On Nonstrict Means

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Abstract

The general form of continuous, symmetric, increasing, idempotent solutions of the bisymmetry equation is established and the family of sequences of functions which are continuous, symmetric, increasing, idempotent, decomposable is described.

Keywords: nonstrict mean values, bisymmetry equation, decomposability property, ordinal sums.

1 Introduction

Kolmogoroff [6] and Nagumo [8] established a fundamental result about mean values. In their definition a mean value is a sequence $(M^{(m)})_{m\in\mathbb{N}_0}$ of functions $M^{(m)} : [a,b]^m \to [a,b]$ (where $[a,b]$ is a closed real interval) which are continuous, symmetric, strictly increasing in each argument, and idempotent (i.e. $M^{(m)}(x,\ldots,x) = x$ for all $x \in [a,b]$). These functions are also linked by a *pseudo-associativity* called the *decomposability* property by several authors (see e.g. [3, Chapter 5]):

$$M^{(m)}(x_1,\ldots,x_k,x_{k+1},\ldots,x_m) = M^{(m)}(M_k,\ldots,M_k,x_{k+1},\ldots,x_m)$$

for all $m \in \mathbb{N}_0$, $k \in \{1,\ldots,m\}$, $x_1,\ldots,x_m \in [a,b]$, with $M_k = M^{(k)}(x_1,\ldots,x_k)$.

The corresponding result of Kolmogoroff and Nagumo states that these conditions are necessary and sufficient for the existence of a continuous strictly monotonic real function f such that

$$M^{(m)}(x_1,\ldots,x_m) = f^{-1}\left[\frac{1}{m}\sum_{i=1}^{m} f(x_i)\right] \ \forall m \in \mathbb{N}_0.$$

Such an expression is called the *generalized mean.*

On the other hand, Aczél [1] (see also [2]) proved that a function M of two variables defined on $[a, b]$ is continuous, symmetric, strictly increasing in each argument, idempotent and fulfils the *bisymmetry equation*

$$M[M(x_{11}, x_{12}), M(x_{21}, x_{22})] = M[M(x_{11}, x_{21}), M(x_{12}, x_{22})] \tag{1}$$

if and only if

$$M(x, y) = f^{-1}\left[\frac{f(x) + f(y)}{2}\right], \quad x, y \in [a, b]$$

with some continuous strictly monotonic function f.

Note that Horváth [5] investigated the connection between the two concepts of bisymmetry and decomposability.

The aim of this paper is to study *nonstrict* means in an elementary way. That is, we investigate means satisfying either the conditions of Aczél's theorem or the conditions of Kolmogoroff and Nagumo's theorem above, except strict monotonicity. We describe the family of continuous, symmetric, increasing, idempotent, bisymmetric functions (Section 2) and also the family of sequences of continuous, symmetric, increasing, idempotent and decomposable functions (Section 3). The structure of both families is very similar to that of *ordinal sums* well-known in the theory of the associativity functional equation (see e.g. [7]).

2 Nonstrict solutions of the bisymmetry equation

The bisymmetry equation (1), which can be considered also as a generalization of simultaneous commutativity and associativity, has been investigated by several authors. For a list of references see [2].

A function $M : [a, b]^2 \to [a, b]$ is called

- *symmetric*, if $M(x, y) = M(y, x)$ for all $x, y \in [a, b]$;
- *increasing*, if $x \le x', y \le y'$ imply $M(x, y) \le M(x', y')$;
- *strictly increasing*, if $x < x'$ implies $M(x, y) < M(x', y)$ and the same for $y < y'$;
- *idempotent*, if $M(x, x) = x$ for all $x \in [a, b]$;
- *Archimedean*, if $M(a, x) < x < M(x, b)$ for all $x \in (a, b)$;
- *internal*, if $x < M(x, y) < y$ for $x, y \in (a, b)$, $x < y$.

As we said above, our goal is to describe the general form of continuous, symmetric, increasing, idempotent solutions of the bisymmetry equation (1). In their structures, these solutions are very similar to ordinal sums which are well-known in the theory of semigroups, see e.g. [7].

Aczél [1] proved the following result.

Theorem 1 *$M : [a, b]^2 \to [a, b]$ is a continuous, symmetric, strictly increasing, idempotent, bisymmetric function if and only if*

$$M(x, y) = f^{-1}\left[\frac{f(x) + f(y)}{2}\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a, b]$.

We also know that this result still holds for intervals of the form (a,b), $[a,b)$, $(a,b]$ or even for any unbounded interval of the real line (see [2], pp 250–251, 280).

The following lemma will be useful in the sequel.

Lemma 1 *If $M:[a,b]^2 \to [a,b]$ is a continuous, symmetric, increasing, idempotent, bisymmetric function then the following conditions are equivalent:*

$$\begin{array}{rl} i) & M \text{ is Archimedean} \\ ii) & M \text{ is internal} \\ iii) & M \text{ is strictly increasing on } (a,b)^2. \end{array}$$

Proof.

ii) $\Rightarrow$ *i)*. For all $x \in (a,b)$, there exists $u, v \in (a,b)$ such that $a < u < x < v < b$. From *ii)* we have $M(a,x) \leq M(u,x) < x < M(x,v) \leq M(x,b)$.

