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TORSION AND PURITY ON NON-INTEGRAL SCHEMES AND SINGULAR SHEAVES IN THE FINE SIMPSON MODULI SPACES OF ONE-DIMENSIONAL SHEAVES ON THE PROJECTIVE PLANE

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Standard notations	
Symbol	**Definition**
$\mathbb{N}$	natural numbers: $\{1, 2, \ldots\}$
$\mathbb{N}_0$	natural numbers including 0: $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{K}$	algebraically closed field of characteristic zero
$\mathbb{P}_2$	projective plane, defined by $\mathbb{P}(\mathbb{K}^3)$
$(x_0 : x_1 : x_2)$	homogeneous coordinates of a point in $\mathbb{P}_2$
$\bar{x}$ or $[x]$	equivalence classes in a quotient
X, Y, Z, X_i	formal variables in polynomial rings
${}^tA$	transpose of a matrix A
$\overline{U}$	closure of a set U in a topological space
$\mathrm{GL}_n(\mathbb{K})$	general linear group of invertible $n \times n$–matrices
$\mathrm{SL}_n(\mathbb{K})$	special linear group of matrices with determinant 1

for an R-module M		
Symbol	**Definition**	
$r * m$	multiplication of $m \in M$ by $r \in R$	
$N \leq M$	submodule of M	
$\langle m_1, \ldots, m_n \rangle$	submodule of M generated by $m_1, \ldots, m_n \in M$	
$\mathrm{Hom}_R(M, N)$	space of R-module homomorphisms $M \to N$	
$\widetilde{M}$	quasi-coherent sheaf on $\mathrm{Spec}\, R$ associated to M	p. 11
$\mathcal{T}_R(M)$	torsion submodule of M	p. 25
$\mathrm{Fitt}_0(M)$	0^{th} Fitting ideal of M	p. 31
$\mathrm{Sky}_x(M)$	skyscraper sheaf with value M at a point $x \in \mathcal{X}$	p. 161
$S^{-1}M$	localization of M with respect to a multiplicative	
	subset $S \subset R$	p. 336
M_P	localization of M at a prime ideal $P \trianglelefteq R$	p. 336
M_r	localized module at an element $r \in R$	p. 336
$\mathrm{Ann}_R(M)$	annihilator ideal of M	p. 349
$\mathrm{Ann}_R(m)$	annihilator ideal of an element $m \in M$	p. 349
$\mathrm{Ass}_R(M)$	set of associated primes of M	p. 363
$\mathrm{supp}\, M$	support of M (as a topological space)	p. 366
M^*	dual of M, defined by $\mathrm{Hom}_R(M, R)$	p. 374
$K_R(M)$	kernel of the canonical morphism $M \to M^{**}$	p. 379
M^I	direct product of copies of M by an index set I	p. 409
$M^{(I)}$	direct sum of copies of M by an index set I	p. 409

for a scheme $\mathcal{X}$		
Symbol	**Definition**	
$\dim \mathcal{X}$	dimension of $\mathcal{X}$ as a topological space	
$\operatorname{codim}_{\mathcal{X}}(Y)$	codimension of a closed subset $Y \subseteq \mathcal{X}$	
$(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ or $\mathcal{X}$	scheme	p. 5
$\mathcal{O}_{\mathcal{X}}$	structure sheaf of a scheme $\mathcal{X}$	p. 5
$\mathcal{O}_{\mathcal{X},x}$	stalk at $x \in \mathcal{X}$	p. 6
$\mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$	category of $\mathcal{O}_{\mathcal{X}}$–modules	p. 7
$\operatorname{Pic}(\mathcal{X})$	Picard group of $\mathcal{X}$	p. 10
$\mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$	category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$–modules	p. 11
$\mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$	category of coherent $\mathcal{O}_{\mathcal{X}}$–modules	p. 11
$\mathcal{K}_{\mathcal{X}}$	sheaf of meromorphic functions on $\mathcal{X}$	p. 79
$\mathcal{X}_s$	fiber over a closed point $s \in S$,	
	defined by $\mathcal{X} \times_S \operatorname{Spec} \kappa(s)$	p. 166
$\mathcal{X}_S$	fiber product of $\mathcal{X}$ and S over $\operatorname{Spec} \mathbb{K}$,	
	defined by $\mathcal{X} \times_{\mathbb{K}} S$	p. 166

for a (quasi-)coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{F}$		
Symbol	**Definition**	
$\operatorname{Hom}(\mathcal{F}, \mathcal{G})$	space of morphisms of sheaves $\mathcal{F} \to \mathcal{G}$	p. 7
$\mathcal{F}_x$	stalk of $\mathcal{F}$ at $x \in \mathcal{X}$	p. 8
$[s]_x$	germ of a section $s \in \mathcal{F}(U)$ at $x \in \mathcal{X}$	p. 8
s_x	given germ in the stalk $\mathcal{F}_x$	p. 8
$\Gamma(U, \mathcal{F})$	sections of $\mathcal{F}$ on an open subset $U \subseteq \mathcal{X}$,	
	defined by $\mathcal{F}(U)$	p. 8
$f_*\mathcal{F}$	direct image of $\mathcal{F}$ by a morphism $f : \mathcal{X} \to \mathcal{Y}$	p. 9
$f^*\mathcal{F}$	inverse image of $\mathcal{F}$ by a morphism $f : \mathcal{Y} \to \mathcal{X}$	p. 9

Symbol	Definition	
$\mathcal{H}om(\mathcal{F},\mathcal{G})$	internal Hom of two sheaves $\mathcal{F}$ and $\mathcal{G}$	p. 9
$\mathcal{F}\|_U$	restriction of $\mathcal{F}$ to an open subset $U \subseteq \mathcal{X}$	p. 9
$\mathcal{F}^*$	dual sheaf of $\mathcal{F}$, defined by $\mathcal{H}om(\mathcal{F},\mathcal{O}_\mathcal{X})$	p. 9
$\mathrm{Ext}^i(\mathcal{F},\,\cdot\,)$	i^{th} right derived functor of $\mathrm{Hom}(\mathcal{F},\,\cdot\,)$	p. 9
$\mathrm{Tor}_i(\mathcal{F},\,\cdot\,)$	i^{th} left derived functor of $\mathcal{F}\otimes\cdot$	p. 10
$H^i(\mathcal{X},\mathcal{F})$	i^{th} sheaf cohomology of $\mathcal{F}$	p. 10
$\mathrm{Fitt}_0(\mathcal{F})$	Fitting ideal sheaf of $\mathcal{F}$	p. 31
$\mathrm{supp}\,\mathcal{F}$	support of $\mathcal{F}$ (as a topological space)	p. 31
$\mathcal{Z}_a(\mathcal{F})$	annihilator support of $\mathcal{F}$	p. 32
$\mathcal{Z}_f(\mathcal{F})$	Fitting support of $\mathcal{F}$	p. 32
$\mathcal{T}(\mathcal{F})$	torsion subsheaf of $\mathcal{F}$	p. 51
$\mathrm{Ass}(\mathcal{F})$	set of associated points of $\mathcal{F}$	p. 90
$\dim\mathcal{F}$	dimension of $\mathrm{supp}\,\mathcal{F}$ as a topological space	p. 102
$T_i(\mathcal{F})$	subsheaf of $\mathcal{F}$ with sections supported	
	in dimension $\leq i$	p. 113
$\chi(\mathcal{X},\mathcal{F})$	Euler characteristic of $\mathcal{F}$	p. 159
$h^i(\mathcal{F})$	dimension of $H^i(\mathcal{X},\mathcal{F})$ as a $\mathbb{K}$-vector space	p. 159
$\mathcal{F}(m)$	twisted sheaf of $\mathcal{F}$ by $\mathcal{O}(m)$, $m\in\mathbb{Z}$	p. 159
$P_\mathcal{F}$	Hilbert polynomial of $\mathcal{F}$	p. 159
$\alpha_d(\mathcal{F})$	multiplicity of $\mathcal{F}$ for $\dim\mathcal{F}=d$	p. 162
$\mu(\mathcal{F})$	slope of $\mathcal{F}$	p. 162
$p_\mathcal{F}$	reduced Hilbert polynomial of $\mathcal{F}$	p. 162
$gr(\mathcal{F})$	graded sheaf of $\mathcal{F}$	p. 163
$\mathcal{F}\|_s$	restriction of $\mathcal{F}$ to the fiber $\mathcal{X}_s$	p. 166
$\mathcal{F}^{\mathrm{D}}$	dual sheaf of $\mathcal{F}$ when $\dim\mathcal{F}<\dim\mathcal{X}$	p. 185
$\mathcal{E}xt^i(\mathcal{F},\,\cdot\,)$	i^{th} right derived functor of $\mathcal{H}om(\mathcal{F},\,\cdot\,)$	p. 185

other notations		
Symbol	**Definition**	
$\mathbb{A}_n$	(classical) affine space of dimension n	p. 7
$\mathbb{A}^n_{\mathbb{K}}$	affine space of dimension n,	
	defined by $\operatorname{Spec}\mathbb{K}[X_1,\ldots,X_n]$	p. 7
$\mathbb{P}_n$	(classical) projective space of dimension n	p. 7
$\mathbb{P}^n_{\mathbb{K}}$	projective space of dimension n,	
	defined by $\operatorname{Proj}\mathbb{K}[X_1,\ldots,X_n]$	p. 7
$\mathcal{O}(1)$	very ample twisting sheaf (on a projective scheme)	p. 158
$\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(1)$	Serre's twisting sheaf on $\mathbb{P}^n_{\mathbb{K}}$	p. 158
$\mathtt{Sch}(\mathbb{K})$	category of Noetherian schemes of finite type	
	over $\operatorname{Spec}\mathbb{K}$	p. 165
$\mathtt{Sch}_{\mathtt{c}}(\mathbb{K})$	full subcategory of connected schemes in $\mathtt{Sch}(\mathbb{K})$	p. 168
$\mathcal{M}_P$	Simpson moduli functor with Hilbert	
	polynomial $P \in \mathbb{Q}[m]$	p. 169
$M_P(\mathcal{X})$	Simpson moduli space on $\mathcal{X}$ with Hilbert	
	polynomial $P \in \mathbb{Q}[m]$	p. 174
$M^s_P(\mathcal{X})$	open subscheme of isoclasses of stable sheaves	
	in $M_P(\mathcal{X})$	p. 174
$\mathcal{M}^s_P$	Simpson moduli functor for stable sheaves	p. 177
$\mathcal{O}_{\mathbb{P}_2}(1)$	Serre's twisting sheaf on $\mathbb{P}_2$	p. 179
$Z(f_1,\ldots,f_k)$	common vanishing set of (homogeneous)	
	polynomials in $\mathbb{A}_n$ or $\mathbb{P}_n$	p. 182
M_{am+b}	Simpson moduli space on $\mathbb{P}_2$ with Hilbert	
	polynomial $am+b$	p. 183
M^s_{am+b}	open subvariety of isoclasses of stable sheaves	
	in M_{am+b}	p. 183
$\langle f\rangle$	vector of coefficients of a polynomial f of	
	degree d	p. 189
$\mathcal{C}_d(\mathbb{P}_2) \cong \mathbb{P}_{N-1}$	Hilbert scheme of all curves in $\mathbb{P}_2$ of degree d,	
	$N = \binom{d+2}{2}$	p. 189

Symbol	Definition	
M'_{am+b}	closed subvariety of isoclasses of singular sheaves in M^s_{am+b}	p. 194
$\mathcal{T}_{\mathbb{P}_n}$	tangent sheaf on $\mathbb{P}_n$	p. 198
$\mathbb{P}_2^{[l]}$	Hilbert scheme of l points on $\mathbb{P}_2$, defined by $\mathrm{Hilb}^l(\mathbb{P}_2)$	p. 284
$\mathrm{Sing}(C)$	set of singular points of a curve C	p. 304
$\mathrm{Stab}_G(x)$	stabilizer of an element x under an action of a group G	p. 420
$O(x)$	orbit of x under a G-action	p. 420

Notations used in Chapter 5		
Symbol	**Definition**	
$n\,,\,d$	$d = n+1$, $n \geq 3$ fixed	p. 225
$\mathbb{W} \cong \mathbb{A}_w$	affine space of morphisms from (5.9)	p. 234
G'	non-reductive group of automorphisms acting on (5.8)	p. 235
Γ'	$\Gamma' = \left\{ (\lambda\,\mathrm{id}_n, \lambda\,\mathrm{id}_n) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G'$	p. 235
$\mathbb{P}G'$	$\mathbb{P}G' = G'/\Gamma'$	p. 235
$\mathbb{V} \cong \mathbb{A}_v$	affine space of Kronecker modules	p. 238
G	$G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$	p. 238
Γ	$\Gamma = \left\{ \left(\lambda\,\mathrm{id}_{n-1}\,,\,\lambda\,\mathrm{id}_n\right) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G$	p. 238
$\mathbb{P}G$	$\mathbb{P}G = G/\Gamma$	p. 238
$\mathbb{U}_2$	$\mathbb{U}_2 = n\,\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$	p. 239
$\mathbb{V}^s$	open subset of stable Kronecker modules in $\mathbb{V}$	p. 239
M	$M = M_{dm-1}$	p. 241
M'	$M' = M'_{dm-1}$	p. 241
M_0	open subvariety of sheaves in M without global sections	p. 241
M'_0	$M'_0 = M' \cap M_0$	p. 241
$\mathbb{W}_0$	open subvariety of $\mathbb{W}$ parametrizing stable sheaves in (5.8)	p. 241

Symbol	Definition	
w	$w = 3n(n+1)$	p. 242
v	$v = 3n(n-1)$	p. 243
$E_i(\lambda)$	elementary $n \times n$–matrix, $\lambda \in \mathbb{K}^*$, $i \in \{1, \ldots, n\}$	p. 247
$F_{ij}(\mu)$	elementary $n \times n$–matrix, $\mu \in \mathbb{K}$, $i, j \in \{1, \ldots, n\}$	p. 247
$\mathbb{V}_0$	open subvariety of Kronecker modules with coprime	
	maximal minors	p. 264
$\gamma : \mathbb{V}^s \to N$	geometric quotient: $N = \mathbb{V}^s / \mathbb{P}G$	p. 272
$\mathbb{U}_1$	$\mathbb{U}_1 = (n-1)\,\Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$	p. 272
F	morphism of vector bundles	
	$F \;:\; \mathbb{V} \times \mathbb{U}_1 \to \mathbb{V} \times \mathbb{U}_2 \;:\; (\Phi, L) \mapsto (\Phi, L \cdot \Phi)$	p. 273
$\mathbb{E}$	cokernel of F	p. 275
$\mathbb{P}\mathbb{E}$	projectivisation of $\mathbb{E}$	p. 280
$\nu : \mathbb{B} \to N$	projective bundle with fiber $\mathbb{P}_{3n+2}$	p. 282
$\mathbb{X}$	$\mathbb{X} = (\mathbb{V}^s \times \mathbb{U}_2) \setminus \operatorname{im} F$	p. 282
$\eta : \mathbb{X} \to \mathbb{B}$	geometric quotient: $\mathbb{B} = \mathbb{X} / \mathbb{P}G'$	p. 282
N_0	open subset of N given by the quotient of $\mathbb{V}_0$	p. 283
$\mathbb{B}_0$	$\mathbb{B}_0 = \mathbb{B}\vert_{N_0}$	p. 283
l	$l = \binom{n}{2} = \frac{n(n-1)}{2}$	p. 284
$H = \mathbb{P}_2^{[l]}$	Hilbert scheme of l points on $\mathbb{P}_2$	p. 283
$H_0 \cong N_0$	open subvariety of H of l points not lying on a	
	curve of degree $n-2$	p. 284
N_c	open subset of N_0 given by Kronecker modules whose	
	maximal minors define a configuration of l points	p. 304
$H_c \cong N_c$	open subvariety of H_0 of a configuration of l points	
	which do not lie on a curve of degree $d-3$	p. 304
$\mathbb{B}_c$	$\mathbb{B}_c = \mathbb{B}\vert_{N_c}$	p. 305
N_1	open subset of $N_0 \setminus N_c$ corresponding to $l-2$ simple	
	points and one double point	p. 310
$\mathbb{B}_1$	$\mathbb{B}_1 = \mathbb{B}\vert_{N_1}$	p. 310

Introduction

Abstract

This thesis consists of two individual parts, each one having an interest in itself, but which are also related to each other.
In Part I we analyze the general notions of the torsion of a module over a non-integral ring and the torsion of a sheaf on a non-integral scheme. We give an explicit definition of the torsion subsheaf of a quasi-coherent $\mathcal{O}_{\mathcal{X}}$–module and prove a condition under which it is also quasi-coherent. Using the associated primes of a module and the primary decomposition of ideals in Noetherian rings, we review the main criteria for torsion-freeness and purity of a sheaf that have been established by Grothendieck and Huybrechts-Lehn. These allow to study the relations between both concepts. It turns out that they are equivalent in "nice" situations, but they can be quite different as soon as the scheme does not have equidimensional components. We illustrate the main differences on various examples. We also discuss some properties of the restriction of a coherent sheaf to its annihilator and its Fitting support and finally prove that sheaves of pure dimension are torsion-free on their support, no matter which closed subscheme structure it is given.
Part II deals with the problem of determining "how many" sheaves in the fine Simpson moduli spaces $M = M_{dm-1}(\mathbb{P}_2)$ of stable sheaves on the projective plane $\mathbb{P}_2$ with linear Hilbert polynomial $dm - 1$ for $d \geq 4$ are not locally free on their support. Such sheaves are called singular and form a closed subvariety $M' \subset M$. Using results of Maican and Drézet, the open subset M_0 of sheaves in M without global sections may be identified with an open subvariety of a projective bundle

over a variety of Kronecker modules N. By the Theorem of Hilbert-Burch we can describe sheaves in an open subvariety of M_0 as twisted ideal sheaves of curves of degree d. In order to determine the singular ones, we look at ideals of points on planar curves. In the case of simple and fat curvilinear points, we characterize free ideals in terms of the absence of two coefficients in the polynomial defining the curve. This allows to show that a generic fiber of $M_0 \cap M'$ over N is a union of projective subspaces of codimension 2 and finally that M' is singular of codimension 2.

Motivation and main results

We say that a module over a (commutative unital) ring is torsion-free if its non-zero elements can only be annihilated by zero-divisors of the ring. This definition can be extended to sheaves on a locally Noetherian scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ by saying that a coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{F}$ is torsion-free if the stalks $\mathcal{F}_x$ are torsion-free modules over the local rings $\mathcal{O}_{\mathcal{X},x}$ for all $x \in \mathcal{X}$.
Another notion we are interest in is the so-called concept of purity. For this we recall that the support of a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, denoted by $\operatorname{supp} \mathcal{F}$, is the closed topological subspace of $\mathcal{X}$ defined by all points $x \in \mathcal{X}$ such that the stalk $\mathcal{F}_x$ is non-zero. Let d be the dimension of $\operatorname{supp} \mathcal{F}$ as a topological space. Then we say that $\mathcal{F}$ is pure of dimension d if the support of every non-zero proper coherent subsheaf of $\mathcal{F}$ also has dimension d.
As $d \leq \dim \mathcal{X}$, we may restrict $\mathcal{F}$ to its support and consider it as a sheaf on a d-dimensional space. For this it is however necessary to introduce a scheme structure on $\operatorname{supp} \mathcal{F}$ which defines a closed subscheme of $\mathcal{X}$. Here there is no canonical choice, but there are two structures which are by definition more relevant than the other ones. The annihilator support can be seen as the minimal closed subscheme since its structure sheaf is obtained by dividing out the functions that vanish on $\operatorname{supp} \mathcal{F}$. The Fitting support is defined via an ideal sheaf which is locally generated by the minors of a finite free presentation of $\mathcal{F}$ and thus encodes the relations between its local generators. These scheme-theoretic supports are denoted by $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$ respectively. In general $\mathcal{Z}_f(\mathcal{F})$ con-

tains $\mathcal{Z}_a(\mathcal{F})$ as a proper closed subscheme as it in addition takes care of the locally free resolution of $\mathcal{F}$. For the precise definitions of $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$ we refer to Section 1.4.1.

Our main goal of Part I is to show that every coherent sheaf $\mathcal{F}$ of pure dimension is a torsion-free sheaf on $\mathcal{Z}_f(\mathcal{F})$. The big problem occurring here is that the Fitting support is a scheme which is in general neither integral, nor reduced. The motivation for this study is to check torsion-freeness on the support of the sheaves in the Simpson moduli spaces, which will be defined in Part II.

Torsion-freeness being a local property, it suffices to prove the statement in the case of affine schemes. It is a well-known result from [35] that if R is a Noetherian ring, then there is a 1-to-1 correspondence between coherent sheaves on the affine scheme $\mathcal{X} = \operatorname{Spec} R$ and finitely generated modules over R. The bijection is denoted by

$$\operatorname{Mod}^f(R) \xrightarrow{\sim} \mathtt{Coh}(\mathcal{O}_\mathcal{X}) \; : \; M \longmapsto \widetilde{M} \, . \tag{1}$$

Now we can state our first main result.

Theorem 3.5.3. *Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring R and M be a finitely generated module over R. Assume that the coherent $\mathcal{O}_\mathcal{X}$–module $\mathcal{F} = \widetilde{M}$ is pure of dimension $d \leq \dim \mathcal{X}$. We denote $I = \operatorname{Fitt}_0(M)$ and $\mathcal{Z} = V(I) \cong \operatorname{Spec}(R/I)$. Then $\mathcal{F}$ is a torsion-free $\mathcal{O}_\mathcal{Z}$–module.*

This is a rather obvious result for integral schemes. Our achievement was to show that the statement actually remains true for **every** Noetherian ring ; it may be non-integral, non-reduced or even have embedded primes. We also point out in Proposition 3.5.1 and Remark 3.5.4 that the torsion-freeness remains true for every closed subscheme structure that the support may be endowed with.

Torsion naturally shows up in the context of modules and thus appears in almost every branch of Algebraic Geometry. For proving our theorem, it became necessary to study the notion of torsion of a module over a non-integral ring. While doing this we however experienced a lack of references in the literature. Classical textbooks on Commutative Algebra that are discussing torsion, such as Atiyah-MacDonald [2], Bourbaki [4] & [6], Eisenbud [16], Hartshorne [35] and Matsumura [54], only treat the case of modules over integral domains.

Our goal is to generalize or disprove some of the classical results about torsion in the case where the ring contains zero-divisors and/or nilpotent elements. Actually some statements have already been proven by various people, e.g. in [11] or [53], and are part of the mathematical folklore. This is where we point out the two important aspects of Part I of this thesis. On one hand it serves as a compilation from different sources of facts which are known but have not yet been written down in a concrete context ; on the other hand it provides a more deep understanding of torsion in general by improving and extending the classical theory.

Torsion of a module is mainly discussed in Chapter 1, but also in Appendix C. Alexander Grothendieck started to mention the torsion of a coherent sheaf on non-integral schemes in his last volume of EGA [33], but only developed it as a tool. For this it is necessary to understand the notion of the torsion subsheaf $\mathcal{T}(\mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on a scheme. We give an explicit description and the main properties of $\mathcal{T}(\mathcal{F})$ in Chapter 2. It turns out that quasi-coherence of $\mathcal{T}(\mathcal{F})$ is one of the main issues. Indeed we show

Theorem 2.2.8. *Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{X}$ given by $\mathcal{F} \cong \widetilde{M}$ for some R-module M. Then*

$$\mathcal{T}(\mathcal{F}) \textit{ is quasi-coherent } \Leftrightarrow \big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P) \ , \quad \forall\, P \in \operatorname{Spec} R \ .$$

If $\mathcal{F}$ is coherent, the same equivalence holds true with $\mathcal{T}(\mathcal{F})$ being coherent.

One of the main tools we are using in our computations are the associated primes $P_1, \ldots, P_\alpha$ of a Noetherian ring R. These are prime ideals $P_i \trianglelefteq R$ which can be written as $\operatorname{Ann}_R(r_i)$ for some $r_i \in R$. An associated prime is called embedded if it is not minimal. It turns out that embedded primes are sources of unpleasant problems. Their absence often has nice consequences, such as

Theorem 2.2.13. *Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F}$ a coherent, resp. quasi-coherent $\mathcal{O}_{\mathcal{X}}$–module. If R has no embedded primes, then the torsion subsheaf $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is coherent, resp. quasi-coherent.*

To prove this statement we use a result from Epstein-Yao [21], which allows to construct global non zero-divisors from local ones, and hence global torsion elements from local ones, if there are no embedded primes. On the other hand there are examples of torsion-free modules whose corresponding coherent sheaf is not torsion-free. Indeed in Section 2.3 we present the case of

$$R = \mathbb{K}[X,Y,Z]/\langle\, XY, X^2, XZ\,\rangle \qquad \text{with} \qquad M = R/\langle\, \bar{Y}\bar{Z}\,\rangle\ , \tag{2}$$

and show that the torsion subsheaf of $\mathcal{F} = \widetilde{M}$ has dense support in $\operatorname{supp}\mathcal{F}$. In particular $\mathcal{T}(\mathcal{F})$ is not coherent. Hence even though there is a 1-to-1 correspondence between modules over a ring and quasi-coherent sheaves on the corresponding affine scheme, there is in general no bijection as in (1) between torsion-free modules and torsion-free sheaves. However it holds true again under the assumption that the scheme has no embedded components ; this is the content of Corollary 2.2.22.
The torsion of a sheaf $\mathcal{F}$ is also related to the notion of a meromorphic function in the sense of Grothendieck. Using an alternative description of $\mathcal{T}(\mathcal{F})$, he proved a powerful criterion for torsion-freeness of a coherent sheaf on a Noetherian scheme $\mathcal{X}$ in [33] by only looking at the associated points of $\mathcal{X}$ and $\mathcal{F}$. We repeat that statement in Theorem 2.5.8. This result is one of the main tools we are going to use in order to prove Theorem 3.5.3.

Next we are interested in describing torsion in geometric terms. The relation between torsion and dimension is the leading idea of Chapter 3 and Section 1.4. Our main occupation is to check whether torsion is supported in smaller dimension. This is indeed satisfied in the coherent case.

Theorem 1.4.23. *Let M be a finitely generated module over a Noetherian ring R. Denote $\mathcal{F} = \widetilde{M}$, $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{X}_i = V(P_i)$ for all i, where $P_1, \dots, P_\alpha$ are the associated primes of R. Then M is a torsion module if and only if the codimension of $\operatorname{supp}\mathcal{F}$ is positive along each irreducible component:*

$$\operatorname{codim}_{\mathcal{X}_i}\big((\operatorname{supp}\mathcal{F}) \cap \mathcal{X}_i\big) \geq 1\ , \quad \forall\, i \in \{1, \dots, \alpha\}\ .$$

The example (2) however shows that Theorem 1.4.23 does not hold true if the torsion sheaf is not coherent. Questions about dimension immediately motivate us to speak about pure sheaves. A priori it is not clear how torsion and purity are related in the non-integral case. Here we obtain

Theorem 3.1.17. *Let $\mathcal{X} = \operatorname{Spec} R$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. Assume that $\dim \mathcal{F} = \dim \mathcal{X} = d$ and that $\mathcal{X}$ has equidimensional components. Then $\mathcal{F}$ is pure of dimension d if and only if $\mathcal{F}$ is torsion-free on $\mathcal{X}$.*

A criterion for purity (Theorem 3.1.11) has been stated by D. Huybrechts and M. Lehn in [38]. Similarly as the one of Grothendieck, it only uses the associated points of $\mathcal{X}$ and $\mathcal{F}$. In particular they illustrate that torsion-freeness of a sheaf depends on the considered ambient space while purity does not (Proposition 3.2.5). The unpleasant aspect of these criteria however is that the non-minimal associated primes of a module have in general no geometric interpretation. So we are looking for more "visual" criteria by considering the support of the sheaf as a new ambient space. It turns out this heavily depends on the chosen subscheme structure of the support ; for example there are fundamental differences between the annihilator support $\mathcal{Z}_a(\mathcal{F})$ and the Fitting support $\mathcal{Z}_f(\mathcal{F})$. For the first one we can say

Proposition 3.2.12. *Let $\mathcal{X} = \operatorname{Spec} R$ be affine and $\mathcal{F} \cong \widetilde{M}$ be coherent with $d = \dim \mathcal{F}$. If the annihilator support $\mathcal{Z}_a(\mathcal{F})$ has a component of dimension $< d$, then $\mathcal{F}$ is not pure.*

On the other hand this is only a partially satisfactory result since the converse does not hold true and a similar statement for the Fitting support does not exist at all ; in other words, the Fitting support of a pure sheaf may have embedded components (as e.g. illustrated in Example 3.4.18). On the other hand it turns out that the statement of Proposition 3.2.12 is sufficient in order to prove Theorem 3.5.3. Indeed we show in Proposition 3.5.1 that a coherent sheaf which is torsion-free on its support endowed with a scheme structure that has no embedded components is also torsion-free on its support when it is given any other subscheme structure. Thus torsion-freeness of a pure sheaf on its annihilator support is enough. Together with Theorem 3.1.17 this provides the proof of Theorem 3.5.3.

Finally we construct explicit counter-examples in Section 3.4.3 to illustrate that our intuition from Theorem 1.4.23 for a geometric interpretation of torsion is completely ruined if a scheme has embedded components and/or components of different dimensions. Moreover there is in general no clear relation between torsion-freeness of a sheaf on the different subscheme structures of its support. An overview of the important results is given in Table 1 on p. xxix as a list of implications.

After having developed the theory of non-integral torsion, we start Part II by discussing the Simpson moduli spaces. In his monumental and influential paper [65] from 1994 Carlos T. Simpson showed that for an arbitrary projective scheme $\mathcal{X}$ over an algebraically closed field $\mathbb{K}$ of characteristic zero and for an arbitrary numerical polynomial $P \in \mathbb{Q}[m]$ there is a coarse moduli space $M_P(\mathcal{X})$ of semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P, which turns out to be a projective scheme.
It is a well-known fact that there is no moduli space which classifies all coherent sheaves on a projective scheme. The way out of this problem is to introduce the notion of semistability. A coherent sheaf is called semistable if the ratio, Hilbert polynomial to its leading coefficient, is asymptotically greater or equal than the corresponding ratio for each non-zero proper coherent subsheaf (for the exact definition we refer to Section 4.1.2). If this inequality is even strict, we say that the sheaf is stable. Historically the moduli spaces of semistable sheaves have first been studied by D. Giesecker in [27] and M. Maruyama in [50] and [51]. However they both required semistable sheaves to be in addition torsion-free. Simpson generalized the definition of semistability by replacing the condition on torsion-freeness by purity and also proved existence of non-trivial moduli spaces in the case where $\deg P < \dim \mathcal{X}$. Indeed it is known from classical cohomology theory that sheaves with Hilbert polynomial P are supported in dimension $d = \deg P$, hence sheaves with $d < \dim \mathcal{X}$ cannot be torsion-free if $\mathcal{X}$ is e.g. integral.
For our work we restrict ourselves to the projective plane $\mathbb{P}_2$ over an algebraically closed field $\mathbb{K}$ and linear Hilbert polynomials $P(m) = am + b \in \mathbb{Z}[m]$ with integer coefficients and $a \geq 1$. The moduli spaces $M_{am+b}(\mathbb{P}_2)$ of 1-dimensional sheaves have been studied for a long time by many algebraic geometers in various contexts. J. Le Potier proved for example general properties such as smoothness, irreducibility and the dimension of the spaces in terms of a and b in [47].

M. Maican on the other hand established some isomorphisms in [49] which reduce the studies to finitely many values of b for a given a. Following the ideas of H.G. Freiermuth developed in [23] he also used Beilinson sequences in [48] and [15] together with J.-M. Drézet in order to decompose M with $a \leq 6$ into several strata, each of which can be described as a quotient of a certain space of matrices. Such descriptions have however not yet been established for $a > 6$ and it is not known how to characterize all semistable sheaves with a given Hilbert polynomial $am + b$.

Our interest will be the following. The sheaves in $M_{am+b}(\mathbb{P}_2)$ are supported on curves of degree a and are hence torsion sheaves on $\mathbb{P}_2$. But one may restrict them to their Fitting support and consider them as sheaves on a 1-dimensional variety. Part I ensures that these restrictions are torsion-free. It turns out that most of the sheaves are even locally free on their support. For proving this one proceeds as in [23] by looking at those whose support is smooth and applying the Structure Theorem of finitely generated modules over principal ideal domains, which implies that freeness and torsion-freeness of the stalks are equivalent. Hence "almost all" stable sheaves in $M_{am+b}(\mathbb{P}_2)$ can be seen as vector bundles on a curve.

In general the Simpson moduli spaces $M_{am+b}(\mathbb{P}_2)$ are not fine and their closed points are not in 1-to-1 correspondence with isomorphism classes of stable sheaves. However it is shown in [47] that this is the case for coprime values of a and b. So it is in particular satisfied for linear Hilbert polynomials of the form $dm - 1$ for some $d \in \mathbb{N}$. We denote $M := M_{dm-1}(\mathbb{P}_2)$. The sheaves that are locally free on their support constitute a dense open subvariety in M whose complement M', consisting of sheaves that are not vector bundles on their support, is in general non-empty. According to the vocabulary introduced by Le Potier in [47], sheaves from the boundary M' are called singular.

We are interested in describing M' and finding some interesting properties, such as smoothness, irreducibility and its codimension. In Section 4.5.5 and Section 4.6 we briefly review the cases for $d \leq 3$. The case of $3m + 1$ has been discussed by H.G. Freiermuth and G. Trautmann in [25] and a summary of this can be found in [41]. O. Iena showed in [39] that $\operatorname{codim}_M M' = 2$ for $4m-1$. Moreover it has been proven that M'_{3m+1} is a smooth and irreducible subvariety, whereas M'_{4m-1} is singular and path-connected. Our main result of Part II is the following generalization of [39], which has been obtained in joint work with Dr Oleksandr Iena.

Theorem 5.5.18. *For any integer $d \geq 4$, let $M = M_{dm-1}(\mathbb{P}_2)$ be the Simpson moduli space of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm-1$. If $M' \subset M$ denotes the closed subvariety of singular sheaves in M, then M' is singular and of codimension 2.*

Let $n = d-1$. The study of 1-dimensional stable sheaves is immediately related to the study of so-called Kronecker modules, which can be represented as $(n-1) \times n$–matrices with entries in linear forms. The affine space of Kronecker modules is denoted by $\mathbb{V}$. For these objects one can define the notion of stability in the abstract sense of Geometric Invariant Theory, developed by D. Mumford and J. Fogarty in [58]. General facts about GIT (which we recall in Appendix D.4) show that there exists a geometric quotient $N = \mathbb{V}^s/G$ of the open subset of stable Kronecker modules $\mathbb{V}^s$ by the reductive group of matrices $G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$. Drézet proved a concrete and easy-to-use characterization of the stability of Kronecker modules in [13]. Using this one we are going to show in Proposition 5.2.14 that Kronecker modules with linearly independent maximal minors are stable.

