the blow-up $\widetilde{\mathbb{B}} := \mathrm{Bl}_{\mathbb{B}\setminus M_0}\mathbb{B}$ and give a geometric "visual" interpretation of this statement. This provides a different and a slightly more straightforward way (compared to [1]) of computing the Poincaré polynomial of M.

2010 *Mathematics Subject Classification.* 14D20.

Key words and phrases. Simpson moduli spaces, 1-dimensional sheaves, blow-up, blow-down, Poincaré polynomial.

We were kindly informed by professor Kiryong Chung, that the statement of Theorem 3.1 coincides with the statement of [3, Theorem 3.1], which must of course be given a priority because it appeared earlier. Our methods are however significantly different.

Structure of the paper. In Section 1 we review the description of the strata of M_{4m-1} from [4] and give a description of the degenerations to the closed stratum. In Section 2 we present a geometric description of the fibres of the bundle $\mathbb{B} \to N$ and construct local charts around the closed subvariety $\mathbb{B}' := \mathbb{B} \setminus M_0$. This allows us to give a geometric description of the blow up $\widetilde{\mathbb{B}} = \mathrm{Bl}_{\mathbb{B}'} \mathbb{B}$ and to "see" the main result, Theorem 3.1, in Section 3 just by looking at the geometric data involved. The computation of the Poincaré polynomial of M is given here as a direct corollary from Theorem 3.1. In Section 4 we rigorously prove Theorem 3.1 by constructing a family of $(4m-1)$-sheaves on $\widetilde{\mathbb{B}}$.

1. M_{4m-1} as a union of two strata

As shown in [10] $M = M_{4m-1}$ is a smooth projective variety of dimension 17. By [4] M is a disjoint union of two strata M_1 and M_0 such that M_1 is a closed subvariety of M of codimension 2 and M_0 is its open complement.

1.1. **Closed stratum.** The closed stratum M_1 is a closed subvariety of M of codimension 2 given by the condition $h^0(\mathcal{E}) \neq 0$.

The sheaves from M_1 possess a locally free resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{E} \to 0, \tag{1}$$

with linear independent linear forms z_1 and z_2 on $\mathbb{P}_2$. M_1 is a geometric quotient of the variety of injective matrices $\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)$ as above by the non-reductive group

$$(\mathrm{Aut}(2\mathcal{O}_{\mathbb{P}_2}(-3)) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}))/\mathbb{C}^*.$$

The points of M_1 are the isomorphism classes of sheaves that are non-trivial extensions

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_{\{p\}} \to 0, \tag{2}$$

where C is a plane quartic given by the determinant of $\left(\begin{smallmatrix} z_1 & q_1 \\ z_2 & q_2 \end{smallmatrix}\right)$ from (1) and $p \in C$ a point on it given as the common zero set of z_1 and z_2.

This describes M_1 as the universal plane quartic, the quotient map is given by

$$\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto (C, p), \quad C = Z(z_1 q_2 - z_2 q_1), \quad p = Z(z_1, z_2).$$

M_1 is smooth of dimension 15.

Let M_{11} be the closed subvariety of M_1 defined by the condition that p is contained on a line L contained in C. Equivalently, in terms of the matrices (1) this conditions reads as vanishing of q_1 or q_2. The dimension of M_{11} is 12.

Lemma 1.1. *The sheaves in M_{11} are non-trivial extensions*

$$0 \to \mathcal{O}_L(-2) \to \mathcal{F} \to \mathcal{O}_{C'} \to 0,$$

where C' is a cubic and L is a line.

Proof. Consider the isomorphism class of $\mathcal{F}$ with resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\left(\begin{smallmatrix} l & 0 \\ w & h \end{smallmatrix}\right)} \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0. \tag{3}$$

This gives the commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}_2}(-3) \arrow[rr, "l"] \arrow[d, "(1\ 0)"] & & \mathcal{O}_{\mathbb{P}_2}(-2) \arrow[r] \arrow[d, "(1\ 0)"] & \mathcal{O}_L(-2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}_2}(-3) \arrow[rr, "\left(\begin{smallmatrix} l & 0 \\ w & h \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & & \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}_2}(-3) \arrow[rr, "h"] \arrow[d] & & \mathcal{O}_{\mathbb{P}_2} \arrow[r] \arrow[d] & \mathcal{O}_{C'} \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

Therefore, $\mathcal{F}$ is an extension

$$0 \to \mathcal{O}_L(-2) \to \mathcal{F} \to \mathcal{O}_{C'} \to 0,$$

which is nontrivial since $\mathcal{F}$ is stable. This proves the required statement. $\square$

Let M_{10} denote the open complement of M_{11} in M_1.

1.2. **Open stratum.** The open stratum M_0 is the complement of M_1 given by the condition $h^0(\mathcal{E}) = 0$, it consists of the isomorphism classes of the cokernels of the injective morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} 3\mathcal{O}_{\mathbb{P}_2}(-1) \tag{4}$$

such that the (2×2)-minors of the linear part $\left(\begin{smallmatrix} z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$ of $A = \left(\begin{smallmatrix} q_0 & q_1 & q_2 \\ z_0 & z_1 & z_2 \\ w_0 & w_1 & w_2 \end{smallmatrix}\right)$ are linear independent.

1.2.1. *M_0 as a geometric quotient.* M_0 is an open subvariety in the geometric quotient $\mathbb{B}$ of the variety $\mathbb{W}^s$ of stable matrices as in (4) (see [11, Proposition 7.7] for details) by the group

$$\mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 2\mathcal{O}_{\mathbb{P}_2}(-2)) \times \mathrm{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1)).$$

Its complement in $\mathbb{B}$ is a closed subvariety $\mathbb{B}'$ corresponding to the matrices with zero determinant.

1.2.2. *Extensions.* If the maximal minors of the linear part of A have a linear common factor, say l, then $\det(A) = l \cdot h$ and $\mathcal{E}_A$ is in this case a non-split extension

$$0 \to \mathcal{O}_L(-2) \to \mathcal{E}_A \to \mathcal{O}_{C'} \to 0, \tag{5}$$

where $L = Z(l)$, $C' = Z(h)$.

The subvariety M_{01} of such sheaves is closed in M_0 and locally closed in M. Its boundary coincides with M_{11}.