i) $\Rightarrow$ *ii)*. Assume firstly that there exists $x_0, y_0 \in (a,b), x_0 < y_0$ such that $M(x_0,y_0) = y_0$. Define

$$X = \{x \in [a,b] : x \leq y_0 \text{ and } M(x,y_0) = y_0\}.$$

On the one hand, it is clear that $X \neq \emptyset$ since $x_0 \in X$. On the other hand, because of continuity of M, X is closed. Introducing $x^* = \inf X$, we have $a < x^* \leq x_0 < y_0$ since, from *i)*, we have $a \notin X$. Moreover, since M is increasing, $[x^*, y_0] \subseteq X$. We should have $x^* > M(a,y_0)$. Indeed, if $x^* \leq M(a,y_0)$, then, since $M(a,y_0) < y_0$ by hypothesis, we have $M(a,y_0) \in X$, i.e. $M(M(a,y_0),y_0) = y_0$ and

$$\begin{aligned} M(M(a,x^*),y_0) &= M(M(a,x^*),M(y_0,y_0)) = M(M(a,y_0),M(x^*,y_0)) \\ &= M(M(a,y_0),y_0) = y_0. \end{aligned}$$

Since $M(a,x^*) \leq x^* < y_0$, we have $M(a,x^*) \in X$ and, by the definition of x^*, we have $M(a,x^*) = x^*$, which contradicts *i)*.

It follows that $M(a,y_0) < x^* < y_0 = M(x^*,y_0)$ and, by continuity of M, there exists $z \in (a,x^*)$ such that $x^* = M(z,y_0)$. Consequently,

$$M(M(z,x^*),y_0) = M(M(z,x^*),M(y_0,y_0)) = M(M(z,y_0),M(x^*,y_0)) = M(x^*,y_0) = y_0.$$

Since $M(z,x^*) \leq x^* < y_0$, we have $M(z,x^*) \in X$ and, by the definition of x^*, we have $M(z,x^*) = x^*$. Finally, we have

$$\begin{aligned} x^* &= M(x^*,x^*) = M(x^*,M(z,y_0)) = M(M(x^*,x^*),M(z,y_0)) \\ &= M(M(x^*,z),M(x^*,y_0)) = M(x^*,y_0) = y_0, \end{aligned}$$

a contradiction. Consequently, we have $M(x,y) < y \ \forall\ x,y \in (a,b),\ x < y$. One can prove in a similar way that $x < M(x,y)$.

ii) $\Leftrightarrow$ *iii)*. Aczél has proved that, under the assumptions of this lemma, the condition *ii)* is equivalent to

$$M(x,y) = f^{-1}\left[\frac{f(x)+f(y)}{2}\right] \quad \forall x,y \in (a,b)$$

where f is any continuous strictly monotonic function on (a,b) (see [2, pages 281–284]), which is sufficient. ■

Now define $\mathcal{B}_{a,b,\theta}$ as the set of functions $M : [a,b]^2 \to [a,b]$ which are continuous, symmetric, increasing, idempotent, bisymmetric and such that $M(a,b) = \theta$, θ being a given number in $[a,b]$. A general element of a class $\mathcal{B}_{a,b,\theta}$ is usually denoted by $M_{a,b,\theta}$ in the sequel. Before stating the following important result we need to introduce the so-called *median* operation. Consider three real numbers $x, y, z \in \mathbb{R}$. Then their median (denoted as $\mathrm{median}(x,y,z)$) is defined by

$$\mathrm{median}(x,y,z) = \begin{cases} x & \text{if } \min(y,z) \leq x \leq \max(y,z) \\ y & \text{if } \min(x,z) \leq y \leq \max(x,z) \\ z & \text{if } \min(x,y) \leq z \leq \max(x,y) \end{cases} .$$

Theorem 2 *$M : [a,b]^2 \to [a,b]$ is a continuous, symmetric, increasing, idempotent, bisymmetric function if and only if there exist two numbers α and β fulfilling $a \leq \alpha \leq \beta \leq b$ such that*

$$\begin{array}{lll} i) & M(x,y) = M_{a,\alpha,\alpha}(x,y) & \text{if } x,y \in [a,\alpha]; \\ ii) & M(x,y) = M_{\beta,b,\beta}(x,y) & \text{if } x,y \in [\beta,b]; \\ iii) & M(x,y) = f^{-1}\left[\dfrac{f[\mathrm{median}(\alpha,x,\beta)] + f[\mathrm{median}(\alpha,y,\beta)]}{2}\right] & \text{otherwise} \end{array}$$

with some $M_{a,\alpha,\alpha} \in \mathcal{B}_{a,\alpha,\alpha}$, $M_{\beta,b,\beta} \in \mathcal{B}_{\beta,b,\beta}$, and f is any continuous strictly monotonic function on $[\alpha,\beta]$.

Proof. "$\Leftarrow$". Indeed, we can easily show that the functions M defined in the statement are continuous, symmetric, increasing, idempotent and bisymmetric.

"$\Rightarrow$". Assume that $M : [a,b]^2 \to [a,b]$ is a continuous, symmetric, increasing, idempotent and bisymmetric function. Define

$$X_a = \{x \in [a,b] \mid M(a,x) = x\} \quad \text{and} \quad X_b = \{x \in [a,b] \mid M(x,b) = x\}.$$

On the one hand, it is clear that $X_a \neq \emptyset$ and $X_b \neq \emptyset$ since $a \in X_a$ and $b \in X_b$. On the other hand, by continuity of M, X_a and X_b are closed. Introducing $\alpha = \sup X_a$ and $\beta = \inf X_b$, we have $\alpha \leq \beta$, otherwise we would have

$$M(a,b) \geq M(a,\alpha) = \alpha > \beta = M(\beta,b) \geq M(a,b),$$

a contradiction.