Maican has shown in [48] that sheaves $\mathcal{F} \in M$ satisfying $h^0(\mathcal{F}) = 0$ (i.e. sheaves without global sections) are exactly those which have a free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0\,, \tag{5.8}$$

where $A = \binom{Q}{\Phi}$ is a $n \times n$–matrix with stable Kronecker module $\Phi \in \mathbb{V}^s$ (see Theorem 5.1.28) and Q is a row vector of n quadratic forms. Moreover $\det A \neq 0$. We denote the quasi-affine variety of all such matrices by $\mathbb{W}_0$. Sheaves $\mathcal{F}$ as given in (5.8) form an open subset $M_0 \subseteq M$ and their isomorphism classes are obtained by dividing out the non-reductive group G' of automorphisms that is acting on the exact sequence. More precisely this gives a geometric quotient $M_0 = \mathbb{W}_0/G'$. By eliminating the action of the non-reductive part of G', Maican and Drézet constructed by descent in [48] a projective bundle $\mathbb{B}$ over N, which is also a geometric quotient by G'. This way M_0 may be seen as an open subvariety of $\mathbb{B}$. We will reproduce this construction in Section 5.3.

Next we restrict ourselves to stable Kronecker modules whose maximal minors are coprime. This open subset is denoted by $\mathbb{V}_0 \subseteq \mathbb{V}^s$. We also set $N_0 = \mathbb{V}_0/G'$ and $\mathbb{B}_0 = \mathbb{B}|_{N_0}$. It has been shown by Yuan in [70] that the codimension of the complement of $\mathbb{B}_0$ in M is at least 2. Hence in order to prove Theorem 5.5.18 it suffices to show that $\mathrm{codim}_{\mathbb{B}_0}(M' \cap \mathbb{B}_0) = 2$. This is especially useful since sheaves in $\mathbb{B}_0 \subseteq M_0$ can be described explicitly as twisted ideal sheaves of curves of degree d. More precisely, motivated by the corresponding results of Drézet and Maican in [14] and [15], we establish

Proposition 5.3.31. *The sheaves $\mathcal{F}$ in $\mathbb{B}_0$ are exactly the twisted ideal sheaves $\mathcal{I}_{Z\subseteq C}(d-3)$ given by a short exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(d-3) \longrightarrow \mathcal{O}_Z \longrightarrow 0 \ , \tag{5.27}$$

where $Z \subseteq C$ is a 0-dimensional subscheme of length $l = \binom{n}{2}$ lying on a curve C of degree d such that Z is not contained in a curve of degree $d-3$.

The proof is a slight variation of the ones in [14] and [15]. We apply the Theorem of Hilbert-Burch to a Kronecker module $\Phi \in \mathbb{V}_0$ and the 0-dimensional subscheme $Z \subset \mathbb{P}_2$ of length $l = \binom{n}{2}$ defined by the vanishing set of its coprime maximal minors $d_1, \ldots, d_n$. This way we show in Proposition 5.2.23 and Corollary 5.2.44 that we obtain an exact sequence

$$0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0 \ ,$$

where $\varphi = {}^t(d_1, \ldots, d_n)$ and the points of $Z = Z(d_1, \ldots, d_n)$ do not lie on a curve of degree $n-2$. Here we use ^{t} to denote the transpose, i.e. φ is a column vector.

Sequence (5.27) is our motivation for studying ideals of points in local rings. Indeed we shall determine under which conditions the ideal $\mathcal{F}_p$ for $p \in C$ is a free module over $\mathcal{O}_{C,p}$. This is true for smooth points and for $p \in C \setminus Z$. Thus $\mathcal{F}$ can only be singular at singular points of C which also lie in Z. In the case where $p \in Z$ is a simple point (i.e. a point of multiplicity 1), we notice the following elementary fact.

Lemma 5.4.1. *Let $R = \mathcal{O}_{C,p}$ be the local Noetherian ring of a curve $C \subset \mathbb{P}_2$ at a point $p \in C$ with unique maximal ideal $\mathfrak{M}$. Consider the exact sequence of R-modules*

$$0 \longrightarrow \mathfrak{M} \longrightarrow R \longrightarrow \Bbbk_p \longrightarrow 0 \ .$$

Then $\mathfrak{M}$ is free (of rank 1) if and only if R is regular, i.e. if and only if p is a smooth point of C.

In general one also has to take care of the multiplicity of p as a point in Z. This is where we establish the following characterization of free ideals over double points. Together with Lemma 5.4.1 it is the key point for proving Theorem 5.5.18.

Proposition 5.4.11. *Let $f \in \mathbb{K}[X, Y]$ be a non-constant polynomial defining a curve $C = Z(f)$ in $\mathbb{A}_2$. Assume that $p = (0, 0)$ is a singular point of C and denote $R = \mathcal{O}_{C,p}$. Let x, y denote the classes of X, Y in the local ring R. If $I = \langle\, x, y^2 \,\rangle \trianglelefteq R$ is the ideal defining the subscheme of a double point $\{p\} \hookrightarrow C$ by the exact sequence of R-modules*

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \ ,$$

then I is a free R-module (of rank 1) if and only if f contains the monomial Y^2.

To prove this we first notice that if I is free, then it is generated by one element because of the inclusion $I \hookrightarrow R$. As p is a singular point we also conclude that the order of f is at least 2. A proof by contradiction with straight-forward computations then implies that f must contain Y^2. Vice-versa if f contains Y^2, one shows that I is necessarily generated by x and that $R \to I : r \mapsto r \cdot x$ is an isomorphism of R-modules. The geometric interpretation of Proposition 5.4.11 is that I is free if and only if the tangent cone of C at p consists of 2 lines (with multiplicities) not containing the line $X = 0$.

Actually we even prove a characterization for all fat curvilinear points of multiplicity n in Proposition 5.4.17 ; by a "fat point" we mean a scheme whose underlying topological space only consists of one point and whose global sections form a $\mathbb{K}$-vector space of dimension n. On the other hand that one has no geometric interpretation, so at this point we only mention the two criteria above. It also turns out that the latter are actually sufficient in order to provide a proof of Theorem 5.5.18. We denote by $N_c \subseteq N_0$ the open subvariety given by Kronecker modules that define a configuration (i.e. l different points) and by $N_1 \subseteq N_0 \setminus N_c$ the subvariety of those where Z from (5.27) consists of one double point and $l-2$ simple points.

Lemma 5.4.1 and Proposition 5.4.11 characterize non-free ideals in terms of the absence of two coefficients in the polynomial that defines the curve (since R is not regular). Hence the subvariety of sheaves in $\mathbb{B}_0$ over $N_c \cup N_1$ that are singular at a given point $p \in Z$ is of codimension 2. A sheaf being singular if and only if there exists a point in Z at which it is singular, we combine the individual conditions at each point and obtain in Corollary 5.5.8 and Corollary 5.5.15 that the fibers of $\mathbb{B}_0 \cap M'$ over N_c, resp. N_1 are unions of l, resp. $l-1$ different linear subspaces of codimension 2.

In particular this shows that M' is singular. Finally we also compute the singular locus of M' over the space of configurations $N_c \subseteq N_0$ in Proposition 5.5.22. Here we find that the smooth points are those sheaves which are singular at only one of the points in Z.

Structure of the dissertation

Throughout this thesis R is used to denote a commutative unital ring (usually also assumed to be Noetherian) and $\mathbb{K}$ is a fixed algebraically closed field of characteristic zero, for example $\mathbb{K} = \mathbb{C}$. An exhaustive list with our notations, including a reference to the page on which they are introduced, is given on the pages v – xii after the table of contents.

We assume the reader to be familiar with basic Commutative Algebra of rings and modules, Scheme Theory, Categories, Functors and Abelian Sheaf Theory.

Outcomes of this thesis to which the author has contributed are indicated by adding the surname (Leytem) on top of the statement. If a result is taken from another source, the precise reference is added afterwards. If the author incorporated new elements to an already established result (e.g. by modifying the statement or changing the assumptions), the shortcut "cf." is added before the reference. Assertions without a caption are either direct consequences of preceding results or auxiliary facts for which a proof had to be provided.

Part I

Torsion on non-integral schemes and relations with purity

Throughout Part I we illustrate a lot of concepts, statements and algorithms on several instructive examples, which recurrently occur at many places. They are denoted by E.1 – E.7 and summarized in Appendix E. When studying these examples, we suggest the reader to simultaneously look at the summary in the appendix for a better visualization and understanding, especially since we don't recall the notations at each time.

In **Chapter 1** we study the relation between the irreducible components of an affine scheme $\operatorname{Spec} R$ and the torsion submodule $\mathcal{T}_R(M)$ of an R-module M.

Section 1.1 contains short reviews of the basics of schemes and sheaves of $\mathcal{O}_{\mathcal{X}}$–modules. We define coherence and the standard operations on sheaves. Theorem 1.1.13 illustrates the relation between coherent sheaves on affine Noetherian schemes and finitely generated modules over the ring of global sections.

Section 1.2 uses the theory of Primary Ideal Decomposition explained in Appendix B.2 to decompose a scheme into irreducible components. We illustrate such a decomposition on an example and point out the problems occurring if we want to look at non-reduced structures. In the same way we explain how to find the connected components of the scheme and discuss the difference between the product and the intersection of two ideals.

Section 1.3 gives characterizations of torsion-free modules (Proposition 1.3.3) and torsion modules (Proposition 1.3.5) in terms of the associated primes of the ring. Moreover we study the behaviour of torsion under localization in Proposition 1.3.8 and prove that an element is torsion if and only if it is a torsion element in all localizations.

Section 1.4 discusses the fact that torsion modules are supported in smaller dimension. We start by showing in Proposition 1.4.4 that the support of a coherent sheaf $\mathcal{F}$ is closed. Then we define its annihilator support $\mathcal{Z}_a(\mathcal{F})$ and the Fitting support $\mathcal{Z}_f(\mathcal{F})$. We also recall some facts about dimensions in rings and state Krull's Height Theorem. In Proposition 1.4.21 and Theorem 1.4.23 it is then shown that the codimension of the support of a coherent torsion sheaf is positive in each irreducible component of the scheme. Finally we illustrate this result on some examples.

Chapter 2 is entirely dedicated to the torsion subsheaf of a quasi-coherent sheaf $\mathcal{F}$ on a locally Noetherian scheme $\mathcal{X}$. We are particularly interested in what it means for $\mathcal{F}$ to be torsion-free.

In Section 2.1 we give a detailed definition of the torsion subsheaf $\mathcal{T}(\mathcal{F})$ and compute its sections and stalks (Proposition 2.1.12 and Proposition 2.1.17).

The aim of Section 2.2 is to determine under which conditions the torsion subsheaf is quasi-coherent. Theorem 2.2.8 says that this is the case if and only if all local torsion elements come from global ones. Using a result from Epstein-Yao [21] we then show in Theorem 2.2.13 that this condition is satisfied if the ring has no embedded primes. As most of the proofs are constructive, we also apply the methods to a concrete example. Finally we briefly explain the relation between embedded primes and Serre's conditions in Proposition 2.2.28.

In Section 2.3 we present an example which illustrates that the torsion subsheaf does not need to be coherent and may even have dense support. This heavily contradicts our idea of a geometric interpretation of torsion from Theorem 1.4.23. In particular it is also an example of a non-coherent subsheaf of a coherent sheaf on an affine Noetherian scheme.

Section 2.4 analyzes the definition of the sheaf $\mathcal{K}_{\mathcal{X}}$ of meromorphic functions on a Noetherian scheme following the ideas of Kleiman in [43]. We compute its sections (Proposition 2.4.14), its stalks (Proposition 2.4.16) and improve a result from Murfet [59] to show that $\mathcal{K}_{\mathcal{X}}$ is quasi-coherent when there are no embedded primes (Theorem 2.4.19). We also state the relation between torsion and meromorphic functions in Theorem 2.4.22.

The main result of Section 2.5 is Grothendieck's criterion for torsion-freeness of a sheaf, which claims that a quasi-coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{F}$ on a locally Noetherian scheme $\mathcal{X}$ is torsion-free if and only if all associated points of $\mathcal{F}$ are associated points of $\mathcal{X}$.

Finally in Section 2.6 we provide an alternative proof of the fact that the dimension of the support of a sheaf given by a torsion module drops in all components of the scheme. The proof being constructive we again apply it to an example for better illustration.

Chapter 3 compares the notions of torsion-freeness and purity. Moreover we point out the main differences between the supports $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$ and prove that pure sheaves are torsion-free on their Fitting support.

In Section 3.1 we define the concept of a pure sheaf and state the criterion of Huybrechts-Lehn (Theorem 3.1.11). Then we show in Theorem 3.1.17 that a sheaf $\mathcal{F}$ on a scheme $\mathcal{X}$ with equidimensional components of dimension d and $\dim \mathcal{F} = \dim \mathcal{X} = d$ is torsion-free if and only if it is pure. These assumptions moreover give a characterization of torsion modules in terms of the dimension of their support (Corollary 3.1.25). We also study the relation between the torsion subsheaf and the torsion filtration; here Proposition 3.1.33 claims that, still under these assumptions, the torsion subsheaf $\mathcal{T}(\mathcal{F})$ is equal to the subsheaf $T_{d-1}(\mathcal{F})$ of sections that are supported in smaller dimension. But this equality may fail if there are embedded components.

Section 3.2 explains how an $\mathcal{O}_{\mathcal{X}}$–module can be considered as a sheaf on its support. In particular we see in Proposition 3.2.2 that the associated primes do not change and that the notion of being pure does not depend on the "ambient space" (Proposition 3.2.5). Torsion-freeness on the other hand does, as can easily be seen by Example 3.2.8. We also show that the annihilator support of a pure sheaf has equidimensional components (Proposition 3.2.12), but the converse is false. Finally Corollary 3.2.23 gives a condition under which sheaves that are torsion-free on a reduced scheme $\mathcal{X}$ are torsion-free on their support. However this statement may fail in the non-reduced case.

In Section 3.3 we review the examples E.3 – E.7 and analyze them for torsion-freeness and purity. Inspired by these results we prove in Proposition 3.3.9 that whenever a component has torsion, this torsion generically remains on the maximal ideals. More precisely, if P is a prime ideal such that M_P has torsion, then $M_{\mathfrak{M}}$ also has torsion for almost all maximal ideals $\mathfrak{M}$ containing P. Vice-versa, if M_P is torsion-free, then $M_{\mathfrak{M}}$ is also torsion-free for a generic $\mathfrak{M} \in V(P)$.

The main goal of Section 3.4 is to compare the properties of the annihilator support and the Fitting support of a sheaf $\mathcal{F}$. Both have the same minimal components (Lemma 3.4.1), but $\mathcal{Z}_a(\mathcal{F})$ has in general "nicer" properties, in the sense that $\mathrm{Ann}_R(M)$ is more closely related to $\mathrm{Ass}_R(M)$ and we have effective results such as Proposition 3.2.12 and Proposition 3.4.6. The Fitting support of a pure sheaf may for example have embedded components (Example 3.4.18). On the other hand one generally prefers $\mathcal{Z}_f(\mathcal{F})$ because of functoriality (Proposition 3.4.13). Then we give many examples to illustrate that there is no clear relation between torsion-freeness on $\mathcal{Z}_a(\mathcal{F})$ and torsion-freeness on $\mathcal{Z}_f(\mathcal{F})$ as soon as one of them has embedded components.

Finally we prove our first main result in Section 3.5. For this we first prove Proposition 3.5.1, which states that a sheaf which is torsion-free on its support $\mathcal{Z}$ with a subscheme structure that has no embedded components is torsion-free on all possible subscheme structures of $\mathcal{Z}$. Together with Proposition 3.2.12 and Theorem 3.1.17, this provides the proof of Theorem 3.5.3. After this we formulate a series of remaining open questions.

All the results from Part I are summarized in Table 1 below. We always ask the question

$$\text{assumptions} \quad \overset{?}{\Rightarrow} \quad \text{statement}$$

If the answer is Yes, we refer to the statement where it is proven. In the negative case, we refer to a counter-example. If we want to impose an additional assumption, it is added below. A question mark indicates that we haven't found an answer yet.

Table 1: (Non-exhaustive) List of implications

Let R be a Noetherian ring, $\mathcal{X} = \operatorname{Spec} R$ the corresponding affine Noetherian scheme, $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$, $\mathcal{Z}_a$ its annihilator support and $\mathcal{Z}_f$ its Fitting support. If we write $\mathcal{Z}$, the subscheme structure of the support is not specified.

Question	**Answer**	**Reference**
$\mathcal{F}$ torsion on $\mathcal{X} \ \Rightarrow\ \dim \mathcal{F} < \dim \mathcal{X}$	Yes	1.4.21
$\dim \mathcal{F} < \dim \mathcal{X} \ \Rightarrow\ \mathcal{F}$ torsion on $\mathcal{X}$	No	1.4.25 & 1.4.26
if $\mathcal{X}$ equidimensional	Yes	3.1.25
$\mathcal{T}(\mathcal{F})$ coherent	No	2.3.1
if $\mathcal{X}$ no embedded	Yes	2.2.13
$\mathcal{T}(\mathcal{F}) \neq 0$ coherent $\ \Rightarrow\ \mathcal{X}$ no embedded	No	3.3.5
$\mathcal{F}$ pure $\ \Rightarrow\ \mathcal{F}$ torsion-free on $\mathcal{X}$	No	3.3.3
if $\mathcal{X}$ integral	No	3.2.8
if $\mathcal{X}$ no embedded and $\dim \mathcal{F} = \dim \mathcal{X}$	Yes	3.1.17
$\mathcal{F}$ torsion-free on $\mathcal{X} \ \Rightarrow\ \mathcal{F}$ pure	No	3.1.19
if $\mathcal{X}$ equidimensional	Yes	3.1.17 & 3.1.25
$\dim \mathcal{F} = \dim \mathcal{X} \ \Rightarrow\ \mathcal{Z} = \mathcal{X}$ (as schemes)	No	3.1.21
if $\mathcal{X}$ is irreducible or reduced	No	3.1.21 & 3.1.22
if $\mathcal{X}$ is integral	Yes	3.1.23
$\mathcal{F}$ pure on $\mathcal{X} \ \Leftrightarrow\ \mathcal{F}$ pure on $\mathcal{Z}$	Yes	3.2.5

Question	**Answer**	**Reference**
$\mathcal{F}$ pure $\Rightarrow$ $\mathcal{Z}_a$ equidimensional	Yes	3.2.12
$\mathcal{Z}_a$ equidimensional $\Rightarrow$ $\mathcal{F}$ pure	No	3.2.17
$\mathcal{F}$ torsion-free on $\mathcal{X}$ $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{Z}_a$	No	3.2.24
if $\mathcal{X}$ reduced and $\mathcal{Z}_a$ no embedded	Yes	3.2.23
$\mathcal{F}$ torsion-free on $\mathcal{Z}_a$ $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{Z}_f$	No	3.4.23
if $\mathcal{Z}_a$ no embedded	Yes	3.5.1
$\mathcal{F}$ torsion-free on $\mathcal{Z}_f$ $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{Z}_a$	No	3.4.25
if $\mathcal{Z}_f$ no embedded	Yes	3.5.1
$\mathcal{F}$ pure $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{Z}$	Yes	3.5.3 & 3.5.4
$\mathcal{F}$ torsion-free on $\mathcal{Z}_a$ $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{X}$	No	3.2.8
if $\dim \mathcal{F} = \dim \mathcal{X}$	No	3.4.23
$\mathcal{X}$ equidimensional $\Rightarrow$ $\mathcal{F}$ pure	No	3.2.17
if $\dim \mathcal{F} = \dim \mathcal{X}$	No	3.2.19
$\mathcal{F}$ pure $\Rightarrow$ $\mathcal{X}$ equidimensional	No	3.3.4
$\mathcal{F}$ torsion-free on $\mathcal{X}$ $\Rightarrow$ $\mathcal{X}$ equidimensional	No	3.3.4
$\mathcal{F}$ torsion-free on $\mathcal{Z}$ $\Rightarrow$ $\mathcal{Z}$ equidimensional	No	3.4.29
$\mathcal{Z}_a$ equidimensional $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{Z}_a$	No	3.4.25
$\mathcal{Z}_a$ no embedded $\Rightarrow$ $\mathcal{Z}_f$ no embedded	No	3.4.18 & 3.4.25
if $\mathcal{X}$ integral	No	3.4.21
$\mathcal{Z}_f$ no embedded $\Rightarrow$ $\mathcal{Z}_a$ no embedded	No	3.4.27
$\mathcal{F}$ torsion-free on $\mathcal{Z}_f$ $\Rightarrow$ $\mathcal{F}$ torsion-free on $\mathcal{X}$	No	3.2.8
if $\dim \mathcal{F} = \dim \mathcal{X}$	?	
$\mathcal{F}$ pure $\Rightarrow$ $\mathcal{Z}_f$ equidimensional	No	3.4.18
if $\mathcal{X}$ reduced	?	

Part II

Singular sheaves in the fine Simpson moduli spaces of 1-dimensional sheaves

Chapter 4 is a reminder of the construction and properties of the Simpson moduli spaces $M_P(\mathcal{X})$ of semistable sheaves on a projective scheme $\mathcal{X}$ with fixed Hilbert polynomial P. We do this by reviewing classical results from Simpson [65], Maican [48] and Le Potier [47]. Reproving some results of Freiermuth and Trautmann in [25] we also describe the moduli spaces M_{am+b} for $a \leq 3$.

Section 4.1 deals with the necessary preliminaries which we will need for the rest of the studies, such as Hilbert polynomials, semistability, s-equivalence and flatness.

In Section 4.2 we define the Simpson moduli functor $\mathcal{M}_P$ and explain the concepts of representability, fine and coarse moduli spaces. Then we state Simpson's Theorem which claims the existence of a projective scheme $M_P(\mathcal{X})$ which corepresents the functor and briefly explain its construction. Moreover there exists an open subscheme $M_P^s(\mathcal{X})$ whose closed points parametrize isomorphism classes of stable sheaves in $M_P(\mathcal{X})$. Finally we illustrate that there cannot exist a fine moduli space when there are properly semistable sheaves with Hilbert polynomial P.

The aim of Section 4.3 is to give an overview of the properties of the moduli spaces M_{am+b} of semistable sheaves on the projective plane with linear Hilbert polynomial. In particular we see in Corollary 4.3.8 that we obtain a fine moduli space for coprime a and b. More advanced results are the Theorem of Le Potier, which gives information about the dimension, irreducibility and smoothness, and the Duality Theorems of Maican, which gives isomorphisms $M_{am+b} \cong M_{am+a+b}$ and $M_{am+b} \cong M_{am-b}$. This restricts the studies to finitely many values of $b \in \mathbb{Z}$ for fixed a.

In Section 4.4 we apply our results from Part I to show that semistable sheaves are torsion-free on their support (Proposition 4.4.1). Then we prove in Proposition 4.4.5 that the support of a sheaf with Hilbert polynomial $am+b$ is a curve of degree a. The final results are Proposition 4.4.16 and Corollary 4.4.21, which state that stable sheaves in M_{am+b} with smooth support are locally free on their support (i.e. non-singular) and hence that a generic sheaf in M_{am+b} is a vector bundle over a smooth curve of degree a. For this one uses that the set of smooth curves (of degree d) is open and dense in the Hilbert schemes of all curves (of degree d) on $\mathbb{P}_2$.

Section 4.5 starts by introducing syzygies and explains how they can be used in order to compute global resolutions of coherent sheaves on $\mathbb{P}_2$. We compute the cokernel of a morphism between direct sums of line bundles (Proposition 4.5.9) and use this in Proposition 4.5.14 to prove that a generic $\mathcal{F} \in M^s_{am+b}$ is a locally free sheaf of rank 1 on its support. Then we reprove that $M_{m+1} \cong \mathbb{P}_2$ and $M_{2m+1} \cong \mathbb{P}_5$. Moreover all of such sheaves are isomorphic to the structure sheaf of their support, hence there are no singular sheaves in both cases.

In Section 4.6 we review the case of the fine moduli space M_{3m+1}. Sheaves $\mathcal{F} \in M_{3m+1}$ can be described by an exact sequence (Proposition 4.6.4) which is used to obtain a criterion for $\mathcal{F}$ to be singular in Proposition 4.6.10. We also illustrate how a non-reductive group of matrices G acts on this exact sequence and that M_{3m+1} is a geometric quotient of a quasi-affine parameter space $X \subset \mathbb{A}_{18}$ by the group G (Theorem 4.6.15). Finally we recall in Theorem 4.6.17 that M_{3m+1} is isomorphic to the universal cubic curve on $\mathbb{P}_2$ and obtain in Proposition 4.6.21 that the subset of singular sheaves M'_{3m+1} is isomorphic to its universal singular locus, which is smooth, irreducible and of codimension 2.

In **Chapter 5** we are going to prove our second main result, which states that the subvariety M' of singular sheaves in $M = M_{dm-1}$ for $d \geq 4$ is singular and of codimension 2.

In Section 5.1 we are describing sheaves in a dense open subset $M_0 \subseteq M$ to which we are going to restrict our computations in the following. For this we define the affine space of Kronecker modules $\mathbb{V}$ and state the characterization of stable sheaves that has been established by Drézet in [13]. In Theorem 5.1.28 we review that sheaves in M without global sections are exactly those of the resolution (5.8) and given as cokernels of injective matrices containing a stable Kronecker module. The parameter space of such morphisms, denoted by $\mathbb{W}_0$, is acted on by a non-reductive group G' of automorphisms. Using that there is a 1-to-1 correspondence between isomorphism classes of sheaves given by (5.8) and the orbits of the G'-action (Corollary 5.1.15), Maican then has shown in [48] that M_0 is a geometric quotient of $\mathbb{W}_0$ by G'.

Section 5.2 is dedicated to determining properties of the maximal minors of a Kronecker module. In particular we prove a formula in Proposition 5.2.5 that describes how these minors change under linear transformations of the rows and columns. This one allows to show in Proposition 5.2.14 that Kronecker modules with linearly independent maximal minors are stable. Next we apply the Theorem of Hilbert-Burch to a Kronecker module Φ with coprime maximal minors $d_1, \ldots, d_n$ in order to obtain a resolution of the structure sheaf of the 0-dimensional subscheme $Z = Z(d_1, \ldots, d_n)$ defined by the common vanishing set of $d_1, \ldots, d_n$ (Proposition 5.2.23). Moreover it is shown in Corollary 5.2.44 that Z is of length $l = \binom{n}{2}$ and does not lie on a curve of degree $n-2$.

In Section 5.3 we reproduce Maican's proof from [48] in order to eliminate the action of the non-reductive part of G' on $\mathbb{W}_0$. This way one obtains by descent a projective bundle $\mathbb{B} \to N$ with fiber $\mathbb{P}_{3d-1}$ (Proposition 5.3.22), where N is a geometric quotient of $\mathbb{V}^s$. Using that $\mathbb{B}$ is also a geometric quotient by the group G', we may see M_0 as an open subvariety of $\mathbb{B}$ (Proposition 5.3.24 and Corollary 5.3.27). Next we restrict the bundle to $\mathbb{B}_0 = \mathbb{B}|_{N_0}$, where the subset $N_0 \subseteq N$ is given by Kronecker modules with coprime maximal minors. Proposition 5.3.30 gives an inclusion of open sets $\mathbb{B}_0 \subseteq M_0$. In Proposition 5.3.31 we finally use Hilbert-Burch and the Snake Lemma to describe sheaves in $\mathbb{B}_0$ as twisted ideals sheaves of curves of degree d in (5.27).

The sequence (5.27) is the main motivation of Section 5.4. First we prove a characterization of free ideals of a simple point on a curve C in Lemma 5.4.1, which claims that the maximal ideal of a local ring R is free if and only if R is a regular local ring. After this we consider the case of the ideal of a double point at the origin given by $\langle X, Y^2 \rangle$. In Proposition 5.4.11 we show that it is equivalent to say that such an ideal is free, that the homogeneous polynomial defining the curve C contains the monomial Y^2 and that the tangent cone of C at the origin does not contain the line $X = 0$. In Proposition 5.4.17 we also consider the case of a fat curvilinear point of multiplicity n. Here a similar statement holds true, but it has no geometric interpretation.

Section 5.5 finally provides the proof of Theorem 5.5.18. We start by observing that sheaves in $\mathbb{B}_0$ can only be singular at singular points of C which simultaneously lie in Z. Then we distinguish the studies according to the constellation of the l points in Z. For a configuration $N_c \subseteq N_0$, we obtain in Proposition 5.5.6 that the fibers of $\mathbb{B}_0$ over N_c are unions of l different projective subspaces of $\mathbb{P}_{3d-1}$ of codimension 2. Similarly we obtain in Proposition 5.5.13 that the sheaves which are singular at a double point also constitute a closed linear projective subspace of codimension 2. After this we compute the smooth locus of M' over N_c in Proposition 5.5.22.

To close the thesis we present some examples (with explicit computations) in Section 5.6 which illustrate inter alia that there exist stable Kronecker modules with linearly dependent maximal minors (Example 5.6.1) and that a stable sheaf may even be non-singular at a double point which is a singular point of the support (Example 5.6.3). We also give a quick interpretation of our main result and explain how the study of M can be applied in other research fields. Finally we again formulate a few open questions for future research.

Appendices

Appendix A is a summary of basic facts of localization of rings and modules. In particular we explain functoriality and exactness of the localization and analyze under which conditions it commutes with the Hom-functor. Moreover we study some local properties.

In Appendix B we develop the theory of Primary Ideal Decomposition in Noetherian rings. We state the Prime Avoidance Lemma and prove that the set of zero-divisors and nilpotent elements of a ring can be described by means of the associated primes $P_1, \ldots, P_\alpha$ of the zero ideal. We also analyze how the associated primes, which can be written as annihilator ideals, behave under localization. In the second part we define the associated primes of a module M over a ring and study their relation with the support of M.

Appendix C contains complements on torsion and modules over a ring. First we define what it means for a module to be torsion-free, resp. torsionless and explain that both notions are in general not equivalent. Then we state some properties of reflexive and projective modules. Finally we study the case of integral domains, over which torsion-freeness and torsionlessness of finitely generated modules are equivalent. We also give a characterization of reflexive modules in that case and illustrate that this one does not hold true in the non-integral case.

Appendix D consists of 4 completely independent parts. In D.1 we collect several well-known results from Commutative Algebra that we are using throughout the thesis. This way the reader can immediately look up the exact statement. D.2 develops some facts about intersection of ideals, while D.3 illustrates an application of essential ideals to prove torsion-freeness of a module on its support. Finally D.4 contains basic facts and definitions of GIT which we are mostly using for constructing quotients in Chapter 5.

In Appendix E we summarize the examples E.1 – E.7 from Part I. We write down the primary decompositions of the ideals, the associated primes of the modules and the decompositions of $\mathcal{X}$ into (possibly non-reduced) irreducible components. For a better visualization we also provide figures of the schemes we are working with.

Appendix F illustrates what happens in case of the moduli space M_{2m+2}, which is not fine. Theorem F.1.14 states that s-equivalence classes of sheaves in M_{2m+2} can be identified with their support and that the singular sheaves are those which correspond to reducible conics. This allows to show in Corollary F.2.6 that the subset of singular sheaves M'_{2m+2} is singular and of codimension one. Here one however has to use a modified definition for a sheaf to be singular since Example F.1.7 shows that s-equivalence classes may simultaneously contain singular and non-singular sheaves.

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Part I

Torsion on non-integral schemes and relations with purity

Chapter 1

Torsion and irreducible components

The first aim of this chapter is to find a natural method for decomposing an affine reducible scheme into finitely many irreducible components. The components of this decomposition should in particular take care of the topology and of the dimension of the scheme. Moreover they should encode the multiple structure if the scheme is not reduced. The main tool for doing this will be Primary Ideal Decomposition in Noetherian rings. In particular we can apply such a decomposition to the support of a coherent sheaf.
By using the associated primes we then give characterizations of torsion-free modules (Proposition 1.3.3) and torsion modules (Proposition 1.3.5). These will be useful in order to obtain a geometric description of torsion. Indeed it turns out that the torsion of the module is related to the dimension of the components of its support. This result is the content of Proposition 1.4.21 and Theorem 1.4.23, in which we show that the codimension of the support of a coherent torsion sheaf is positive in each irreducible component of the scheme.

1.1 Reminder on schemes and $\mathcal{O}_{\mathcal{X}}$–modules

We start by summarizing some foundations of schemes and sheaves of modules in general. The aim of this section is just to recall the main results and to fix the most important notations as we will use them all the time later on. Most of the statements are given without further explanations. The advanced reader may immediately skip to Section 1.2. First we refer to Appendix A for some basic facts on localizations of rings and modules.

1.1.1 Schemes as locally ringed spaces

First we are going to explain the notions of the spectrum of a ring, its Zariski topology and its structure sheaf, turning it into a locally ringed space. We also explain the correspondence between closed subschemes of the spectrum and quotients of the ring. Good references on this topic are e.g. Hartshorne [35], Section II.2 and [11], Chapter 10.16.

The *spectrum* of a ring R is defined as the set of all prime ideals in R and denoted by $\operatorname{Spec} R$. If $I \trianglelefteq R$ is an ideal we also denote by $V(I)$ the subset of all prime ideals containing I. It has the following properties: if $J \trianglelefteq R$ is another ideal and $\{I_i\}_i$ is a family of ideals, then $V(I) = V\big(\operatorname{Rad}(I)\big)$,

$$V(I) \cup V(J) = V(I \cdot J) = V(I \cap J) \qquad \text{and} \qquad \bigcap_i V(I_i) = V\big(\textstyle\sum_i I_i\big) \ .$$

Moreover $V(I) \subseteq V(J)$ if and only if $\operatorname{Rad}(J) \subseteq \operatorname{Rad}(I)$. This allows to define the *Zariski topology* on $\operatorname{Spec} R$ by saying that the closed sets are those that are of the form $V(I)$ for some $I \trianglelefteq R$. The closure in this topology of a point $P \in \operatorname{Spec} R$ is

$$\overline{\{P\}} = V(P) \ .$$

So the closed points of $\operatorname{Spec} R$ are exactly the maximal ideals of R. The dimension of $\operatorname{Spec} R$ as a topological space is equal to the Krull dimension of the ring R. For $r \in R$, we define the *distinguished* open set $D(r)$ to be the set of all prime ideals in R which do not contain r. They satisfy $D(r) \cap D(s) = D(rs)$, $\forall\, r, s \in R$, hence open sets of the form $D(r)$ form a basis for the Zariski topology on $\operatorname{Spec} R$. Moreover they allow to prove that $\operatorname{Spec} R$ is a compact[1] Kolmogorov space.

[1]Some authors say that $\operatorname{Spec} R$ is only quasi-compact as they require compact spaces in addition to be Hausdorff, which is usually not the case for spectra. We do not adopt this convention.

If $\varphi : R \to T$ is a ring homomorphism, it induces a continuous map

$$\phi \ : \ \operatorname{Spec} T \longrightarrow \operatorname{Spec} R \ : \ Q \longmapsto \varphi^{-1}(Q) \ .$$

Let $I \trianglelefteq R$. Using that there is a 1-to-1 correspondence between ideals in the quotient R/I and ideals in R containing I, one also gets a homeomorphism of topological spaces $V(I) \cong \operatorname{Spec}(R/I)$.