1.2.3. *Twisted ideals of* 3 *points on a quartic.* Let M_{00} denote the open complement of M_{01} in M_0. In this case the maximal minors of the linear part of A are coprime, and the cokernel $\mathcal{E}_A$ of (4) is a part of the exact sequence

$$0 \to \mathcal{E}_A \to \mathcal{O}_C(1) \to \mathcal{O}_Z \to 0,$$

where C is a planar quartic curve given by the determinant of A from (4) and Z is the zero dimensional subscheme of length 3 given by the maximal minors of the linear submatrix of A. Notice that in this case the subscheme Z does not lie on a line.

1.3. Degenerations to the closed stratum.

Proposition 1.2. *1) Every sheaf in M_1 is a degeneration of sheaves from M_{00}. This corresponds to a degeneration of $Z \subseteq C$, where Z is a zero-dimensional scheme of length 3 not lying on a line and C is a quartic curve, to a flag $Z \subseteq C$ with Z contained in a line L that is not included in C. The limit corresponds to the point in M_1 described by the point $(L \cap C) \setminus Z$ on the quartic curve C.*
2) The sheaves from M_1 given by pairs (C, p) such that p belongs to a line L contained in C are degenerations of sheaves from M_{01}. This corresponds to degenerations of extensions (5) without sections to extensions with sections.

The proof follows from the considerations below.

1.3.1. *Degenerations along M_{00}.* Fix a curve $C \subseteq \mathbb{P}_2$ of degree 4, $C = Z(f)$, $f \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(4))$. Let $Z \subseteq C$ be a zero-dimensional scheme of length 3 contained in a line $L = Z(l)$, $l \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. Let $\mathcal{F} = \mathcal{I}_Z(1)$ be the twisted ideal sheaf of Z in C so that there is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_C(1) \to \mathcal{O}_Z \to 0.$$

Lemma 1.3. *In the notations as above, the twisted ideal sheaf $\mathcal{F} = \mathcal{I}_Z(1)$ is semistable if and only if L is not contained in C.*

Proof. First of all notice that if L is contained in C, then the inclusion of subschemes $Z \subseteq L \subseteq C$ provides a destabilizing subsheaf $\mathcal{I}_L(1) \subseteq \mathcal{I}_Z(1)$.

Let us construct now a locally free resolution of $\mathcal{F}$. Let $g \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(3))$ such that $\mathcal{O}_Z$ is given by the resolution

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-4) \xrightarrow{(l\ g)} \mathcal{O}_{\mathbb{P}_2}(-3) \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\binom{-g}{l}} \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_Z \to 0.$$

Since $Z = Z(l, g)$ is contained in $C = Z(f)$, one concludes that $f = lh - wg$ for some $w \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ and $h \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(3))$. This gives the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}_2}(-3) \arrow[r, equal] \arrow[d, "(1\ 0)"] & \mathcal{O}_{\mathbb{P}_2}(-3) \arrow[r] \arrow[d, "(l\ g)"] & 0 \arrow[d] & \\
0 \arrow[r] & 2\mathcal{O}_{\mathbb{P}_2}(-3) \arrow[r, "\begin{pmatrix} l & g \\ w & h \end{pmatrix}"] \arrow[d, "\binom{0}{1}"] & \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \arrow[r] \arrow[d, "\binom{-g}{l}"] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}_2}(-3) \arrow[r, "lh-wg"] \arrow[d] & \mathcal{O}_{\mathbb{P}_2}(1) \arrow[r] \arrow[d] & \mathcal{O}_C(1) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{O}_Z \arrow[r, equal] \arrow[d] & \mathcal{O}_Z \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Therefore, $\mathcal{F}$ possesses a locally free resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\begin{pmatrix} l & g \\ w & h \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0.$$

In particular, if l and w are linear independent, which is true if and only if f is not divisible by l, this is a resolution of type (1), hence $\mathcal{F}$ is a sheaf from M_1.

If l and w are linear dependent, then without loss of generality we can assume that $w = 0$, which gives an extension

$$0 \to \mathcal{O}_{C'} \to \mathcal{F} \to \mathcal{O}_L(-2) \to 0, \quad C' = Z(h),$$

and thus a destabilizing subsheaf $\mathcal{O}_{C'}$ of $\mathcal{F}$. This concludes the proof. □

Let $H(3,4)$ be the flag Hilbert scheme of zero-dimensional schemes of length 3 on plane projective curves $C \subseteq \mathbb{P}_2$ of degree 4. Let $H' \subseteq H(3,4)$ be the subscheme of those flags $Z \subseteq C$ such that Z lies on a linear component of C. Using the universal family on $H(3,4)$, one obtains a natural morphism

$$H(3,4) \setminus H' \to M,$$

whose image coincides with $M \setminus M_{01}$.

Its restriction to the open subvariety $H_0(3,4)$ of $H(3,4)$ of flags $Z \subseteq C \subseteq \mathbb{P}_2$ such that Z does not lie on a line gives an isomorphism

$$H_0(3,4) \to M_{00}.$$

Over M_1 one gets one-dimensional fibres: over an isomorphism class in M_1, which is uniquely defined by a point $p \in C$ on a curve of degree 4, the fibre can be identified with the variety of lines through p that are not contained in C, i. e., with a projective line without up to 4 points.

Remark 1.4. Notice that the subvariety H' is a $\mathbb{P}_3$-bundle over $\mathbb{P}V^* \times \mathbb{P}S^3V^*$, the fibre over the pair (L, C') of a line L and a cubic curve C' is the Hilbert scheme $L^{[3]}$.

As shown in [1, Theorem 3.3 and Proposition 4.4], the blow-up of $H(3,4)$ along H' can be blown down along the fibres $L^{[3]}$ to the blow-up $\widetilde{M} := \mathrm{Bl}_{M_1} M$.

1.3.2. *Degenerations along* M_{01}. For a fixed line L and a fixed cubic curve C' one can compute $\mathrm{Ext}^1(\mathcal{O}_{C'}, \mathcal{O}_L(-2)) \cong \mathbb{k}^3$. Therefore, using [9] one gets a projective bundle P over $\mathbb{P}V^* \times \mathbb{P}S^3V^*$ with fibre $\mathbb{P}_2$ and a universal family of extensions on it parameterizing the extensions

$$0 \to \mathcal{O}_L(-2) \to \mathcal{F} \to \mathcal{O}_{C'} \to 0, \quad L \in \mathbb{P}V^*, C' \in \mathbb{P}S^3V^*.$$

This provides a morphism $P \to M$ and describes the degenerations of sheaves from M_{01} to sheaves in M_{11}.