Let $(x,y) \in [a,b]^2$. There are three mutually exclusive cases:

1. If $x,y \in [a,\alpha]$, then we have $M(x,y) = M_{a,\alpha,\alpha}(x,y)$, where $M_{a,\alpha,\alpha} \in \mathcal{B}_{a,\alpha,\alpha}$.
2. If $x,y \in [\beta,b]$, then we have $M(x,y) = M_{\beta,b,\beta}(x,y)$, where $M_{\beta,b,\beta} \in \mathcal{B}_{\beta,b,\beta}$.
3. Otherwise:
   - If $\alpha = \beta$, then we have

     $$\alpha = M(a,\alpha) \leq M(x,y) \leq M(\alpha,b) = \alpha,$$

     that is $M(x,y) = \alpha$.

- If $\alpha < \beta$, then we have

$$\begin{aligned} M(a,y) &= M(\alpha,y) \;\; \forall y \in [\alpha, M(\alpha,b)], & (2)\\ M(x,b) &= M(x,\beta) \;\; \forall x \in [M(a,\beta),\beta]. & (3) \end{aligned}$$

Indeed, if $y \in [\alpha, M(\alpha,b)]$ then, by continuity of M, there exists $z \in (\alpha,b)$ such that $y = M(\alpha,z)$. So, we have

$$\begin{aligned} M(a,y) &= M(M(a,a),M(\alpha,z)) = M(M(a,\alpha),M(a,z))\\ &= M(M(\alpha,\alpha),M(a,z)) = M(M(\alpha,a),M(\alpha,z))\\ &= M(\alpha,y), \end{aligned}$$

which proves (2). We can show that (3) is true by using the same argument. Moreover, we have

$$M(\alpha,\beta) = M(\alpha,b) = M(a,\beta) = M(a,b). \quad (4)$$

Indeed, setting $\theta = M(a,b)$, we have

$$\alpha = M(a,\alpha) \le M(a,\beta) \le \theta \le M(\alpha,b) \le M(\beta,b) = \beta$$

and we can apply (2) and (3). Therefore, we have

$$\begin{aligned} \theta &= M(M(a,b),\theta) = M(M(a,\theta),M(\theta,b)) = M(M(\alpha,\theta),M(\theta,\beta))\\ &= M(M(\alpha,\beta),\theta) = M(M(a,\alpha),M(\beta,b)) = M(\alpha,\beta), \end{aligned}$$

and

$$\begin{aligned} M(\alpha,b) &= M(M(a,\alpha),b) = M(M(\alpha,b),\theta) = M(M(\alpha,b),M(\alpha,\beta))\\ &= M(\alpha,\beta), \end{aligned}$$

and

$$\begin{aligned} M(a,\beta) &= M(a,M(\beta,b)) = M(\theta,M(a,\beta)) = M(M(\alpha,\beta),M(a,\beta))\\ &= M(\alpha,\beta), \end{aligned}$$

which proves (4).
We also have

$$\begin{aligned} M(a,x) &= M(\alpha,x) \;\; \forall x \in [\alpha,\beta], & (5)\\ M(x,b) &= M(x,\beta) \;\; \forall x \in [\alpha,\beta]. & (6) \end{aligned}$$

By (2)–(4), it sufficies to prove that $M(a,x) = M(\alpha,x)$ for all $x \in [\theta,\beta]$, and $M(x,b) = M(x,\beta)$ for all $x \in [\alpha,\theta]$.
M is continuous, thus for any $x \in [\theta,\beta]$ there exists $z \in [a,b]$ such that $x = M(\beta,z)$. Thus we have

$$\begin{aligned} M(a,x) &= M(a,M(\beta,z)) = M(M(a,\beta),M(a,z))\\ &= M(M(\alpha,\beta),M(a,z)) = M(M(\beta,z),\alpha)\\ &= M(\alpha,x) \end{aligned}$$

which proves (5). We can prove (6) similarly.

For any $x \leq \alpha$, $y \geq \beta$, $M(x,y) = \theta$ holds. Indeed, from (4), we have $\theta = M(a,\beta) \leq M(x,y) \leq M(\alpha,b) = \theta$.

Finally, by Theorem 1 and Lemma 1, it sufficies to show that $M(\alpha,x) < x < M(x,\beta)$ for all $x \in (\alpha,\beta)$. Suppose the first inequality is not true. Then, from (5), there exists $x \in (\alpha,\beta)$ such that $M(a,x) = M(\alpha,x) = x$, which contradict the definition of α. We can prove the second inequality in a similar way.

■

Now, our task consists in describing the two families $\mathcal{B}_{a,b,a}$ and $\mathcal{B}_{a,b,b}$. Before going on, we prove a lemma.

Lemma 2 *A function $M \in \mathcal{B}_{a,b,a}$ (resp. $\mathcal{B}_{a,b,b}$) is strictly increasing on $(a,b)^2$ if and only if*

$$M(x,y) = g^{-1}\sqrt{g(x)g(y)} \quad \text{for } x,y \in [a,b],$$

where g is any continuous strictly increasing (resp. decreasing) function on $[a,b]$, with $g(a) = 0$ (resp. g(b)=0).

Proof. Let us consider the case $\mathcal{B}_{a,b,a}$, the other one is symmetric.

"$\Leftarrow$". Easy.

"$\Rightarrow$". Let $M \in \mathcal{B}_{a,b,a}$ be strictly increasing on $(a,b)^2$. From Theorem 1, there exists a function f which is continuous and strictly monotonic on (a,b), such that

$$2f(M(x,y)) = f(x) + f(y) \quad \forall x,y \in (a,b). \tag{7}$$

Replacing f by $-f$, if necessary, we can assume that f is strictly increasing on (a,b). By continuity of M, we have

$$\lim_{x \to a^+} M(x,y) = M(a,y) = a \quad \forall y \in (a,b).$$

Then assume that $\lim_{x \to a^+} f(x) = \theta \in \mathbb{R}$. From (7), we have $f(y) = \theta \quad \forall y \in (a,b)$, which is impossible since f is strictly increasing on (a,b). Therefore, $\lim_{x \to a^+} f(x) = -\infty$.