Proposition 1.1.1. [[11], 10.16.5]
If $S \subset R$ is a multiplicatively closed set, the ring homomorphism $i_S : R \to S^{-1}R$ (see Definition A.1.1) induces a homeomorphism of topological spaces

$$\operatorname{Spec}(S^{-1}R) \xrightarrow{\sim} \left\{ P \in \operatorname{Spec} R \;\middle|\; P \cap S = \emptyset \right\} \ : \ Q \longmapsto i_S^{-1}(Q)$$

with inverse map $P \mapsto S^{-1}P$. In particular there is a 1-to-1 correspondence between prime ideals in the localization $S^{-1}R$ and prime ideals in R which do not intersect S.

In order to turn $\operatorname{Spec} R$ into a locally ring space, we equip it with a structure sheaf $\mathcal{O}_R = \mathcal{O}_{\operatorname{Spec} R}$ which satisfies $\mathcal{O}_{R,P} \cong R_P$ for all $P \in \operatorname{Spec} R$ and $\mathcal{O}_R\big(D(r)\big) \cong R_r$, $\forall\, r \in R$. In particular its global sections are $\mathcal{O}_R(\operatorname{Spec} R) \cong R$. The continuous map ϕ induced by a ring homomorphism $\varphi : R \to T$ can now be extended to a morphism of locally ring spaces

$$(\phi, \phi^{\#}) \ : \ (\operatorname{Spec} T, \mathcal{O}_T) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_R) \quad , \quad \phi^{\#}_U \ : \ \mathcal{O}_R(U) \to \mathcal{O}_T(\phi^{-1}(U))$$

for $U \subseteq \operatorname{Spec} R$ open, giving rise to a local homomorphism $\phi^{\#}_P : \mathcal{O}_{T,\phi(P)} \to \mathcal{O}_{R,P}$ which respects the corresponding maximal ideals. Hence we have a contravariant functor from the category of rings to the category of locally ringed spaces. Moreover this functor is fully faithful, in the sense that

$$\operatorname{Hom}_{\texttt{Ring}}(R,T) \ \cong \ \operatorname{Hom}_{\texttt{LRS}}\big((\operatorname{Spec} T, \mathcal{O}_T), (\operatorname{Spec} R, \mathcal{O}_R)\big) \ .$$

Definition 1.1.2. We say that a locally ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is an *affine scheme* if it is isomorphic (as locally ringed spaces) to the spectrum of a ring. A *scheme* is a locally ringed space which is locally isomorphic to an affine scheme. Thus there is an open covering $\mathcal{X} = \bigcup_i U_i$ such that

$$(U_i, \mathcal{O}_{\mathcal{X}}|_{U_i}) \ \cong \ (\operatorname{Spec} R_i, \mathcal{O}_{R_i})$$

for some rings R_i, $\forall\, i$. The cover $\{U_i\}_i$ is also called an *affine covering* of $\mathcal{X}$.

If $\mathcal{X} = \operatorname{Spec} R$ is affine and $r \in R$, then $D(r)$ is again an affine scheme. More precisely, we have the isomorphism of schemes

$$\big(D(r)\,,\,\mathcal{O}_R|_{D(r)}\big) \;\cong\; \big(\operatorname{Spec} R_r, \mathcal{O}_{R_r}\big)\;.$$

Note that the homeomorphism of topological spaces already follows from Proposition 1.1.1. One also says that $D(r)$ is an affine *open subscheme* of $\operatorname{Spec} R$.
A *closed subscheme* of a scheme $\mathcal{X}$ is a scheme which can be embedded into $\mathcal{X}$ as a closed topological subspace. If $\mathcal{X} = \operatorname{Spec} R$ is affine, then all closed subschemes of $\mathcal{X}$ are of the form $V(I) \cong \operatorname{Spec}(R/I)$ for some ideal $I \trianglelefteq R$, where the injection $\operatorname{Spec}(R/I) \hookrightarrow \operatorname{Spec} R$ is induced by $R \to R/I$.
But depending on the chosen ideal, the scheme structure of a closed subscheme may change while the underlying topological space does not. For example, if I is not radical, then $V(I)$ and $V(\operatorname{Rad}(I))$ define two different subschemes of $\operatorname{Spec} R$, even though they are homeomorphic as topological spaces.

Definition 1.1.3. A scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is called
– *connected* / *irreducible* if it is connected / irreducible as a topological space.
– *integral* if $\mathcal{O}_\mathcal{X}(U)$ is an integral domain for all $U \subseteq \mathcal{X}$ open.
– *reduced* if $\mathcal{O}_\mathcal{X}(U)$ is an reduced ring for all $U \subseteq \mathcal{X}$ open.
– *locally Noetherian* if it can be covered by spectra of Noetherian rings.
– *Noetherian* if it is locally Noetherian and compact.

Remark 1.1.4. Thus the stalks $\mathcal{O}_{\mathcal{X},x}$ of an integral, resp. reduced scheme $\mathcal{X}$ are integral domains, resp. reduced rings for all $x \in \mathcal{X}$.
The converse is false: [[11], 91.5] provides an example of an non-integral ring R such that the localizations R_P are integral domains for all $P \in \operatorname{Spec} R$.

Lemma 1.1.5. *1) A scheme is integral if and only if it is reduced and irreducible.*
2) Let $\mathcal{X} = \operatorname{Spec} R$ *be an affine scheme. Then* $\mathcal{X}$ *is*
a) integral if and only if R *is an integral domain.*
b) reduced if and only if R *is a reduced ring.*
c) irreducible if and only if the nilradical $\operatorname{nil}(R)$ *is a prime ideal.*
d) Noetherian if and only if R *is a Noetherian ring.*
3) The underlying topological space of a Noetherian scheme is a Noetherian topological space.

Example 1.1.6. Let $n \in \mathbb{N}$. The *affine n-space* is defined as the spectrum of the polynomial ring in n variables:

$$\mathbb{A}^n_{\mathbb{K}} := \mathrm{Spec}\left(\mathbb{K}[X_1, \ldots, X_n]\right) .$$

Since $\mathbb{K}$ is algebraically closed, its closed points (the maximal ideals) are in 1-to-1 correspondence with points of the classical affine space $\mathbb{A}_n$. We also denote the *projective n-space* by

$$\mathbb{P}^n_{\mathbb{K}} = \mathrm{Proj}\left(\mathbb{K}[X_1, \ldots, X_n]\right) .$$

Proj is the set of all homogeneous prime ideals in $\mathbb{K}[X_1, \ldots, X_n]$ not containing $\mathfrak{M} = \langle X_1, \ldots, X_n \rangle$, endowed with a similar Zariski topology as above. $\mathbb{P}^n_{\mathbb{K}}$ is a scheme as it can be covered by $n+1$ copies of $\mathbb{A}^n_{\mathbb{K}}$. Moreover the homogeneous maximal ideals which are different from $\mathfrak{M}$ are in 1-to-1 correspondence with the points of the classical projective space $\mathbb{P}_n$.

1.1.2 Sheaves of $\mathcal{O}_{\mathcal{X}}$–modules

Next we want to introduce sheaves on a scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which are compatible with the scheme structure of $\mathcal{X}$. In particular we are interested in so-called coherent and locally free sheaves. The main result about coherence is that there is a correspondence between coherent sheaves and finitely generated modules, which allows to restrict the study of coherent sheaves to the one of modules over a ring. References here are Hartshorne [35], Section II.5 and [11], Chapter 17.

Definition 1.1.7. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a scheme. A *sheaf of $\mathcal{O}_{\mathcal{X}}$–modules* (or an *$\mathcal{O}_{\mathcal{X}}$–module*) is the data of a sheaf $\mathcal{F}$ on $\mathcal{X}$ such that $\mathcal{F}(U)$ is a module over the ring $\mathcal{O}_{\mathcal{X}}(U)$ for all open set $U \subseteq \mathcal{X}$ and the module structure commutes with the restrictions of $\mathcal{F}$ and $\mathcal{O}_{\mathcal{X}}$. We denote $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$. In particular $\mathcal{O}_{\mathcal{X}} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ since every ring defines a module over itself.
An *$\mathcal{O}_{\mathcal{X}}$–submodule* of $\mathcal{F}$ is a subsheaf $\mathcal{F}' \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ such that $\mathcal{F}'(U)$ is an $\mathcal{O}_{\mathcal{X}}(U)$–submodule of $\mathcal{F}(U)$ for all $U \subseteq \mathcal{X}$ open. An *ideal sheaf* is a subsheaf of the structure sheaf $\mathcal{O}_{\mathcal{X}}$. For $\mathcal{F}, \mathcal{G} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$, a *morphism* of $\mathcal{O}_{\mathcal{X}}$–modules $\varphi : \mathcal{F} \to \mathcal{G}$ is the data of a homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ of modules over $\mathcal{O}_{\mathcal{X}}(U)$ for all open sets $U \subseteq \mathcal{X}$ which commutes with the restrictions of $\mathcal{F}$ and $\mathcal{G}$. The space of such morphisms is denoted by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

The stalk $\mathcal{F}_x$ of $\mathcal{F} \in \texttt{Mod}(\mathcal{O}_{\mathcal{X}})$ is a module over the local ring $\mathcal{O}_{\mathcal{X},x}$ for all $x \in \mathcal{X}$. Its elements are denoted by $s_x \in \mathcal{F}_x$ and represented by a *section* $s \in \mathcal{F}(U)$ for some open neighborhood U of x such that its *germ* $[s]_x$ is equal to s_x.
Every morphism $\varphi : \mathcal{F} \to \mathcal{G}$ induces a homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ of modules over $\mathcal{O}_{\mathcal{X},x}$ on the stalks via $\varphi_x(s_x) = [\varphi_U(s)]_x$. φ is said to be *injective*, resp. *surjective* if the induced module homomorphism φ_x is injective, resp. surjective for all $x \in \mathcal{X}$.

For all $\mathcal{F}, \mathcal{G} \in \texttt{Mod}(\mathcal{O}_{\mathcal{X}})$, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ and a subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, one can construct the *direct sum* $\mathcal{F} \oplus \mathcal{G}$, the *tensor product* $\mathcal{F} \otimes \mathcal{G}$, the *quotient* $\mathcal{F}/\mathcal{F}'$, the *kernel* $\ker \varphi$ and the *cokernel* $\operatorname{coker} \varphi$ by using sheafification if necessary. The stalks of these constructions behave nicely and we get

$$(\mathcal{F} \oplus \mathcal{G})_x \cong \mathcal{F}_x \oplus \mathcal{G}_x \quad , \quad (\mathcal{F} \otimes \mathcal{G})_x \cong \mathcal{F}_x \otimes \mathcal{G}_x \quad , \quad (\mathcal{F}/\mathcal{F}')_x \cong \mathcal{F}_x/\mathcal{F}'_x$$
$$(\ker \varphi)_x \cong \ker(\varphi_x) \quad , \quad (\operatorname{coker} \varphi)_x \cong \operatorname{coker}(\varphi_x) \tag{1.1}$$

for all $x \in \mathcal{X}$. All of them are again sheaves of $\mathcal{O}_{\mathcal{X}}$–modules, so the category $\texttt{Mod}(\mathcal{O}_{\mathcal{X}})$ admits kernels and cokernels. Moreover it is abelian. By (1.1) a sequence of $\mathcal{O}_{\mathcal{X}}$–modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is *exact* if and only if the induced sequence on the stalks

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is an exact sequence of modules over $\mathcal{O}_{\mathcal{X},x}$, $\forall\, x \in \mathcal{X}$. Using the corresponding results in the case of modules over a ring, one obtains that the following (bi)functors are left or right exact:
• $(\,\cdot\,)_x : \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X},x}) : \mathcal{F} \mapsto \mathcal{F}_x$ is exact.
• $\mathrm{Hom} : \texttt{Mod}(\mathcal{O}_{\mathcal{X}})^{\mathrm{op}} \times \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}(X)) : (\mathcal{F}, \mathcal{G}) \mapsto \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is left exact (contravariant in the first argument).
• $\oplus : \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \times \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) : (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \oplus \mathcal{G}$ is exact.
• $\otimes : \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \times \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) : (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes \mathcal{G}$ is right exact.
• $\Gamma(U, \cdot\,) : \texttt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}(U)) : \mathcal{F} \mapsto \mathcal{F}(U)$ is left exact, $\forall\, U \subseteq \mathcal{X}$ open.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of schemes with $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ and $\mathcal{G} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{Y}})$. One defines the *direct image* $f_*\mathcal{F}$ by pushing forward $\mathcal{F}$ on $\mathcal{Y}$ and the *inverse image* $f^*\mathcal{G}$ by pulling $\mathcal{G}$ back on $\mathcal{X}$. This way we obtain the pair of functors $f_* : \mathtt{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathtt{Mod}(\mathcal{O}_{\mathcal{Y}})$ and $f^* : \mathtt{Mod}(\mathcal{O}_Y) \to \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ which are adjoint, i.e.

$$\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) ,$$

functorially with respect to $\mathcal{F}$ and $\mathcal{G}$. In particular f_* is left exact, f^* is right exact and there are canonical morphisms $\mathcal{G} \to f_*f^*\mathcal{G}$ and $f^*f_*\mathcal{F} \to \mathcal{F}$.

Definition 1.1.8. Let $\mathcal{F}, \mathcal{G} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$. The *internal Hom* of $\mathcal{F}$ and $\mathcal{G}$ is defined by the assignement

$$\mathcal{H}\mathfrak{om}(\mathcal{F}, \mathcal{G}) \; : \; U \longmapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) ,$$

where $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to the open subscheme $(U, \mathcal{O}_{\mathcal{X}}|_U)$. It is again an $\mathcal{O}_{\mathcal{X}}$–module and hence defines a bifunctor which is contravariant in the first argument. It induces a module homomorphism $\big(\mathcal{H}\mathfrak{om}(\mathcal{F}, \mathcal{G})\big)_x \to \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$ for all $x \in \mathcal{X}$, which is not necessarily an isomorphism. Finally we also have the adjunction

$$\mathrm{Hom}\left(\mathcal{F} \otimes \mathcal{G} \, , \, \mathcal{H}\right) \cong \mathrm{Hom}\left(\mathcal{F} \, , \, \mathcal{H}\mathfrak{om}(\mathcal{G}, \mathcal{H})\right) , \tag{1.2}$$

functorially with respect to $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$. If the second entry is given by $\mathcal{O}_{\mathcal{X}}$, we denote $\mathcal{F}^* := \mathcal{H}\mathfrak{om}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})$ and call it the *dual sheaf* of $\mathcal{F}$.

Now we introduce the derived functors Ext^i, Tor_i and H^i that are related to Hom, $\otimes$ and Γ. Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be a short exact sequences of $\mathcal{O}_{\mathcal{X}}$–modules. For $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ we then have the following long exact sequences. Left exactness and contravariance of Hom in the first argument give

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathcal{F}_3, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{F}_2, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{F}_3, \mathcal{F}) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{F}_2, \mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{F}) \longrightarrow \mathrm{Ext}^2(\mathcal{F}_3, \mathcal{F}) \longrightarrow \dots \; . \end{aligned}$$

Covariance and right exactness of the tensor product $\otimes$ yields

$$\ldots \longrightarrow \mathrm{Tor}_2(\mathcal{F},\mathcal{F}_3) \longrightarrow \mathrm{Tor}_1(\mathcal{F},\mathcal{F}_1) \longrightarrow \mathrm{Tor}_1(\mathcal{F},\mathcal{F}_2) \longrightarrow \mathrm{Tor}_1(\mathcal{F},\mathcal{F}_3) \\ \longrightarrow \mathcal{F}\otimes\mathcal{F}_1 \longrightarrow \mathcal{F}\otimes\mathcal{F}_2 \longrightarrow \mathcal{F}\otimes\mathcal{F}_3 \longrightarrow 0\ ,$$

and left exactness of $\Gamma(U,\,\cdot\,)$ for all $U\subseteq\mathcal{X}$ open implies

$$0 \longrightarrow \Gamma(U,\mathcal{F}_1) \longrightarrow \Gamma(U,\mathcal{F}_2) \longrightarrow \Gamma(U,\mathcal{F}_3) \\ \longrightarrow H^1(U,\mathcal{F}_1) \longrightarrow H^1(U,\mathcal{F}_2) \longrightarrow H^1(U,\mathcal{F}_3) \longrightarrow H^2(U,\mathcal{F}_1) \longrightarrow \ldots\ .$$

$H^i(U,\mathcal{F})$ is called the *i^{th} cohomology space* of $\mathcal{F}$ on U. We say that $\mathcal{F}$ is *acyclic* if all its higher cohomology spaces vanish, i.e. if $H^i(\mathcal{X},\mathcal{F})=\{0\}$ for all $i\geq 1$.

1.1.3 Coherent sheaves

Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be a scheme. In order that the sheaves of our interest behave somehow "nicely" we want them to be of a certain type.

Definition 1.1.9. We say that a sheaf $\mathcal{F}\in\mathtt{Mod}(\mathcal{O}_\mathcal{X})$ is *locally free* if it is locally isomorphic to a direct sum of structure sheaves, i.e. if there is an affine covering $\{U_i\}_i$ such that

$$\mathcal{F}|_{U_i} \cong \mathcal{O}_\mathcal{X}^{(I_i)}|_{U_i}$$

for some index sets I_i, $\forall\, i$. If the cardinality of these index sets is finite and constant, say n, then $\mathcal{F}$ is said to be *locally free of rank n*. If $n=1$, $\mathcal{F}$ is called *invertible*. Constant cardinality of the index sets is e.g. satisfied if the scheme is connected. If $\mathcal{E}\in\mathtt{Mod}(\mathcal{O}_\mathcal{X})$ is locally free of finite rank, we moreover have

$$\mathcal{E}^*\otimes\mathcal{F} \cong \mathcal{H}\mathtt{om}(\mathcal{E},\mathcal{F})\ , \quad \forall\,\mathcal{F}\in\mathtt{Mod}(\mathcal{O}_\mathcal{X})$$

Proposition 1.1.10. *The set consisting of isomorphism classes of invertible sheaves $\mathcal{L}\in\mathtt{Mod}(\mathcal{O}_\mathcal{X})$ forms an abelian group with respect to the tensor product, the structure sheaf $\mathcal{O}_\mathcal{X}$ as neutral element and inverse $\mathcal{L}^*$ because $\mathcal{L}\otimes\mathcal{L}^*\cong\mathcal{O}_\mathcal{X}$. This group is called the Picard group of $\mathcal{X}$ and denoted by* $\mathrm{Pic}(\mathcal{X})$.

The category of locally free sheaves is not abelian since, as in the case of vector bundles, kernels and cokernels of morphisms between locally free sheaves may no longer be locally free. This is why we want to look for a "bigger" category which naturally includes the one of locally free sheaves.

Definition 1.1.11. We say that a sheaf $\mathcal{F}$ is *of finite type* if it can locally be generated by finitely many sections, i.e. if there is an affine covering $\{U_i\}_i$ and exact sequences $\mathcal{O}_{\mathcal{X}}^{n_i}|_{U_i} \to \mathcal{F}|_{U_i} \to 0$ for some $n_i \in \mathbb{N}$ (where $n_i = n_j$ on the intersection $U_i \cap U_j$). If moreover there are $m_i \in \mathbb{N}$ such that the sequences

$$\mathcal{O}_{\mathcal{X}}^{m_i}|_{U_i} \longrightarrow \mathcal{O}_{\mathcal{X}}^{n_i}|_{U_i} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0$$

are exact for all i, then $\mathcal{F}$ is said to be *of finite presentation*.
If there only exist such local sequences with infinite direct sums, $\mathcal{F}$ is called *quasi-coherent*. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted by $\mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$.
Finally we say that $\mathcal{F}$ is *coherent* if it is of finite type and if for every $p \in \mathbb{N}$, every open subset $U \subseteq \mathcal{X}$ and every morphism $\varphi : \mathcal{O}_{\mathcal{X}}^p|_U \to \mathcal{F}|_U$, $\ker\varphi$ is also of finite type. In particular coherent sheaves are thus of finite presentation. Intuitively this says that coherent sheaves are locally generated by finitely many sections and these sections only have finitely many relations between each other. The category of coherent sheaves on $\mathcal{X}$ is denoted by $\mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$.

Proposition 1.1.12. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally Noetherian scheme. Then*
1) The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is coherent (as an $\mathcal{O}_{\mathcal{X}}$–module).
2) A sheaf $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ is coherent if and only if it is of finite presentation.
3) If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$, then the canonical morphism $\mathcal{H}\mathtt{om}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$ is an isomorphism for all $x \in \mathcal{X}$.

The previous definitions of coherence and quasi-coherence actually hold true on any non-trivial locally ringed space. On schemes we however have an equivalent description which allows to connect coherent and quasi-coherent sheaves to classical modules over a ring.
If R is a ring, there exists a functor $\sim\, : \mathrm{Mod}(R) \to \mathtt{Mod}(\mathcal{O}_R) : M \mapsto \widetilde{M}$ which associated a sheaf on the affine scheme $\mathcal{X} = \mathrm{Spec}\, R$ to every R-module M with the following properties. For $M = R$, we obtain $\widetilde{R} = \mathcal{O}_R$. If M and N are R-modules, $P \in \mathrm{Spec}\, R$ and $r \in R$, then

$$\big(\widetilde{M}\big)_P \cong M_P \quad , \quad \widetilde{M}\big(D(r)\big) \cong M_r \quad , \quad \widetilde{M}(\mathrm{Spec}\, R) \cong M$$
$$\widetilde{M \oplus N} \cong \widetilde{M} \oplus \widetilde{N} \quad , \quad \widetilde{M \otimes_R N} \cong \widetilde{M} \otimes \widetilde{N}\,.$$

If $u: M \to N$ is an R-module homomorphism, it induces a morphism $\widetilde{u}: \widetilde{M} \to \widetilde{N}$ by functoriality with

$$\widetilde{\ker u} \cong \ker \widetilde{u} \qquad \text{and} \qquad \widetilde{\operatorname{coker} u} \cong \operatorname{coker} \widetilde{u} \,.$$

A ring homomorphism $\varphi: R \to T$ induces a morphism of schemes in the opposite direction $\phi: \operatorname{Spec} T \to \operatorname{Spec} R$, which satisfies

$$\phi^*(\widetilde{M}) \cong \widetilde{M \otimes_R T} \qquad \text{and} \qquad \phi_*(\widetilde{L}) \cong \widetilde{L} \,, \tag{1.3}$$

where $L \in \operatorname{Mod}(T)$ is also a module over R because of the morphism φ (see Lemma D.1.2).

Theorem 1.1.13. *1) If $M \in \operatorname{Mod}(R)$, then $\widetilde{M} \in \mathtt{QCoh}(\mathcal{O}_R)$.*
2) The functor $\sim$: $\operatorname{Mod}(R) \to \mathtt{QCoh}(\mathcal{O}_R) : M \mapsto \widetilde{M}$ is fully faithful and exact.
3) A sheaf $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_R)$ is quasi-coherent if and only if $\exists\, M \in \operatorname{Mod}(R)$ such that $\mathcal{F} \cong \widetilde{M}$.
4) If R is a Noetherian ring, then $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_R)$ is coherent if and only if there exists a finitely generated R-module M such that $\mathcal{F} \cong \widetilde{M}$.
5) The functor $\sim$ is a left adjoint of the left exact functor of global sections $\Gamma(\operatorname{Spec} R, \cdot\,)$, i.e. we have

$$\operatorname{Hom}\left(\widetilde{M}, \mathcal{F}\right) \cong \operatorname{Hom}_R\left(M, \mathcal{F}(\operatorname{Spec} R)\right) \,, \tag{1.4}$$

functorially with respect to $M \in \operatorname{Mod}(R)$ and $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_R)$.

Hence the functor $\sim$ yields a categorical equivalence between the category of R-modules and the category of quasi-coherent sheaves on the affine scheme Spec R. If R is Noetherian, this restricts to an equivalence between the category of finitely generated R-modules and the category of coherent sheaves. As it is fully faithful, every morphism of quasi-coherent sheaves is induced by some morphism between the corresponding modules. Exactness and the fact that the stalks of $\widetilde{M}$ are localizations of the module M moreover imply that a sequence of quasi-coherent $\mathcal{O}_R$–modules

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{L} \longrightarrow 0$$

is exact if and only if the sequence of R-modules $0 \to M \to N \to L \to 0$ is exact (since the localized sequence $0 \to M_P \to N_P \to L_P \to 0$ is exact for all prime ideals P, see Corollary A.2.14). In particular every injective, resp. surjective morphism of quasi-coherent sheaves on $\operatorname{Spec} R$ is induced by an injective, resp. surjective homomorphism of modules and quasi-coherent subsheaves of $\widetilde{M}$ are given by submodules of M.

Since an arbitrary scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is covered by affine schemes, we get the following criterion:
If $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$, then $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$–module if and only if there exists an affine covering $\{U_i\}_i$ of $\mathcal{X}$ with $U_i \cong \operatorname{Spec} R_i$ and R_i-modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for all i. If all R_i are Noetherian rings, then moreover $\mathcal{F}$ is coherent if and only if there are finitely generated R_i-modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$, $\forall\, i$. Finally the properties of $\sim$ also imply that the categories $\mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$ and $\mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ are closed under taking direct sums, tensor products, kernels, cokernels and inverse images ; in the case of coherent sheaves, one needs the additional assumption that $\mathcal{X}$ is locally Noetherian (by (1.3) we however see that coherent sheaves are in general not closed under taking direct images). In particular both categories are abelian.

Remark 1.1.14. By convention we always assume a scheme to be locally Noetherian as soon as we mention coherent sheaves. Indeed coherence is a property which does not behave well in the non-Noetherian case as finitely generated modules are not necessarily of finite presentation and may have submodules which are no longer finitely generated (compare with Proposition D.1.5).

1.2 Decomposition of a scheme into irreducible components

In general an arbitrary (affine) Noetherian scheme is a rather complicated object as it may be reducible and not even connected. This is why we want to decompose it "somehow naturally" into several "easier" (i.e. irreducible) components which should also take care of the non-reduced scheme structure, if there is any. Let us refer to Appendix B.2 for some general theory about Primary Ideal Decomposition. This will be the main tool we are going to use in order to decompose

affine schemes into irreducible components.
In the following we first give the general procedure of how to find such an irreducible decomposition and then apply it to a concrete example. This one should give the reader a detailed illustration of the process once and for all as we do not repeat the details in applications later on. Moreover we will state several problems of the decomposition that are related to the non-reduced structure and explain how to solve them.

Remark 1.2.1. We recall that the zero ideal in a Noetherian ring is decomposable (Theorem B.2.23) and can be written as an intersection $\{0\} = Q_1 \cap \ldots \cap Q_\alpha$ of finitely many primary ideals $Q_i \trianglelefteq R$. Their radicals $P_i = \mathrm{Rad}(Q_i)$ are prime ideals and called the associated primes of R. By Corollary B.2.16 they give the set of zero-divisors and nilpotent elements in R as

$$\mathrm{ZD}(R) = P_1 \cup \ldots \cup P_\alpha \qquad \text{and} \qquad \mathrm{nil}(R) = P_1 \cap \ldots \cap P_\alpha \,. \tag{1.5}$$

Finally if $r \in R$ and $P \in \operatorname{Spec} R$, then Corollary B.2.22 gives the sets of zero-divisors in the localizations by

$$\mathrm{ZD}(R_r) = (P_1)_r \cup \ldots \cup (P_{\gamma_1})_r \qquad , \qquad \mathrm{ZD}(R_P) = (P_1)_P \cup \ldots \cup (P_{\gamma_2})_P \tag{1.6}$$

for some $\gamma_1, \gamma_2 \leq \alpha$ and the primes are numbered such that $r \notin P_1, \ldots, P_{\gamma_1}$, resp. $P_1, \ldots, P_{\gamma_2} \subseteq P$.

1.2.1 Irreducible components of an affine scheme

Let R be a Noetherian ring with associated primes $P_1, \ldots, P_\alpha$. Similarly as in (B.7) we compute

$$\operatorname{Spec} R = V\big(\{0\}\big) = V\big(\textstyle\bigcap_i Q_i\big) = \textstyle\bigcup_i V(Q_i) = \textstyle\bigcup_i V(P_i) \,.$$

So if we denote $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{X}_i = V(P_i)$ for all $i \in \{1, \ldots, \alpha\}$, we get a decomposition

$$\mathcal{X} = \bigcup_{i=1}^{\alpha} \mathcal{X}_i \tag{1.7}$$

of $\mathcal{X}$ into irreducible subschemes (at least on the topological level) since we have $\mathcal{X}_i \cong \mathrm{Spec}(R/P_i)$ where each R/P_i is an integral domain, see Lemma 1.1.5.

Remark 1.2.2. This decomposition allows to explain the terminology of minimal and embedded primes of R. Indeed if $P_i \subsetneq P_j$ are associated primes, then $\mathcal{X}_j \subsetneq \mathcal{X}_i$ and the component $\mathcal{X}_j$ is included in $\mathcal{X}_i$. The minimal primes correspond to the (maximal) irreducible components and the embedded primes correspond to irreducible subschemes that are embedded in the irreducible components. If the ring R is reduced, Lemma B.2.18 implies that it has no embedded primes, and hence that $\operatorname{Spec} R$ has no embedded components.
Topologically one may forget about the embedded components since they are not maximal. However they are important since they encode a non-reduced structure which cannot be seen on the topological level. We illustrate this fact on the following example.

1.2.2 Example

Consider the Noetherian ring $R = \mathbb{K}[X,Y]/\langle\, Y^2, XY\,\rangle$; we will call this Example E.1. Let us find its associated primes. We have

$$\{\bar{0}\} = \langle\, \bar{X}\,\rangle \cap \langle\, \bar{Y}\,\rangle\ ,$$

where $Q_1 = \langle\, \bar{X}\,\rangle$ is primary since $R/\langle\, \bar{X}\,\rangle \cong \mathbb{K}[Y]/\langle\, Y^2\,\rangle$ is a ring in which all zero-divisors are nilpotent and $Q_2 = \langle\, \bar{Y}\,\rangle$ is prime since $R/\langle\, \bar{Y}\,\rangle \cong \mathbb{K}[\bar{X}]$ is an integral domain. The corresponding radicals are $P_1 = \operatorname{Rad}(Q_1) = \langle\, \bar{X}, \bar{Y}\,\rangle$ and $P_2 = Q_2 = \langle\, \bar{Y}\,\rangle$. Hence

$$\operatorname{Ass}\big(\{\bar{0}\}\big) = \big\{\, P_1 = \langle\, \bar{X}, \bar{Y}\,\rangle\,,\, P_2 = \langle\, \bar{Y}\,\rangle\,\big\}\ ,$$

where P_2 is minimal and $P_2 \subsetneq P_1$. Thus we have $\operatorname{ZD}(R) = P_1 \cup P_2 = \langle\, \bar{X}, \bar{Y}\,\rangle$. This can also be seen directly since

$$\bar{Y} \cdot \big(\bar{f}\bar{X} + \bar{g}\bar{Y}\big) = \bar{0}\ , \quad \forall\, f, g \in \mathbb{K}[X,Y]\ .$$

As $P_2 \subsetneq P_1$, one may omit the component $V(P_1) \subsetneq V(P_2)$ and according to (1.7) the affine scheme $\operatorname{Spec} R = V(P_2)$ is already irreducible. Actually $V(P_1)$ just defines a point lying on the affine line

$$V(P_2) \cong \operatorname{Spec}\big(R/\langle\, \bar{Y}\,\rangle\big) \cong \operatorname{Spec}\big(\mathbb{K}[\bar{X}]\big) \cong \mathbb{A}^1_{\mathbb{K}}\ .$$

However we cannot omit P_1 if we want to describe all zero-divisors of R. Alternatively irreducibility of $\operatorname{Spec} R$ can be seen by the fact that $\operatorname{nil}(R) = \langle \bar{Y} \rangle = P_2$ (by (1.5) it is the intersection of all minimal primes), hence the nilradical is a prime ideal. More precisely, we have

Lemma 1.2.3.

$$\operatorname{Spec} R = \left\{ \langle \bar{Y} \rangle \,,\, \langle \bar{X} - a, \bar{Y} \rangle \;\middle|\; a \in \mathbb{K} \right\} . \tag{1.8}$$

Proof. Denote $I = \langle Y^2, XY \rangle$; we have to find all prime ideals P such that $I \subseteq P \subsetneq \mathbb{K}[X,Y]$. Note that I is not a prime ideal since $Y^2 \in I$, but $Y \notin I$. This implies that $Y \in \operatorname{Rad}(I)$. $I \subseteq P$ then implies that $\operatorname{Rad}(I) \subseteq \operatorname{Rad}(P)$, i.e. we have $Y \in P$. But for this $\langle Y \rangle$ and $\langle X - a, Y \rangle$ are all possibilities. Indeed, assume that $P = \langle Y, f_1, \dots, f_k \rangle$ for $k \geq 1$ such that $\gcd(f_1, \dots, f_k) \neq 1$. Hence $\gcd(f_i, f_j) \neq 1$, $\forall\, i, j$. Using Y we may moreover assume that they only depend on X. If all f_i are irreducible, we need that $k = 1$, otherwise their gcd will be 1. If some f_j is reducible, then P must contain at least one of its factors, otherwise P is not prime. By induction, P must contain an irreducible polynomial g which divides f_j. So we can replace f_j by g. But this g then needs to divide all other f_i, otherwise the gcd will be 1. In the end, all f_i will be multiples of g. In both cases we thus have $P = \langle Y, f \rangle$ with $f \in \mathbb{K}[X]$ irreducible, i.e. f is of degree 1 and $\exists\, a \in \mathbb{K}$ such that $f = X - a$ since $\mathbb{K}$ is algebraically closed. $\square$

Remark 1.2.4. Consider P_1 and P_2 as points in $\operatorname{Spec} R$. We want to see what the inclusion $P_2 \subset P_1$ means on the topological level. P_1 is a maximal ideal since $R/P_1 \cong \mathbb{K}$, hence the point $\{P_1\}$ is closed. But $\{P_2\}$ is not closed; actually it is a generic point since

$$\overline{\{P_2\}} = V(P_2) = \left\{ P \in \operatorname{Spec} R \;\middle|\; P_2 \subseteq P \right\} = \mathcal{X} \,.$$

In particular, P_1 belongs to the closure of P_2. Hence $P_2 \subset P_1$ can be rewritten as $P_1 \in \overline{\{P_2\}}$.