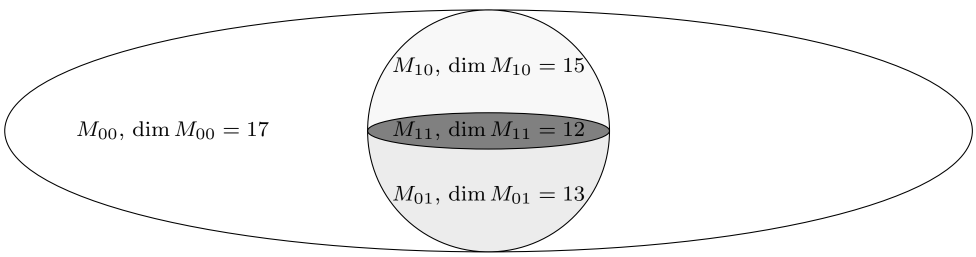

FIGURE 1. Moduli space $M = M_{4m-1}(\mathbb{P}_2)$.

2. Description of $\mathbb{B}$.

$\mathbb{B}$ is a projective bundle associated to a vector bundle of rank 12 over the moduli space $N = N(3;2,3)$ of stable (2×3) Kronecker modules, i. e., over the GIT-quotient of the space $\mathbb{V}^s$ of stable (2×3)-matrices of linear forms on $\mathbb{P}_2$ by $\mathrm{Aut}(2\mathcal{O}_{\mathbb{P}_2}(-2))\times\mathrm{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1))$.

The projection $\mathbb{B}\to N$ is induced by

$$\begin{pmatrix} q_0 & q_1 & q_2\\ z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{pmatrix}\mapsto\begin{pmatrix} z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{pmatrix}.$$

For more details see [11, Proposition 7.7].

2.1. **The base N.** The subvariety $N'\subseteq N$ corresponding to the matrices whose minors have a common linear factor is isomorphic to $\mathbb{P}_2^*=\mathbb{P}V^*$, the space of lines in $\mathbb{P}_2$, such that a line corresponds to the common linear factor of the minors of the corresponding Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{smallmatrix}\right)$.

The blow up of N along N' is isomorphic to the Hilbert scheme $H=\mathbb{P}_2^{[3]}$ of 3 points in $\mathbb{P}_2$ (cf. [6]). The exceptional divisor $H'\subseteq H$ is a $\mathbb{P}_3$-bundle over N', whose fibre over $\langle l\rangle\in\mathbb{P}_2^*$ is the Hilbert scheme $L^{[3]}$ of 3 points on $L=Z(l)$. The class in N of a Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{smallmatrix}\right)$ with coprime minors corresponds to the subscheme of 3 non-collinear points in $\mathbb{P}_2$ defined by the minors of the matrix.

2.2. **The fibres of $\mathbb{B}\to N$.** A fibre over a point from $N\setminus N'$ can be seen as the space of plane quartics through the corresponding subscheme of 3 non-collinear points. Indeed, consider a point from $N\setminus N'$ given by a Kronecker module $\left(\begin{smallmatrix} z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{smallmatrix}\right)$ with coprime minors d_0,d_1,d_2. The fibre over such a point consists of the orbits of injective matrices

$$\begin{pmatrix} q_0 & q_1 & q_2\\ z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{pmatrix},\quad q_0,q_1,q_2\in S^2V^*,$$

under the group action of

$$\mathrm{Aut}(\mathcal{O}_{\mathbb{P}_2}(-3)\oplus 2\mathcal{O}_{\mathbb{P}_2}(-2))\times\mathrm{Aut}(3\mathcal{O}_{\mathbb{P}_2}(-1)).$$

If two matrices

$$\begin{pmatrix} q_0 & q_1 & q_2\\ z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{pmatrix},\quad\begin{pmatrix} Q_0 & Q_1 & Q_2\\ z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{pmatrix}$$

lie in the same orbit of the group action, then their determinants are equal up to a multiplication by a non-zero constant. Vice versa, if the determinants of two such matrices are equal, $q-Q=(q_0-Q_0,q_1-Q_1,q_2-Q_2)$ lies in the syzygy module of $\left(\begin{smallmatrix} d_0\\ d_1\\ d_2\end{smallmatrix}\right)$, which is generated by the rows of $\left(\begin{smallmatrix} z_0 & z_1 & z_2\\ w_0 & w_1 & w_2\end{smallmatrix}\right)$ by Hilbert-Burch theorem. This implies that $q-Q$ is a combination of the rows and thus the matrices lie on the same orbit.

A fibre over $\langle l\rangle\in N'$ can be seen as the join $J(L^*,\mathbb{P}S^3V^*)\cong\mathbb{P}_{11}$ of $L^*\cong\mathbb{P}H^0(L,\mathcal{O}_L(1))\cong\mathbb{P}_1$ and the space of plane cubic curves $\mathbb{P}(S^3V^*)\cong\mathbb{P}_9$. To see this assume $l=x_0$, i. e., $\langle x_0\rangle$ is considered as the class of

$$\begin{pmatrix} -x_2 & 0 & x_0\\ x_1 & -x_0 & 0\end{pmatrix}.$$

Then the fibre over $[\left(\begin{smallmatrix} -x_2 & 0 & x_0\\ x_1 & -x_0 & 0\end{smallmatrix}\right)]$ is given by the orbits of matrices

$$\begin{pmatrix} q_0(x_0,x_1,x_2) & q_1(x_1,x_2) & q_2(x_1,x_2)\\ -x_2 & 0 & x_0\\ x_1 & -x_0 & 0\end{pmatrix}\tag{6}$$

and can be identified with the projective space $\mathbb{P}(2H^0(L,\mathcal{O}_L(2))\oplus S^2V^*)$. Rewrite the matrix (6) as

$$\begin{pmatrix} q_0(x_0,x_1,x_2) & q_1(x_1,x_2)-x_2\cdot w & q_2(x_2)+x_1\cdot w \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}, \quad w(x_1,x_2)=\gamma x_1+\delta x_2, \quad \gamma,\delta\in\Bbbk.$$

Its determinant equals

$$x_0\cdot(x_0\cdot q_0(x_0,x_1,x_2)+x_1\cdot q_1(x_1,x_2)+x_2\cdot q_2(x_2)).$$

This allows to reinterpret the fibre as the projective space

$$\mathbb{P}(H^0(L,\mathcal{O}_L(1))\oplus S^3V^*)\cong J(L^*,\mathbb{P}S^3V^*).$$

$J(L^*,\mathbb{P}(S^3V^*))\setminus L^*$ is a rank 2 vector bundle over $\mathbb{P}(S^3V^*)$, whose fibre over a cubic curve $C'\in\mathbb{P}S^3V^*$ is identified with the isomorphism classes of sheaves defined by (5) from M_{01} with fixed L and C', it corresponds to the projective plane joining C' with L^* inside the join $J(L^*,\mathbb{P}(S^3V^*))$. In the notations of the example above, γ and δ are the coordinates of this plane.