From (7), we also have $\lim_{y \to b^-} f(y) \in \mathbb{R}$. Then let $g(x)$ be the continuous extension of the function $\exp f(x)$ on $[a,b]$, that is, $g(a) = 0$ and $g(x) = \exp f(x)$ on $(a,b]$. The function g thus defined is continuous and strictly increasing on $[a,b]$ and (7) becomes

$$\log g[M(x,y)] = \frac{\log g(x) + \log g(y)}{2} \quad \forall x,y \in (a,b]$$

and so we have

$$M(x,y) = g^{-1}\sqrt{g(x)g(y)}$$

on $(a,b]^2$ and even on $[a,b]^2$ since M is continuous. ■

Now, we present a description of the two families $\mathcal{B}_{a,b,a}$ and $\mathcal{B}_{a,b,b}$. The next two theorems deal with this issue.

Theorem 3 *$M \in \mathcal{B}_{a,b,a}$ if and only if*

- *either*

$$M(x,y) = \min(x,y),$$

- *or*

$$M(x,y) = g^{-1}\sqrt{g(x)g(y)},$$

where g is any continuous strictly increasing function on $[a,b]$, with $g(a) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a,b]$ such that*

$$M(x,y) = \begin{cases} g_k^{-1}\sqrt{g_k[\min(x,b_k)]g_k[\min(y,b_k)]} & \text{if there exists } k \in K \text{ such that} \\ & \min(x,y) \in (a_k,b_k), \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where g_k is any continuous strictly increasing function on $[a_k,b_k]$, with $g_k(a_k) = 0$.

Proof. "$\Leftarrow$". One can easily check that the functions M defined in the statement belong to $\mathcal{B}_{a,b,a}$.

"$\Rightarrow$". Let $x, y \in [a,b]$ and $M \in \mathcal{B}_{a,b,a}$. Define a set $X \subseteq [a,b]$ by

$$X = \{x \in [a,b] : M(x,b) = x\}.$$

It is clear that X is closed and nonempty. Thus $Y = [a,b] \backslash X$ is open and bounded. In fact $Y = \emptyset$ if and only if $M(x,b) = x \ \ \forall x \in [a,b]$, i.e.

$$M(x,y) = \min(x,y)$$

since assuming $x \leq y, x, y \in [a,b]$, we have $M(x,y) \leq M(x,b) = x = M(x,x) \leq M(x,y)$.

In the other extreme case we have $Y = (a,b)$, i.e. $X = \{a,b\}$, if and only if $x < M(x,b) \ \ \forall x \in (a,b)$. However $M(a,a) = a$ and $M(a,b) = a$ imply $M(a,x) = a < x \ \ \forall x \in (a,b)$. It follows from Lemma 1 that $M(x,y)$ is strictly increasing on $(a,b)^2$ and from Lemma 2 that

$$M(x,y) = g^{-1}\sqrt{g(x)g(y)},$$

where g is any continuous strictly increasing function on $[a,b]$, with $g(a) = 0$.

Consider the remaining case, that is $\emptyset \subset Y \subset (a,b)$. Then there exists a countable index set K and a class of pairwise disjointed open intervals $\{(a_k,b_k) : k \in K\}$ of $[a,b]$ such that

$$Y = \bigcup_{k \in K} (a_k, b_k).$$

For all $k \in K$, we obviously have $M(a_k,b) = a_k$ and $M(b_k,b) = b_k$ since $a_k, b_k \in X$, but also

$$M(x,b) > x \ \ \forall x \in (a_k,b_k), \tag{8}$$

$$M(a_k,x) = a_k \ \ \forall x \in [a_k,b], \tag{9}$$

$$M(b_k,x) = b_k \ \ \forall x \in [b_k,b], \tag{10}$$

In order to establish (8), we can notice that $x \in (a_k,b_k)$ implies $x \notin X$. For (9) and (10), we obviously have

$$a_k = M(a_k,a_k) \leq M(a_k,x) \leq M(a_k,b) = a_k \ \ \forall x \in [a_k,b]$$

$$b_k = M(b_k,b_k) \leq M(b_k,x) \leq M(b_k,b) = b_k \ \ \forall x \in [b_k,b].$$

If $\min(x,y) \in X$, then

$$M(x,y) = \min(x,y).$$

If $\min(x,y) \in Y$, i.e. $\min(x,y) \in (a_k,b_k)$ for one $k \in K$, then, assuming that $x \in (a_k,b_k)$ and $y \in [b_k,b]$, we have

$$M(x,y) = M(x,b_k). \tag{11}$$

Indeed, since (9) implies $M(a_k,b_k) = a_k$ and since $M(b_k,b_k) = b_k$, then, by continuity of M, there exists $z \in (a_k,b_k)$ such that $x = M(z,b_k)$. Then, from (10) we deduce

$$\begin{aligned} M(x,y) &= M(M(z,b_k),M(y,y)) = M(M(z,y),M(b_k,y)) \\ &= M(M(z,y),b_k) = M(M(z,y),M(b_k,b_k)) = M(M(z,b_k),M(y,b_k)) \\ &= M(x,b_k). \end{aligned}$$