Remark 1.2.5. Note that the primary decomposition of $\{\bar{0}\}$ in R can be recovered from the one of I in $\mathbb{K}[X,Y]$. Indeed when looking at the generators of

I, one sees $\operatorname{Spec} R$ intuitively as a subscheme of $\mathbb{A}^2_{\mathbb{K}}$ given by the horizontal line $Z(Y)$ with a "vertical" double point at the origin $(0,0)$. Consider the individual ingredients $L = \langle\, Y\,\rangle$ for the line and $D = \langle\, X, Y^2\,\rangle$ for the double point. Then

$$I = L \cap D$$

is a primary decomposition of I in $\mathbb{K}[X,Y]$. Indeed, D is primary since

$$\mathbb{K}[X,Y]/D \cong \mathbb{K}[Y]/\langle\, Y^2\,\rangle$$

as rings with $\operatorname{Rad}(D) = \langle\, X, Y\,\rangle$ and L is a prime ideal. Moreover $Y^2 \in L \cap D$ and $XY \in L \cap D$, so $I \subseteq L \cap D$. Conversely, let $f \in L \cap D$, i.e. $\exists\, a, b, g \in \mathbb{K}[X,Y]$ such that $f = gY = aX + bY^2$. Hence

$$gY = aX + bY^2 \;\Leftrightarrow\; aX = (g - bY)\,Y\ .$$

Since $\langle\, Y\,\rangle$ is a prime ideal with $X \notin \langle\, Y\,\rangle$, we need that $a = h \cdot Y$ for some $h \in \mathbb{K}[X,Y]$. Therefore

$$aX = (g - bY)\,Y \;\Leftrightarrow\; hX \cdot Y = (g - bY) \cdot Y \;\Rightarrow\; g - bY = hX\ ,$$

so that $f = gY = (hX + bY)\,Y = hXY + bY^2 \in I$.

Problem 1

However the descriptions (1.7) and (1.8) have several problems. The first one is that (1.8) does not allow to see $\operatorname{Spec} R$ as a line with a double point as a closed subscheme of $\mathbb{A}^2_{\mathbb{K}}$ since it only describes the spectrum topologically. In fact, (1.7) already implies that all structures in $\mathcal{X}$ are integral since we only consider prime ideals.

A solution for this problem is to go back to the actual primary ideal decomposition of $\{\bar{0}\}$ (or I in $\mathbb{K}[X,Y]$) instead of just looking at the associated primes. This gives the decomposition

$$\mathcal{X} = \operatorname{Spec} R = \textstyle\bigcup_i V(Q_i) \;\cong\; \bigcup_i \operatorname{Spec}(R/Q_i)\ .$$

As $P_i = \operatorname{Rad}(Q_i)$, this is topologically the same decomposition as the one in (1.7), but with a richer scheme structure on each component because $V(Q_i)$ is given by the quotient R/Q_i instead of R/P_i, which is an integral domain. In particular we can have non-reduced structures since all zero-divisors in R/Q_i are nilpotent. Applying this to our example, we find

$$\begin{aligned}\mathcal{X} = \operatorname{Spec} R = \operatorname{Spec}\big(\mathbb{K}[X,Y]/I\big) &\cong V(I) = V(L) \cup V(D) \\ &\cong \operatorname{Spec}\big(\mathbb{K}[X,Y]/L\big) \cup \operatorname{Spec}\big(\mathbb{K}[X,Y]/D\big) \\ &\cong \operatorname{Spec}\big(\mathbb{K}[X]\big) \cup \operatorname{Spec}\big(\mathbb{K}[Y]/\langle\, Y^2 \,\rangle\big) \ ,\end{aligned}$$

thus $\mathcal{X} = \mathbb{A}^1_{\mathbb{K}} \cup \{dp\}$, where $\{dp\}$ is a double point.[2]

Lemma 1.2.6. *$\mathbb{A}^1_{\mathbb{K}} \hookrightarrow \mathcal{X}$ and $\{dp\} \hookrightarrow \mathcal{X}$ are both maximal irreducible closed subsets in $\mathcal{X} \hookrightarrow \mathbb{A}^2_{\mathbb{K}}$.*

Proof. $\mathbb{A}^1_{\mathbb{K}} \cong V(L) \subset \mathbb{A}^2_{\mathbb{K}}$, thus it is closed in $\mathcal{X} \cong V(I)$. Moreover it is irreducible since $\mathbb{K}[X]$ is an integral domain. To see that it is maximal, we have to show that there does not exist an ideal $J \trianglelefteq \mathbb{K}[X,Y]$ such that $V(J)$ is irreducible and $\mathbb{A}^1_{\mathbb{K}} \subsetneq V(J) \subsetneq \mathcal{X}$. But

$$\begin{aligned}\mathbb{A}^1_{\mathbb{K}} \subseteq V(J) \subseteq \mathcal{X} \ &\Leftrightarrow \ V(L) \subseteq V(J) \subseteq V(I) \\ &\Leftrightarrow \ \operatorname{Rad}(I) \subseteq \operatorname{Rad}(J) \subseteq \operatorname{Rad}(L) \ \Leftrightarrow \ \langle\, Y \,\rangle \subseteq \operatorname{Rad}(J) \subseteq \langle\, Y \,\rangle \ .\end{aligned}$$

Hence $\mathbb{A}^1_{\mathbb{K}} \hookrightarrow \mathcal{X}$ is a maximal irreducible closed subset. For the double point we have $\{dp\} \cong V(D) \subset \mathbb{A}^2_{\mathbb{K}}$ and $\operatorname{nil}\big(\mathbb{K}[Y]/\langle\, Y^2 \,\rangle\big) = \langle\, \bar{Y} \,\rangle$, which is a prime ideal, so it is closed and irreducible. For maximality, let $J \trianglelefteq \mathbb{K}[X,Y]$ be such that $V(J)$ is irreducible and $V(D) \subseteq V(J) \subseteq V(I)$. As above, we get

$$\operatorname{Rad}(I) \subseteq \operatorname{Rad}(J) \subseteq \operatorname{Rad}(D) \ \Leftrightarrow \ \langle\, Y \,\rangle \subseteq \operatorname{Rad}(J) \subseteq \langle\, X,Y \,\rangle \ .$$

Note that there does not exist prime ideals between $\langle\, Y \,\rangle$ and $\langle\, X,Y \,\rangle$ since $\mathbb{K}[X,Y]$ has Krull dimension 2. However there exist arbitrary ideals, and even radical ideals between them, e.g.

$$\langle\, Y \,\rangle \subsetneq \langle\, Y, X^2 \,\rangle \subsetneq \langle\, X,Y \,\rangle \qquad \text{or} \qquad \langle\, Y \,\rangle \subsetneq \langle\, Y, X^2+X \,\rangle \subsetneq \langle\, X,Y \,\rangle \ .$$

[2]Right now it suffices to imagine a “double point” as two points being infinitesimally close to each other. More precise explanations will be given in Section 1.4.1 and Example 1.4.9.

But there are none if we require that $V(J)$ should be irreducible (e.g. the second ideal defines 2 simple points, so it is not an admissible choice). As $Y \in \operatorname{Rad}(J)$, let $\operatorname{Rad}(J) = \langle\, Y, f_1, \ldots, f_k \,\rangle$, where the f_i only depend on X. Since $\operatorname{Rad}(J)$ is in $\langle\, X, Y \,\rangle$, we also need that no f_i has a constant term, i.e. $\gcd(f_1, \ldots, f_k)$ is at least X and we can rewrite $\operatorname{Rad}(J) = \langle\, Y, g_1X, \ldots, g_kX \,\rangle$. Let

$$N := \operatorname{nil}\left(\mathbb{K}[X,Y]/\operatorname{Rad}(J)\right) ,$$
$$V(J) = V\big(\operatorname{Rad}(J)\big) \cong \operatorname{Spec}\left(\mathbb{K}[X,Y]/\operatorname{Rad}(J)\right) .$$

Since the quotient by a radical ideal is a reduced ring, we have $N = \{\bar{0}\}$. To get irreducibility we need it to be a prime ideal, i.e. the quotient must be an integral domain. But we have the relations $\bar{g}_i \cdot \bar{X} = \bar{0}$, $\forall\, i$, which lead to the following possibilities:

If $k = 0$, then $\operatorname{Rad}(J) = \langle\, Y \,\rangle$, so let $k \geq 1$. If $\bar{X} = \bar{0}$, then $X \in \operatorname{Rad}(J)$, hence $\operatorname{Rad}(J) = \langle\, X, Y \,\rangle$. If $\bar{X} \neq \bar{0}$, then $\bar{g}_i = \bar{0}$, so $g_i \in \operatorname{Rad}(J)$ for all i. But since the g_i do not depend on Y, this means that they are again multiples of X, i.e. $g_i = h_iX$, $\forall\, i$. Then the relation $(\bar{h}_i\bar{X}) \cdot \bar{X} = \bar{0}$ implies again that $h_i \in \operatorname{Rad}(J)$ for all i, otherwise $\bar{X}$ would be a zero-divisor. By induction, this yields

$$g_i = c_i \cdot X^{n_i}$$

for some $c_i \in \mathbb{K}$ and $n_i = \deg g_i$. If $n_i > 0$, we need that $c_i = 0$, otherwise $\bar{X}$ would be nilpotent. So we obtain that all g_i are constant and $\operatorname{Rad}(J) = \langle\, X, Y \,\rangle$. Hence $\{dp\} \hookrightarrow \mathcal{X}$ is maximal too. □

Remark 1.2.7. Thus $\mathcal{X} = \mathbb{A}^1_{\mathbb{K}} \cup \{dp\}$ defines a decomposition of $\mathcal{X}$ into irreducible components which is essentially different from (1.7). In particular no component is superfluous, even though $\{dp\}$ is included in $\mathbb{A}^1_{\mathbb{K}}$ as a topological space.

The difference between the decompositions $\mathcal{X} = V(P_2)$ and $\mathcal{X} = V(L) \cup V(D)$ into irreducible components is that the second one considers the possible non-reduced structures on the components defined by the embedded primes, whose corresponding irreducible components are not maximal and hence disappear on the topological level.

Problem 2

Since P_1 is an embedded prime, its corresponding primary ideal is not unique (compare with Proposition B.2.19). Similarly the primary ideal decomposition of I in $\mathbb{K}[X,Y]$ is not unique, e.g. one checks that $I = \langle\, Y^2, XY \,\rangle$ writes as

$$I = \langle\, Y \,\rangle \cap \langle\, X, Y^2 \,\rangle = \langle\, Y \,\rangle \cap \langle\, X^2, XY, Y^2 \,\rangle = \langle\, Y \,\rangle \cap \langle\, Y^2, X+Y \,\rangle$$

with

$$\text{Rad}\left(\langle\, X, Y^2 \,\rangle\right) = \text{Rad}\left(\langle\, X^2, XY, Y^2 \,\rangle\right) = \text{Rad}\left(\langle\, Y^2, X+Y \,\rangle\right) = \langle\, X, Y \,\rangle\ ,$$

i.e. all decompositions define the same topological space, but they give different scheme structures.

1) Denote $J = \langle\, X^2, XY, Y^2 \,\rangle$; then Spec $\left(\mathbb{K}[X,Y]/J\right)$ defines a triple point since the $\mathbb{K}$-vector space $\mathbb{K}[X,Y]/J \cong \mathbb{K} \oplus \mathbb{K}\bar{X} \oplus \mathbb{K}\bar{Y}$ is 3-dimensional. With this we get a decomposition $\mathcal{X} = V(L) \cup V(J)$ into (maximal) irreducible components given by line $Z(Y) \cong \mathbb{A}^1_{\mathbb{K}}$ and a triple point, which is essentially different from the decomposition $\mathbb{A}^1_{\mathbb{K}} \cup \{dp\}$. Intuitively this also makes sense since "one part of the triple point is included in the line", so we don't get "more" than for the double point. In the quotient $R = \mathbb{K}[X,Y]/I$ this decomposition becomes

$$I = \langle\, Y \,\rangle \cap \langle\, X^2, XY, Y^2 \,\rangle \quad \Rightarrow \quad \{\bar{0}\} = \langle\, \bar{Y} \,\rangle \cap \langle\, \bar{X}^2 \,\rangle\ ,$$

where $\langle\, \bar{X}^2 \,\rangle \trianglelefteq R$ is also primary since $R/\langle\, \bar{X}^2 \,\rangle \cong \mathbb{K}[X,Y]/J$ is a ring in which all zero-divisors are nilpotent (of order 2):

$$(\bar{f}\bar{X} + \bar{g}\bar{Y})^2 = \bar{f}^2\bar{X}^2 + 2\bar{f}\bar{g}\bar{X}\bar{Y} + \bar{g}^2\bar{Y}^2 = \bar{0}$$

for all $\bar{f}, \bar{g} \in \mathbb{K}[X,Y]/J$.

2) Denote $K_1 = \langle\, Y^2, X+Y \,\rangle$; this gives a decomposition of $\mathcal{X} = V(L) \cup V(K_1)$ into a line and a double point. Whether it is different from $\mathcal{X} = V(L) \cup V(D)$ depends on how we want to see our scheme. If we consider $V(K_1)$ and $V(D)$ as independent schemes (i.e. as spectra of rings), both decompositions will be the same since there is just 1 double point without embedding. But if we see them as closed subschemes of $\mathbb{A}^2_{\mathbb{K}}$ (i.e. as modules over the polynomial ring given by

the quotients $\mathbb{K}[X,Y]/K_1$ and $\mathbb{K}[X,Y]/D$), the decompositions will be different. Actually for any $a \in \mathbb{K}$, the ideal

$$K_a = \langle\, Y^2, X + aY \,\rangle$$

will give another primary decomposition $I = L \cap K_a$, which yields a decomposition $\mathcal{X} = \mathbb{A}^1_{\mathbb{K}} \cup \{dp\}$ where the double point depends on the "angle" at the origin. For $a = 0$, we recover the "vertical" double point.

Remark 1.2.8. Hence there is (still) no canonical way of decomposing $\mathcal{X}$ into maximal irreducible components that takes care of non-reduced structures on the embedded components. All we can say is that we have the topological decomposition given by (1.7), but this one only sees the reduced structures. In order to give the component a non-reduced structure, if it is intended to exist, we have to choose one of the primary ideals in the initial primary decomposition of $\{\bar{0}\}$ (or I).

1.2.3 Computing primary decompositions in polynomial rings

In practise it is not always easy to find the primary decomposition of an ideal in a Noetherian ring. However it can be done quite quickly for Noetherian rings which are finitely generated $\mathbb{K}$-algebras. Basically this means that such rings are just quotients of polynomial rings. This is also useful in applications since spectra of quotients of polynomial ring describe subschemes of the affine space $\mathbb{A}^n_{\mathbb{K}}$, which are easy to visualize.

The computer algebra system `Singular` [12], developed by the University of Kaiserslautern[3], allows to decompose ideals in polynomial rings over $\mathbb{Q}$. For our purposes this will mostly be sufficient. So in practice we may assume that every ideal $I \trianglelefteq R = \mathbb{K}[X_1, \ldots, X_n]$ can be decomposed. For an ideal in a quotient of the form $\bar{I} \trianglelefteq R/J$, one computes the primary decomposition of $I + J$ in R and then projects down again to the quotient (compare Remark B.2.11).

[3]A program simulating `Singular` is available at https://cloud.sagemath.com

Remark 1.2.9. Using a similar method one can also compute primary decompositions in sums of polynomial rings and quotients. Let us describe the idea of this method by the following examples.[4]

Example 1.2.10. We want to find the primary decomposition of $\{0\}$ in the Noetherian ring

$$R = \mathbb{K}[X,Y] \oplus \Big(\mathbb{K}[X,Y]/\langle\, X,Y \,\rangle\Big) .$$

The idea is to consider R as a finitely generated $\mathbb{K}$-algebra and to find the relations between its generators so that we can rewrite it as a quotient of a bigger polynomial ring. Then it will be of the form $R \cong \mathbb{K}[X_1,\ldots,X_n]/A$ for some ideal $A \trianglelefteq \mathbb{K}[X_1,\ldots,X_n]$ that describes the relations. We e.g. set the generators of R to be

$$1 \leftrightarrow (1,\bar{0}) \quad , \quad X \leftrightarrow (X,\bar{0}) \quad , \quad Y \leftrightarrow (Y,\bar{0}) \quad , \quad U \leftrightarrow (1,\bar{1}) \; .$$

There are no relations between $1, X, Y$. For the last one we get $1 \cdot U = U$, $X \cdot U = 0$, $Y \cdot U = 0$ and $U^2 = U$. So

$$R \;\cong\; \mathbb{K}[X,Y,U]/A = \mathbb{K}[X,Y,U]/\langle\, XU, YU, U(U-1) \,\rangle$$

and the primary decomposition is $A = \langle\, U \,\rangle \cap \langle\, X, Y, U-1 \,\rangle$. Hence $\operatorname{Spec} R \subset \mathbb{A}^3_{\mathbb{K}}$ is given by the plane $V(U)$ together with the (simple) point $(0,0,1)$. This is also the intuitive picture (the definition of R suggests that $\operatorname{Spec} R$ consists of a plane and a point).

Example 1.2.11. Now we consider the Noetherian ring

$$R = \Big(\mathbb{K}[X,Y] \oplus \mathbb{K}[X,Y]\Big)/\langle\, (X,Y) \,\rangle \; .$$

Here the computations are a bit more complicated since we take the direct sum of 2 polynomial rings and divide out the principal ideal generated by the element $(X,Y) = (X,0) + (0,Y)$. Let

$$1 \leftrightarrow (1,1) \quad , \quad U_1 \leftrightarrow (1,0) \quad , \quad U_2 \leftrightarrow (0,1) \; ,$$
$$X_1 \leftrightarrow (X,0) \quad , \quad X_2 \leftrightarrow (0,X) \quad , \quad Y_1 \leftrightarrow (Y,0) \quad , \quad Y_2 \leftrightarrow (0,Y) \; .$$

[4] We point out that the technique is not indispensable for the rest of our work ; we just present it as a complement.

Apart from the relations $U_1+U_2=1$ and $X_1+Y_2=0$ (dividing out the principal ideal) we have

$$\begin{aligned} &U_1^2=U_1 \quad , \quad U_1U_2=0 \quad , \quad U_1X_1=X_1 \quad , \quad U_1X_2=0 \quad , \quad U_1Y_1=Y_1 \ , \\ &U_1Y_2=0 \quad , \quad U_2^2=U_2 \quad , \quad U_2X_1=0 \quad , \quad U_2X_2=X_2 \quad , \quad U_2Y_1=0 \ , \\ &U_2Y_2=Y_2 \quad , \quad X_1X_2=0 \quad , \quad X_1Y_2=0 \quad , \quad X_2Y_1=0 \quad , \quad Y_1Y_2=0 \ . \end{aligned}$$

Computing a Gröbner basis of all these relations (there are 17 in total) with `Singular` gives

$$A=\langle\, Y_2\, ,\, X_1+Y_2\, ,\, U_1+U_2-1\, ,\, X_2Y_1\, ,\, U_2Y_1\, ,\, X_2(U_2-1)\, ,\, U_2^2-U_2\,\rangle\ ,$$

hence we can divide out the variables Y_2, X_1 and U_1. Renaming $U_2=X$, $X_2=Y$ and $Y_1=Z$, we finally obtain

$$R\cong\mathbb{K}[X,Y,Z]/\langle\, YZ, XZ, XY-Y, X^2-X\,\rangle$$

and `Singular` gives the primary decomposition $\langle\, X,Y\,\rangle\cap\langle\, X-1,Z\,\rangle$. Thus $\operatorname{Spec} R$ is equal to the union of the 2 skew lines $\{X=Y=0\}$ and $\{X=1,Z=0\}$ in $\mathbb{A}^3_{\mathbb{K}}$.

1.2.4 Connected components of a scheme

Similarly as for the irreducible components of a scheme, one may ask how to find its connected components. First note that every irreducible component is connected itself (as a topological space), but they are not necessarily maximal, e.g. two components may intersect, so that the connected component to which they belong is bigger than the irreducible ones.
This already describes the idea of how to find the connected components of $\operatorname{Spec} R$: first we determine the irreducible ones by using the primary decomposition of $\{0\}\trianglelefteq R$, then we analyze which of them intersect and take the union in case of a non-empty intersection. To know whether two (or more) components intersect or not, we compute $V(I)\cap V(J)=V(I+J)$ and

$$V(I+J)=\emptyset\ \Leftrightarrow\ \nexists\, P\in\operatorname{Spec} R \text{ such that } I+J\subseteq P\ .$$

As every proper ideal is contained in a maximal ideal, we hence get

$$V(I) \cap V(J) = \emptyset \Leftrightarrow I + J = R \, .$$

So we know that two irreducible components do not intersect if the sum of their defining ideals is equal to R. The ones which do not intersect any other one are thus connected components. For those which intersect, we have to take their union in order to obtain the connected component to which they belong. However there is a problem when taking this union because

$$V(I) \cup V(J) = V(I \cap J) = V(I \cdot J)$$

and both descriptions give the same topological space. But since $I \cdot J \subseteq I \cap J$, the one defined by $I \cdot J$ has a richer subscheme structure. So one is tempted to choose $I \cdot J$ as it will define a "bigger" scheme (similarly as we preferred primary ideals instead of prime ideals so that we can obtain non-reduced structures).
On the other hand, it may happen that the structure defined by $I \cdot J$ is even richer than the one we started with, see e.g. Example 1.2.12 below. This should of course not happen as we don't want to "create new structures" in the initial scheme that we are supposed to study. Hence in order to get the connected components of a scheme, one always takes the intersection of the ideals defining the irreducible components that intersect.
Some general facts about intersections of ideals can be found in Appendix D.2. It includes a criterion (Proposition D.2.7) to decide under which conditions the product and the intersection agree.

Example 1.2.12. 1) If we take $I = J$, we get $I^2 \subseteq I$. For $I = \langle X \rangle$ in $R = \mathbb{K}[X]$, we then find $\langle X^2 \rangle \subsetneq \langle X \rangle$. This would e.g. create a double point out of a simple point.
2) Let $I = \langle X \rangle$ and $J = \langle X, Y \rangle$ in $R = \mathbb{K}[X, Y]$, so that $I \cdot J = \langle X^2, XY \rangle$ and $I \cap J = \langle X \rangle$.
3) An example where no ideal is included in the other one is e.g. $I = \langle X \rangle$ and $J = \langle X^2, Y \rangle$ in $R = \mathbb{K}[X, Y]$. Then one gets $I \cdot J = \langle X^3, XY \rangle$ and $I \cap J = \langle X^2, XY \rangle$. Note that this corresponds to Example E.1 (by interchanging the variables X and Y). The primary decomposition

$$\langle X^2, Y \rangle = \langle X \rangle \cap \langle X^2, Y \rangle$$

decomposes the scheme into two irreducible components. As

$$\langle X \rangle + \langle X^2, Y \rangle = \langle X, Y \rangle \neq R \ ,$$

both components intersect, so we only have one connected component. But the product has the primary decomposition

$$\langle X \rangle \cdot \langle X^2, Y \rangle = \langle X^3, XY \rangle = \langle X \rangle \cap \langle X^3, Y \rangle \ ,$$

and thus gives a line with an embedded triple point. Hence taking the product of the ideals defining the components instead of the intersection gives a bigger scheme than the initial one.

1.3 Modules, torsion and their associated primes

In this section we want to show that there are relations between the associated primes of a module and its torsion. Recall that the torsion submodule of an R-module M is given by

$$\mathcal{T}_R(M) = \left\{ m \in M \ \middle|\ \exists\, r \in R,\ r \neq 0 \text{ such that } r \text{ is a NZD and } r * m = 0 \right\} .$$

Basic facts and properties about $\mathcal{T}_R(M)$ are given in Appendix C.

1.3.1 A criterion for torsion-freeness

It is possible to see whether an R-module M is torsion-free by only looking at the associated primes of M and R. Our first goal is to establish such a criterion. For this let us refer to Appendix B.3 for the definition and basic facts of $\mathrm{Ass}_R(M)$.

Proposition 1.3.1. [[4], II.§7.n°10.Cor.2, p.115] and [[53], 60762]
The assignment $\mathcal{T}_R : \mathrm{Mod}(R) \to \mathrm{Mod}(R) : M \mapsto \mathcal{T}_R(M)$ defines an additive covariant and left exact functor. Moreover, if R is Noetherian and M is finitely generated, then $\mathcal{T}_R(M)$ is finitely generated as well and we get a left exact functor

$$\mathcal{T}_R \ :\ \mathrm{Mod}^f(R) \longrightarrow \mathrm{Mod}^f(R) \ :\ M \longmapsto \mathcal{T}_R(M) \ .$$

Proof. Additivity is proven in Proposition C.1.4 and functoriality follows from Lemma C.1.9. Since an R-module homomorphism $\varphi : M \to N$ moreover satisfies $\varphi\big(\mathcal{T}_R(M)\big) \subseteq \mathcal{T}_R(N)$, it induces a morphism

$$\mathcal{T}_R(\varphi) = \varphi_{|\mathcal{T}_R(M)} \ : \ \mathcal{T}_R(M) \longrightarrow \mathcal{T}_R(N) \ .$$

To check left exactness, let $0 \longrightarrow M \stackrel{\varphi}{\longrightarrow} N \stackrel{\psi}{\longrightarrow} L$ be an exact sequence of R-modules. It induces the sequence

$$0 \longrightarrow \mathcal{T}_R(M) \stackrel{\varphi}{\longrightarrow} \mathcal{T}_R(N) \stackrel{\psi}{\longrightarrow} \mathcal{T}_R(L) \ ,$$

where $\psi \circ \varphi = 0$ and φ is still injective (restriction of an injective map). It remains to show that a torsion element in $\ker \psi$ is the image of a torsion element under φ. Let $n \in \mathcal{T}_R(N)$ such that $\psi(n) = 0$. Then $\exists\, m \in M$ such that $n = \varphi(m)$ by exactness of the initial sequence. From $n \in \operatorname{im} \varphi$ and injectivity of φ, it follows again from Lemma C.1.9 that $m \in \mathcal{T}_R(M)$. □

Remark 1.3.2. In general, the functor is not right exact. Consider for example the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \stackrel{\cdot 2}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \ .$$

Then $\mathbb{Z}$ is torsion-free (over itself), but $\mathcal{T}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) \neq \{0\}$ by Example C.1.6.

Proposition 1.3.3. [[21], 3.8, p.7] and [[11], 15.16.8]
If R is a Noetherian ring, we have the following criterion for torsion-freeness of an R-module M:

M *is torsion-free*

$\Leftrightarrow$ *all associated primes of M are contained in the associated primes of R*

$\Leftrightarrow$ $\forall\, P \in \operatorname{Ass}_R(M),\ \exists\, Q \in \operatorname{Ass}_R(R)$ *such that* $P \subseteq Q$.

where $\operatorname{Ass}_R(R) = \operatorname{Ass}\big(\{0\}\big)$. *In particular, torsion-free modules over integral domains are exactly those whose only associated prime is* $\{0\}$.

Proof. Denote the associated primes of R by $P_1, \ldots, P_\alpha$. By Remark B.3.2 they correspond to the associated primes of the zero ideal $\{0\} \trianglelefteq R$.

$\underline{\Rightarrow}$: Assume that M is torsion-free and let $P = \text{Ann}_R(x)$ for some $x \in M$, $x \neq 0$ be an associated prime of M. P cannot contain a NZD, otherwise this NZD annihilates x, which would thus be a non-zero torsion element. Hence P only contains zero-divisors and $P \subseteq P_1 \cup \ldots \cup P_\alpha$, which by Prime Avoidance (Lemma B.1.3) implies that $P \subseteq P_i$ for some $i \in \{1, \ldots, \alpha\}$.

$\underline{\Leftarrow}$: By contraposition, assume that M has a non-zero torsion element m with NZD $r \in R$. Denote $A := \text{Ann}_R(m)$. Thus $r \in A$. Now consider the injection $R/A \hookrightarrow M : \bar{a} \mapsto a * m$. We have $A \neq R$ as $1 * m \neq 0$, hence $R/A \neq \{0\}$ and $\text{Ass}_R(R/A) \neq \emptyset$ by Proposition B.3.4 since R is Noetherian. Let $P = \text{Ann}_R(\bar{c})$ for some $\bar{c} \in R/A$ be an associated prime of R/A. Proposition B.3.5 and the exact sequence $0 \to R/A \to M$ imply that $P \in \text{Ass}_R(M)$ as well. On the other hand, $r \in P$ since $r * \bar{c} = \bar{r} \cdot \bar{c} = \bar{0}$ because $r \in A$. Thus P is an associated prime of M and contains the NZD r. So it cannot be contained in any of the P_i, otherwise it would only consist of zero-divisors. □

Remark 1.3.4. $A = \text{Ann}_R(m)$ is an annihilator ideal, but it does not need to be prime. So we do not necessarily have $A \in \text{Ass}_R(M)$.

1.3.2 Characterization of torsion modules

After discussing the case of torsion-freeness we are now interested in finding a criterion that indicates under which conditions we have a torsion module. Again it suffices to look at the associated primes.

Proposition 1.3.5. *Let $P_1, \ldots, P_\alpha$ be the associated primes of a Noetherian ring R and M a finitely generated R-module. The following conditions are equivalent:*

1) M is a torsion module.

2) $\text{Ann}_R(M)$ contains a NZD.

3) $\text{Ann}_R(M) \nsubseteq P_i$ for all $i \in \{1, \ldots, \alpha\}$.

Proof. The equivalence 1) $\Leftrightarrow$ 2) is proven in Lemma C.1.2 since M is finitely generated.

2) $\Rightarrow$ 3) : Let $r \in \text{Ann}_R(M)$ be a NZD. Since $P_1 \cup \ldots \cup P_\alpha$ is the set of all zero-divisors in R, we have $r \notin P_i$, $\forall\, i$. Hence $\text{Ann}_R(M) \nsubseteq P_i$ for all i.

3) $\Rightarrow$ 2) : If $\mathrm{Ann}_R(M)$ only contains zero-divisors, then $\mathrm{Ann}_R(M) \subseteq P_1 \cup \ldots \cup P_\alpha$, hence by Prime Avoidance we have $\mathrm{Ann}_R(M) \subseteq P_j$ for some j, which is a contradiction. So $\mathrm{Ann}_R(M)$ must contain a NZD. □

1.3.3 Behaviour of torsion under localization

Let M be an R-module and $P \in \operatorname{Spec} R$. By Definition A.2.1 we know that the localization M_P is a module over R_P via $\frac{r}{s} * \frac{m}{a} = \frac{r*m}{s \cdot a}$. In this section we want to analyze how the torsion of M behaves under localization. Let us again denote the associated primes of R by $P_1, \ldots, P_\alpha$.

Proposition 1.3.6. [[43], p.204] and [[66], p.8]
Let R be a Noetherian ring an $r \in R$ with $r \neq 0$. Then r is a NZD if and only if $\frac{r}{1} \in R_P$ is a NZD for all $P \in \operatorname{Spec} R$.

Proof. $\underline{\Rightarrow}$: (also works in the non-Noetherian case) r being a NZD means that $\nexists\, t \in R$, $t \neq 0$ such that $r \cdot t = 0$. Fix any $P \in \operatorname{Spec} R$. Then $\frac{r}{1} \neq 0$ since $b \cdot r \neq 0$, $\forall\, b \notin P$. And if there is an element $\frac{s}{a} \in R_P$ such that $\frac{r}{1} \cdot \frac{s}{a} = 0$, then $\exists\, b \notin P$ such that $brs = bs \cdot r = 0$. Since r is a NZD, we need $b \cdot s = 0$, which means that $\frac{s}{a} = 0$. It follows that $\frac{r}{1}$ is a NZD as well.
$\underline{\Leftarrow}$: Let r be a zero-divisor; we show that in this case there exists a prime ideal P such that $\frac{r}{1}$ remains a zero-divisor in R_P. As r is a zero-divisor, we have $r \in P_1 \cup \ldots \cup P_\alpha$, hence $r \in P_i$ for some i. But then $\frac{r}{1}$ is a zero-divisor in R_{P_i} because of (1.6); the zero-divisors in R_{P_i} are given by the localizations of the assciated primes that are contained in P_i, so in particular for P_i itself. Thus the statement follows by contraposition. □

Remark 1.3.7. If $\frac{r}{1}$ is a NZD in R_P for some P, this does not imply that r is a NZD in R. In Example 1.4.29 we will see that there exist zero-divisors that become NZDs in the localization, e.g. if the associated primes are

$$P_1 \cup \ldots \cup P_\gamma \cup P_{\gamma+1} \cup \ldots \cup P_\alpha \,,$$

where $P_1, \ldots, P_\gamma \subseteq P$, then a zero-divisor in P_j with $j > \gamma$ becomes a NZD in the localization R_P.

Proposition 1.3.8. cf. [[52], 35328]
Let M be a module over a Noetherian ring R and $m \in M$. Then

$$m \in \mathcal{T}_R(M) \;\Leftrightarrow\; \tfrac{m}{1} \in \mathcal{T}_{R_P}(M_P)\,,\; \forall\, P \in \operatorname{Spec} R\,. \tag{1.9}$$

Proof. $\underline{\Rightarrow}$: (also works in the non-Noetherian case) Let $m \in \mathcal{T}_R(M)$ and $r \in R$, $r \neq 0$ be a NZD such that $r * m = 0$. Thus $\frac{r}{1} * \frac{m}{1} = 0$ and from Proposition 1.3.6 we know that $\frac{r}{1} \in R_P$ is a NZD for each prime ideal P, hence $\frac{m}{1}$ (which may be zero for some P's, but not for all, except if $m = 0$, see Corollary A.2.12) is a torsion element in each M_P.
$\underline{\Leftarrow}$: Let $m \notin \mathcal{T}_R(M)$; we show that in this case there exists a prime ideal Q such that $\frac{m}{1}$ is not a torsion element in M_Q. As m is not a torsion element in R, it cannot be annihilated by a NZD, hence its annihilator $\operatorname{Ann}_R(m)$ is contained in the set $P_1 \cup \ldots \cup P_\alpha$ of all zero-divisors. By Prime Avoidance we thus have $\operatorname{Ann}_R(m) \subseteq P_i$ for some $i \in \{1, \ldots, \alpha\}$. Let $Q := P_i$; then $\frac{m}{1} \notin \mathcal{T}_{R_Q}(M_Q)$. Indeed, first note that $\frac{m}{1} \neq 0$ since all elements that annihilate m belong to $\operatorname{Ann}_R(m)$ and are hence in Q, thus $a * m \neq 0$, $\forall\, a \in R \setminus Q$. Next assume that $\frac{r}{s} * \frac{m}{1} = 0$ for some $\frac{r}{s} \in R_Q$, which means that

$$b * (r * m) = 0 \;\Leftrightarrow\; (b \cdot r) * m = 0$$

for some $b \notin Q$. This implies $b \cdot r \in \operatorname{Ann}_R(m) \subseteq Q$, hence $r \in Q$ since Q is prime and it follows again from (1.6) that $\frac{r}{1}$ is a zero-divisor in R_Q. Thus $\frac{m}{1}$ is a non-zero element that can only be annihilated by zero-divisors, i.e. it is not a torsion element in M_Q. □

Remark 1.3.9. The equivalence (1.9) is true for arbitrary modules since the proof does not need an assumption on M to be finitely generated. However it does not hold in the non-Noetherian case. An example is given in [[52], 35328]. In the proof of Proposition 1.3.8 we use the fact that R is Noetherian to ensure existence of the associated primes.