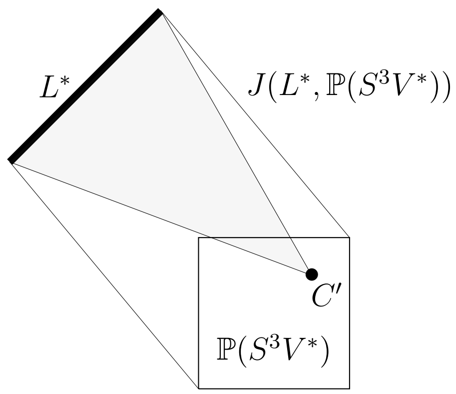

Figure 2. The fibre of $\mathbb{B}$ over $L\in N'$.

The points of $J(L^*,\mathbb{P}(S^3V^*))\setminus L^*$ parameterize the extensions (5) from M_{01} with fixed L.

2.3. **Description of $\mathbb{B}'$.** $\mathbb{B}'$ is a union of lines L^* from each fibre over N' (as explained above). It is isomorphic to the tautological $\mathbb{P}_1$-bundle over $N'=\mathbb{P}_2^*$

$$\{(L,p)\in\mathbb{P}_2^*\times\mathbb{P}_2 \mid L\in\mathbb{P}_2^*, p\in L\}. \tag{7}$$

Equivalently (cf. [4, p. 36]), $\mathbb{B}'$ is isomorphic to the projective bundle associated to the tangent bundle $\mathrm{T}\,\mathbb{P}_2^*$. The fibre $\mathbb{P}_1$ of $\mathbb{B}'$ over, say, line $L=Z(x_0)\subseteq\mathbb{P}_2$ can be identified with the space of classes of matrices (4) with zero determinant

$$\begin{pmatrix} 0 & -x_2\cdot w & x_1\cdot w \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}, \quad w=\gamma x_1+\delta x_2, \quad \langle\gamma,\delta\rangle\in\mathbb{P}_1. \tag{8}$$

2.4. **Local charts around $\mathbb{B}'$.**

Lemma 2.1. *Let $L\in N'$ be the class of the Kronecker module*

$$\begin{pmatrix} -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}.$$

Then there is an open neighbourhood of L that can be identified with an open neighbourhood U of zero in the affine space $\Bbbk^6$ via

$$\Bbbk^6 \supseteq U \to N, \quad (\alpha, \beta, a, b, c, d) \mapsto \left[\begin{pmatrix} -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}\right], \quad \bar{x}_0 = x_0 + \alpha x_1 + \beta x_2,$$

which establishes a local section of the quotient $\mathbb{V}^s \to N$.

Proof. In some open neighbourhood U of zero in $\Bbbk^6$ the morphism

$$U \to \mathbb{V}^s, \quad (\alpha, \beta, a, b, c, d) \mapsto \begin{pmatrix} -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}$$

is well-defined. Notice that two Kronecker modules of the form

$$\begin{pmatrix} -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}$$

can lie in the same orbit of the group action if and only if the matrices are equal. Therefore, the morphism

$$(\alpha, \beta, a, b, c, d) \mapsto \left[\begin{pmatrix} -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}\right]$$

is injective. □

Remark 2.2. By abuse of notation we identify U with its image in N.

Lemma 2.3. *N' is cut out in U by the equations $a = b = c = d = 0$.*

Proof. The maximal minors of $\left(\begin{smallmatrix} -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0+ax_1+bx_2 & dx_2 \end{smallmatrix}\right)$ are

$$cdx_1x_2 + \bar{x}_0(\bar{x}_0 - ax_1 - bx_2), \quad -dx_2^2 - \bar{x}_0 x_1, \quad x_2(\bar{x}_0 - ax_1 - bx_2) - cx_1^2.$$

Clearly these minors have a common linear factor if a, b, c, d vanish. On the other hand the condition $c = d = 0$ is necessary to ensure the reducibility of these quadratic forms. If $c = d = 0$, the conditions $a = b = 0$ are necessary for the minors to have a common factor. □

Lemma 2.4. *The restriction of $\mathbb{B}$ to U is a trivial $\mathbb{P}_{11}$-bundle. Identifying $\mathbb{P}_{11}$ with the projective space*

$$\mathbb{P}(S^2V^* \oplus 2\,\mathrm{Span}(x_1^2, x_1x_2, x_2^2)),$$

i. e., a point in $\mathbb{P}_{11}$ is identified with the class of the triple of quadratic forms

$$(q_0(x_0, x_1, x_2), q_1(x_1, x_2), q_2(x_1, x_2)),$$

one can identify $U \times \mathbb{P}_{11}$, and hence $\mathbb{B}|_U$, with the classes of matrices

$$\begin{pmatrix} q_0(x_0, x_1, x_2) & q_1(x_1, x_2) & q_2(x_1, x_2) \\ -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}. \tag{9}$$

Assuming one of the coefficients of q_0, q_1, q_2 equal to 1, we get local charts of the form $U \times \Bbbk^{11}$ and local sections of the quotient $\mathbb{W}^s \to \mathbb{B}$.