Now, we can show that if $x,y \in (a_k,b_k)$, then

$$M(x,y) = g_k^{-1}\sqrt{g_k(x)g_k(y)}$$

where g_k is any continuous strictly increasing function on $[a_k,b_k]$, with $g_k(a_k) = 0$. It is sufficient, from Lemma 1 and Lemma 2, to show that

$$M(a_k,x) < x < M(x,b_k) \quad \forall x \in (a_k,b_k).$$

The first inequality comes from (9). For the second one, we notice that if $x = M(x,b_k)$ for one $x \in (a_k,b_k)$, then, from (11), we would have $x = M(x,b_k) = M(x,b)$, which contradicts (8). ■

Theorem 4 *$M \in \mathcal{B}_{a,b,b}$ if and only if*

- *either*

$$M(x,y) = \max(x,y),$$

- *or*

$$M(x,y) = g^{-1}\sqrt{g(x)g(y)},$$

  where g is any continuous strictly decreasing function on $[a,b]$, with $g(b) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k,b_k) : k \in K\}$ of $[a,b]$ such that*

$$M(x,y) = \begin{cases} g_k^{-1}\sqrt{g_k[\max(a_k,x)]g_k[\max(a_k,y)]} & \text{if there exists } k \in K \text{ such that} \\ & \max(x,y) \in (a_k,b_k), \\ \max(x,y) & \text{otherwise,} \end{cases}$$

  where g_k is any continuous strictly decreasing function on $[a_k,b_k]$, with $g_k(b_k) = 0$.

Proof. Similar to the previous one. ■

3 Extended Kolmogoroff-means

We show now that the results obtained in the previous section can be extended to the mean values by replacing bisymmetry by decomposability. According to Fodor and Roubens [3], we may view any mean value M as an aggregation operator:

$$M : \Lambda = \bigcup_{m=1}^{\infty} [a,b]^m \to [a,b] \ , \quad (x_1, \dots, x_m) \mapsto M^{(m)}(x_1, \dots, x_m)^1.$$

Such an operator M is said to be

- *continuous*, if for all $m \in \mathbb{N}_0$, $M^{(m)}$ is a continuous function on $[a,b]^m$;
- *symmetric*, if for all $m \in \mathbb{N}_0$, $M^{(m)}$ is a symmetric function on $[a,b]^m$:

$$M^{(m)}(x_1, \dots, x_m) = M^{(m)}(x_{\sigma(1)}, \dots, x_{\sigma(m)})$$

  where σ is a permutation of $\{1, \dots, m\}$ and $(x_1, \dots, x_m) \in [a,b]^m$;
- *increasing*, if for all $m \in \mathbb{N}_0$, $M^{(m)}$ is increasing in each argument:

$$x_i < x_i' \Rightarrow M^{(m)}(x_1, \dots, x_i, \dots, x_m) \le M^{(m)}(x_1, \dots, x_i', \dots, x_m), \ i = 1, \dots, m;$$

- *strictly increasing*, if for all $m \in \mathbb{N}_0$, $M^{(m)}$ is strictly increasing in each argument:

$$x_i < x_i' \Rightarrow M^{(m)}(x_1, \dots, x_i, \dots, x_m) < M^{(m)}(x_1, \dots, x_i', \dots, x_m), \ i = 1, \dots, m;$$

- *idempotent*, if for all $m \in \mathbb{N}_0$, $M^{(m)}$ satisfies

$$M^{(m)}(x, \dots, x) = x, \quad \forall x \in [a,b];$$

- *decomposable*, if for all $m \in \mathbb{N}_0$ and all $k \in \{1, \dots, m\}$, the following equality holds:

$$M^{(m)}(x_1, \dots, x_k, x_{k+1}, \dots, x_m) = M^{(m)}(M_k, \dots, M_k, x_{k+1}, \dots, x_m)$$

  where $M_k = M^{(k)}(x_1, \dots, x_k)$.

Kolmogoroff [6] established the following two results.

Lemma 3 *If M is an aggregation operator defined on Λ, continuous, symmetric, increasing, idempotent and decomposable, then there exists a function ψ which is continuous on $(0,1)$ and increasing on $[0,1]$, with $\psi(0) = a$ and $\psi(1) = b$, such that, for all $m \in \mathbb{N}_0$, we have*

$$M^{(m)}(\psi(t_1), \dots, \psi(t_m)) = \psi\left(\frac{1}{m}\sum_i t_i\right)$$

for all $(t_1, \dots, t_m) \in [0,1]^m$.

[1]We will often write $M(x_1, \dots, x_m)$ instead of $M^{(m)}(x_1, \dots, x_m)$.

Theorem 5 *An aggregation operator M, defined on Λ, is continuous, symmetric, strictly increasing, idempotent and decomposable if and only if for all $m \in \mathbb{N}_0$,*

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a, b]$.

Theorem 5 still holds for sets Λ of the form $\bigcup_{m=1}^{\infty}(a, b)^m$, $\bigcup_{m=1}^{\infty}[a, b)^m$, $\bigcup_{m=1}^{\infty}(a, b]^m$, even if $a = -\infty$ and/or $b = +\infty$.