Corollary 1.3.10. *If M is a module over a Noetherian ring R such that the localization M_P is a torsion-free R_P-module for all $P \in \operatorname{Spec} R$, then M is torsion-free module over R.*

Proof. Let $m \in M$ be such that $r * m = 0$ for some NZD $r \in R$. Then $\frac{r}{1} \in R_P$ is also a NZD by Proposition 1.3.6, hence $\frac{r}{1} * \frac{m}{1} = 0$ for all $P \in \operatorname{Spec} R$. Since all M_P are torsion-free, we need that $\frac{m}{1} = 0$ for all P, hence $m = 0$ by Corollary A.2.12 and thus $\mathcal{T}_R(M) = \{0\}$. □

Remark 1.3.11. If R is an integral domain, the converse of Corollary 1.3.10 holds true as well, see Proposition C.4.14. However it is false in general! If M is torsion-free, then there may exist prime ideals $P \trianglelefteq R$ such that M_P has non-trivial torsion; we will see an example in Section 2.3.
However the statement that torsion-freeness of M implies torsion-freeness of the localizations also holds true in a more general setting; this will be the aim of Proposition 2.2.20 and Corollary 2.2.22.

1.4 Relation between torsion and dimension of the support

Now we are ready to prove the main result of the first chapter. We want to show that there is a relation between the torsion of a finitely generated module and the support of the coherent sheaf that it defines. Indeed in Proposition 1.4.21 we see that torsion modules are supported in smaller dimension and Theorem 1.4.23 will prove that the dimension of the support of a torsion module actually drops in all irreducible components of the scheme. Finally we will also illustrate these results on several examples.

1.4.1 Scheme structure of the support of a sheaf

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally Noetherian scheme scheme and $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$. The goal of this section is to define the support of $\mathcal{F}$ on $\mathcal{X}$ by putting a suitable subscheme structure on it. Here there is no canonical choice, but there are two structures which are more relevant than the other ones.

Definition 1.4.1. [[16], p.492-493]
Let M be an R-module of finite presentation with a generating set $\{m_1, \dots, m_n\}$

and consider

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

for some $m \in \mathbb{N}$. The matrix A is a morphism between free modules and its entries are elements in R which encode the relations $r_1 * m_1 + \ldots + r_n * m_n = 0$ between the generators. Here we consider the vectors of the free modules as rows and multiply them by the matrix on the right. The *Fitting ideal* of M, denoted by $\mathrm{Fitt}_0(M)$, is defined as the ideal generated by all minors of A (determinants of submatrices) of order n.[5] If $m < n$, we set $\mathrm{Fitt}_0(M) := \{0\}$ by definition.

Lemma 1.4.2 (Fitting's Lemma)**.** [[16], p.493-495] and [[18], V-9, p.219-220]
1) The Fitting ideal does not depend on the generators or the presentation of M.
2) $\mathrm{Ann}_R(M)^n \subseteq \mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}_R(M)$.
3) In particular, the Fitting ideal and the annihilator ideal have the same radical.

Now we apply the same idea to coherent sheaves on a scheme. Recall that every closed subscheme of an affine scheme $\operatorname{Spec} R$ is of the form $V(I) \cong \operatorname{Spec}(R/I)$ for some ideal $I \trianglelefteq R$ and thus given by an ideal sheaf $\widetilde{I} \subseteq \mathcal{O}_R$. Vice-versa, every ideal sheaf defines a closed subscheme. On an arbitrary scheme, the same construction works locally.

Definition 1.4.3. [[18], V-10, p.220]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ and $\{U_i\}_i$ with $U_i \cong \operatorname{Spec} R_i$ be an affine covering such that $\mathcal{F}|_{U_i}$ is given by some finitely generated R_i-module M_i. The Fitting ideals $\mathrm{Fitt}_0(M_i)$ coincide on the intersections $U_i \cap U_j$ and hence define an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_\mathcal{X}$. We denote $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{F})$.

As a topological space the *support* of $\mathcal{F}$ is defined as the set

$$\operatorname{supp} \mathcal{F} = \left\{ x \in \mathcal{X} \;\middle|\; \mathcal{F}_x \neq \{0\} \right\} ,$$

on which we now want to put the structure of a scheme. Note that if $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$, then we have a homeomorphism $\operatorname{supp}(\mathcal{F}|_{U_i}) \cong \operatorname{supp}(M_i)$, where the support of M_i is as in Definition B.3.10.

[5] This is actually the definition of the 0^{th} Fitting ideal of M. As we do not need the other ones, we omit the general definition.

Proposition 1.4.4. cf. [[35], II, Ex. 5.6, p.124] , [[6], II.§4.n°4.Prop.17, p.133]
Let R be a Noetherian ring, M a finitely generated R-module and consider the coherent sheaf $\mathcal{F} = \widetilde{M}$ on $\mathcal{X} = \operatorname{Spec} R$. Then $\operatorname{supp} \mathcal{F}$ is closed and, as a topological space, given by $V(\operatorname{Ann}_R(M))$.

Proof. The statement already follows from Proposition B.3.11 since

$$\begin{aligned}\operatorname{supp} M &= \left\{ P \in \operatorname{Spec} R \;\middle|\; M_P \neq \{0\} \right\} \\ &= \left\{ P \in \operatorname{Spec} R \;\middle|\; \operatorname{Ann}_R(M) \subseteq P \right\} = V\big(\operatorname{Ann}_R(M)\big) .\end{aligned}$$

However let us also prove directly that $Z = \operatorname{supp} \mathcal{F}$ is closed. For this we show that $U := \mathcal{X} \setminus Z$ is open, i.e. if $M_P = \{0\}$ for some $P \in \operatorname{Spec} R$, there exists an open neighborhood V of P such that $M_Q = \{0\}$ for all $Q \in V$.
Let $P \in U$ be fixed. If $M_P = \{0\}$, then every generator m_i satisfies $\frac{m_i}{1} = 0$, i.e. $\exists\, r_i \in R \backslash P$ such that $r_i * m_i = 0, \forall\, i \in \{1, \ldots, n\}$. Define $V := D(r_1) \cap \ldots \cap D(r_n)$. Then V is open and any element $Q \in V$ satisfies $r_i \notin Q, \forall\, i$ (thus $P \in V$). Hence $\forall\, i$, $\frac{m_i}{s} = 0$ in M_Q for such Q. It follows that $\frac{m}{s} = 0$ for all $m \in M$ since M is finitely generated and $M_Q = \{0\}$ for all $Q \in V$. Thus $V \subseteq U$. □

Thus any ideal $I \trianglelefteq R$ satisfying $\operatorname{Rad}(I) = \operatorname{Rad}(\operatorname{Ann}_R(M))$ defines the same topological subspace of $\operatorname{Spec} R$. In particular $\operatorname{Rad}(\operatorname{Ann}_R(M))$ is the biggest ideal that introduces a closed subscheme structure on the support. Moreover, as a radical ideal, Lemma 1.1.5 implies that it is the only one which defines a reduced scheme structure. By Lemma 1.4.2, we know that $\operatorname{Fitt}_0(M)$ and $\operatorname{Ann}_R(M)$ have the same radical. This motivates

Definition 1.4.5. Let R be a Noetherian ring, M a finitely generated R-module and $\mathcal{F} = \widetilde{M}$. Denote $I = \operatorname{Ann}_R(M)$ and $I' = \operatorname{Fitt}_0(M)$. We define the *annihilator support* of $\mathcal{F}$ as the closed subscheme $V(I) \cong \operatorname{Spec}(R/I)$ and the *Fitting support* of $\mathcal{F}$ by $V(I') \cong \operatorname{Spec}(R/I')$.
If $\mathcal{X}$ is an arbitrary locally Noetherian scheme and $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_\mathcal{X})$, the same definition applies locally on an affine covering where $\mathcal{F}$ is described by finitely generated modules whose Fitting ideals glue to the ideal sheaf $\operatorname{Fitt}_0(\mathcal{F}) \subseteq \mathcal{O}_\mathcal{X}$.
We denote the annihilator support of $\mathcal{F}$ by $\mathcal{Z}_a(\mathcal{F})$ and its Fitting support by $\mathcal{Z}_f(\mathcal{F})$. $\mathcal{Z}_a(\mathcal{F})$ can be seen as the minimal closed subscheme structure which can

be put on $\operatorname{supp}\mathcal{F}$ since its structure sheaf is obtained by dividing out the functions that vanish on the support. $\mathcal{Z}_f(\mathcal{F})$ is defined via $\operatorname{Fitt}_0(\mathcal{F})$, which is locally generated by the minors of a finite free presentation of $\mathcal{F}$, thus it encodes the relations between the local generators and takes care of the locally free resolution of $\mathcal{F}$.

Remark 1.4.6. In general $\mathcal{Z}_a(\mathcal{F})$ is a proper closed subscheme of $\mathcal{Z}_f(\mathcal{F})$ and the structure of the Fitting support may be richer than the one of the annihilator support, see Example 1.4.9. We will point out some other essential differences between both supports in Section 3.4.

Example 1.4.7. Let $I \trianglelefteq R$ be an ideal and consider the R-module $M = R/I$ with $\mathcal{F} = \widetilde{M}$. Then

$$\mathcal{Z}_a(\mathcal{F}) = \mathcal{Z}_f(\mathcal{F}) = V(I) \cong \operatorname{Spec}(R/I) \ .$$

Indeed we have $\operatorname{Ann}_R(M) = \operatorname{Ann}_R(R/I) = I$. For the Fitting support, note that M is generated by $\bar{1}$, hence a relation $r * \bar{1} = \bar{0}$ implies that r annihilates all elements in M and vice-versa since every $m \in M$ writes as $m = a * \bar{1}$ for some $a \in R$. Thus $\operatorname{Fitt}_0(M) = I$ as well.

This example already illustrates the following fact.

Lemma 1.4.8. *If $M \in \operatorname{Mod}(R)$ is generated by 1 element, then*

$$\operatorname{Ann}_R(M) = \operatorname{Fitt}_0(M) \qquad \textit{and} \qquad \mathcal{Z}_a(\widetilde{M}) = \mathcal{Z}_f(\widetilde{M}) \ .$$

Proof. If M is generated by some $m \in M$, then an element $r \in R$ defines a relation $r * m = 0$ if and only if r belongs to $\operatorname{Ann}_R(M)$ since all elements in M are multiples of m. □

Example 1.4.9. [[18], p.220]
Let $R = \mathbb{K}[X]$, $I = \langle X \rangle$, $J = \langle X^2 \rangle$. Consider the affine line $\mathcal{X} = \operatorname{Spec} R = \mathbb{A}^1_{\mathbb{K}}$ with the closed subschemes $\mathcal{Y}_1 = V(I)$ and $\mathcal{Y}_2 = V(J)$. As topological spaces we have $\mathcal{Y}_1 = \mathcal{Y}_2$ since

$$\mathcal{Y}_1 \cong \operatorname{Spec}(R/I) \cong \operatorname{Spec}\mathbb{K} \qquad \text{and} \qquad \mathcal{Y}_2 \cong \operatorname{Spec}(R/J) \cong \operatorname{Spec}\big(\mathbb{K}[\varepsilon]\big) \ ,$$

where $\varepsilon^2 = 0$, so both just consist of 1 point. However the schemes are different since $R/I \cong \mathbb{K}$ and $R/J \cong \mathbb{K} \oplus \mathbb{K}\varepsilon$, so $\mathcal{O}_{\mathcal{Y}_2} = \mathcal{O}_{R/J}$ has more sections than $\mathcal{O}_{\mathcal{Y}_1} = \mathcal{O}_{R/I}$. Indeed $\mathcal{Y}_1$ is a single point and $\mathcal{Y}_2$ is a double point, both sitting in the affine line $\mathbb{A}^1_{\mathbb{K}}$. The projection $R/J \to R/I$ actually implies that $\mathcal{Y}_1$ is a proper closed subscheme of $\mathcal{Y}_2$ (even though their underlying topological spaces are equal). Now consider the sheaves

$$\mathcal{F} := \mathcal{O}_{\mathcal{Y}_2} \qquad \text{and} \qquad \mathcal{G} := \mathcal{O}_{\mathcal{Y}_1} \oplus \mathcal{O}_{\mathcal{Y}_1} \ .$$

Since $\operatorname{Ann}_R(R/J) = J$ and $\operatorname{Ann}_R(R/I \oplus R/I) = I$, we obtain $\mathcal{Z}_a(\mathcal{F}) = \mathcal{Y}_2$ and $\mathcal{Z}_a(\mathcal{G}) = \mathcal{Y}_1$. Multiplying by the generators of the annihilators we moreover obtain the resolutions

$$R \xrightarrow{\varphi} R \longrightarrow R/J \longrightarrow 0 \qquad \text{and} \qquad R \oplus R \xrightarrow{A} R \oplus R \longrightarrow R/I \oplus R/I \longrightarrow 0 \ ,$$

where $\varphi = X^2$ and

$$A = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \ .$$

Since $\det \varphi = \det A = X^2$, we get $\operatorname{Fitt}_0(R/J) = \operatorname{Fitt}_0(R/I \oplus R/I) = J$ and therefore the Fitting supports are $\mathcal{Z}_f(\mathcal{F}) = \mathcal{Z}_f(\mathcal{G}) = \mathcal{Y}_2$. In particular we see that $\mathcal{Z}_a(\mathcal{G})$ is a proper subscheme of $\mathcal{Z}_f(\mathcal{G})$.

1.4.2 Application: irreducible components of the support

Let M be a finitely generated R-module defining a coherent sheaf $\mathcal{F} = \widetilde{M}$ on $\operatorname{Spec} R$. We want to decompose its support into irreducible components in the same way as described in Section 1.2.

Let $I \trianglelefteq R$ be any ideal such that $V(I) = \operatorname{supp} M$ as topological spaces (i.e. we choose a structure on the support of $\mathcal{F}$). If we want to find a decomposition of $V(I) \cong \operatorname{Spec}(R/I)$, we shall find a primary decomposition of the zero ideal $\{\bar{0}\}$ in the ring R/I. By Proposition B.2.10 this can be done by finding the primary decomposition of I in R. This one then goes down to a primary decomposition of $\{\bar{0}\}$ in R/I and gives the associated primes which define the irreducible components of $\operatorname{Spec}(R/I)$.

Note that the decomposition depends on the chosen ideal I, e.g. the irreducible components of the Fitting support and the annihilator support are in general not the same.

1.4.3 Some facts about dimensions of ideals

In order to prove our main theorems about torsion and dimension, we need some classical results about the dimension of ideals in a Noetherian ring, which we briefly state in this subsection.

Definition 1.4.10. [[54], 12.A, p.71]
Let R be a ring. If $P \trianglelefteq R$ is a prime ideal, we define its height, denoted by $\mathrm{ht}(P)$, to be the biggest number of proper inclusions in a chain of prime ideals contained in P. This notion extends to arbitrary ideals: if $I \trianglelefteq R$ is any ideal, we define the *height* of I to be

$$\mathrm{ht}(I) := \inf \left\{ \mathrm{ht}(P) \;\middle|\; P \in \operatorname{Spec} R \text{ such that } I \subseteq P \right\} .$$

The *Krull dimension* of R is given by $\dim R = \sup\{ \mathrm{ht}(P) \mid P \in \operatorname{Spec} R \}$.

Example 1.4.11. It is known that the Krull dimension of the polynomial ring $\mathbb{K}[X_1, \ldots, X_n]$ is n. If R is Noetherian, then every prime ideal in R has finite height. But there also exist Noetherian rings of infinite Krull dimension.

Lemma 1.4.12. [[54], 12.A, p.72]
If $P \in \operatorname{Spec} R$, then $\mathrm{ht}(P)$ is equal to the Krull dimension of the localization R_P.

Proof. This follows from Proposition 1.1.1 which claims that prime ideals in a localization $S^{-1}R$ are in 1-to-1 correspondence with prime ideals in R which do not intersect S via $P \mapsto S^{-1}P$. Taking $S = R \setminus P$, a maximal chain of prime ideals in R which are contained in P thus corresponds to a maximal chain of prime ideals in R_P, and vice-versa. □

Proposition 1.4.13. [[54], 12.A, p.72]
For any ideal $I \trianglelefteq R$, we have

$$\dim(R/I) + \mathrm{ht}(I) \leq \dim R \,. \tag{1.10}$$

Proof. First note that the projection $\pi : R \twoheadrightarrow R/I$ gives a 1-to-1 correspondence between prime ideals in R/I and prime ideals in R containing I (if $\bar{P} \trianglelefteq R/I$ is prime, then so is $\pi^{-1}(\bar{P})$ and it contains I and by surjectivity of π, the image of a prime ideal $P \trianglelefteq R$ containing I is still an ideal and also prime because

$$\begin{aligned} \bar{a} \cdot \bar{b} \in \pi(P) \;&\Leftrightarrow\; \exists\, r \in P \text{ such that } \overline{a \cdot b} = \pi(r) = \bar{r} \\ &\Leftrightarrow\; a \cdot b - r \in \ker \pi = I \subseteq P \;\Rightarrow\; a \cdot b \in P \;) . \end{aligned}$$

Hence a maximal chain of prime ideals in R/I corresponds to a maximal chain of prime ideals in R containing I. Denote $\dim(R/I) = m$ and let $P_0 \trianglelefteq R$ be a prime ideal containing I which gives a maximal chain of inclusions:

$$I \subseteq P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_m \subsetneq R .$$

Let $\operatorname{ht}(P_0) = \ell$. So there is another chain of prime ideals $P'_\ell \subsetneq \ldots \subsetneq P'_1 \subsetneq P_0$. Adding both chains, we obtain a chain of prime ideals in R of length $m + \ell$, thus $\dim R \geq m + \ell$. But since $I \subseteq P_0$, we get $\ell \geq \operatorname{ht}(I)$. Finally

$$\dim(R/I) + \operatorname{ht}(I) \leq m + \ell \leq \dim R .$$

□

Remark 1.4.14. In some cases, one can even obtain equality, e.g. it is shown in [[54], 14.H, p.92] that if R is an integral domain which is also a finitely generated $\mathbb{K}$-algebra, then every prime ideal $P \trianglelefteq R$ satisfies $\dim(R/P) + \operatorname{ht}(P) = \dim R$.

Theorem 1.4.15 (Krull's Height Theorem)**.** [[54], 12.I, p.77] , [[55], 13.5, p.100] *Let R be a Noetherian ring and $I = \langle\, r_1, \ldots, r_k \,\rangle$ an ideal generated by k elements. Then every minimal prime ideal containing I (i.e. a prime that is minimal among all primes containing I) has height at most k. In particular,* $\operatorname{ht}(I) \leq k$.

Originally this theorem has been proven by Krull on induction. The base case $k = 1$ is the hardest part and is itself an important theorem.

Theorem 1.4.16 (Krull's Principal Ideal Theorem)**.** [[2], 11.17 & 11.18, p.122] *Let R be a Noetherian ring and $a \in R$ such that $a \neq 0$ and a is not a unit. Then the principal ideal $\langle\, a \,\rangle \neq R$ has height at most 1. Moreover it has height 1 if a is a NZD (the converse is false).*

Proof. We only prove the statements about NZDs. Let a be a NZD and assume that the height of $\langle a \rangle$ is 0. Let P be a prime ideal containing $\langle a \rangle$ and which is minimal for this property. Thus $\mathrm{ht}(P) = 0$. On the other hand, P must contain a minimal prime associated to $\{0\}$, see Proposition B.2.19. But this inclusion cannot be strict as P is of height 0. Thus P is a minimal prime, which contradicts that $a \in P$ as (1.5) implies that minimal primes only contain zero-divisors. It follows that $\langle a \rangle$ must be of height 1. □

Remark 1.4.17. cf. [[53], 334340]
If a is a zero-divisor, then it is still possible that $\langle a \rangle$ has height 1.[6] Consider e.g. Example E.1 and the zero-divisor $a = \bar{X}$. The only prime ideal in R containing $\bar{X}$ is $P_1 = \langle \bar{X}, \bar{Y} \rangle$, see (1.8), and this one is of height 1 as it contains $\langle \bar{Y} \rangle$. Thus

$$\mathrm{ht}\big(\langle \bar{Y} \rangle\big) = 0 \qquad , \qquad \mathrm{ht}\big(\langle \bar{X} \rangle\big) = \mathrm{ht}\big(\langle \bar{X}, \bar{Y} \rangle\big) = 1 \ .$$

In particular, this example shows that $\langle a \rangle$ being of height 1 does not necessarily mean that $\langle a \rangle$ contains a prime ideal itself (as it is the case for prime ideals).

Remark 1.4.18. The key element for the failure above is that $a = \bar{X}$ does not belong to any minimal prime. Indeed the converse is true if R does not have embedded primes (e.g. if R is reduced). If this is the case, then any zero-divisor $a \in R$ belongs to a minimal prime, which does not contain any other prime ideal, hence the height of $\langle a \rangle$ is 0.

Lemma 1.4.19. *Let P_j be an embedded prime of a Noetherian ring R. If we denote $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{X}_j = V(P_j)$, then $\dim \mathcal{X}_j < \dim \mathcal{X}$.*

Proof. Let $P_i \subsetneq P_j$ be a minimal prime contained in P_j and consider the projection $\varphi : R \to R/P_j$. If we have a maximal chain of prime ideals

$$\{\bar{0}\} \subsetneq Q_1 \subsetneq Q_2 \subsetneq \ldots \subsetneq Q_\ell$$

in the integral domain R/P_j, then

$$P_i \subsetneq P_j = \varphi^{-1}\big(\{\bar{0}\}\big) \subsetneq \varphi^{-1}(Q_1) \subsetneq \varphi^{-1}(Q_2) \subsetneq \ldots \subsetneq \varphi^{-1}(Q_\ell)$$

is a longer chain of prime ideals in R, hence $\dim R > \dim(R/P_j)$. □

[6] So there is an error on this webpage:
http://mathworld.wolfram.com/KrullsPrincipalIdealTheorem.html

Remark 1.4.20. In general, nothing can be said about the dimension of a component $\mathcal{X}_i = V(P_i)$ if P_i is a minimal prime. It may be equal to $\dim \mathcal{X}$ (Example 1.4.25) or drop as well (Example 1.4.26).

1.4.4 Theorem and examples

Now we prove that coherent sheaves given by torsion modules are supported in smaller dimension.

Proposition 1.4.21. *Let M be a finitely generated module over a Noetherian ring R. Denote $\mathcal{F} = \widetilde{M}$ and $\mathcal{X} = \operatorname{Spec} R$. If M is a torsion module, then*

$$\dim(\operatorname{supp} \mathcal{F}) < \dim \mathcal{X} \ . \tag{1.11}$$

The converse is true if R is an integral domain.

Proof. By Proposition 1.4.4 we have

$$\operatorname{supp} \mathcal{F} = V\big(\operatorname{Ann}_R(M)\big) \cong \operatorname{Spec}\big(R/\operatorname{Ann}_R(M)\big) \ .$$

Hence in order to prove (1.11), it suffices to show that the Krull dimensions satisfy

$$\dim\big(R/\operatorname{Ann}_R(M)\big) < \dim R \ .$$

Since M is a finitely generated torsion module, we know by Proposition 1.3.5 that there is a NZD $a \in \operatorname{Ann}_R(M)$ with $a \neq 0$. By Krull's Principal Ideal Theorem, $\langle\, a \,\rangle$ has height 1 and thus (1.10) implies that

$$\dim\big(R/\langle\, a \,\rangle\big) + 1 \leq \dim R \ .$$

Since $\langle\, a \,\rangle \subseteq \operatorname{Ann}_R(M)$, we moreover have a projection $R/\langle\, a \,\rangle \twoheadrightarrow R/\operatorname{Ann}_R(M)$, which implies that

$$\dim\big(R/\langle\, a \,\rangle\big) \geq \dim\big(R/\operatorname{Ann}_R(M)\big) \ .$$

Hence $\dim(R/\operatorname{Ann}_R(M)) \leq \dim R - 1 < \dim R$.
Conversely, let R be an integral domain and assume that M is not a torsion module. Proposition 1.3.5 then says that $\operatorname{Ann}_R(M)$ only contains zero-divisors, i.e. $\operatorname{Ann}_R(M) = \{0\}$, so the Krull dimensions are equal. By contraposition: if $\dim\big(R/\operatorname{Ann}_R(M)\big) < \dim R$, then M must be a torsion module. $\square$

Remark 1.4.22. Actually the converse of Proposition 1.4.21 also holds true in a more general case. We will see that one in Corollary 3.1.25. But in general it may fail (in the non-integral case) even if M is torsion-free. We will illustrate this in Example 1.4.25 and Example 1.4.26.

So we see that the dimension of the support of a sheaf given by a torsion module always drops at least by 1, i.e. $\operatorname{codim}_{\mathcal{X}}(\operatorname{supp}\mathcal{F}) \geq 1$. However there is an even stronger result, which has mainly been pointed out by O. Iena.

Theorem 1.4.23. *Let M be a finitely generated module over a Noetherian ring R. Denote $\mathcal{F} = \widetilde{M}$, $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{X}_i = V(P_i)$ for all i, where $P_1, \dots, P_\alpha$ are the associated primes of R. Then M is a torsion module if and only if the codimension of $\operatorname{supp}\mathcal{F}$ is positive along each irreducible component:*

$$\operatorname{codim}_{\mathcal{X}_i}\big((\operatorname{supp}\mathcal{F}) \cap \mathcal{X}_i\big) \geq 1 \ , \quad \forall\, i \in \{1, \dots, \alpha\} \ . \tag{1.12}$$

By negation, M is not a torsion module if and only if there is at least 1 component in which the dimension does not drop: $\exists\, i \in \{1, \dots, \alpha\}$ such that

$$\dim\big((\operatorname{supp}\mathcal{F}) \cap \mathcal{X}_i\big) = \dim \mathcal{X}_i \ . \tag{1.13}$$

Proof. Assume that M is not a torsion module, so by Proposition 1.3.5 we have $\operatorname{Ann}_R(M) \subseteq P_i$ for some $i \in \{1, \dots, \alpha\}$. This P_i may be a minimal or an embedded prime. But then

$$\begin{aligned} \operatorname{Ann}_R(M) \subseteq P_i \ &\Leftrightarrow\ \operatorname{Rad}\big(\operatorname{Ann}_R(M)\big) \subseteq P_i \\ &\Leftrightarrow\ V(P_i) \subseteq V\big(\operatorname{Ann}_R(M)\big) \ \Leftrightarrow\ \mathcal{X}_i \subseteq \operatorname{supp}\mathcal{F} \ . \end{aligned}$$

Hence M is not completely torsion if and only if there is a component $\mathcal{X}_i$ which is completely contained in the support of $\mathcal{F}$, i.e. all stalks of $\mathcal{F}$ on $\mathcal{X}_i$ are non-zero. By negation, this means:
M is a torsion module if and only if no component $\mathcal{X}_i$ is completely included in $\operatorname{supp}\mathcal{F}$, so on each $\mathcal{X}_i$ there are stalks of $\mathcal{F}$ which are zero. By coherence of $\mathcal{F}$ this implies that there is an open neighborhood $U \subseteq \mathcal{X}$ with $U \cap \mathcal{X}_i \neq \emptyset$ on which all stalks of $\mathcal{F}$ are zero. Since $\mathcal{X}_i$ is an irreducible component, its underlying topological space is irreducible and $U \cap \mathcal{X}_i$ is dense in $\mathcal{X}_i$. It follows

that the complement $(\operatorname{supp}\mathcal{F}) \cap \mathcal{X}_i$ is closed and proper in $\mathcal{X}_i$. Thus it must be of codimension ≥ 1. □

Remark 1.4.24. Note that (1.12) is a much stronger condition than (1.11) as it says that the dimension drops in every component! In particular this is important if the components of $\operatorname{Spec} R$ have different dimensions, see e.g. Example 1.4.26. It may also be possible that the codimension in one (or more) of the components is strictly bigger than 1. We will e.g. encounter the case of codimension 2 at the end of Example 1.4.27.

Example 1.4.25. Consider again Example E.1. R is not reduced and we have an embedded prime $P_2 \subsetneq P_1$. Let us first compute the dimensions of $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{X}_1 = V(P_1)$ and $\mathcal{X}_2 = V(P_2)$.
Since $\mathbb{K}[X,Y] \twoheadrightarrow R$ and $\mathbb{K}[X,Y]$ is an integral domain, we have $\dim R < 2$. The chain $\langle \bar{Y} \rangle \subsetneq \langle \bar{X}, \bar{Y} \rangle$ then implies that $\dim R = 1$ (note that $\{\bar{0}\}$ and $\langle \bar{X} \rangle$ are not prime ideals). Since $R/P_1 \cong \mathbb{K}$ and $R/P_2 \cong \mathbb{K}[\bar{X}]$, we obtain $\dim(R/P_1) = 0$ and $\dim(R/P_2) = 1$. Finally $\dim \mathcal{X} = 1$, $\dim \mathcal{X}_1 = 0$ and $\dim \mathcal{X}_2 = 1$ (which is intuitively clear as $\mathcal{X}$ consists of a line with an embedded double point).

Now consider $M = \mathbb{K}$. We give it an R-module structure by first considering it as $\mathbb{K} \cong \mathbb{K}[X,Y]/\langle X, Y \rangle$ as a module over $\mathbb{K}[X,Y]$. Then

$$\langle Y^2, XY \rangle \subseteq \operatorname{Ann}_{\mathbb{K}[X,Y]}(\mathbb{K}) = \langle X, Y \rangle ,$$

so $\mathbb{K}$ is also a module over R; the structure is given by $\bar{f} * \lambda = \lambda \cdot f(0)$. Actually this is the same as $\mathbb{K} \cong R/\langle \bar{X}, \bar{Y} \rangle$. So we also see that $\mathbb{K}$ is generated by 1 (as an R-module), hence $\mathcal{F} = \widetilde{\mathbb{K}}$ is a coherent sheaf on $\mathcal{X}$. To find its support, we have to compute $\operatorname{Ann}_R(\mathbb{K})$. Let $\bar{f} \in R$ and $\lambda \in \mathbb{K}$ such that $\bar{f} * \lambda = 0$, i.e.

$$\bar{f} * \lambda = 0 \;\Leftrightarrow\; \lambda \cdot f(0) = 0 \;\Leftrightarrow\; \lambda = 0 \text{ or } f(0) = 0 \;\Leftrightarrow\; \lambda = 0 \text{ or } \bar{f} \in \langle \bar{X}, \bar{Y} \rangle ,$$

and we get $\operatorname{Ann}_R(\mathbb{K}) = \langle \bar{X}, \bar{Y} \rangle$. But $\langle \bar{X}, \bar{Y} \rangle = P_1 \cup P_2$ is the set of all zero-divisors in R. It follows by Proposition 1.3.5 that $\mathbb{K}$ is not a torsion module over R. More precisely, we even have $\mathcal{T}_R(\mathbb{K}) = \{0\}$ since only zero-divisors can annihilate non-zero elements, i.e. $\mathbb{K}$ is torsion-free over R. This can also be seen by computing the associated primes of M. By Remark B.3.2 we have

$$\operatorname{Ass}_R(M) = \operatorname{Ass}_R\left(R/\langle \bar{X}, \bar{Y} \rangle\right) = \operatorname{Ass}\left(\langle \bar{X}, \bar{Y} \rangle\right) = \left\{ \langle \bar{X}, \bar{Y} \rangle \right\}$$

since $\langle \bar{X}, \bar{Y} \rangle$ is already a prime ideal in R. Proposition 1.3.3 now implies that M is torsion-free over R since $\langle \bar{X}, \bar{Y} \rangle \subseteq P_1$. As $\operatorname{Ann}_R(\mathbb{K}) = P_1$ is an embedded prime, we also see that

$$\operatorname{supp} \mathcal{F} = V\big(\operatorname{Ann}_R(\mathbb{K})\big) = V(P_1) = \mathcal{X}_1$$

is 0-dimensional. Thus the converse of Proposition 1.4.21 is false: even sheaves that are given by torsion-free modules can have supports of lower dimension if the scheme is not integral. But (1.13) is satisfied because $(\operatorname{supp} \mathcal{F}) \cap \mathcal{X}_1 = \mathcal{X}_1$, so the dimension along the (embedded) component $\mathcal{X}_1$ did not drop.

Example 1.4.26. We want to analyze what happens in a space where the minimal components have different dimensions. Consider the space given by a plane in $\mathbb{A}^3_{\mathbb{K}}$ and a line passing through this plane. In coordinates, we thus need either $Z = 0$ or $X = Y = 0$, i.e. the ideal $\langle ZX, ZY \rangle$. Let $R := \mathbb{K}[X, Y, Z]/\langle ZX, ZY \rangle$, which is a reduced Noetherian ring, but not an integral domain. This will be called Example E.2.

The line is given by the ideal $L = \langle \bar{X}, \bar{Y} \rangle$ and the plane by $P = \langle \bar{Z} \rangle$. Their intersection gives $\{\bar{0}\} = L \cap P$, which is also the primary ideal decomposition of $\{\bar{0}\}$. Note that L and P are both prime ideals since $R/L \cong \mathbb{K}[\bar{Z}]$ and $R/P \cong \mathbb{K}[\bar{X}, \bar{Y}]$ are integral domains. In addition both are minimal, so there are no embedded primes (which was clear since R is reduced).

Next we compute $\dim R$: from $\mathbb{K}[X, Y, Z] \twoheadrightarrow R$, we get $\dim R < 3$. Moreover we have the chain of prime ideals

$$\langle \bar{Z} \rangle \subsetneq \langle \bar{X}, \bar{Z} \rangle \subsetneq \langle \bar{X}, \bar{Y}, \bar{Z} \rangle ,$$

so that $\dim R = 2$ (note that $\{\bar{0}\}$, $\langle \bar{X} \rangle$ and $\langle \bar{Y} \rangle$ are not prime). By the above, $\dim(R/L) = 1$ and $\dim(R/P) = 2$. In particular, we see that the dimension of the component $\mathcal{X}_L = V(L)$ is strictly smaller than the one of $\mathcal{X} = \operatorname{Spec} R$, even though L is a minimal prime. On the other hand, the dimension of $\mathcal{X}_P = V(P)$ is equal to $\dim \mathcal{X}$.

Now we consider $M = R/\langle \bar{X}, \bar{Y} \rangle \cong \mathbb{K}[\bar{Z}]$ which is an R-module via $\bar{f} * [\bar{g}] = [\bar{f} \cdot \bar{g}]$. As it is generated by $[\bar{1}]$, the sheaf $\mathcal{F} = \widetilde{M}$ is coherent on $\mathcal{X}$. Let us compute

$\mathrm{Ann}_R(M)$ and $\mathcal{T}_R(M)$. The set of all zero-divisors in R is given by

$$P \cup L = \langle \bar{Z} \rangle \cup \langle \bar{X}, \bar{Y} \rangle \ .$$

Note that $M = R/\langle \bar{X}, \bar{Y} \rangle$, seen as a ring, is an integral domain. Hence for all $\bar{f} \in R$ and $[\bar{g}] \in M$ with $[\bar{g}] \neq [\bar{0}]$ we have

$$\bar{f} * [\bar{g}] = [\bar{0}] \ \Leftrightarrow \ [\bar{f} \cdot \bar{g}] = [\bar{0}] \ \Leftrightarrow \ [\bar{f}] \cdot [\bar{g}] = [\bar{0}] \ \Leftrightarrow \ [\bar{f}] = [\bar{0}] \ \Leftrightarrow \ \bar{f} \in \langle \bar{X}, \bar{Y} \rangle \ ,$$

so that $\mathrm{Ann}_R(M) = \langle \bar{X}, \bar{Y} \rangle$ and $\mathcal{T}_R(M) = \{0\}$ since only zero-divisors can annihilate non-zero elements. This is again verified by the associated primes since $\mathrm{Ass}_R(M) = \{ \langle \bar{X}, \bar{Y} \rangle \}$ and we have $\langle \bar{X}, \bar{Y} \rangle \subseteq L$. Thus M is torsion-free, but the support

$$\mathrm{supp}\, \mathcal{F} = V\big(\mathrm{Ann}_R(M)\big) = V(L) = \mathcal{X}_L$$

is 1-dimensional. This is an example in a reduced ring which shows that the converse of Proposition 1.4.21 is not true. On the other hand (1.13) is still satisfied. Note in addition that $\mathcal{F} \cong \mathcal{O}_{\mathcal{X}_L}$ since $V(L) \cong \mathrm{Spec}(R/L)$, i.e. $\mathcal{F}$ is nothing but the structure sheaf of the component $\mathcal{X}_L$.