Proof. It is enough to notice that as in (6) one can get rid of x_0 in the expressions of q_1 and q_2. □

In order to get charts around $[A] \in \mathbb{B}'$,

$$A = \begin{pmatrix} 0 & -x_2 \cdot w & x_1 \cdot w \\ -x_2 & 0 & \bar{x}_0 \\ x_1 & -\bar{x}_0 & 0 \end{pmatrix}, \quad w = \gamma x_1 + \delta x_2, \quad \langle \gamma, \delta \rangle \in \mathbb{P}_1,$$

rewrite (9) as

$$\begin{pmatrix} q_0(x_0,x_1,x_2) & q_1(x_1,x_2) - x_2 \cdot w & q_2(x_2) + x_1 \cdot w \\ -x_2 & cx_1 & \bar{x}_0 \\ x_1 & -\bar{x}_0 + ax_1 + bx_2 & dx_2 \end{pmatrix}. \tag{10}$$

Putting $\gamma = 1$ or $\delta = 1$, we get charts around $\mathbb{B}'$, each isomorphic to $U \times \Bbbk \times \Bbbk^{10}$. Denote them by $\mathbb{B}(\gamma)$ and $\mathbb{B}(\delta)$ respectively. Their coordinates are those of U together with δ respectively γ and the coefficients of q_i, $i = 0, 1, 2$.

The equations of $\mathbb{B}'$ are those of N' in U, i. e., $a = b = c = d = 0$, and the conditions imposed by vanishing of q_0, q_1, q_2.

Remark 2.5. Notice that these equations generate the ideal given by the vanishing of the determinant of (10).

3. Main result

Consider the blow-up $\widetilde{\mathbb{B}} = \mathrm{Bl}_{\mathbb{B}'} \mathbb{B}$. Let D denote its exceptional divisor.

Theorem 3.1. *$\widetilde{\mathbb{B}}$ is isomorphic to the blow-up $\widetilde{M} := \mathrm{Bl}_{M_1} M$. The exceptional divisor of $\widetilde{M}$ corresponds to D under this isomorphism. The fibres of the morphism $D \to M_1$ over the point of M_1 represented by a point p on a quartic curve C is identified with the projective line of lines in $\mathbb{P}_2$ passing through p.*

3.1. **A rather intuitive explanation.** Before rigorously proving this, let us explain how to arrive to Theorem 3.1 and "see" it just by looking at the geometric data involved. What follows in not quite rigorous but provides, in our opinion, a nice geometric picture.

Blowing up $\mathbb{B}$ along $\mathbb{B}'$ substitutes $\mathbb{B}'$ by the projective normal bundle of $\mathbb{B}'$. So a point of $\mathbb{B}'$ represented by a line $L \in \mathbb{P}_2^*$ and a point $p \in L$, which is encoded by some $\langle w \rangle \in \mathbb{P}H^*(L, \mathcal{O}_L(1))$, is substituted by the projective space $D_{(L,p)}$ of the normal space $\mathrm{T}_{(L,p)}\,\mathbb{B} / \mathrm{T}_{(L,p)}\,\mathbb{B}'$ to $\mathbb{B}'$ at (L,p).

As $\mathbb{B}$ is a projective bundle over N, and $\mathbb{B}'$ is a $\mathbb{P}_1$-bundle over N', the normal space is a direct sum of the normal spaces along the base and along the fibre. Therefore, $D_{(L,p)}$ is the join of the corresponding projective spaces: of $\mathbb{P}_3 = L^{[3]}$ (normal projective space to N' in N at $L \in N'$) and $\mathbb{P}_9 = \mathbb{P}(S^3V^*)$ (normal projective space to L^* in $J(L^*, \mathbb{P}(S^3V^*))$ at $p \in L \cong L^*$; notice that the normal bundle of $L^* \subseteq J(L^*, \mathbb{P}(S^3V^*))$, i. e., $\mathbb{P}_1 \subseteq \mathbb{P}_{11}$, is trivial).

The space $L^{[3]}$ is naturally identified with the projective space of cubic forms on L^* whereas $\mathbb{P}(S^3V^*)$ is clearly the space of cubic curves on $\mathbb{P}_2$.

Assume $L = Z(x_0)$ such that $\{x_0, x_1, x_2\}$ is a basis of $V^* = H^0(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$. Identifying x_1 and x_2 with their images in $H^0(L, \mathcal{O}_L(1))$, $\{x_1, x_2\}$ is a basis of $H^0(L, \mathcal{O}_L(1))$.

We conclude that the join of $L^{[3]}$ and $\mathbb{P}S^3V^*$ can be identified with the projective space corresponding to the vector space

$$\{\lambda \cdot x_0 h + \lambda' \cdot wg \mid h(x_0,x_1,x_2) \in S^3V^*, g(x_1,x_2) \in H^0(L, \mathcal{O}_L(3)), \langle \lambda, \lambda' \rangle \in \mathbb{P}_1\},$$

i. e., the space of planar quartic curves through the point $p = Z(x_0, w)$.

So the exceptional divisor of the blow-up $\mathrm{Bl}_{\mathbb{B}'} \mathbb{B}$ is a projective bundle with fibre over (L,p) being interpreted as the space of quartic curves through p.

The fibre of $\mathbb{B} \to N$ over $L \in N'$ is substituted by the fibre that consists of two components: the first component is the blow-up of $J(L^*, \mathbb{P}(S^3V^*))$ along L^*, the second one is a projective bundle over L^* with the fibre $\mathbb{P}_{13} = J(L^{[3]}, \mathbb{P}(S^3V^*))$, the components intersect along $L^* \times \mathbb{P}(S^3V^*)$.

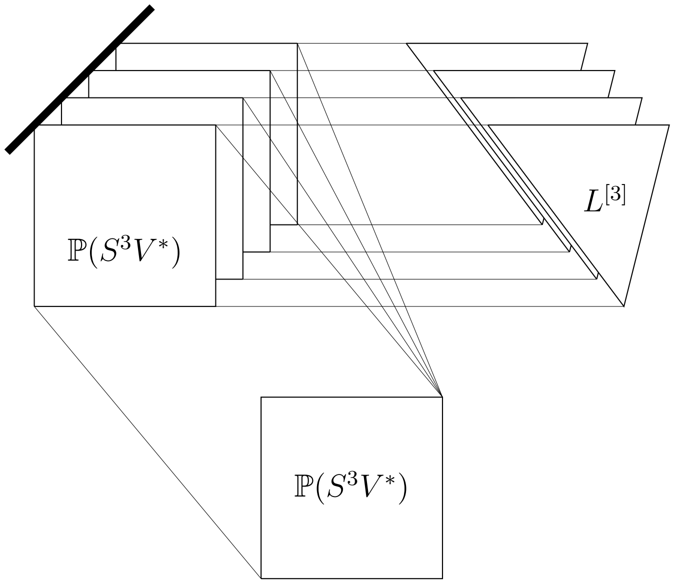

FIGURE 3. The fibre of $\widetilde{\mathbb{B}}$ over $L \in N'$.