Given $\theta \in [a, b]$, $\mathcal{D}_{a,b,\theta}$ is the set of aggregation operators $M : \Lambda = \bigcup_{m=1}^{\infty}[a, b]^m \to [a, b]$ which are continuous, symmetric, increasing, idempotent, decomposable and such that for all $k_1, k_2 \in \mathbb{N}_0$,

$$M^{(k_1+k_2)}(\underbrace{a, \ldots, a}_{k_1\ times}, \underbrace{b, \ldots, b}_{k_2\ times}) = \theta.$$

Then we have the following result:

Theorem 6 *An aggregation operator M, defined on Λ, is continuous, symmetric, increasing, idempotent and decomposable if and only if there exists two numbers α and β fulfilling $a \le \alpha \le \beta \le b$, such that, for all $m \in \mathbb{N}_0$, $(x_1, \ldots x_m) \in [a, b]^m$*

$$\begin{array}{lll} i) & M(x_1, \ldots, x_m) = M_{a,\alpha,\alpha}(x_1, \ldots, x_m), & \text{if } \max_i x_i \in [a, \alpha]; \\ ii) & M(x_1, \ldots, x_m) = M_{\beta,b,\beta}(x_1, \ldots, x_m), & \text{if } \min_i x_i \in [\beta, b]; \\ iii) & M(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f[median(\alpha, x_i, \beta)]\right] & \text{otherwise,} \end{array}$$

where $M_{a,\alpha,\alpha} \in \mathcal{D}_{a,\alpha,\alpha}$, $M_{\beta,b,\beta} \in \mathcal{D}_{\beta,b,\beta}$, and f is any continuous strictly monotonic function on $[\alpha, \beta]$.

Proof. First of all, consider the following practical notation

$$M(n_1 \cdot x_1, \ldots, n_m \cdot x_m) = M(\underbrace{x_1, \ldots, x_1}_{n_1\ times}, \ldots, \underbrace{x_m, \ldots, x_m}_{n_m\ times}), \quad n_1, \ldots, n_m \in \mathbb{N}_0.$$

"$\Leftarrow$". Indeed, we can easily show that M satisfies the announced properties.

"$\Rightarrow$". According to Lemma 3, there exists a function ψ which is continuous on $(0, 1)$ and increasing on $[0, 1]$, with $\psi(0) = a$ and $\psi(1) = b$, such that, for all $m \in \mathbb{N}_0$, we have

$$M^{(m)}(\psi(t_1), \ldots, \psi(t_m)) = \psi\left(\frac{1}{m}\sum_i t_i\right)$$

for all $(t_1, \ldots, t_m) \in [0, 1]^m$. Define α and β in the following way:

$$a \le \alpha = \lim_{t \to 0^+} \psi(t) \le \lim_{t \to 1^-} \psi(t) = \beta \le b.$$

Then, for all $k_1, k_2 \in \mathbb{N}_0$, we have

$$M(k_1 \cdot a, k_2 \cdot \alpha) = \alpha, \tag{12}$$

$$M(k_1 \cdot \beta, k_2 \cdot b) = \beta. \tag{13}$$

Indeed, according to Lemma 3 and since ψ and M are continuous, we have

$$M(k_1 \cdot a, k_2 \cdot \alpha) = \lim_{t \to 0^+} M(k_1 \cdot \psi(0), k_2 \cdot \psi(t)) = \lim_{t \to 0^+} \psi\left(\frac{k_2 t}{k_1 + k_2}\right) = \alpha$$

and

$$M(k_1 \cdot \beta, k_2 \cdot b) = \lim_{t \to 1^-} M(k_1 \cdot \psi(t), k_2 \cdot \psi(1)) = \lim_{t \to 1^-} \psi\left(\frac{k_1 t + k_2}{k_1 + k_2}\right) = \beta.$$

Then let $m \in \mathbb{N}_0$ and $(x_1, \ldots, x_m) \in [a, b]^m$. There are three mutually exclusive cases:

1. If $\max_i x_i \in [a, \alpha]$, then, from (12), we have $M(x_1, \ldots, x_m) = M_{a,\alpha,\alpha}(x_1, \ldots, x_m)$, where $M_{a,\alpha,\alpha} \in \mathcal{D}_{a,\alpha,\alpha}$.
2. If $\min_i x_i \in [\beta, b]$, then, from (13), we have $M(x_1, \ldots, x_m) = M_{\beta,b,\beta}(x_1, \ldots, x_m)$, where $M_{\beta,b,\beta} \in \mathcal{D}_{\beta,b,\beta}$.
3. Otherwise, we have:
   - If $\alpha = \beta$, then, from (12) and (13), we have

     $$\alpha = M((m-1) \cdot a, \alpha) \leq M(x_1, \ldots, x_m) \leq M(\alpha, (m-1) \cdot b) = \alpha,$$

     i.e. $M(x_1, \ldots, x_m) = \alpha$.
   - If $\alpha < \beta$, then ψ is strictly increasing on [0,1]. Suppose it is not true and there exists $t_1, t_2 \in (0,1), t_1 < t_2$, such that $\psi(t_1) = \psi(t_2)$. Then, for all $p, q \in \mathbb{N}, p \leq q, q \neq 0$,

     $$M(p \cdot \psi(t_1), (q-p) \cdot \psi(0)) = M(p \cdot \psi(t_2), (q-p) \cdot \psi(0)),$$

     i.e. from Lemma 3,

     $$\psi\left(\frac{p}{q} t_1\right) = \psi\left(\frac{p}{q} t_2\right).$$

     Therefore, for any rational number $r \in [0,1]$, we have $\psi(rt_1) = \psi(rt_2)$, which still holds, because of the continuity of ψ, for all real number $r \in [0,1]$. Choosing $r = t_1/t_2 \in (0,1)$, the previous equality becomes $\psi(rt_1) = \psi(t_1) = \psi(t_2)$. By iteration, we get $\psi(r^n t_1) = \psi(t_2)$, $\forall n \in \mathbb{N}_0$, and because of the continuity of ψ, $\alpha = \lim_{n \to +\infty} \psi(r^n t_1) = \psi(t_2)$. One can show, in a similar way, that $\psi(t_1) = \beta$. Indeed we have, for all $p, q \in \mathbb{N}, p \leq q, q \neq 0$,

     $$M(p \cdot \psi(t_1), (q-p) \cdot \psi(1)) = M(p \cdot \psi(t_2), (q-p) \cdot \psi(1)),$$