Example 1.4.27. We want to construct an example where we have a non-trivial torsion submodule. The idea is to consider a subscheme of $\mathbb{A}^3_{\mathbb{K}}$ that is made up of 2 perpendicular planes and a line that only passes through one of the planes, e.g. the planes described by the equations $Z = 0$ and $X = 0$ with the line $\{X = Y = 0\}$. This can be obtained by taking

$$R = \mathbb{K}[X,Y,Z]\big/\big\langle YZ(X-1), XZ(X-1) \big\rangle \ , \tag{1.14}$$

which is a reduced Noetherian ring. (1.14) will from now on be called Example E.3. The primary decomposition of the defining ideal is

$$\begin{aligned} I = \langle YZ(X-1)\, ,\, XZ(X-1) \rangle &= \langle Z \rangle \cap \langle X-1 \rangle \cap \langle X, Y \rangle \\ \Rightarrow \ \{\bar{0}\} &= \langle \bar{Z} \rangle \cap \langle \bar{X}-1 \rangle \cap \langle \bar{X}, \bar{Y} \rangle \ , \end{aligned}$$

where all the ideals in the decomposition in R are already prime (dividing out anyone of them will give an integral domain). Hence the associated primes are $P_1 = \langle \bar{Z} \rangle$, $P_2 = \langle \bar{X}-1 \rangle$, $P_3 = \langle \bar{X}, \bar{Y} \rangle$ and the set of zero-divisors in R reads

$$\mathrm{ZD}(R) = \langle \bar{Z} \rangle \cup \langle \bar{X}-1 \rangle \cup \langle \bar{X}, \bar{Y} \rangle \ .$$

Denote $\mathcal{X} = \operatorname{Spec} R$ with the irreducible components $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ defined by the associated primes. As expected, we obtain $\dim \mathcal{X} = 2$ because of $\mathbb{K}[X,Y,Z] \twoheadrightarrow R$ and we have the chain of prime ideals $\langle \bar{Z} \rangle \subsetneq \langle \bar{Z}, \bar{X} \rangle \subsetneq \langle \bar{Z}, \bar{X}, \bar{Y} \rangle$. Moreover $\dim \mathcal{X}_1 = 2$, $\dim \mathcal{X}_2 = 2$ and $\dim \mathcal{X}_3 = 1$ since

$$R/P_1 \cong \mathbb{K}[\bar{X}, \bar{Y}] \qquad , \qquad R/P_2 \cong \mathbb{K}[\bar{Y}, \bar{Z}] \qquad , \qquad R/P_3 \cong \mathbb{K}[\bar{Z}] \ .$$

So we get another example where not all irreducible components of $\mathcal{X}$ are of the same dimension.

Next we consider the R-module $M = R/\langle \bar{Y}\bar{Z} \rangle$ and the sheaf $\mathcal{F} = \widetilde{M}$. M is generated by $[\bar{1}] \in M$, so $\mathcal{F}$ is coherent. Intuitively the support of $\mathcal{F}$ consists of the intersection of $\operatorname{Spec} R$ with the union of the 2 planes $V(Z)$ and $V(Y)$. This gives the union of a plane and 2 lines:

$$\operatorname{supp} \mathcal{F} = \{Z = 0\} \cup \{X = Y = 0\} \cup \{X = 1, Y = 0\} \ .$$

To prove this rigorously, we use the method from Section 1.4.2 (here $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$ coincide since M is generated by 1 element) and compute the primary decomposition of $\operatorname{Ann}_R(M) = \langle \bar{Y}\bar{Z} \rangle$ in R.
Note that it is not $\langle \bar{Y}\bar{Z} \rangle = \langle \bar{Y} \rangle \cap \langle \bar{Z} \rangle$ since $\langle \bar{Y} \rangle$ is not a primary ideal (the quotient contains zero-divisors which are not nilpotent). Moreover we cannot take the primary decomposition of $\langle YZ \rangle$ in $\mathbb{K}[X,Y,Z]$ since this ideal does not contain the defining ideal I (compare Remark B.2.11). One has to decompose

$$\begin{aligned} \big\langle YZ \,,\, YZ(X-1) \,,\, XZ(X-1) \big\rangle &= \big\langle YZ \,,\, XZ(X-1) \big\rangle \\ &= \langle Z \rangle \cap \langle X-1, Y \rangle \cap \langle X, Y \rangle \ , \end{aligned}$$

which yields $\langle \bar{Y}\bar{Z} \rangle = \langle \bar{Z} \rangle \cap \langle \bar{X} - 1, \bar{Y} \rangle \cap \langle \bar{X}, \bar{Y} \rangle$ and gives the support we intuitively found before since

$$\operatorname{supp} \mathcal{F} = V\big(\operatorname{Ann}_R(M)\big) \cong \operatorname{Spec}\big(R/\operatorname{Ann}_R(M)\big) \ ,$$

so the decomposition of $\langle \bar{Y}\bar{Z} \rangle$ gives the primary decomposition of the zero ideal in $R/\operatorname{Ann}_R(M)$. We denote these components by $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ respectively. By

Remark B.3.2 we again have

$$\begin{aligned}\operatorname{Ass}_R(M) &= \operatorname{Ass}\big(\operatorname{Ann}_R(M)\big) = \operatorname{Ass}_R\left(\langle \bar{Y}\bar{Z}\rangle\right)\\ &= \left\{ P_1' = \langle \bar{Z} \rangle \ , \ P_2' = \langle \bar{X}-1, \bar{Y} \rangle \ , \ P_3' = \langle \bar{X}, \bar{Y} \rangle \right\} .\end{aligned}$$

In particular, we see that the support along the components $\mathcal{X}_1$ and $\mathcal{X}_3$ did not drop ($\mathcal{Z}_1 = \mathcal{X}_1$ and $\mathcal{Z}_3 = \mathcal{X}_3$), which means that M is not a torsion module by Theorem 1.4.23. This can also be seen by the fact that $\operatorname{Ann}_R(M)$ only contains zero-divisors (Proposition 1.3.5).

On the other hand, this does not imply that M is torsion-free. Proposition 1.3.3 actually implies that it is not since P_2' is an associated prime of M which is not contained in any of the P_i's. So let us find the torsion submodule $\mathcal{T}_R(M)$. $[\bar{g}] \in M$ is a torsion element if there exists a NZD $\bar{f} \in R$ such that

$$\bar{f} * [\bar{g}] = [\bar{0}] \ \Leftrightarrow \ [\bar{f} \cdot \bar{g}] = [\bar{0}] \ \Leftrightarrow \ \bar{f} \cdot \bar{g} \in \langle \bar{Y}\bar{Z} \rangle = \langle \bar{Z} \rangle \cap \langle \bar{X}-1, \bar{Y} \rangle \cap \langle \bar{X}, \bar{Y} \rangle \ .$$

Now $\bar{f} \cdot \bar{g} \in \langle \bar{Z} \rangle$ implies that either $\bar{f}$ or $\bar{g}$ is a multiple of $\bar{Z}$ since $\langle \bar{Z} \rangle$ is a prime ideal. But $\bar{f} \notin \langle \bar{Z} \rangle$ since it is a NZD, hence $\bar{g} \in \langle \bar{Z} \rangle$. Similarly we obtain $\bar{g} \in \langle \bar{X}, \bar{Y} \rangle$. The remaining ideal does not give additional information. So we can already say that

$$\bar{g} \in \langle \bar{Z} \rangle \cap \langle \bar{X}, \bar{Y} \rangle = \langle \bar{X}\bar{Z}, \bar{Y}\bar{Z} \rangle \ ,$$

and hence $[\bar{g}] \in \langle [\bar{X}\bar{Z}] \rangle$. But this necessary form is also sufficient since $[\bar{X}\bar{Z}]$ is a torsion element:

$$\big(\bar{Y}+\bar{X}-1\big) * [\bar{X}\bar{Z}] = \bar{Y} * [\bar{X}\bar{Z}] + (\bar{X}-1) * [\bar{X}\bar{Z}] = [\bar{X}\bar{Y}\bar{Z}] + [\bar{X}\bar{Z}(\bar{X}-1)] = [\bar{0}] \ ,$$

where $\bar{Y}+\bar{X}-1 \notin P_1 \cup P_2 \cup P_3$ is a NZD. It follows that $\mathcal{T}_R(M) = \langle [\bar{X}\bar{Z}] \rangle$. Denote $T = \mathcal{T}_R(M)$ and let us compute the support of the sheaf associated to this torsion module. Intuitively the support of $\widetilde{T}$ just consists of the line $\mathcal{Z}_2$ since it vanishes on $\{XZ = 0\} = V(X) \cup V(Z)$ inside of $\operatorname{supp} \mathcal{F}$. More precisely: for $P \in \operatorname{Spec} R$,

$$\begin{aligned}T_P = \{0\} \ \Leftrightarrow \ \tfrac{[\bar{X}\bar{Z}]}{\bar{1}} = 0 \ &\Leftrightarrow \ \exists\, \bar{f} \notin P \text{ such that } \bar{f} * [\bar{X}\bar{Z}] = [\bar{0}]\\ &\Leftrightarrow \ \exists\, \bar{f} \notin P \text{ such that } \bar{f}\bar{X}\bar{Z} \in \langle \bar{Y}\bar{Z} \rangle \ . \qquad (1.15)\end{aligned}$$

But $\langle \bar{Y}\bar{Z} \rangle \subseteq \langle \bar{X}-1, \bar{Y} \rangle$ which is prime and neither contains $\bar{X}$, nor $\bar{Z}$. Thus $\bar{f}\bar{X}\bar{Z} \in \langle \bar{Y}\bar{Z} \rangle$ if and only if $\bar{f} \in \langle \bar{X}-1, \bar{Y} \rangle$ and we get the criterion

$$T_P = \{0\} \;\Leftrightarrow\; \exists\, \bar{f} \in \langle \bar{X}-1, \bar{Y} \rangle \setminus P \,,$$

hence

$$T_P \neq \{0\} \;\Leftrightarrow\; \nexists\, \bar{f} \in \langle \bar{X}-1, \bar{Y} \rangle \setminus P \;\Leftrightarrow\; \langle \bar{X}-1, \bar{Y} \rangle \subseteq P \;\Leftrightarrow\; V(P) \subseteq \mathcal{Z}_2 \,,$$

i.e. non-zero localizations can only appear inside of the line $\mathcal{Z}_2$. This is also compatible with (1.12) since the dimension of the support of $\widetilde{T}$ dropped in each component of $\mathcal{X}$:

$$(\operatorname{supp} T) \cap \mathcal{X}_1 = \text{point} \quad , \quad (\operatorname{supp} T) \cap \mathcal{X}_2 = \text{line} \quad , \quad (\operatorname{supp} T) \cap \mathcal{X}_3 = \emptyset \,.$$

In particular, we see that the dimension even dropped by 2 in the plane $\mathcal{X}_1$.

Remark 1.4.28. Let us apply the process described in Section 1.2.4 to Example E.3 in order to find its connected components. The intuitive picture (union of 2 planes and a perpendicular line) already illustrates that $\mathcal{X}$ should be connected. If we consider the ideals, we see that $P_2 + P_3 = R$, so $\mathcal{X}_2$ and $\mathcal{X}_3$ do not intersect. However $P_1 + P_2 \neq R$ and $P_1 + P_3 \neq R$, so $\mathcal{X}_1$ intersects both of them. Hence we shall take the intersection of all 3 ideals to get $P_1 \cap P_2 \cap P_3 = \{\bar{0}\}$ and

$$\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 = V(P_1 \cap P_2 \cap P_3) = V\big(\{\bar{0}\}\big) = \operatorname{Spec} R = \mathcal{X} \,.$$

Example 1.4.29. Let us also check the results from Section 1.3.3 about localization of torsion on Example E.3. We know that $[\bar{X}\bar{Z}] \in \mathcal{T}_R(M)$. How do behave its localizations ? We denote

$$[\bar{X}\bar{Z}]_P := \tfrac{[\bar{X}\bar{Z}]}{\bar{1}} \in M_P \;, \quad \forall\, P \in \operatorname{Spec} R \,.$$

According to Proposition 1.3.8, these should still be torsion elements for all P. First note that $[\bar{X}\bar{Z}]_P = 0$ if and only if $\langle \bar{X}-1, \bar{Y} \rangle \nsubseteq P$ by (1.15), so in this case $[\bar{X}\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$ anyway. Hence let $\langle \bar{X}-1, \bar{Y} \rangle \subseteq P$, i.e. we only consider prime ideals P such that $V(P)$ is contained in the line $\mathcal{Z}_2$. Then $P_3 \nsubseteq P$ since $\bar{X} \notin P$ (otherwise $\bar{1} = \bar{X} - (\bar{X}-1) \in P$), which means by (1.6) that the localizations from elements in P_3 become NZDs in R_P. In particular, $\bar{Y}_P$ is a NZD and it is non-zero since $\bar{Z}(\bar{X}-1) \in P$. In other words, the relation $\bar{Y} \cdot \bar{Z}(\bar{X}-1) = \bar{0}$ which makes $\bar{Y}$ a zero-divisor in R disappears in the localization R_P since

$$\bar{X}\bar{Z}(\bar{X}-1) = \bar{0} \text{ with } \bar{X} \notin P \;\Rightarrow\; \bar{Z}_P(\bar{X}-1)_P = 0 \; .$$

Therefore $\bar{Y}_P \in R_P$ is a NZD whereas $\bar{Y} \in R$ is not (compare Remark 1.3.7). Finally we get

$$\bar{Y}_P * [\bar{X}\bar{Z}]_P = [\bar{Y}\bar{X}\bar{Z}]_P = 0 \; ,$$

i.e. $[\bar{X}\bar{Z}]_P$ is a torsion element in M_P for all $P \in \operatorname{Spec} R$.

Chapter 2

The torsion subsheaf

After having discussed the notion of torsion of a finitely generated module in Chapter 1, we are now interested in the torsion of a (quasi-)coherent sheaf. For this we are going to introduce the notion of the torsion subsheaf. The main idea is to define a subsheaf of a (quasi-)coherent sheaf $\mathcal{F}$ whose stalks consists of the torsion submodules of the stalks $\mathcal{F}_x$.

In the non-integral case not much on this topic can be found in the classical literature. Torsion-free sheaves on non-integral schemes have first been introduced by A. Grothendieck in [33], but he only used it as a tool and did not study the torsion of a sheaf itself. This is why we dedicated a whole chapter to the definition and properties of the torsion subsheaf in the Noetherian case.

Our first main result of this chapter is a criterion for (quasi-)coherence of the torsion subsheaf $\mathcal{T}(\mathcal{F})$ of $\mathcal{F}$ (Theorem 2.2.8). Then we show that this condition is satisfied if the ring defining an affine scheme has no embedded primes (Theorem 2.2.13). We also present a counter-example which shows that $\mathcal{T}(\mathcal{F})$ does not need to be coherent and may have dense support (Section 2.3). Another aim is to study the relation of $\mathcal{T}(\mathcal{F})$ with the sheaf of meromorphic functions (Theorem 2.4.19 and Theorem 2.4.22). Finally we reprove Grothendieck's criterion for torsion-freeness in Theorem 2.5.8.

2.1 Definition and properties

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a scheme and $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_{\mathcal{X}})$. The main idea of the torsion subsheaf is to define a subsheaf of $\mathcal{F}$ whose stalks consists of the torsion submodules of the stalks $\mathcal{F}_x$. The naive idea for doing this is to take the sheafification of the assignment

$$U \longmapsto \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big) \tag{2.1}$$

for all $U \subseteq \mathcal{X}$ open. However this is not correct since (2.1) is not a presheaf. It may happen that the restriction map is not well-defined, e.g. if $V \subseteq U$ is open and $s \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big)$. Thus there is a NZD $f \in \mathcal{O}_{\mathcal{X}}(U)$ such that $f * s = 0$, but it is not sure whether the restriction $s_{|V} \in \mathcal{F}(V)$ belongs to $\mathcal{T}_{\mathcal{O}_{\mathcal{X}}(V)}\big(\mathcal{F}(V)\big)$ since $f_{|V}$ may be zero or a zero-divisor. To get the correct definition, one has to start differently.

2.1.1 Definition

Remark 2.1.1. In practise we only work with coherent sheaves on a locally Noetherian scheme. However a lot of the following results are true in some more generality, so a priori we only need that $\mathcal{F}$ is a quasi-coherent sheaf on an arbitrary scheme. If a coherence or Noetherian condition is needed, it will be pointed out.

Lemma 2.1.2. *Let R be a ring.*
1) If $S \subset R$ is a multiplicatively closed subset and $r \in R$ is a NZD, then $\frac{r}{1} \in S^{-1}R$ is also a NZD.
2) If M is any R-module, then torsion elements in M remain torsion elements in $S^{-1}M$. More precisely,

$$m \in \mathcal{T}_R(M) \;\Rightarrow\; \tfrac{m}{1} \in \mathcal{T}_{S^{-1}R}(S^{-1}M)\ .$$

Proof. 1) The proof is similar as the one of Proposition 1.3.6. If $\exists\, \frac{a}{s} \in S^{-1}R$ such that $\frac{a}{s} \cdot \frac{r}{1} = 0$, then $\exists\, b \in S$ such that $ba \cdot r = 0$, i.e. $ba = 0$ since r is a NZD and thus $\frac{a}{s} = 0$.
2) because the NZD that annihilates them remains a NZD in the localization. □

Corollary 2.1.3. *Let $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_\mathcal{X})$ and $U \cong \operatorname{Spec} R$ be affine with $\mathcal{F}|_U \cong \widetilde{M}$. Let $r \in R$ and consider a distinguished open set $V = D(r) \cong \operatorname{Spec}(R_r)$. Then the restriction*

$$\mathcal{T}_{\mathcal{O}_\mathcal{X}(U)}\big(\mathcal{F}(U)\big) \longrightarrow \mathcal{T}_{\mathcal{O}_\mathcal{X}(V)}\big(\mathcal{F}(V)\big) \;\Leftrightarrow\; \mathcal{T}_R(M) \longrightarrow \mathcal{T}_{R_r}(M_r) \;:\; m \mapsto \tfrac{m}{1}$$

is well-defined.

Proof. As $(U, \mathcal{O}_\mathcal{X}|_U) \cong (\operatorname{Spec} R, \mathcal{O}_R)$ and $(V, \mathcal{O}_\mathcal{X}|_V) \cong (\operatorname{Spec} R_r, \mathcal{O}_R|_{D(r)})$, we get

$$\begin{aligned} \mathcal{O}_\mathcal{X}(U) = \mathcal{O}_\mathcal{X}|_U(U) \cong \mathcal{O}_R(\operatorname{Spec} R) \cong R \quad &, \quad \mathcal{O}_\mathcal{X}(V) \cong \mathcal{O}_R\big(D(r)\big) \cong R_r \;, \\ \mathcal{F}(U) = \mathcal{F}|_U(U) \cong \widetilde{M}(\operatorname{Spec} R) \cong M \quad &, \quad \mathcal{F}(V) \cong \widetilde{M}\big(D(r)\big) \cong M_r \;. \end{aligned}$$

Well-definedness of the map $m \mapsto \frac{m}{1}$ follows from Lemma 2.1.2. □

Hence restriction of torsion elements behaves well over affine open sets. This motivates the following definition of seeing the torsion of a sheaf as local torsion elements over affines.

Definition 2.1.4. [[52], 35328]
Let $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_\mathcal{X})$, $U \subseteq \mathcal{X}$ be open and $s \in \mathcal{F}(U)$. We say that s is a *torsion section* of $\mathcal{F}$ if there exist an affine open covering $U = \bigcup_i U_i$ such that

$$s_{|U_i} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(U_i)}\big(\mathcal{F}(U_i)\big), \;\forall\, i \;.$$

In other words, we have $U_i \cong \operatorname{Spec} R_i$ and elements $f_i \in \mathcal{O}_\mathcal{X}(U_i) \cong R_i$ which are NZDs such that $f_i * s_{|U_i} = 0$, $\forall\, i$. The set of all torsion sections of $\mathcal{F}$ over U is denoted is by $\mathcal{T}(\mathcal{F})(U)$.

Remark 2.1.5. In particular if U is affine, then

$$\mathcal{T}_{\mathcal{O}_\mathcal{X}(U)}\big(\mathcal{F}(U)\big) \subseteq \mathcal{T}(\mathcal{F})(U) \tag{2.2}$$

since we don't need a covering in this case. The other inclusion is a priori not clear (but true in the Noetherian case, see Proposition 2.1.12).

Remark 2.1.6. Let $W \subseteq \mathcal{X}$ be open, $s \in \mathcal{F}(W)$ and $V \subseteq U \subseteq W$ both be affine with $U \cong \operatorname{Spec} R$, $V = D(r)$ for some $r \in R$ and $\mathcal{F}|_U \cong \widetilde{M}$. Then Corollary 2.1.3 also implies that

$$s_{|U} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(U)}\big(\mathcal{F}(U)\big) \cong \mathcal{T}_R(M) \qquad \Rightarrow \qquad s_{|V} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(V)}\big(\mathcal{F}(V)\big) \cong \mathcal{T}_{R_r}(M_r) ,$$

i.e. a section that is torsion over an affine open set remains torsion when restricting it to a smaller affine open set.

Lemma 2.1.7. *Let $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_\mathcal{X})$. For all inclusions of open sets $V \subseteq U \subseteq \mathcal{X}$, the restriction map $\mathcal{T}(\mathcal{F})(U) \to \mathcal{T}(\mathcal{F})(V) : s \mapsto s_{|V}$ is well-defined.*

Proof. Let $s \in \mathcal{T}(\mathcal{F})(U)$ with affine covering $\bigcup_i U_i$. An open covering of V is given by $\bigcup_i (V \cap U_i)$, where each $V \cap U_i$ can be covered by distinguished open sets V_{ij}, hence we get an affine open cover $V = \bigcup_{ij} V_{ij}$. As the restrictions of the $s_{|U_i}$ to the affines V_{ij} are still torsion elements, Remark 2.1.6 gives

$$(s_{|V})_{|V_{ij}} = (s_{|U_i})_{|V_{ij}} = s_{|V_{ij}} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(V_{ij})}\big(\mathcal{F}(V_{ij})\big), \forall\, i,j \quad \Rightarrow \quad s_{|V} \in \mathcal{T}(\mathcal{F})(V) . \qquad \square$$

Lemma 2.1.8. *If $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_\mathcal{X})$, then $\mathcal{T}(\mathcal{F})(U)$ is an $\mathcal{O}_\mathcal{X}(U)$–submodule of $\mathcal{F}(U)$, $\forall\, U \subseteq \mathcal{X}$ open.*

Proof. Let $s, t \in \mathcal{T}(\mathcal{F})(U)$ with affine open coverings $\bigcup_i U_i$ and $\bigcup_j U'_j$ as in Definition 2.1.4 respectively. In [[68], 5.3.1, p.157-158] it is shown that $U_i \cap U'_j$ can be covered by affine open sets V_{ij}^k which are distinguished for both U_i and U'_j. Hence we get an affine covering

$$U = \big(\textstyle\bigcup_i U_i\big) \cap \big(\bigcup_j U'_j\big) = \bigcup_{ij} \big(U_i \cap U'_j\big) = \bigcup_{ijk} V_{ij}^k .$$

As the V_{ij}^k are distinguished for both U_i and U'_j, we obtain as above

$$\begin{aligned} s_{|V_{ij}^k} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(V_{ij}^k)}\big(\mathcal{F}(V_{ij}^k)\big) \qquad &, \qquad t_{|V_{ij}^k} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(V_{ij}^k)}\big(\mathcal{F}(V_{ij}^k)\big) , \\ \Rightarrow \qquad (s+t)_{|V_{ij}^k} &\in \mathcal{T}_{\mathcal{O}_\mathcal{X}(V_{ij}^k)}\big(\mathcal{F}(V_{ij}^k)\big), \forall\, i,j,k , \end{aligned}$$

i.e. $s + t \in \mathcal{T}(\mathcal{F})(U)$. Moreover $f * s \in \mathcal{T}(\mathcal{F})(U)$ for all $f \in \mathcal{O}_\mathcal{X}(U)$ follows from the fact that $f_{|U_i} * s_{|U_i} \in \mathcal{T}_{\mathcal{O}_\mathcal{X}(U_i)}\big(\mathcal{F}(U_i)\big)$, $\forall\, i$. $\square$

Definition 2.1.9. Hence the assignment $\mathcal{T}(\mathcal{F}) : U \mapsto \mathcal{T}(\mathcal{F})(U)$ defines a presheaf of $\mathcal{O}_{\mathcal{X}}$–modules on $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. It is also separated since its restrictions are those from the sheaf $\mathcal{F}$. Moreover it satisfies the gluing axiom as all defining conditions are local. Indeed let $U = \bigcup_i U_i$ be any open covering and $s_i \in \mathcal{T}(\mathcal{F})(U_i)$ sections that agree on intersections. Since $s_i \in \mathcal{F}(U_i)$ for all i, these glue to a section $s \in \mathcal{F}(U)$, which also belongs to $\mathcal{T}(\mathcal{F})(U)$ because each U_i has an affine open covering $\bigcup_j U_{ij}$ such that $s_{i|U_{ij}}$ are torsion elements for all j, thus $\bigcup_{ij} U_{ij}$ is an affine open covering of U such that $s_{|U_{ij}} = s_{i|U_{ij}}$ is torsion, $\forall\, i, j$.
It follows that $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathcal{X}}$–modules; it is called the *torsion subsheaf* of $\mathcal{F}$.

Proposition 2.1.10. *The assignment* $\mathcal{T} : \mathtt{QCoh}(\mathcal{O}_{\mathcal{X}}) \to \mathtt{Mod}(\mathcal{O}_{\mathcal{X}}) : \mathcal{F} \mapsto \mathcal{T}(\mathcal{F})$ *defines an additive covariant and left exact functor.*

Proof. Everything already follows from the corresponding statement about modules. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism, then $\mathcal{T}(\varphi) : \mathcal{T}(\mathcal{F}) \to \mathcal{T}(\mathcal{G})$ is constructed as follows : let $U \subseteq \mathcal{X}$ be open, $s \in \mathcal{F}(U)$ and take an affine open covering $U = \bigcup_i U_i$ such that $s_{|U_i}$ is torsion in $\mathcal{F}(U_i)$, $\forall\, i$. Then $\varphi_{U_i}(s_{|U_i})$ is torsion in $\mathcal{G}(U_i)$ by Lemma C.1.9, hence $\varphi_U(s) \in \mathcal{T}(\mathcal{G})(U)$, i.e. $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ and

$$\mathcal{T}(\varphi)_U \; : \; \mathcal{T}(\mathcal{F})(U) \xrightarrow{\varphi_U} \mathcal{T}(\mathcal{G})(U) \; : \; s \longmapsto \varphi_U(s) \; .$$

This gives an $\mathcal{O}_{\mathcal{X}}(U)$–module homomorphism $\mathcal{T}(\mathcal{F})(U) \to \mathcal{T}(\mathcal{G})(U)$ for all open subsets $U \subseteq \mathcal{X}$. Additivity follows from Proposition C.1.4 because

$$\begin{aligned}
&(s,t) \in \mathcal{T}(\mathcal{F} \oplus \mathcal{G})(U) \\
&\quad\Leftrightarrow\; (s,t)_{|U_i} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big((\mathcal{F} \oplus \mathcal{G})(U_i)\big) = \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big) \oplus \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{G}(U_i)\big),\ \forall\, i \\
&\quad\Leftrightarrow\; s_{|U_i} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big) \text{ and } t_{|U_i} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{G}(U_i)\big),\ \forall\, i \\
&\quad\Leftrightarrow\; s \in \mathcal{T}(\mathcal{F})(U) \text{ and } t \in \mathcal{T}(\mathcal{G})(U) \;\Leftrightarrow\; (s,t) \in \mathcal{T}(\mathcal{F})(U) \oplus \mathcal{T}(\mathcal{G})(U) \; .
\end{aligned}$$

Hence $\mathcal{T}(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{T}(\mathcal{F})(U) \oplus \mathcal{T}(\mathcal{G})(U)$ for all $U \subseteq \mathcal{X}$ and this commutes with restrictions, so we get $\mathcal{T}(\mathcal{F} \oplus \mathcal{G}) = \mathcal{T}(\mathcal{F}) \oplus \mathcal{T}(\mathcal{G})$ as sheaves. Finally for left exactness, let

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

be an exact sequence of quasi-coherent $\mathcal{O}_{\mathcal{X}}$–modules. Left exactness of $\Gamma(U,\,\cdot\,)$ for all $U \subseteq \mathcal{X}$ open gives the exact sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)\ ,$$

hence $\mathcal{T}(\varphi)$ is injective as it is just a restriction of φ. Finally, if $t \in \mathcal{T}(\mathcal{G})(U)$ is such that $\psi_U(t) = 0$, then $\exists\, s \in \mathcal{F}(U)$ such that $t = \varphi_U(s)$. The $t_{|U_i} = \varphi_{U_i}(s_{|U_i})$ are torsion over some affine covering $U = \bigcup_i U_i$, thus injectivity of φ_{U_i} and Proposition 1.3.1 imply that the $s_{|U_i}$ are torsion as well, i.e. $s \in \mathcal{T}(\mathcal{F})(U)$. □

2.1.2 Properties: sections and stalks

An equality $\mathcal{T}(\mathcal{F})(U) = \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big)$ for all affine open sets $U \subseteq \mathcal{X}$ does not immediately follow from the definition because affine sets may a priori still need a covering. However we will show that the inverse inclusion of (2.2) holds true if we are working with Noetherian rings. After this we are going to prove that the stalks of $\mathcal{T}(\mathcal{F})$ are indeed the torsion submodules of the stalks of $\mathcal{F}$.

Lemma 2.1.11. *Let $S \subset R$ be a multiplicatively closed subset and M an R-module. For any $m \in M$, we have*

$$S^{-1}\big(\operatorname{Ann}_R(m)\big) = \operatorname{Ann}_{S^{-1}R}\left(\tfrac{m}{1}\right)\ . \tag{2.3}$$

Proof. $\subseteq$: If $\frac{r}{s}$ is such that $s \in S$ and $r \in \operatorname{Ann}_R(m)$, i.e. $r * m = 0$, then $\frac{r}{s} * \frac{m}{1} = 0$ as well.

$\supseteq$: If $\frac{r}{s}$ is such that $\frac{r}{s} * \frac{m}{1} = 0$, then $\exists\, b \in S$ such that $b * (r * m) = 0$, i.e. $(b \cdot r) * m = 0$, so we get

$$\frac{r}{s} = \frac{b \cdot r}{b \cdot s}$$

with $b \cdot r \in \operatorname{Ann}_R(m)$ and $b \cdot s \in S$, hence $\frac{r}{s} \in S^{-1}\big(\operatorname{Ann}_R(m)\big)$. □

Proposition 2.1.12. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be locally Noetherian and $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$. Then for affine open sets $U \cong \operatorname{Spec} R$ with $\mathcal{F}|_U \cong \widetilde{M}$, we get*

$$\mathcal{T}(\mathcal{F})(U) = \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big) \cong \mathcal{T}_R(M)\ .$$

Proof. The isomorphism holds because of $\mathcal{O}_{\mathcal{X}}(U) \cong R$ and $\mathcal{F}(U) \cong M$.
Now assume that $U \cong \operatorname{Spec} R$ where R is Noetherian and let $U = \bigcup_i U_i$ be an affine open covering with $U_i = D(r_i)$ for some $r_i \in R$ (note that finitely many of them are sufficient since $\operatorname{Spec} R$ is compact). According to (2.2), we only have to prove the inclusion $\subseteq$. Let $s \in \mathcal{T}(\mathcal{F})(U)$ be a section such that

$$s_i := s_{|U_i} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big),\ \forall\, i\ .$$

We shall show that s is torsion itself. Denote the localized rings by $R_i := R_{r_i}$, so that $U_i \cong \operatorname{Spec} R_i$, $\mathcal{F}(U_i) \cong M_{r_i}$ and $s_i \in \mathcal{T}_{R_i}(M_{r_i})$ for all i. Hence $s_i = \frac{s}{1}$ seen as an element in M_{r_i} and we are left to prove that $s \in \mathcal{T}_R(M)$.
For this we need that $\operatorname{Ann}_R(s)$ contains a NZD. Let $P_1, \ldots, P_\alpha$ be the associated primes of R. If $\operatorname{Ann}_R(s)$ only contains zero-divisors, then $\operatorname{Ann}_R(s) \subseteq P_k$ for some $k \in \{1, \ldots, \alpha\}$ by Prime Avoidance. As $P_k \in \operatorname{Spec} R$, $\exists\, j$ such that $P_k \in D(r_j)$, i.e. $r_j \notin P_k$. From (1.6), we hence get that all elements from $(P_k)_{r_j}$ are zero-divisors in R_j. In particular all elements from $\operatorname{Ann}_R(s)$ become zero-divisors after localization, which is a contradiction by (2.3) since $\frac{s}{1} = s_j \in \mathcal{T}_{R_j}(M_{r_j})$ is torsion and its annihilator contains a NZD. $\square$

Remark 2.1.13. Thus the attempt of a definition $U \mapsto \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big)$ from (2.1) is correct in a locally Noetherian scheme when only ranging over affines.

Next we determine the stalks of $\mathcal{T}(\mathcal{F})$. Being a subsheaf of $\mathcal{F}$, we know that they must be submodules of the stalks of $\mathcal{F}$. First an observation which illustrates the local nature of torsion sections.