Consider the map $\mathbb{B}' \to \mathbb{P}_2$ given by the projection to the second factor (cf. (7)). The fibre $\mathbb{B}'_p$ over a point $p \in \mathbb{P}_2$ is a projective line $\{(L,p) \mid p \in L\}$ in $\mathbb{B}'$, every two points (L,p) and (L',p) of $\mathbb{B}'_p \subseteq \mathbb{B}'$ are substituted by the projective spaces $J(L^{[3]}, \mathbb{P}(S^3V^*))$ and $J(L'^{[3]}, \mathbb{P}(S^3V^*))$ respectively, each of which is naturally identified with the space of quartics through p. Assume without loss of generality $p = \langle 0,0,1 \rangle$.

The fibre $\mathbb{B}'_p$ in this case is identified with the space of lines in $\mathbb{P}_2$ through p, i. e., with the projective line in $N' = \mathbb{P}_2^* = \mathbb{P}V^*$ consisting of classes of linear forms $\alpha x_0 + \beta x_1$, $\langle \alpha, \beta \rangle \in \mathbb{P}_1$. The fibre has a standard covering $\mathbb{B}'_{p,0} = \{x_0 + \beta x_1\}$, $\mathbb{B}'_{p,1} = \{\alpha x_0 + x_1\}$, which is induced by the standard covering of $\mathbb{P}_2^*$. The elements of the fibre corresponding to the points of $\mathbb{B}'_{p,0}$ are the equivalence classes of matrices

$$A_0 = \begin{pmatrix} 0 & x_1 \cdot x_2 & -x_1 \cdot x_1 \\ -x_2 & 0 & x_0 + \beta x_1 \\ x_1 & -(x_0 + \beta x_1) & 0 \end{pmatrix}.$$

The elements of the fibre corresponding to the points of $\mathbb{B}'_{p,1}$ are the equivalence classes of matrices

$$A_1 = \begin{pmatrix} 0 & x_0 \cdot x_2 & -x_0 \cdot x_0 \\ -x_2 & 0 & \alpha x_0 + x_1 \\ x_0 & -(\alpha x_0 + x_1) & 0 \end{pmatrix}.$$

This way we have chosen so to say the normal forms for the representatives of the points in the fibre $\mathbb{B}'_p$. For $\beta = \alpha^{-1}$, i. e. on the intersection of $\mathbb{B}'_{p,0}$ and $\mathbb{B}'_{p,0}$, these matrices are equivalent. One computes that $gA_1h = A_0$ for matrices

$$g = \begin{pmatrix} \alpha^3 & \alpha^2 x_0 - \alpha x_1 & \alpha^2 x_2 \\ 0 & 1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ \alpha^{-1} & -\alpha^{-2} & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$$

with determinants

$$\det g = -\alpha^4, \quad \det h = -\alpha^{-3}.$$

Consider the automorphism $\mathbb{W}^s \xrightarrow{\xi} \mathbb{W}^s$, $A \mapsto gAh$. Then

$$\det(\xi(A_1 + B_1)) = \alpha \cdot \det(A_1 + B_1). \tag{11}$$

From (11) it follows that the restriction of the ideal sheaf of D to a fibre of the morphisms $D \to M_1$ is $\mathcal{O}_{\mathbb{P}_1}(1)$. By [13, 7, 8], this means that one can blow down D in $\widetilde{\mathbb{B}}$ along the map $D \to M_1$. This gives the blow down $\mathrm{Bl}_{\mathbb{B}'} \mathbb{B} \to M$ that contracts the exceptional divisor of $\mathrm{Bl}_{\mathbb{B}'} \mathbb{B}$ along all lines $\mathbb{B}'_p$.

3.2. **Poincaré polynomial of M.** Theorem 3.1 provides an easy way to compute the Poincaré polynomial $P(M)$ of M. Though our computation is similar to the one from [1, Corollary 5.2], it is a bit more straightforward. Notice that $P(M)$ has been also computed using a torus action on M in [2, Theorem 1.1].

Corollary 3.2. *The Poincaré polynomial of M equals* $1+2q+6q^2+10q^3+14q^4+15q^5+16q^7+16q^8+16q^9+16q^{10}+16q^{11}+15q^{12}+14q^{13}+10q^{14}+6q^{15}+2q^{16}+q^{17}$.

Proof. By [5, page 90] the Poincaré polynomial of H is $P(H) = 1+2q+5q^2+6q^3+5q^4+2q^5+q^6$. Then $P(N) = P(H) - P(\mathbb{P}_2) \cdot P(\mathbb{P}_3) + P(\mathbb{P}_2)$. Since $\mathbb{B}$ is a projective bundle over N with fibre $\mathbb{P}_{11}$, one gets $P(\mathbb{B}) = P(N) \cdot P(\mathbb{P}_{11})$. By blowing up $\mathbb{B}$ along $\mathbb{B}'$ and blowing down the result to M one substitutes $\mathbb{B}'$ by M_1, which is a $\mathbb{P}_{13}$-bundle over $\mathbb{P}_2$. Therefore, $P(M) = P(\mathbb{B}) - P(\mathbb{P}_2) \cdot P(\mathbb{P}_1) + P(\mathbb{P}_2) \cdot P(\mathbb{P}_{13})$. Using $P(\mathbb{P}_n) = \frac{1-q^n}{1-q}$ we get the result. $\square$

4. The proof

Now let us properly prove Theorem 3.1.

4.1. **Exceptional divisor D and quartic curves.** Notice that the subvariety $\mathbb{W}'$ in $\mathbb{W}^s$ parameterizing $\mathbb{B}'$ is given by the condition $\det A = 0$. This way we obtain a morphism $\widetilde{\mathbb{B}} \to \mathbb{P}S^4V^*$.

Lemma 4.1. *1) The restriction of $\widetilde{\mathbb{B}} \to \mathbb{P}S^4V^*$ to D maps a point of D lying over a point $p \in \mathbb{P}_2$ (via the map $D \to \mathbb{B}' \to \mathbb{P}_2$) to a quartic curve through p, i. e., there is a morphism $D \to M_1 \subseteq \mathbb{P}_2 \times \mathbb{P}S^4V^*$.*

2) The fibre $D_{(L,p)}$ of $D \to \mathbb{B}'$ over $(L,p) \in \mathbb{P}_2 \times \mathbb{P}_2^*$, $p \in L$, is isomorphic via the map $\widetilde{\mathbb{B}} \to \mathbb{P}S^4V^*$ to the linear subspace in $\mathbb{P}S^dV^*$ of curves through p.