     that is, from Lemma 3, $\psi(1 - r(1-t_1)) = \psi(1 - r(1-t_2))$ for all $r \in [0,1]$. Choosing $r = (1-t_2)/(1-t_1) \in (0,1)$, the previous equality implies

     $$\psi(1 - (1-t_1)) = \psi(t_1) = \psi(t_2) = \psi(1 - r(1-t_1)).$$

     By iteration, we get $\psi(t_1) = \psi(1 - r^n(1-t_1))$, $\forall n \in \mathbb{N}_0$, and because of the continuity of ψ, $\psi(t_1) = \lim_{n \to +\infty} \psi(1 - r^n(1-t_1)) = \beta$. Finally, we have $\alpha = \beta$, a contradiction. Consequently, ψ is strictly increasing on (0,1) and thus on [0,1].

Since ψ is continuous on (0,1), its inverse ψ^{-1} is defined on $(\alpha, \beta) \cup \{a, b\}$ and is continuous on (α, β). Now, investigate the following expression

$$M(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1, \ldots, z_{m_3}), \quad m_1 + m_2 + m_3 = m,$$

with

$$\begin{cases} x_1, \ldots, x_{m_1} \in [a, \alpha], & m_1 < m, \\ y_1, \ldots, y_{m_2} \in (\alpha, \beta), & \\ z_1, \ldots, z_{m_3} \in [\beta, b], & m_3 < m. \end{cases}$$

Since M is continuous, we have

$$\begin{aligned} & M(m_1 \cdot a, y_1, \ldots, y_{m_2}, m_3 \cdot \beta) \\ = \; & \lim_{t \to 1^-} M\left[m_1 \cdot \psi(0), \psi\psi^{-1}(y_1), \ldots, \psi\psi^{-1}(y_{m_2}), m_3 \cdot \psi(t)\right] \\ = \; & \lim_{t \to 1^-} \psi\left[\frac{m_1}{m} 0 + \frac{1}{m} \sum_{i=1}^{m_2} \psi^{-1}(y_i) + \frac{m_3}{m} t\right] \quad \text{(lemma 3)} \\ = \; & \psi\left[\frac{m_1}{m} 0 + \frac{1}{m} \sum_{i=1}^{m_2} \psi^{-1}(y_i) + \frac{m_3}{m} 1\right] \quad (\text{since } m_3 < m) \end{aligned}$$

Since $m_1 < m$, this last expression is also equal to $M(m_1 \cdot \alpha, y_1, \ldots, y_{m_2}, m_3 \cdot b)$ and thus finally to $M(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1, \ldots, z_{m_3})$ because, since M is increasing, we have

$$\begin{aligned} M(m_1 \cdot a, y_1, \ldots, y_{m_2}, m_3 \cdot \beta) & \leq M(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1, \ldots, z_{m_3}) \\ & \leq M(m_1 \cdot \alpha, y_1, \ldots, y_{m_2}, m_3 \cdot b). \end{aligned}$$

Then, let $f(x)$ be the continuous extension on $[\alpha, \beta]$ of the function $\psi^{-1}(x)$, i.e. $f(\alpha) = 0$, $f(\beta) = 1$ and $f(x) = \psi^{-1}(x)$ on (α, β). The function f is thus continuous and strictly monotonic on $[\alpha, \beta]$ and we have

$$\begin{aligned} & M(x_1, \ldots, x_{m_1}, y_1, \ldots, y_{m_2}, z_1, \ldots, z_{m_3}) \\ & = f^{-1}\left[\frac{m_1}{m} f(\alpha) + \frac{1}{m} \sum_{i=1}^{m_2} f(y_i) + \frac{m_3}{m} f(\beta)\right]. \end{aligned}$$

■

The next lemma, due to Nagumo [8], will be very useful in the sequel.

Lemma 4 *If M is an aggregation operator which is symmetric, idempotent and decomposable, then* (1) *holds. Moreover, for all $m \in \mathbb{N}_0$, $m \geq 3$, we have*

$$M^{(m)}(x_1, \ldots, x_m) = M^{(m)}(x'_1, \ldots, x'_m),$$

where $x'_i = M^{(m-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$, $i = 1, \ldots, m$ (the argument x_i being omitted).

Now, we describe the two families $\mathcal{D}_{a,b,a}$ and $\mathcal{D}_{a,b,b}$.

Theorem 7 *$M \in \mathcal{D}_{a,b,a}$ if and only if*

- *either, for all* $m \in \mathbb{N}_0$,

$$M(x_1, \ldots, x_m) = \min(x_1, \ldots, x_m),$$

- *or, for all* $m \in \mathbb{N}_0$,

$$M(x_1, \ldots, x_m) = g^{-1} \sqrt[m]{\prod_i g(x_i)},$$

where g *is any continuous strictly increasing function on* $[a, b]$*, with* $g(a) = 0$*,*

- *or there exists a countable index set* K *and a family of disjoint subintervals* $\{(a_k, b_k) : k \in K\}$ *of* $[a, b]$ *such that, for all* $m \in \mathbb{N}_0$,

$$M(x_1, \ldots, x_m) = \begin{cases} g_k^{-1} \sqrt[m]{\prod_i g_k[\min(x_i, b_k)]} & \text{if there exists } k \in K \text{ such that} \\ & \min_i x_i \in (a_k, b_k), \\ \min_i x_i & \text{otherwise,} \end{cases}$$

where g_k *is any continuous strictly increasing function on* $[a_k, b_k]$*, with* $g_k(a_k) = 0$*.*

Proof. "$\Leftarrow$". One can easily check that the operators M defined in the statement belong to $\mathcal{D}_{a,b,a}$.