Lemma 2.1.14. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be locally Noetherian, $U \subseteq \mathcal{X}$ open and $s \in \mathcal{F}(U)$. Then*

$$s \in \mathcal{T}(\mathcal{F})(U) \ \Leftrightarrow\ [s]_x \in \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)\,,\ \forall\, x \in U\ .$$

Proof. follows from the definition and Proposition 1.3.8
$\underline{\Rightarrow}$: Let $V \cong \operatorname{Spec} R$ be an affine open neighborhood of a fixed $x \in U$ such that $\mathcal{F}|_V \cong \widetilde{M}$ and $s_{|V} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(V)}\big(\mathcal{F}(V)\big)$. x corresponds to some prime ideal $P \in \operatorname{Spec} R$. Hence $s_{|V} \in \mathcal{T}_R(M)$ and (1.9) implies that

$$[s]_x = [s]_P = \tfrac{s_{|V}}{1} \in \mathcal{T}_{R_P}(M_P) \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)\ .$$

$\Leftarrow$: Let $U = \bigcup_i U_i$ be an affine open covering of U with $U_i \cong \operatorname{Spec} R_i$ and $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. Since the germs satisfy $[s]_P \in \mathcal{T}_{(R_i)_P}\big((M_i)_P\big)$ for all $P \in \operatorname{Spec} R_i$, (1.9) again implies that

$$s_{|U_i} \in \mathcal{T}_{R_i}(M_i) \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big)$$

for all i, i.e. $s \in \mathcal{T}(\mathcal{F})(U)$. □

Proposition 2.1.15. [[11], 30.23.7]
Let R be a Noetherian ring, $P \in \operatorname{Spec} R$ and $a \in R$ be such that $\frac{a}{1}$ is a NZD in R_P. Then $\exists\, r \in R \setminus P$ such that $\frac{a}{1}$ is also a NZD in R_r.

Proof. If a is a NZD in R, then $\frac{a}{1}$ is a NZD in $S^{-1}R$ for every multiplicatively closed subset $S \subset R$. So we may assume that $a \in R$ is a zero-divisor. Now there are 2 ways to prove the statement.
method 1 : Being a zero-divisor, a belongs to one of the associated primes $P_1, \ldots, P_\alpha$ of R. Let $P_1, \ldots, P_\gamma$ be the ones that are included in P. a cannot belong to some P_i with $i \leq \gamma$, otherwise $\frac{a}{1}$ would be a zero-divisor in R_P by (1.6). Thus $a \in P_j$ for some $j > \gamma$. Let $\mathcal{P}$ be the set of all associated primes that contain a, so that

$$\forall\, P_j \in \mathcal{P} \;\Rightarrow\; j > \gamma \;\Rightarrow\; P_j \nsubseteq P \;\Rightarrow\; \exists\, r_j \in P_j \setminus P\ .$$

Let $r := \prod_j r_j$; then $r \in P_j$, $\forall\, P_j \in \mathcal{P}$ with $r \notin P$. By (1.6) again, we know that the zero-divisors in R_r are given by the localizations of associated primes that do not contain r. a cannot belong to such a prime since all those that contain a also contain r (by construction of r). Hence $\frac{a}{1}$ becomes a NZD in R_r.
method 2 : Let $I = \operatorname{Ann}_R(a)$; since a is a zero-divisor, we know that $I \neq \{0\}$. By Lemma 2.1.11 we have

$$I_P = S^{-1}\big(\operatorname{Ann}_R(a)\big) = \operatorname{Ann}_{S^{-1}R}\left(\tfrac{a}{1}\right) = \operatorname{Ann}_{R_P}\left(\tfrac{a}{1}\right)$$

for $S = R \setminus P$. Since $\frac{a}{1}$ is a NZD, we thus get $I_P = \{0\}$. On the other hand I is finitely generated since R is Noetherian. Let $I = \langle\, a_1, \ldots, a_k \,\rangle$; then for all

$i \in \{1, \ldots, k\}$, $\exists\, r_i \notin P$ such that $r_i \cdot a_i = 0$. Take $r := \prod_i r_i$, so that $r \notin P$ and $r \cdot I = \{0\}$. Then $\frac{a}{1}$ is a NZD in R_r. Indeed,

$$\begin{aligned} \tfrac{a}{1} \cdot \tfrac{b}{r^n} = 0 \;&\Leftrightarrow\; \exists\, m \in \mathbb{N} \text{ such that } r^m \cdot a \cdot b = 0 \\ &\Rightarrow\; r^m b \in I \;\Rightarrow\; r \cdot r^m b = 0 \;\Rightarrow\; \tfrac{b}{1} = 0\,. \end{aligned}$$

□

Remark 2.1.16. Geometrically this means that if $f \in \mathcal{O}_{\mathcal{X}}(U)$ is a section such that $[f]_x$ is a NZD in the stalk $\mathcal{O}_{\mathcal{X},x}$, then there exists a distinguished open subset $V = D(r) \subseteq U$ which is a neighborhood of x and such that $f_{|V}$ is also a NZD in $\mathcal{O}_{\mathcal{X}}(V)$.

Proposition 2.1.17. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be locally Noetherian and $\mathcal{F} \in$ $\texttt{QCoh}(\mathcal{O}_{\mathcal{X}})$. Then the stalks of the torsion subsheaf are $\mathcal{T}(\mathcal{F})_x \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$, $\forall\, x \in \mathcal{X}$. In other words, they are given by the torsion submodules of the stalks of $\mathcal{F}$.*

Proof. It suffices to prove the statement locally. Let $U \cong \operatorname{Spec} R$ be an affine open neighborhood of x with $\mathcal{F}|_U \cong \widetilde{M}$ and $x \leftrightarrow P \in \operatorname{Spec} R$. The set of affine open neighborhoods of x being cofinal, we already get

$$\mathcal{T}(\mathcal{F})_x = \varinjlim_{V \ni x} \mathcal{T}(\mathcal{F})(V) \;\cong\; \varinjlim_{V \ni x} \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(V)}\big(\mathcal{F}(V)\big)\;,$$

where $V \subseteq U$ only runs over affines. Reformulating in terms of spectra, we hence get

$$\mathcal{T}(\mathcal{F})_x = \big(\mathcal{T}(\mathcal{F})|_U\big)_P \cong \varinjlim_{r \notin P} \mathcal{T}_{R_r}(M_r)\;, \tag{2.4}$$

so it remains to show that this inductive limit is equal to $\mathcal{T}_{R_P}(M_P)$. For this we shall prove that for every element in $\mathcal{T}_{R_P}(M_P)$, there exists $r \in R \setminus P$ and a torsion section over $D(r)$ representing it (compare : if $f_x \in \mathcal{F}_x$, then $\exists\, U \ni x$ and $f \in \mathcal{F}(U)$ such that $[f]_x = f_x$). This characterizes the stalk completely. Let $\frac{m}{s} \in \mathcal{T}_{R_P}(M_P)$ with $s \notin P$.

• If $\frac{m}{s} = \frac{m}{1} = 0$, then $\exists\, r \notin P$ such that $r * m = 0$. Hence $P \in D(r)$ and $\frac{m}{1} = 0 \in R_r$ on $D(r)$ as well. We showed: if the germ of a section is zero, then the section is locally zero around that point.

• So now we may assume that $\frac{m}{s} \neq 0$, i.e. $\operatorname{Ann}_R(m) \subseteq P$. Since $\frac{m}{s}$ is torsion, we have a NZD $\frac{a}{t} \in R_P$ such that $\frac{a}{t} * \frac{m}{s} = 0$, i.e. $\exists\, b \notin P$ such that $ba * m = 0$. In particular, we also have $a \in P$.

case 1 : if $ba \in R$ is a NZD, then $m \in \mathcal{T}_R(M)$ and all its localizations are torsion elements too. So it suffices to take the given $s \notin P$ and the element $\frac{m}{s} \in \mathcal{T}_{R_s}(M_s)$ on $D(s)$ will represent $\frac{m}{s} \in \mathcal{T}_{R_P}(M_P)$.

case 2 : if $ba \in R$ is a zero-divisor, then $\frac{ba}{1}$ is still a NZD in R_P since $\frac{a}{t} = \frac{1}{bt} \cdot \frac{ba}{1}$. By Proposition 2.1.15 we can find $r \notin P$ such that $\frac{ba}{1}$ is a NZD in R_r. So $P \in D(r)$ and $r * m \neq 0$ since $\mathrm{Ann}_R(m) \subseteq P$, i.e. $\frac{m}{1} \neq 0$ in M_r. Hence we get $\frac{m}{1} \in \mathcal{T}_{R_r}(M_r)$. Intersecting with $D(s)$, we can construct the non-zero section

$$\frac{r * m}{r \cdot s} = \frac{r}{r \cdot s} * \frac{m}{1} \in \mathcal{T}_{R_{rs}}(M_{rs})$$

on $D(rs)$ since $\frac{m}{1}$ remains torsion when restricting it to smaller affines (see Remark 2.1.6). This one then restricts to $\frac{m}{s}$ in M_P since $r \cdot s \notin P$ (it does not make sense to write $\frac{m}{s}$ in M_{rs}). □

Corollary 2.1.18. *Let R be a Noetherian ring and $m \in M$ an element such that $\frac{m}{1} \in \mathcal{T}_{R_P}(M_P)$ for some $P \in \operatorname{Spec} R$. Then then there exists an (affine) open neighborhood $D(r)$ of P such that $\frac{m}{1} \in \mathcal{T}_{R_r}(M_r)$, so*

$$\tfrac{m}{1} \in \mathcal{T}_{R_Q}(M_Q)\,, \quad \forall\, Q \in D(r)\ .$$

In terms of sheaves: If $s \in \mathcal{F}(U)$ is such that $[s]_x$ is a torsion element in $\mathcal{F}_x$ for some $x \in U$, then there exists an open neighborhood $V \subseteq U$ of x such that $[s]_y$ is a torsion element in $\mathcal{F}_y$, $\forall\, y \in V$, and thus $s_{|V} \in \mathcal{T}_{\mathcal{O}_X(V)}\big(\mathcal{F}(V)\big)$ by Lemma 2.1.14.

Remark 2.1.19. Note that the proofs above are constructive, so the element $r \notin P$ for $D(r)$ can always be obtained explicitly.

Remark 2.1.20. Corollary 2.1.18 already looks like a coherence-condition (if a property is satisfied on a stalk, then it is satisfied on some small open neighborhood around that stalk). However we will see in Theorem 2.2.8 that this is not always true.

Corollary 2.1.21. *Let R be Noetherian and M a finitely generated R-module. For each $P \in \operatorname{Spec} R$, there exists $r \in R \setminus P$ such that*

$$\mathcal{T}_{R_P}(M_P) \;\cong\; \big(\mathcal{T}_{R_r}(M_r)\big)_{P_r}\ ,$$

where the R_P-module structure of the RHS is induced by $(R_r)_{P_r} \cong R_P$.

Proof. For $r \notin P$, we define

$$\rho_r \ : \ \big(\mathcal{T}_{R_r}(M_r)\big)_{P_r} \longrightarrow \mathcal{T}_{R_P}(M_P) \ : \ \frac{m/r^k}{s/r^l} \longmapsto \frac{r^l}{s} * \frac{m}{r^k} = \frac{r^l * m}{s \cdot r^k} \ ,$$

which is well-defined since $r \notin P$, thus torsion elements in M_r remain torsion elements in M_P (take the same power of r on the denominator). Moreover it is injective because it is a restriction of the isomorphism $(M_r)_{P_r} \cong M_P$. ρ_r is however not surjective for all $r \notin P$.
As M is finitely generated, so is M_P (take the localizations of the generators). R_P still being a Noetherian ring (see Proposition D.1.10), the submodule

$$\mathcal{T}_{R_P}(M_P) = \left\langle \tfrac{m_1}{1} , \ldots , \tfrac{m_n}{1} \right\rangle$$

is finitely generated as well (as a module over R_P, so we may assume that all denominators are 1). For each $\frac{m_i}{1}$ we know by Corollary 2.1.18 that there exists $r_i \notin P$ such that $\frac{m_i}{1} \in M_{r_i}$ is torsion over $D(r_i)$. Let $r := r_1 \cdot \ldots \cdot r_n$. Then all $\frac{m_i}{1}$ are torsion elements over $D(r) = D(r_1) \cap \ldots \cap D(r_n)$ by Remark 2.1.6, i.e. $\frac{m_i}{1} \in \mathcal{T}_{R_r}(M_r)$ for all i and these elements restrict to the generators of $\mathcal{T}_{R_P}(M_P)$ in the stalk. In other words, all generators are in the image of ρ_r, which is hence surjective. □

Remark 2.1.22. In order for the torsion subsheaf to be coherent one needs that every torsion element in the stalk can be represented by a global torsion element (see Theorem 2.2.8). But this is not satisfied in general ; only the weaker statement of Corollary 2.1.21 holds true. It says that for every torsion element in the stalk, there exists a representative which is torsion on some affine open neighborhood.

2.2 Torsion-freeness and coherence

In this section we define what it means for a sheaf to be torsion-free and establish a criterion under which the torsion subsheaf is (quasi-)coherent. Here we always assume that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is locally Noetherian, otherwise the torsion subsheaf does not have the necessary "nice" properties on its stalks and over affine sets. In particular, we also obtain that $\mathcal{O}_\mathcal{X}$ is coherent. As before, let $\mathcal{F}$ be quasi-coherent and in the cases where it must be coherent, it will be pointed out.

2.2.1 Definition and examples

Definition 2.2.1. Let $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_\mathcal{X})$. $\mathcal{F}$ is said to be *torsion-free* if $\mathcal{T}(\mathcal{F}) = 0$. Furthermore one says that $\mathcal{F}$ is a *torsion sheaf* if $\mathcal{T}(\mathcal{F}) = \mathcal{F}$.

Remark 2.2.2. A priori these definitions only make sense for quasi-coherent sheaves, otherwise the assignment $\mathcal{T} : \mathcal{F} \mapsto \mathcal{T}(\mathcal{F})$ may not be well-defined. However one often wants to extend the notion of torsion-freeness to arbitrary sheaves as well. For this we simply take the characterizations of the above definitions in the quasi-coherent case, namely:
A sheaf $\mathcal{F} \in \texttt{Mod}(\mathcal{O}_\mathcal{X})$ is torsion-free if and only if all its stalks are torsion-free modules, i.e. if

$$\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) = \{0\}\,,\ \forall\, x \in \mathcal{X}\ .$$

Similarly $\mathcal{F} \in \texttt{Mod}(\mathcal{O}_\mathcal{X})$ is a torsion sheaf if and only if $\mathcal{F}_x$ is a torsion module over $\mathcal{O}_{\mathcal{X},x}$ for all $x \in \mathcal{X}$.

Example 2.2.3. 1) If $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_\mathcal{X})$, then the torsion subsheaf $\mathcal{T}(\mathcal{F})$ is always a torsion sheaf and $\mathcal{F}/\mathcal{T}(\mathcal{F})$ is always torsion-free.
2) Locally free sheaves (not necessarily of finite rank) and reflexive sheaves are torsion-free.

Proof. 1) We have to check that the stalks of $\mathcal{T}(\mathcal{F})$ are torsion modules. Indeed,

$$\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}\big(\mathcal{T}(\mathcal{F})_x\big) \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}\big(\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)\big) = \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \cong \mathcal{T}(\mathcal{F})_x\,,\ \forall\, x \in \mathcal{X}\ .$$

Moreover

$$\big(\mathcal{F}/\mathcal{T}(\mathcal{F})\big)_x \cong \mathcal{F}_x/\mathcal{T}(\mathcal{F})_x \cong \mathcal{F}_x/\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$$

and this module is torsion-free.
2) a) If $\mathcal{F}$ is locally free, then each stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{\mathcal{X},x}$–module and hence torsion-free.
b) As reflexive sheaves are coherent, we have $(\mathcal{F}^*)_x \cong (\mathcal{F}_x)^*$, $\forall\, x \in \mathcal{X}$. So if $\mathcal{F} \cong \mathcal{F}^{**}$, then $\mathcal{F}_x \cong \mathcal{F}_x^{**}$ for all x, i.e. all stalks are reflexive, but reflexive modules are torsion-free (see Definition C.3.1). Note that one still needs to check that all isomorphisms agree. □

Remark 2.2.4. We point out that one should not write $\mathcal{T}\big(\mathcal{T}(\mathcal{F})\big) = \mathcal{T}(\mathcal{F})$ or $\mathcal{T}\big(\mathcal{F}/\mathcal{T}(\mathcal{F})\big) = 0$ since $\mathcal{T} : \mathcal{F} \mapsto \mathcal{T}(\mathcal{F})$ is only defined for quasi-coherent sheaves. Until now it is not clear whether $\mathcal{T}(\mathcal{F})$ is always quasi-coherent[1].
But if it is, then so is the quotient $\mathcal{F}/\mathcal{T}(\mathcal{F})$ because an exact sequence of R-modules $0 \to N \to M \to M/N \to 0$ implies that

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M/N} \longrightarrow 0$$

and

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M}/\widetilde{N} \longrightarrow 0 \,,$$

so if $\mathcal{F} \cong \widetilde{M}$ and $\mathcal{T}(\mathcal{F}) \cong \widetilde{N}$ are quasi-coherent, then $\mathcal{F}/\mathcal{T}(\mathcal{F}) \cong \widetilde{M/N}$ is quasi-coherent as well. Similarly if $\mathcal{F}$ is coherent and $\mathcal{T}(\mathcal{F})$ quasi-coherent, then the quotient is again coherent since M/N is finitely generated if M is.

2.2.2 Criteria for (quasi-)coherence of the torsion subsheaf

Now we are going to attack the question under which conditions the torsion subsheaf of a quasi-coherent sheaf is again (quasi-)coherent and, if so, by which module it is given. First some more preliminaries.

Lemma 2.2.5. *Let $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_\mathcal{X})$ and $\mathcal{G} \subseteq \mathcal{F}$ be a quasi-coherent subsheaf which is a torsion sheaf. Then $\mathcal{G} \subseteq \mathcal{T}(\mathcal{F})$. In other words: if $\mathcal{F}$ contains a subsheaf that is torsion, then this one is a subsheaf of the torsion subsheaf. It follows that $\mathcal{T}(\mathcal{F})$ is the biggest torsion subsheaf of $\mathcal{F}$ and the smallest subsheaf $\mathcal{T} \subseteq \mathcal{F}$ such that $\mathcal{F}/\mathcal{T}$ is torsion-free.*

Proof. follows from Proposition 2.1.10 which gives left exactness of the functor $\mathcal{T} : \mathcal{F} \mapsto \mathcal{T}(\mathcal{F})$. If $0 \to \mathcal{G} \to \mathcal{F}$ is exact, then so is $0 \to \mathcal{T}(\mathcal{G}) \to \mathcal{T}(\mathcal{F})$, hence $0 \to \mathcal{G} \to \mathcal{T}(\mathcal{F})$ and $\mathcal{G} \subseteq \mathcal{T}(\mathcal{F})$. □

Lemma 2.2.6.

$$M \in \mathrm{Mod}(R) \textit{ is a torsion module } \Leftrightarrow \widetilde{M} \in \mathtt{QCoh}(\mathcal{O}_R) \textit{ is a torsion sheaf}\,.$$

[1] And indeed it is not, see Theorem 2.2.8.

Proof. follows from Proposition 1.3.8:
$\underline{\Rightarrow}$: if $m \in \mathcal{T}_R(M)$, $\forall\, m \in M$, then $\frac{m}{s} = \frac{1}{s} * \frac{m}{1} \in \mathcal{T}_{R_P}(M_P)$, $\forall\, P \in \operatorname{Spec} R$, hence $\mathcal{T}_{R_P}(M_P) = M_P$.
$\underline{\Leftarrow}$: take any $m \in M$; since $\frac{m}{1} \in M_P = \mathcal{T}_{R_P}(M_P)$ for all $P \in \operatorname{Spec} R$, we get $m \in \mathcal{T}_R(M)$. □

Proposition 2.2.7. *Let $\mathcal{X} = \operatorname{Spec} R$ be affine and $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_R)$ given by $\mathcal{F} \cong \widetilde{M}$.* ***<u>If</u>*** *$\mathcal{T}(\mathcal{F})$ is quasi-coherent, then it is given by the sheaf associated to the submodule $\mathcal{T}_R(M)$. In other words,*

$$\mathcal{T}(\mathcal{F}) = \mathcal{T}(\widetilde{M}) = \widetilde{\mathcal{T}_R(M)}\ .$$

Proof. By Lemma 2.2.5 and Lemma 2.2.6, we have

$$\widetilde{\mathcal{T}_R(M)} \subseteq \mathcal{T}(\mathcal{F}) \tag{2.5}$$

because $\mathcal{T}_R(M) \leq M$ is a torsion submodule, so its associated sheaf is a torsion sheaf. Note that $\mathcal{T}_R(M)$ is also finitely generated if M is finitely generated since R is Noetherian. If we assume that $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is quasi-coherent, there must be a submodule $N \leq M$ such that $\mathcal{T}(\mathcal{F}) \cong \widetilde{N}$. As this is a torsion sheaf, N must be a torsion module, hence $N \subseteq \mathcal{T}_R(M)$. But (2.5) and Theorem 1.1.13 also imply that $\mathcal{T}_R(M) \hookrightarrow N$ since

$$0 \longrightarrow \widetilde{\mathcal{T}_R(M)} \longrightarrow \widetilde{N} \quad \text{is exact if and only if} \quad 0 \longrightarrow \mathcal{T}_R(M) \longrightarrow N \quad \text{is exact}\ .$$

Hence $N = \mathcal{T}_R(M)$ and both sheaves agree. □

Theorem 2.2.8 (Leytem)**.** *Let $\mathcal{X} = \operatorname{Spec} R$ be affine and $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_R)$ given by $\mathcal{F} \cong \widetilde{M}$. Then*

$$\mathcal{T}(\mathcal{F}) \text{ is quasi-coherent} \quad \Leftrightarrow \quad \big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)\,,\ \forall\, P \in \operatorname{Spec} R\ .$$

If $\mathcal{F}$ is coherent, the same equivalence holds true with $\mathcal{T}(\mathcal{F})$ being coherent.

Proof. The inclusion $\big(\mathcal{T}_R(M)\big)_P \subseteq \mathcal{T}_{R_P}(M_P)$ always holds true because of (1.9) and (2.5).

$\Rightarrow$: If $\mathcal{T}(\mathcal{F})$ is quasi-coherent, we know that it is given by the sheaf associated to the torsion submodule $\mathcal{T}_R(M)$, hence

$$\mathcal{T}_{R_P}(M_P) \cong \big(\mathcal{T}(\widetilde{M})\big)_P = \big(\widetilde{\mathcal{T}_R(M)}\big)_P \cong \big(\mathcal{T}_R(M)\big)_P\,,\ \forall\, P \in \operatorname{Spec} R\,.$$

Both R_P-modules are isomorphic and one of them is included in the other one, hence they are equal.

$\Leftarrow$: Assume that the R_P-modules are equal and note that they are the stalks of the sheaves $\widetilde{\mathcal{T}_R(M)}$ and $\mathcal{T}(\mathcal{F})$ respectively. Both sheaves thus have the same stalks and the inclusion (2.5) implies that they are equal.

In the case where $\mathcal{F}$ is coherent (i.e. if M is finitely generated), $\mathcal{T}(\mathcal{F})$ will also be coherent since R is Noetherian, so $\mathcal{T}_R(M) \leq M$ is finitely generated as well. □

Corollary 2.2.9. *Let R be a Noetherian ring and assume that M is an R-module such that $\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)$ for all $P \in \operatorname{Spec} R$. Then*

$$\begin{aligned}\mathcal{T}(\widetilde{M}) = 0 \ &\Leftrightarrow\ \widetilde{M} \text{ is a torsion-free sheaf} \\ &\Leftrightarrow\ M \text{ is a torsion-free module}\,. \end{aligned} \tag{2.6}$$

Proof. The first equivalence is the definition of torsion-freeness. For (2.6):

$\Rightarrow$: always holds true since modules with torsion-free localizations are torsion-free themselves, see Corollary 1.3.10.

$\Leftarrow$: (false in general) if $\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)$ holds for all prime ideals $P \trianglelefteq R$,

$$\begin{aligned}\mathcal{T}(\widetilde{M}) = 0 \ &\Leftrightarrow\ \big(\mathcal{T}(\widetilde{M})\big)_P = 0,\ \forall\, P \ \Leftrightarrow\ \mathcal{T}_{R_P}(M_P) = 0,\ \forall\, P \\ &\Leftrightarrow\ \big(\mathcal{T}_R(M)\big)_P = 0,\ \forall\, P \ \Leftrightarrow\ \mathcal{T}_R(M) = 0\,. \end{aligned}$$

□

Conclusion

Hence knowing whether $\mathcal{T}(\mathcal{F})$ is (quasi-)coherent on an affine scheme comes down to determining under which conditions we have $\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)$ for all $P \in \operatorname{Spec} R$. This equality means that every torsion element in the stalk can be represented by a global torsion element. In Corollary 2.1.21 we have already seen that this is true locally on a neighborhood of the considered stalk. However it is not true globally ; a counter-example will be presented in Section 2.3.

First we want to know under which conditions it actually holds true. For this we need the following important result ; it gives an algorithmic construction of global NZDs from local ones.

Proposition 2.2.10 (Epstein-Yao)**.** [[21], 4.7, p.11]
Let R be a Noetherian ring which has no embedded associated primes and $S \subset R$ a multiplicatively closed subset. If $c \in R$ is such that $\frac{c}{1}$ is a NZD in $S^{-1}R$, then $\exists\, w \in S$ and $\exists\, r \in R$ such that $w \cdot r = 0$ and $wc + r$ is a NZD in R.

Proof. Let $P_1, \ldots, P_\alpha$ be the associated primes of R. If c is a NZD itself, take $w = 1$ and $r = 0$. So we may assume that $c \in P_1 \cup \ldots \cup P_\alpha$ is a zero-divisor.
Let $P_1, \ldots, P_\gamma$ for some $\gamma \leq \alpha$ be the associated primes that have empty intersection with S. R having no embedded primes means that $P_1, \ldots, P_\alpha$ are mutually incomparable. In particular, by Prime Avoidance we have

$$P_i \not\subseteq \bigcup_{k \neq i} P_k \ , \quad \forall\, i \in \{1, \ldots, \alpha\} \ .$$

For all $j > \gamma$, we now have $P_j \cap S \neq \emptyset$, so $\exists\, w_j \in P_j \cap S$ which satisfies $w_j \notin P_1 \cup \ldots \cup P_\gamma$ since those have empty intersection with S. Then we set

$$\widetilde{w} := w_{\gamma+1} \cdot \ldots \cdot w_\alpha \in \big(P_{\gamma+1} \cap \ldots \cap P_\alpha \cap S\big) \setminus \big(P_1 \cup \ldots \cup P_\gamma\big)$$

since S is multiplicatively closed and all ideals are prime. Similarly for all $i \leq \gamma$, we have that $P_i \not\subseteq P_{\gamma+1} \cup \ldots \cup P_\alpha$, hence $\exists\, r_i \in P_i$ such that $r_i \notin P_{\gamma+1} \cup \ldots \cup P_\alpha$ and we get

$$\widetilde{r} := r_1 \cdot \ldots \cdot r_i \in \big(P_1 \cap \ldots \cap P_\gamma\big) \setminus \big(P_{\gamma+1} \cup \ldots \cup P_\alpha\big) \ .$$

Moreover $\widetilde{w} \cdot \widetilde{r} \in P_1 \cap \ldots \cap P_\alpha = \mathrm{nil}(R)$, so $\exists\, n \in \mathbb{N}$ such that $\widetilde{w}^n \cdot \widetilde{r}^n = 0$. Set $w := \widetilde{w}^n$ and $r := \widetilde{r}^n$. These satisfy the same conditions as $\widetilde{w}$ and $\widetilde{r}$. Summarizing we have $w \cdot r = 0$, where

$$w \in \big(P_{\gamma+1} \cap \ldots \cap P_\alpha \cap S\big) \setminus \big(P_1 \cup \ldots \cup P_\gamma\big) \ ,$$
$$r \in \big(P_1 \cap \ldots \cap P_\gamma\big) \setminus \big(P_{\gamma+1} \cup \ldots \cup P_\alpha\big) \ .$$

In addition we have $c \notin P_1 \cup \ldots \cup P_\gamma$, otherwise $\frac{c}{1}$ would be a zero-divisor in $S^{-1}R$. This implies that

$$wc \in \big(P_{\gamma+1} \cap \ldots \cap P_\alpha\big) \setminus \big(P_1 \cup \ldots \cup P_\gamma\big) \ .$$

Now consider the element $wc + r$. To prove that it is a NZD, it suffices to show that it does not belong to any of the associated primes. First consider P_i for some $i \leq \gamma$. Then $wc + r \notin P_i$ since $r \in P_i$ and $wc \notin P_i$. Similarly for P_j with $j > \gamma$; $wc+r \notin P_j$ since $wc \in P_j$ but $r \notin P_j$. It follows that $wc+r \notin P_1 \cup \ldots \cup P_\alpha$ is a NZD. □

Remark 2.2.11. In particular Proposition 2.2.10 can be applied in the case where $S = R \setminus P$ for some $P \in \operatorname{Spec} R$. Thus if $c \in R$ is such that $\frac{c}{1}$ is a NZD in R_P, then $\exists\, w \notin P$, $\exists\, r \in R$ such that $w \cdot r = 0$ and $wc + r$ is a NZD in R.

Remark 2.2.12. In general, the obtained NZD $wc + r$ does not restrict to $\frac{c}{1}$ in $S^{-1}R$. But they only differ by a unit. Indeed $w \cdot r = 0$ with $w \in S$, so $\frac{r}{1} = 0$ in the localization and we are left with $\frac{wc}{1} = \frac{w}{1} \cdot \frac{c}{1}$ where $\frac{w}{1} \in S^{-1}R$ is a unit with inverse $\frac{1}{w}$.

Theorem 2.2.13 (Leytem)**.** *Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F} \cong \widetilde{M}$ a coherent, resp. quasi-coherent $\mathcal{O}_R$–module. If R has no embedded primes, then the torsion subsheaf $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is coherent, resp. quasi-coherent.*

Proof. Fix $P \in \operatorname{Spec} R$. In the sense of Theorem 2.2.8 we shall prove that $\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)$. As the inclusion $\subseteq$ always holds true, we only have to prove $\supseteq$. Let $\frac{m}{s} \in \mathcal{T}_{R_P}(M_P)$. We shall find a global torsion element $n \in \mathcal{T}_R(M)$ and $s' \notin P$ such that

$$\frac{n}{s'} = \frac{m}{s}$$

as elements in M_P. We may assume that $\frac{m}{s} \neq 0$, otherwise $0 \in \mathcal{T}_R(M)$ can be chosen to represent $\frac{m}{s} = 0$. So in particular we may assume that $\operatorname{Ann}_R(m) \subseteq P$. Let $\frac{a}{t} \in R_P$ be a NZD such that $\frac{a}{t} * \frac{m}{s} = 0$, i.e. $\exists\, b \notin P$ such that $ba * m = 0$ (note that $a \in P$). If ba is a NZD, there is no problem and $m \in \mathcal{T}_R(M)$ is torsion itself, so we can choose $n = m$. So let us assume that ba is a zero-divisor. Then $\frac{ba}{1} \in R_P$ is still a NZD since $\frac{a}{t} = \frac{1}{bt} \cdot \frac{ba}{1}$ is a NZD. By Proposition 2.2.10 and Remark 2.2.11, we thus can find elements $w \notin P$ and $r \in R$ such that $w \cdot r = 0$ and $wba + r$ is a NZD in R. Now consider

$$(wba + r) * (w * m) = w^2 * (ba * m) + (r \cdot w) * m = 0 \ ,$$

i.e. $w * m \in \mathcal{T}_R(M)$ and
$$\frac{m}{s} = \frac{w * m}{w \cdot s} ,$$
so we can choose $n = w * m$ and $s' = w \cdot s \notin P$. Note in addition that $w * m \neq 0$ since $w \notin P$. □

2.2.3 Examples

The condition about R not having embedded primes may seem a bit technical, but it is actually satisfied in a lot of cases that appear in practise.

Lemma 2.2.14. [[53], 658589]
Let R be a Noetherian ring. An embedded prime is given by the annihilator of a nilpotent element (more precisely, a nilpotent element of degree 2).

Proof. Let Q be an embedded associated prime of R. By Proposition B.2.25 we know that is given as $Q = \text{Ann}_R(s)$ for some $s \in R$. Thus $q \cdot s = 0, \forall\, q \in Q$. Since Q is embedded, it is not contained in any minimal prime P_i, i.e. $\forall\, i, \exists\, q_i \in Q$ such that $q_i \notin P_i$. But $q_i \cdot s = 0 \in P_i$ for all i, which implies that $s \in P_i$ for all i, so s is nilpotent by (1.5). More precisely, s also belongs to a minimal prime that is contained in Q, so that $s \in Q$ as well and $s^2 = 0$. □

Remark 2.2.15. The converse of Lemma 2.2.14 is false. Minimal primes can also be given by annihilators of nilpotent elements. Consider e.g. $R = \mathbb{K}[X]/\langle X^2 \rangle$. The only associated prime is $P_1 = \langle \bar{X} \rangle = \text{Ann}_R(\bar{X})$, where $\bar{X}$ is nilpotent.

Example 2.2.16. All of the following conditions are sufficient for R not to have embedded primes:
1) R is an integral domain.
2) R is a reduced ring.
3) R has no nilpotent elements of degree 2.
4) The elements whose annihilator define the associated primes are not nilpotent.
5) R is a ring in which every zero-divisor is nilpotent.
6) R is a quotient of a polynomial ring (over a field) by a principal ideal.

Proof. 1) For integral domains, the only associated prime is $P_1 = \{0\}$.
2) For reduced rings, the result follows from Lemma B.2.18 ; existence of embedded primes would contradict minimality of the primary decomposition of $\{0\}$.
3) 4) since embedded primes are given by annihilators of nilpotent elements of degree 2.
5) Rings in which every zero-divisor is nilpotent only have one associated prime. Indeed let P, Q be two arbitrary associated primes. As all zero-divisors are nilpotent, every associated prime is contained in the intersection of all associated primes. In particular,

$$P \subseteq \mathrm{nil}(R) = \bigcap_i P_i \subseteq Q \ ,$$

and similarly $Q \subseteq P$, hence $P = Q$ is the only associated prime.
6) Here we have $R = \mathbb{K}[X_1, \ldots, X_n]/\langle f \rangle$ for some non-constant polynomial f. $\mathbb{K}[X_1, \ldots, X_n]$ being a UFD, there is a unit ε such that we can uniquely factorize f into irreducible components $f = \varepsilon \cdot f_1^{k_1} \cdot \ldots \cdot f_m^{k_m}$, which gives the primary decomposition

$$\langle f \rangle = \langle f_1^{k_1} \rangle \cap \ldots \cap \langle f_m^{k_m} \rangle$$

with radicals $\langle f_i \rangle$. In the quotient we get the associated primes $P_i = \langle \bar{f}_i \rangle$ for all i and none of them is an embedded prime since all f_i are irreducible. □

Remark 2.2.17. We have the chain of implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4). The last one holds true because if $Q = \mathrm{Ann}_R(s)$ is prime and s is nilpotent, then $s^n = 0$ for some $n \in \mathbb{N}$ and $s^n \in Q$ implies that $s \in Q$, hence $s^2 = 0$.