3) The morphism $D \to M_1$ is a $\mathbb{P}_1$-bundle over M_1, its fibre over a point of M_1 given by a pair $p \in C$ can be identified with the fibre of $\mathbb{B}' \to \mathbb{P}_2$ over p.

Proof. 1) Let $[A] \in \mathbb{B}'$ with A as in (8) and let a_0, a_1, a_2 be the rows of A. Let B be a tangent vector at A, which can be identified with a morphism of type (4). Let b_0, b_1, b_2 be its rows. Then, since $\det A = 0$,

$$\det(A + tB) = f_{A,B} \cdot t \mod (t^2), \quad \text{for } f_{A,B} = \det\begin{pmatrix} b_0 \\ a_1 \\ a_2 \end{pmatrix} + \det\begin{pmatrix} a_0 \\ b_1 \\ a_2 \end{pmatrix} + \det\begin{pmatrix} a_0 \\ a_1 \\ b_2 \end{pmatrix}.$$

Then $f_{A,B}$ is a non-zero quartic form if B is normal to $\mathbb{W}'$. One computes

$$f_{A,B} = x_0 \sum_{i=0}^{2} x_i b_{0i} - w(x_1 \sum_{i=0}^{2} x_i b_{1i} + x_2 \sum_{i=0}^{2} x_i b_{2i})$$

and thus $f_{A,B}$ vanishes at p, which is the common zero point of x_0 and w.

2) Since the map $D \to \mathbb{P}S^4V^*$ is injective, it is enough to notice that, for a fixed $A \in \mathbb{W}'$, every quartic form through p can be obtained by varying B. This gives a bijection and thus an isomorphism from $D_{(L,p)}$ to the space of quartics through p.

3) Follows from 1) and 2). $\square$

4.2. **Local charts.** Let us describe $\widetilde{\mathbb{B}}$ over $\mathbb{B}(\delta)$. Around points of D lying over $[A] \in \mathbb{B}(\delta)$ there are 14 charts. For a fixed coordinate t of $\mathbb{B}(\delta)$ different from α, β, γ, denote the corresponding chart of $\mathrm{Bl}_{\mathbb{B}' \cap B(\delta)} \mathbb{B}(\delta)$ by $\widetilde{\mathbb{B}}(t)$. Then $\widetilde{\mathbb{B}}(t)$ can be identified with the variety of triples (A, t, B),

$$
\begin{aligned}
A &= \begin{pmatrix} 0 & -x_2 \cdot (\gamma x_1 + x_2) & x_1 \cdot (\gamma x_1 + x_2) \\ -x_2 & 0 & \bar{x}_0 \\ x_1 & -\bar{x}_0 & 0 \end{pmatrix}, \\
B &= \begin{pmatrix} q_0(x_0, x_1, x_2) & q_1(x_1, x_2) & q_2(x_2) \\ 0 & cx_1 & 0 \\ 0 & ax_1 + bx_2 & dx_2 \end{pmatrix},
\end{aligned} \tag{12}
$$

such that the coefficient of B corresponding to t equals 1 and $A + t \cdot B$ belongs to $\mathbb{B}(\delta)$. The blow-up map $\widetilde{\mathbb{B}}(t) \to \mathbb{B}(t)$ is given under this identification by sending a triple (A, t, B) to $A + t \cdot B$.

4.3. **Family of $(4m-1)$-sheaves on $\widetilde{\mathbb{B}}$.** Notice that the cokernel of (4) is isomorphic to the cokernel of

$$
2\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_0 & q_1 & q_2 \\ 0 & z_0 & z_1 & z_2 \\ 0 & w_0 & w_1 & w_2 \\ 1 & 0 & 0 & 0 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-1).
$$

4.3.1. *Local construction.*

Lemma 4.2. *For $t \neq 0$ consider the matrix*

$$
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -wx_2 & wx_1 \\ 0 & -x_2 & 0 & \bar{x}_0 \\ 0 & x_1 & -\bar{x}_0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_0 & q_1 & q_2 \\ 0 & y_0 & y_1 & y_2 \\ 0 & z_0 & z_1 & z_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}
$$

as a morphism $2\mathcal{O}_{\mathbb{P}_2}(-3) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \to \mathcal{O}_{\mathbb{P}_2}(-2) \oplus 3\mathcal{O}_{\mathbb{P}_2}(-1)$. Then its cokernel is isomorphic to the cokernel of

$$
\begin{pmatrix} \bar{x}_0 & 0 & 0 & 0 \\ w & 0 & 0 & 0 \\ 0 & -x_2 & 0 & \bar{x}_0 \\ 0 & x_1 & -\bar{x}_0 & 0 \\ t & 0 & x_2 & -x_1 \end{pmatrix} + \begin{pmatrix} 0 & x_1y_0 + x_2z_0 & x_1y_1 + x_2z_1 & x_1y_2 + x_2z_2 \\ 0 & q_0 & q_1 & q_2 \\ 0 & ty_0 & ty_1 & ty_2 \\ 0 & tz_0 & tz_1 & tz_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}
$$