"$\Rightarrow$". Let $m \in \mathbb{N}_0$, $(x_1, \ldots, x_m) \in [a, b]^m$ and $M \in \mathcal{D}_{a,b,a}$. From Lemma 4, $M^{(2)} \in \mathcal{B}_{a,b,a}$ and we can use Theorem 3.

- If $M(x, y) = \min(x, y)$ then $M(x_1, \ldots, x_m) = \min_i x_i$. Indeed, suppose the result is true for $m - 1$ ($m \geq 3$) and $x_1 \leq \ldots \leq x_m$. So, from Lemma 4, we have

$$M^{(m)}(x_1, \ldots, x_m) = M^{(m)}(x_2, x_1, \ldots, x_1) = M^{(m)}(x_1, \ldots, x_1) = x_1 = \min_i x_i,$$

and the result is still true for m.

- If $M(x, y) = g^{-1}\sqrt{g(x)g(y)}$, where g is any continuous strictly increasing function on $[a, b]$, with $g(a) = 0$, then $M(x_1, \ldots, x_m) = g^{-1} \sqrt[m]{\prod_i g(x_i)}$. Indeed, suppose the result is true for $m - 1$ ($m \geq 3$). Since M is decomposable, the operator F defined by

$$F^{(q)}(z_1, \ldots, z_q) = f[M^{(q)}(f^{-1}(z_1), \ldots, f^{-1}(z_q))], \quad \forall q \in \mathbb{N}_0,$$

where $f(x) = \log g(x)$ on $(a, b]$, is also decomposable and, since it is also symmetric and idempotent, we have, from Lemma 4,

$$F^{(m)}(z_1, \ldots, z_m) = F^{(m)}(z'_1, \ldots, z'_m)$$

where, from the induction hypothesis,

$$z'_j = F^{(m-1)}(z_1, \ldots, [z_j], \ldots, z_m) = \frac{1}{m-1} \left(\sum_{i=1}^{m} z_i - z_j \right) \quad j = 1, \ldots, m.$$

Consequently, we get (see [8]) $F^{(m)}(z_1, \ldots, z_m) = (1/m) \sum_i z_i$. From that, we have

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1} \left[F^{(m)}(f(x_1), \ldots, f(x_m)) \right] = g^{-1} \sqrt[m]{\prod_i g(x_i)}$$

on $[a, b]^m$. Thus, the result is still true for m.

- In the last case, there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a, b]$ such that

$$M(x,y) = \begin{cases} g_k^{-1}\sqrt{g_k[\min(x,b_k)]g_k[\min(y,b_k)]} & \text{if there exists } k \in K \text{ such that} \\ & \min(x,y) \in (a_k,b_k), \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where g_k is any continuous strictly increasing function on $[a_k, b_k]$, with $g_k(a_k) = 0$. Suppose that there exists $k \in K$ such that $\min_i x_i \in (a_k, b_k)$, then

$$M^{(m)}(x_1,\ldots,x_j,x_{j+1},\ldots,x_m) = M^{(m)}(x_1,\ldots,x_j,b_k,\ldots,b_k)$$

if $x_1, \ldots, x_j \in (a_k, b_k)$ and $x_{j+1}, \ldots, x_m \in [b_k, b]$, $j \in \{1, \ldots, m\}, m \in \mathbb{N}_0$. Indeed, if $m = 2$, and if $x \in (a_k, b_k)$ and $y \in [b_k, b]$ then $M(x, y) = M(x, b_k)$. Suppose the result is true for $m - 1$ ($m \geq 3$) and also $x_1, \ldots, x_j \in (a_k, b_k)$ and $x_{j+1}, \ldots, x_m \in [b_k, b]$, $j \in \{1, \ldots, m\}$. So, from Lemma 4, we deduce

$$\begin{aligned} & M^{(m)}(x_1,\ldots,x_j,x_{j+1},\ldots,x_m) \\ = \; & M^{(m)}(M^{(m-1)}(x_2,\ldots,x_j,b_k,\ldots,b_k),\ldots,M^{(m-1)}(x_1,\ldots,x_j,b_k,\ldots,b_k)) \\ = \; & M^{(m)}(x_1,\ldots,x_j,b_k,\ldots,b_k) \end{aligned}$$

Thus, the result is still true for m.

Eventually we use induction to show that, if $x_1, \ldots, x_m \in (a_k, b_k)$, then

$$M(x_1,\ldots,x_m) = g_k^{-1}\sqrt[m]{\prod_i g_k(x_i)}.$$

■

Theorem 8 $M \in \mathcal{D}_{a,b,b}$ *if and only if*

- *either, for all* $m \in \mathbb{N}_0$,
$$M(x_1,\ldots,x_m) = \max_i x_i,$$

- *or, for all* $m \in \mathbb{N}_0$,
$$M(x_1,\ldots,x_m) = g^{-1}\sqrt[m]{\prod_i g(x_i)},$$
where g is any continuous strictly decreasing function on $[a, b]$, with $g(b) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a, b]$ such that, for all $m \in \mathbb{N}_0$,*

$$M(x_1,\ldots,x_m) = \begin{cases} g_k^{-1}\sqrt[m]{\prod_i g_k[\max(a_k,x_i)]} & \text{if there exists } k \in K \text{ such that} \\ & \max_i x_i \in (a_k,b_k), \\ \max_i x_i & \text{otherwise,} \end{cases}$$

where g_k is any continuous strictly decreasing function on $[a_k, b_k]$, with $g_k(b_k) = 0$.

Proof. Similar to the previous one. ■

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