Proof. Acting by the automorphisms of $2\mathcal{O}_{\mathbb{P}_2}(-3)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-2)$ on the left and by the automorphisms of $\mathcal{O}_{\mathbb{P}_2}(-2)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-1)$ on the right of (13), we transform this matrix as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -wx_2 & wx_1\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} + t\cdot \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & q_0 & q_1 & q_2\\ 0 & y_0 & y_1 & y_2\\ 0 & z_0 & z_1 & z_2\\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0\\ w & 0 & -wx_2 & wx_1\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} + t\cdot \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & q_0 & q_1 & q_2\\ 0 & y_0 & y_1 & y_2\\ 0 & z_0 & z_1 & z_2\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0\\ w & 0 & 0 & 0\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & x_2 & -x_1 \end{pmatrix} + t\cdot \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & q_0 & q_1 & q_2\\ 0 & y_0 & y_1 & y_2\\ 0 & z_0 & z_1 & z_2\\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \bar{x}_0 & 0 & \bar{x}_0x_2 & -\bar{x}_0x_1\\ w & 0 & 0 & 0\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & x_2 & -x_1 \end{pmatrix} + t\cdot \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & q_0 & q_1 & q_2\\ 0 & y_0 & y_1 & y_2\\ 0 & z_0 & z_1 & z_2\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} \bar{x}_0 & 0 & 0 & 0\\ w & 0 & 0 & 0\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & x_2 & -x_1 \end{pmatrix} + t\cdot \begin{pmatrix} 0 & x_1y_0+x_2z_0 & x_1y_1+x_2z_1 & x_1y_2+x_2z_2\\ 0 & q_0 & q_1 & q_2\\ 0 & y_0 & y_1 & y_2\\ 0 & z_0 & z_1 & z_2\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} t^{-1}\bar{x}_0 & 0 & 0 & 0\\ t^{-1}w & 0 & 0 & 0\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 1 & 0 & x_2 & -x_1 \end{pmatrix} + \begin{pmatrix} 0 & x_1y_0+x_2z_0 & x_1y_1+x_2z_1 & x_1y_2+x_2z_2\\ 0 & q_0 & q_1 & q_2\\ 0 & ty_0 & ty_1 & ty_2\\ 0 & tz_0 & tz_1 & tz_2\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} \bar{x}_0 & 0 & 0 & 0\\ w & 0 & 0 & 0\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ t & 0 & x_2 & -x_1 \end{pmatrix} + \begin{pmatrix} 0 & x_1y_0+x_2z_0 & x_1y_1+x_2z_1 & x_1y_2+x_2z_2\\ 0 & q_0 & q_1 & q_2\\ 0 & ty_0 & ty_1 & ty_2\\ 0 & tz_0 & tz_1 & tz_2\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which concludes the proof. □

Evaluating (14) at $t = 0$ gives

$$\begin{pmatrix} \bar{x}_0 & x_1y_0+x_2z_0 & x_1y_1+x_2z_1 & x_1y_2+x_2z_2\\ w & q_0 & q_1 & q_2\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 0 & 0 & x_2 & -x_1 \end{pmatrix}.$$

Lemma 4.3. *The isomorphism class of the cokernel $\mathcal{F}$ of*

$$2\mathcal{O}_{\mathbb{P}_2}(-3)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\begin{pmatrix} \bar{x}_0 & p_0 & p_1 & p_2\\ w & q_0 & q_1 & q_2\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 0 & 0 & x_2 & -x_1 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-1)$$

is a sheaf from M_1 with resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\begin{pmatrix} \bar{x}_0 & g\\ w & h \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2)\oplus\mathcal{O}_{\mathbb{P}_2} \to \mathcal{F} \to 0, \tag{15}$$

if $\bar{x}_0h - wg \neq 0$ for $g = x_0p_0 + x_1p_1 + x_2p_2$, $h = x_0q_0 + x_1q_1 + x_2q_2$.

Proof. Consider the isomorphism class of $\mathcal{F}$ with resolution (15). Then, using the Koszul resolution of $\mathcal{O}_{\mathbb{P}_2}$, one concludes that the kernel of the composition of two surjective morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-2)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\begin{pmatrix} 1 & 0\\ 0 & x_0\\ 0 & x_1\\ 0 & x_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2)\oplus\mathcal{O}_{\mathbb{P}_2} \to \mathcal{F}$$

coincides with the image of

$$2\mathcal{O}_{\mathbb{P}_2}(-3)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\begin{pmatrix} \bar{x}_0 & p_0 & p_1 & p_2\\ w & q_0 & q_1 & q_2\\ 0 & -x_2 & 0 & \bar{x}_0\\ 0 & x_1 & -\bar{x}_0 & 0\\ 0 & 0 & x_2 & -x_1 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-1),$$

which concludes the proof. □

For $A+tB$ with A and B as in (12) we obtain the morphism

$$2\mathcal{O}_{\mathbb{P}_2}(-3)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\begin{pmatrix} \bar{x}_0 & 0 & cx_1^2+ax_1x_2+bx_2^2 & dx_2^2 \\ w & q_0 & q_1 & q_2 \\ 0 & -x_2 & tcx_1 & \bar{x}_0 \\ 0 & x_1 & -\bar{x}_0+t(ax_1+bx_2) & tdx_2 \\ t & 0 & x_2 & -x_1 \end{pmatrix}} \mathcal{O}_{\mathbb{P}_2}(-2)\oplus 3\mathcal{O}_{\mathbb{P}_2}(-1),$$

which defines by Lemma 4.3 a family of $(4m-1)$-sheaves on $\widetilde{\mathbb{B}}(t)$ and therefore a morphism $\widetilde{\mathbb{B}}(t)\to M$. This morphism sends the point of the exceptional divisor represented by $(A,0,B)$ to the point given by the quartic curve $C=Z(f)$,

$$f=\bar{x}_0\cdot(\bar{x}_0q_0(x_0,x_1,x_2)+x_1q_1(x_1,x_2)+x_2q_2(x_2))-w\cdot(cx_1^3+ax_1^2x_2+bx_1x_2^2+dx_2^3)$$

and the point $p=Z(\bar{x}_0,w)$ on C.

4.3.2. *Gluing the morphisms* $\widetilde{\mathbb{B}}(t)\to M$. For different charts $\widetilde{\mathbb{B}}(t)$ and $\widetilde{\mathbb{B}}(t')$ the corresponding morphisms agree on intersections. Therefore we conclude that there exists a morphism $\widetilde{\mathbb{B}}\to M$. It is an isomorphism outside of D. As already mentioned in Lemma 4.1, the restriction of this morphism to D gives a $\mathbb{P}_1$-bundle $D\to M_1$.

Lemma 4.4. *The map* $\widetilde{\mathbb{B}}\to M$ *is the blow-up* $\operatorname{Bl}_{M_1}M\to M$.

Proof. By the universal property of blow-ups, there exists a unique morphism $\widetilde{\mathbb{B}}\xrightarrow{\phi}\operatorname{Bl}_{M_1}M$ over M, which maps D to the exceptional divisor E of $\operatorname{Bl}_{M_1}M$ and is an isomorphism outside of D. This morphism must be surjective as its image is irreducible and contains an open set. Therefore, the fibres of ϕ over E must be zero-dimensional and connected by the Zariski main theorem. This means that ϕ is a bijective morphism of smooth varieties and hence an isomorphism. $\square$

This concludes the proof of Theorem 3.1.

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