Example 2.2.18. Let us apply the procedure of Epstein-Yao which constructs global NZDs from local ones to Example E.3. We localize at the maximal ideal $P = \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle$. Hence $P_1, P_2 \subseteq P$, but $P_3 \nsubseteq P$. Now consider

$$\bar{Y}_P * [\bar{Z}]_P = \frac{\bar{Y}}{\bar{1}} * \frac{[\bar{Z}]}{\bar{1}} = \frac{[\bar{Y}\bar{Z}]}{\bar{1}} = 0$$

in M_P. As explained in Example 1.4.29, $\bar{Y}_P \in R_P$ is a non-zero NZD. Moreover $[\bar{Z}]_P \neq 0$ since

$$\mathrm{Ann}_R([\bar{Z}]) = \langle \bar{Y}(\bar{X}-1), \bar{X}(\bar{X}-1), \bar{Y} \rangle = \langle \bar{X}(\bar{X}-1), \bar{Y} \rangle \subseteq P \ .$$

Thus $[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$ is a non-zero torsion element. According to Remark 2.1.19 we now apply the methods from Proposition 2.1.15, Corollary 2.1.18, Proposition 2.2.10 and Theorem 2.2.13 to
− find $r \notin P$ such that $\frac{[\bar{Z}]}{\bar{1}}$ is a torsion element in $\mathcal{T}_{R_r}(M_r)$ over $D(r)$.
− find a global torsion element in $\mathcal{T}_R(M)$ that represents $[\bar{Z}]_P$, which is possible since R is reduced.
The algorithm from Proposition 2.1.15 requires $r \in P_3 \setminus P$, so we take $r = \bar{X}$. Alternatively we can look for $I = \mathrm{Ann}_R(\bar{Y}) = \langle\, \bar{Z}(\bar{X}-1)\,\rangle$, so that $\bar{X} \cdot I = \{\bar{0}\}$. By (1.6) the zero-divisors in $R_{\bar{X}}$ are the localizations of elements from the primes that do not contain $\bar{X}$. Thus $\frac{\bar{Y}}{\bar{1}} \in R_{\bar{X}}$ is also a NZD and $\bar{X} * [\bar{Z}] \neq 0$, so $[\bar{Z}]$ remains a non-zero torsion element on $D(\bar{X})$:

$$[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P) \quad \Rightarrow \quad \tfrac{[\bar{Z}]}{\bar{1}} \in \mathcal{T}_{R_{\bar{X}}}(M_{\bar{X}}) \ .$$

To find a global torsion element, we already computed $\mathcal{T}_R(M) = \langle\, [\bar{X}\bar{Z}]\,\rangle$ in Example 1.4.27, so it cannot be $[\bar{Z}]$ itself. As in Proposition 2.2.10 we first shall find $w \notin P$ and $r \in R$ such that $w\bar{Y} + r$ is a global NZD. As $P_3 \nsubseteq P_1 \cup P_2 \cup P$, we can take $w = \bar{X}$. $P_1 \nsubseteq P_3$ and $P_2 \nsubseteq P_3$ give $r_1 = \bar{Z}$ and $r_2 = \bar{X} - 1$, hence $r = \bar{Z}(\bar{X}-1)$. Moreover $w \cdot r = \bar{X}\bar{Z}(\bar{X}-1) = \bar{0}$. Here we don't need to take powers since R is reduced. Finally as in the proof of Theorem 2.2.13, we find

$$\big(\bar{X}\bar{Y} + \bar{Z}(\bar{X}-1)\big) * \big(\bar{X} * [\bar{Z}]\big) = \bar{X}\bar{Y} * [\bar{X}\bar{Z}] + \bar{Z}(\bar{X}-1) * [\bar{X}\bar{Z}] = [\bar{0}] \ ,$$

hence $[\bar{X}\bar{Z}] \in \mathcal{T}_R(M)$ is a global torsion element that represents $[\bar{Z}]_P$; since $\bar{X} \notin P$, we can write

$$[\bar{Z}]_P = \frac{[\bar{Z}]}{\bar{1}} = \frac{\bar{X} * [\bar{Z}]}{\bar{X}} = \frac{[\bar{X}\bar{Z}]}{\bar{X}} \in \big(\mathcal{T}_R(M)\big)_P \ .$$

Remark 2.2.19. In this example we even have $[\bar{X}\bar{Z}]_P = [\bar{Z}]_P$ since

$$\begin{aligned} \frac{[\bar{X}\bar{Z}]}{\bar{1}} = \frac{[\bar{Z}]}{\bar{1}} \ &\Leftrightarrow\ \exists\, \bar{f} \notin P \text{ such that } \bar{f} * \big([\bar{X}\bar{Z}] - [\bar{Z}]\big) = [\bar{0}] \\ &\Leftrightarrow\ \exists\, \bar{f} \notin P \ :\ \bar{f}\bar{X}\bar{Z} - \bar{f}\bar{Z} \in \langle\, \bar{Y}\bar{Z}\,\rangle \\ &\Leftrightarrow\ \exists\, \bar{f} \notin P \ :\ \bar{f}\bar{Z}(\bar{X}-1) \in \langle\, \bar{Y}\bar{Z}\,\rangle \ , \end{aligned}$$

so it suffices to take $\bar{f} = \bar{X}$. This is however a coincidence due to the relations in the ring; in general a denominator for the global torsion element is needed.

2.2.4 Other consequences

In a Noetherian ring with no embedded primes, one can also prove some more results which do not hold true in general. For example we have the following generalization of Proposition C.4.14.

Proposition 2.2.20. [[21], 3.8, p.7]
Let R be a Noetherian ring that has no embedded primes, M an R-module and $S \subset R$ a multiplicatively closed subset. If M is torsion-free over R, then $S^{-1}M$ is torsion-free over $S^{-1}R$.

Proof. We use the criterion from Proposition 1.3.3 which says that a module is torsion-free if and only if all its associated primes are contained in some of the $P_1, \ldots, P_\alpha \in \mathrm{Ass}_R(R)$. If there are no embedded primes, this means that a module is torsion-free if and only if all associated primes of M are minimal primes of R (since if $P \subseteq P_i$ for P_i minimal implies by Proposition B.2.19 that there is another minimal prime P_j such that $P_j \subseteq P \subseteq P_i$, so we have equality otherwise P_i would be embedded). Now assume that M is torsion-free, i.e. every prime in $\mathrm{Ass}_R(M)$ is equal to some $P_i \in \mathrm{Ass}_R(R)$. By Proposition B.3.7 the associated primes of the localization are given by

$$\begin{aligned} \mathrm{Ass}_{S^{-1}R}\left(S^{-1}M\right) &= \left\{\, S^{-1}P \;\middle|\; P \in \mathrm{Ass}_R(M),\ P \cap S = \emptyset \,\right\} \\ &\subseteq \left\{\, S^{-1}P_i \;\middle|\; P_i \cap S = \emptyset \,\right\} . \end{aligned}$$

If we want $S^{-1}M$ to be torsion-free, these should be contained in the associated primes of $S^{-1}R$. But the latter are exactly given by the $S^{-1}P_i$ such that P_i and S have empty intersection. Hence all associated primes of $S^{-1}M$ are associated primes of $S^{-1}R$, i.e. $S^{-1}M$ is torsion-free. □

Remark 2.2.21. What goes wrong in this argument if there are embedded primes? If M is torsion-free over R and $P \in \mathrm{Ass}_R(M)$ is such that $P \subsetneq P_j$ for some embedded prime P_j, then it may happen that $P \cap S = \emptyset$, but $P_j \cap S \neq \emptyset$ and so $S^{-1}P$ would no longer be contained in an associated prime of $S^{-1}R$.

Another result one can obtain is e.g. the converse of Corollary 1.3.10.

Corollary 2.2.22. *Let $\mathcal{X} = \operatorname{Spec} R$ for a Noetherian ring R that has no embedded primes and $\mathcal{F} = \widetilde{M}$ be quasi-coherent. Then*

$$M \in \operatorname{Mod}(R) \textit{ is torsion-free } \Leftrightarrow \widetilde{M} \in \mathtt{QCoh}(\mathcal{O}_R) \textit{ is torsion-free }.$$

Proof. Sufficiency is proven in Corollary 1.3.10 since the stalks of $\widetilde{M}$ are the localizations M_P. For necessity we now apply Proposition 2.2.20 with $S = R \setminus P$, which says that torsion-freeness of M over R implies that M_P is torsion-free over R_P for all $P \in \operatorname{Spec} R$. □

2.2.5 Relation with Serre's conditions

The condition of a Noetherian ring having no embedded primes is actually a particular case of a more general concept. We briefly explain the relations in this section.

Definition 2.2.23. [[16], p.241 & 425][2]
Let A be a local Noetherian ring with maximal ideal $\mathfrak{M}$. For all elements $a_1, \ldots, a_n \in R$, denote $I_i = \langle a_1, \ldots, a_i \rangle$. We say that $a_1, \ldots, a_n$ is a *regular sequence* if $I_n \neq A$, a_1 is a NZD and $\bar{a}_{i+1}$ is a NZD in A/I_i, $\forall\, i \geq 1$. The *depth* of A, denoted by $\operatorname{depth}(A)$, is the maximal length of a regular sequence $a_1, \ldots, a_n$ with $a_i \in \mathfrak{M}$ for all i.
We also recall that A is called a *regular* local ring if $\dim A = \dim_{R/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2)$.

Lemma 2.2.24. [[16], 18.2, p.448-449]
For any local Noetherian ring A, we have $\operatorname{depth}(A) \leq \dim A$.

Proof. First look at $I_1 = \langle a_1 \rangle$. Since a_1 is a NZD we know by Krull's Principal Ideal Theorem that I_1 has height 1 and thus by Proposition 1.4.13:

$$\dim(A/I_1) + \operatorname{ht}(I_1) \leq \dim A \;\Leftrightarrow\; \dim(A/I_1) \leq \dim A - 1 .$$

[2]The definitions in Eisenbud [16] may a priori seem to be different since we took them from several chapters, but they are equivalent by combining all of them.

Next we want to divide out I_2. As $\bar{a}_2$ is a NZD in A/I_1, the principal ideal $\langle \bar{a}_2 \rangle$ has height 1 as well and we get again

$$\begin{aligned}\dim(A/I_2) = \dim\big((A/I_1)/(I_2/I_1)\big) &\le \dim\big((A/I_1)/\langle \bar{a}_2 \rangle\big)\\ &\le \dim(A/I_1) - 1 \le \dim A - 2\end{aligned}$$

because $\langle \bar{a}_2 \rangle \subseteq I_2/I_1$. Continuing the same way we obtain that $0 \le \dim(A/I_n) \le \dim A - n$, so $n \le \dim A$. Taking a regular sequence of maximal length n then gives $\operatorname{depth}(A) \le \dim A$. □

Definition 2.2.25. [[16], p.225 & p.452]
1) A *local Cohen-Macaulay ring* is a local Noetherian ring A such that

$$\operatorname{depth}(A) = \dim A \ .$$

2) An arbitrary Noetherian ring R is called *Cohen-Macaulay* if the localizations R_P are local Cohen-Macaulay rings for all prime ideals $P \trianglelefteq R$. By Lemma 1.4.12 we have $\dim(R_P) = \operatorname{ht}(P)$ for any prime ideal P. Hence a Cohen-Macaulay ring R satisfies $\operatorname{depth}(R_P) = \operatorname{ht}(P)$, $\forall\, P \in \operatorname{Spec} R$.

Definition 2.2.26. Fix an integer $k \ge 0$ and let R be a Noetherian ring.
1) R is said to satisfy *Serre's condition* (R_k) if R_P is a regular local ring for all $P \in \operatorname{Spec} R$ such that $\operatorname{ht}(P) \le k$.
2) R satisfies *Serre's condition* (S_k) if $\operatorname{depth}(R_P) \ge \min\{k, \operatorname{ht}(P)\}$ for all prime ideals $P \trianglelefteq R$.

Example 2.2.27. Every ring satisfies (S_0). If R is Cohen-Macaulay, then R satisfies (S_k), $\forall\, k \ge 0$. If R satisfies (R_k), resp. (S_k) for some k, then it satisfies (R_i), resp. (S_i) for all $i \le k$.

We are particularly interested in the condition (S_1). Indeed

Proposition 2.2.28. [[53], 920120]
A Noetherian ring satisfies condition (S_1) if and only if it has no embedded associated primes.

Proof. First note the 2 following facts:
1) P is an associated prime of R if and only if P_P is an associated prime of R_P (since $P \subseteq P$).
2) A local ring $(A, \mathfrak{M})$ satisfies $\text{depth}(A) = 0$ if and only if $\mathfrak{M}$ is an associated prime of A. Indeed if $\mathfrak{M}$ is associated, then it only contains zero-divisors so one cannot choose a regular element $r_1 \in \mathfrak{M}$. Conversely if the depth of A is 0, this means that $\mathfrak{M}$ only contains zero-divisors (otherwise one could choose a regular element), hence is contained in an associated prime of A by Prime Avoidance. Maximality then implies that it is equal to that prime ideal.
$\underline{\Rightarrow}$: Assume that R satisfies (S_1) and let P be an associated prime of R. Then P_P is an associated prime of R_P, hence

$$0 = \text{depth}(R_P) \geq \min\{1, \text{ht}(P)\} \;\Rightarrow\; \text{ht}(P) = 0$$

and P is a minimal prime.
$\underline{\Leftarrow}$: (S_1) is obvious satisfied for minimal primes since these are of height 0. Thus let P be any prime ideal in R that is not minimal, so by assumption it is not associated. Hence P_P is not associated neither and $\text{depth}(R_P) \geq 1$. Therefore all prime ideals in R satisfy $\text{depth}(R_P) \geq \min\{1, \text{ht}(P)\}$. □

Remark 2.2.29. [[37], 4.5.2 & 4.5.3, p.70-71]
For completion let us also mention the following results. One can show that a Noetherian ring is reduced if and only if it satisfies (R_0) and (S_1) and that it is normal (i.e. all localizations are integrally closed domains) if and only if it satisfies (R_1) and (S_2).

2.3 Example of a non-coherent torsion subsheaf

Now we give an example of a Noetherian ring and a finitely generated module such that the torsion subsheaf of the coherent sheaf associated to that module is not coherent anymore. It is similar to the one mentioned in [[21], 3.9, p.7]. By Theorem 2.2.13, we know that such an example must have embedded primes. Consider

$$R := \mathbb{K}[X, Y, Z]/\langle\, XY, X^2, XZ \,\rangle \;.$$

Geometrically this corresponds to the plane given by the equation $X = 0$ with an embedded double point at the origin. From now on this will be called Example E.4.

Example 2.3.1. R is not reduced as $\bar{X}$ is nilpotent of degree 2. The primary decomposition of its defining ideal is

$$\langle XY, X^2, XZ \rangle = \langle X \rangle \cap \langle X^2, Y, Z \rangle ,$$

giving the decomposition $\{\bar{0}\} = \langle \bar{X} \rangle \cap \langle \bar{Y}, \bar{Z} \rangle = Q_1 \cap Q_2$ and the associated primes

$$\begin{aligned} P_1 &= Q_1 = \langle \bar{X} \rangle = \mathrm{Ann}_R(\bar{Y}) = \mathrm{Ann}_R(\bar{Z}) , \\ P_2 &= \mathrm{Rad}(Q_2) = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle = \mathrm{Ann}_R(\bar{X}) . \end{aligned}$$

Hence P_2 is an embedded prime and given by the annihilator of a nilpotent element (Lemma 2.2.14). It describes the embedded double point at $(0, 0, 0)$. The set of zero-divisors is therefore given by $\mathrm{ZD}(R) = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$. If $\mathcal{X} = \mathrm{Spec}\, R$, we get the dimension $\dim \mathcal{X} = \dim R = 2$ since $\mathbb{K}[X, Y, Z] \twoheadrightarrow R$ and because of the chain of prime ideals $\langle \bar{X} \rangle \subsetneq \langle \bar{X}, \bar{Y} \rangle \subsetneq \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$. If $\mathcal{X}_1 = V(P_1)$ and $\mathcal{X}_2 = V(Q_2)$, then $\dim \mathcal{X}_1 = 2$ and $\dim \mathcal{X}_2 = 0$ since $R/P_1 \cong \mathbb{K}[\bar{Y}, \bar{Z}]$ and $R/P_2 \cong \mathbb{K}$.

Now consider the R-module $M = R/\langle \bar{Y}\bar{Z} \rangle$, which is generated by $[\bar{1}]$. Therefore the sheaf $\mathcal{F} = \widetilde{M}$, which is just but the structure sheaf of the "cross" $\{YZ = 0\}$ inside of $\mathcal{X}$, is coherent. In Example C.4.23 it is shown that M is torsion-free, but let us also check this by using Proposition 1.3.3. If we denote

$$J := \langle XY, X^2, XZ, YZ \rangle \trianglelefteq \mathbb{K}[X, Y, Z] ,$$

this gives the primary decompositions $J = \langle X, Z \rangle \cap \langle X, Y \rangle \cap \langle X^2, Y, Z \rangle$ and

$$\langle \bar{Y}\bar{Z} \rangle = \langle \bar{X}, \bar{Z} \rangle \cap \langle \bar{X}, \bar{Y} \rangle \cap \langle \bar{Y}, \bar{Z} \rangle .$$

By Remark B.3.2 we thus have the associated primes

$$\begin{aligned} \mathrm{Ass}_R(M) &= \mathrm{Ass}_R\left(R/\langle \bar{Y}\bar{Z} \rangle\right) = \mathrm{Ass}\left(\langle \bar{Y}\bar{Z} \rangle\right) \\ &= \left\{ P_1' = \langle \bar{X}, \bar{Z} \rangle \ , \ P_2' = \langle \bar{X}, \bar{Y} \rangle \ , \ P_3' = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle \right\} \end{aligned} \tag{2.7}$$

since $\mathrm{Rad}(Q_2) = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$. As every $P'_j \in \mathrm{Ass}_R(M)$ is contained in P_2, we obtain that M is torsion-free. In particular it follows that $\big(\mathcal{T}_R(M)\big)_P = \{0\}$ for all $P \in \operatorname{Spec} R$.

However there are prime ideals such that $\mathcal{T}_{R_P}(M_P) \neq \{0\}$. For example consider $P = \langle \bar{X}, \bar{Y}, \bar{Z} - 1 \rangle$ and the relation $\bar{Y}_P * [\bar{Z}]_P = 0$. Since $P_1 \subseteq P$ and $P_2 \nsubseteq P$, we obtain by (1.6) that $\bar{Y}_P$ is a NZD in R_P. Moreover it is non-zero since $\mathrm{Ann}_R(\bar{Y}) = \langle \bar{X} \rangle \subseteq P$. Note that R_P is actually an integral domain since $\bar{Y}_P$ and $\bar{Z}_P$ are NZDs (being contained in P_2) and $\bar{X}_P = 0$ since $\bar{Z} \cdot \bar{X} = \bar{0}$ with $\bar{Z} \notin P$. Next we have $[\bar{Z}]_P \neq 0$ because

$$\bar{f} * [\bar{Z}] = [\bar{0}] \;\Leftrightarrow\; \bar{f} \cdot \bar{Z} \in \langle \bar{Y}\bar{Z} \rangle = \langle \bar{X}, \bar{Z} \rangle \cap \langle \bar{X}, \bar{Y} \rangle \cap \langle \bar{Y}, \bar{Z} \rangle$$

with $\bar{Z} \notin \langle \bar{X}, \bar{Y} \rangle$, which is prime, so $\bar{f} \in \langle \bar{X}, \bar{Y} \rangle$ and $\mathrm{Ann}_R([\bar{Z}]) = \langle \bar{X}, \bar{Y} \rangle \subseteq P$. Hence the relation $\bar{Y}_P * [\bar{Z}]_P = 0$ implies that $[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$. This non-zero local torsion element cannot be represented by a global one (since there are none: $\mathcal{T}_R(M) = \{0\}$).

However by Corollary 2.1.21 there is always an affine open neighborhood $D(r)$ of P over which it can be represented. Since $P_2 \nsubseteq P$, we can take $r = \bar{Z}$. The zero-divisors in $R_{\bar{Z}}$ are the localizations of the elements in associated primes that do not contain $\bar{Z}$, hence just $\frac{\bar{X}}{\bar{1}}$, but this one is zero because $\bar{Z} \cdot \bar{X} = \bar{0}$. So

$$\frac{\bar{Y}}{\bar{1}} * \frac{[\bar{Z}]}{\bar{1}} = 0$$

as elements in $M_{\bar{Z}}$ where $\frac{\bar{Y}}{\bar{1}} \in R_{\bar{Z}}$ is a non-zero NZD, i.e. $\frac{[\bar{Z}]}{\bar{1}} \in \mathcal{T}_{R_{\bar{Z}}}(M_{\bar{Z}})$ is still torsion over $D(\bar{Z})$.

Remark 2.3.2. We have shown: the torsion subsheaf $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is not coherent, even though $\mathcal{F}$ is coherent. It is not quasi-coherent neither since all submodules of M are finitely generated (R being Noetherian). In particular, the equivalence (2.6) is not satisfied. Although M is a torsion-free R-module, the associated sheaf is not torsion-free as it has torsion in the stalks.

Remark 2.3.3. Let us also analyze $\operatorname{Spec} R$ from the point of view of Section 1.2.4 (connected and irreducible components). As a topological space it is irreducible, thus connected, as $\mathrm{nil}(R) = \langle \bar{X} \rangle$ is a prime ideal and we have $\operatorname{Spec} R = V\big(\langle \bar{X} \rangle\big)$. If in addition we want to take care of the embedded double point, we get

$$\begin{aligned}\operatorname{Spec} R = V\big(\{\bar{0}\}\big) &= V\big(\langle \bar{X} \rangle \cap \langle \bar{X}^2, \bar{Y}, \bar{Z} \rangle\big) = V\big(\langle \bar{X} \rangle\big) \cup V\big(\langle \bar{Y}, \bar{Z} \rangle\big) \\ &\cong \operatorname{Spec}\big(R/\langle \bar{X} \rangle\big) \cup \operatorname{Spec}\big(R/\langle \bar{Y}, \bar{Z} \rangle\big) \\ &\cong \operatorname{Spec} \mathbb{K}[\bar{Y}, \bar{Z}] \cup \operatorname{Spec}\big(\mathbb{K}[X]/\langle X^2 \rangle\big) = \mathbb{A}^2_{\mathbb{K}} \cup \{dp\}\ .\end{aligned}$$

Here again it is important to take the intersection of the ideals in order to get the union. For example, let $I = \langle X \rangle$ and $I' = \langle X^2, Y, Z \rangle$. Then $I + I' = \langle X, Y, Z \rangle$ and

$$I \cap I' = \langle XY, X^2, XZ \rangle \qquad , \qquad I \cdot I' = \langle XY, X^3, XZ \rangle\ ,$$

hence $\operatorname{Spec}\big(\mathbb{K}[X,Y,Z]/\langle XY, X^3, XZ \rangle\big)$ would define a plane with an embedded triple point at the origin, i.e. a scheme with a richer structure than the initial scheme $\operatorname{Spec} R$.

Example 2.3.4. How does the torsion of Example E.4 looks like on the support? First note that $\mathcal{Z}_a(\mathcal{F}) = \mathcal{Z}_f(\mathcal{F})$ since M is generated by 1 element (see Lemma 1.4.8).
We have seen that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ with $\mathcal{X}_1 \cong \mathbb{A}^2_{\mathbb{K}}$, $\mathcal{X}_2 \cong \{dp\}$ (where the double point is invisible on the topological level) and $\dim \mathcal{X}_1 = 2$, $\dim \mathcal{X}_2 = 0$. Let us denote the support of $\mathcal{F}$ by $\mathcal{Z}$. By definition we have $\mathrm{Ann}_R(M) = \langle \bar{Y}\bar{Z} \rangle$, so

$$\mathcal{Z} = V\big(\mathrm{Ann}_R(M)\big) \cong \operatorname{Spec}\big(R/\langle \bar{Y}\bar{Z} \rangle\big) \cong \operatorname{Spec}\big(\mathbb{K}[X,Y,Z]/J\big) \hookrightarrow \operatorname{Spec} R\ .$$

Looking at the primary decomposition of J, we see that $\mathcal{Z}$ is given by the union of the lines

$$\mathcal{Z}_1 = V\big(\langle \bar{X}, \bar{Z} \rangle\big) \qquad , \qquad \mathcal{Z}_2 = V\big(\langle \bar{X}, \bar{Y} \rangle\big)$$

and the embedded double point $\mathcal{Z}_3 = \mathcal{X}_2$, which is exactly the "cross" $\{YZ = 0\}$ inside of $\mathcal{X}$. Intuitively we thus have $\dim \mathcal{Z} = 1$. To prove it rigorously, note that any prime ideal in $\mathcal{Z}$ must contain $\bar{X}$ since $\mathcal{Z} \subset \mathcal{X} = V\big(\langle \bar{X} \rangle\big)$. From $\varphi : \mathbb{K}[X,Y,Z] \twoheadrightarrow \mathbb{K}[X,Y,Z]/J$, we already get $\dim \mathcal{Z} < 3$. To show that the dimension cannot be 2, let $\bar{P}_0 \subsetneq \bar{P}_1 \subsetneq \bar{P}_2$ be a maximal chain of prime ideals in

$\mathbb{K}[X,Y,Z]/J$. As the prime ideals must contain $\bar{X}$, we get $\langle \bar{X} \rangle \subsetneq \bar{P}_0 \subsetneq \bar{P}_1 \subsetneq \bar{P}_2$, where $\langle \bar{X} \rangle$ is not prime. Now consider the chain of preimages under φ, i.e.

$$\{0\} \subsetneq \langle X \rangle \subsetneq \Big(\langle X, YZ \rangle = \varphi^{-1}\big(\langle \bar{X} \rangle\big) \subsetneq \Big)\varphi^{-1}(\bar{P}_0) \subsetneq \varphi^{-1}(\bar{P}_1) \subsetneq \varphi^{-1}(\bar{P}_2) \ ,$$

which gives a chain of length 4 in $\mathbb{K}[X,Y,Z]$: contradiction. Hence $\dim \mathcal{Z} < 2$ and the chain $\langle \bar{X}, \bar{Y} \rangle \subsetneq \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$ finally gives $\dim \mathcal{Z} = 1$.
We point out that $\mathcal{Z} \subseteq \mathcal{X}_1 = V(P_1)$ as topological spaces since every prime ideal in $\mathcal{Z}$ contains $\bar{X}$, but $\mathcal{Z} \hookrightarrow \mathcal{X}_1$ is not a subscheme! The reason is the double point, so there is no surjection

$$\nexists \quad \mathbb{K}[X,Y,Z]/\langle X \rangle \twoheadrightarrow \mathbb{K}[X,Y,Z]/J$$

which could define a closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}_1$. Now we see that

$$\dim(\mathcal{Z} \cap \mathcal{X}_1) = \dim \mathcal{Z} = 1 < 2 = \dim \mathcal{X}_1$$

and $\dim(\mathcal{Z} \cap \mathcal{X}_2) = \dim \mathcal{X}_2 = 0$ since $\langle \bar{X}, \bar{Y}, \bar{Z} \rangle \in \mathcal{Z}$, so the dimension dropped in the component $\mathcal{X}_1$ but not in $\mathcal{X}_2$. By (1.13) and Theorem 1.4.23 this again illustrates that M is not a torsion module.

Now let us compute the support of the torsion subsheaf $\mathcal{T}(\mathcal{F})$. It is given by the set of all prime ideals $P \trianglelefteq R$ such that $\mathcal{T}_{R_P}(M_P) \neq \{0\}$. It suffices to check it for $P \in \mathcal{Z}$ (otherwise $M_P = \{0\}$ anyway). As a set we have

$$\mathcal{Z} = \Big\{ \langle \bar{X}, \bar{Y} \rangle , \langle \bar{X}, \bar{Z} \rangle , \langle \bar{X}, \bar{Y}, \bar{Z} - \lambda \rangle , \langle \bar{X}, \bar{Y} - \mu, \bar{Z} \rangle , \mathfrak{M} \ \Big| \ \lambda, \mu \neq 0 \Big\} \subseteq \mathcal{X} \ .$$

where $\mathfrak{M} = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$. First consider $P \neq \mathfrak{M}$, so that $P_1 \subseteq P$ and $P_2 \nsubseteq P$. Then $\bar{X}_P = 0$ since either $\bar{X} \notin P$ or $\bar{Y} \notin P$ or $\bar{Z} \notin P$. Hence $\bar{Y}_P$ and $\bar{Z}_P$ are NZDs in R_P and they are moreover non-zero since $\mathrm{Ann}_R(\bar{Y}) = \mathrm{Ann}_R(\bar{Z}) = \langle \bar{X} \rangle \subseteq P$. We also get

$$\mathrm{Ann}_R([\bar{Y}]) = \langle \bar{X}, \bar{Z} \rangle \qquad \text{and} \qquad \mathrm{Ann}_R([\bar{Z}]) = \langle \bar{X}, \bar{Y} \rangle \ .$$

If $\bar{Z} \in P$, then $\mathrm{Ann}_R([\bar{Y}]) \subseteq P$, so $[\bar{Y}]_P \neq 0$, $\bar{Z}_P * [\bar{Y}]_P = 0$ and $[\bar{Y}]_P \in \mathcal{T}_{R_P}(M_P)$.
If $\bar{Y} \in P$, then $\mathrm{Ann}_R([\bar{Z}]) \subseteq P$, so $[\bar{Z}]_P \neq 0$, $\bar{Y}_P * [\bar{Z}]_P = 0$ and $[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$.
Thus

$$\langle \bar{X}, \bar{Y} \rangle , \langle \bar{X}, \bar{Z} \rangle , \langle \bar{X}, \bar{Y}, \bar{Z} - \lambda \rangle , \langle \bar{X}, \bar{Y} - \mu, \bar{Z} \rangle \in \operatorname{supp} \mathcal{T}(\mathcal{F})$$

and the only ideal left to check is $\mathfrak{M}$. Here $P_1 \subset P_2 = \mathfrak{M}$, so $\mathrm{ZD}(R_{\mathfrak{M}}) = (P_2)_{\mathfrak{M}}$ implies that $[\bar{X}]_{\mathfrak{M}}, [\bar{Y}]_{\mathfrak{M}}, [\bar{Z}]_{\mathfrak{M}}$ remain zero-divisors in $R_{\mathfrak{M}}$ and none of them is zero because their annihilators are contained in $\mathfrak{M}$. On the other hand, $\mathfrak{M}_{\mathfrak{M}}$ is the unique maximal ideal in the local ring $R_{\mathfrak{M}}$, so all elements from $R_{\mathfrak{M}} \setminus \mathfrak{M}_{\mathfrak{M}}$ (i.e. all NZDs) are units. Thus if there is a torsion element $\frac{m}{s} \in \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}})$ with a NZD $\frac{a}{t} \in R_{\mathfrak{M}}$ annihilating it, then $\frac{a}{t}$ is invertible and multiplying by its inverse in the equation $\frac{a}{t} * \frac{m}{s} = 0$ implies that $\frac{m}{s} = 0$, hence $\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) = \{0\}$. So $\mathfrak{M}$ is the only prime ideal in $\mathcal{Z}$ that is not in the support of the torsion subsheaf. Since it is a maximal ideal, the point $\{\mathfrak{M}\}$ is closed, which means that

$$\operatorname{supp} \mathcal{T}(\mathcal{F}) = \mathcal{Z} \setminus \{\mathfrak{M}\}$$

is open (in $\mathcal{Z}$ and in $\mathcal{X}$). In particular Proposition 1.4.4 implies that $\mathcal{T}(\mathcal{F})$ cannot be coherent because its support is not closed.

Remark 2.3.5. This example shows that torsion can be very strange. The sheaf $\mathcal{F} = \widetilde{M}$ has torsion while the module M itself has none and the torsion subsheaf, which is not coherent, has support which is (topologically) dense in $\operatorname{supp} \mathcal{F}$. On the other hand the example does not contradict our intuition which says that torsion should drop dimension in each component. Indeed we have

$$\dim \big(\operatorname{supp} \mathcal{T}(\mathcal{F}) \cap \mathcal{X}_1 \big) = \dim \big(\mathcal{Z} \setminus \{\mathfrak{M}\} \big) = 1 < 2 = \dim \mathcal{X}_1 \ ,$$
$$\dim \big(\operatorname{supp} \mathcal{T}(\mathcal{F}) \cap \mathcal{X}_2 \big) = \dim \emptyset = -1 < 0 = \dim \mathcal{X}_2 \ ,$$

and even though the torsion is dense in $\operatorname{supp} \mathcal{F}$, it is not dense in each component of the support. For this consider $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$, i.e. the support of $\mathcal{T}(\mathcal{F})$ is dense in the lines $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (just missing the point $\mathfrak{M}$), but not in the double point $\mathcal{Z}_3$.

Remark 2.3.6. The subsheaf $\mathcal{T}(\mathcal{F})$ is also an example of a non-coherent subsheaf of a coherent sheaf on a Noetherian scheme. This is because it is not of finite type in a neighborhood of $\mathfrak{M}$. Indeed assume that there is an open neighborhood $U \subseteq \mathcal{X}$ of $\mathfrak{M}$ such that the sequence

$$(\mathcal{O}_{\mathcal{X}}|_U)^n \longrightarrow \mathcal{T}(\mathcal{F})|_U \longrightarrow 0$$

is exact. So in particular there exist sections $s_1, \ldots, s_n \in \mathcal{O}_{\mathcal{X}}(U)$ whose germs generate all stalks in U, i.e.

$$\big(\mathcal{T}(\mathcal{F})\big)_x \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) = \big\langle\, [s_1]_x, \ldots, [s_n]_x \,\big\rangle \ , \quad \forall\, x \in U \ .$$

Since the stalk at $x = \mathfrak{M}$ is zero, we get $[s_1]_{\mathfrak{M}} = \ldots = [s_n]_{\mathfrak{M}} = 0$, hence there is an open neighborhood $V \subseteq U$ of $\mathfrak{M}$ on which all the s_i are zero (take the intersection of the finitely many open sets where the s_i vanish individually). But this would imply that $\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) = \{0\}$ for all $x \in V$. Assume that $V \cap \mathcal{Z} = \{\mathfrak{M}\}$, hence $\mathcal{Z} \setminus \{\mathfrak{M}\} \subseteq \mathcal{X} \setminus V$. Taking the closure we get $\mathcal{Z} \subseteq \mathcal{X} \setminus V$ since V is open and this contradicts that $\mathfrak{M} \in V \cap \mathcal{Z}$. So we get a contradiction as $\{\mathfrak{M}\} \subsetneq V \cap \mathcal{Z}$, but we have known in Example 2.3.4 that $\mathfrak{M}$ is the only point in $\mathcal{Z}$ where $\mathcal{T}(\mathcal{F})$ has zero stalk.
Not being locally of finite type does not mean that some stalks are not finitely generated. Indeed all stalks of $\mathcal{T}(\mathcal{F})$ are finitely generated since $R_P \cong \mathcal{O}_{\mathcal{X},x}$ is Noetherian and M_P is finitely generated over R_P, $\forall\, P \in \operatorname{Spec} R$, hence so are all its submodules. The crucial fact here is that one cannot find an open neighborhood U of $\mathfrak{M}$ on which <u>the same</u> sections generate <u>all</u> stalks in U.

2.4 Meromorphic functions

In this section we want to give an alternative description of the torsion subsheaf of a coherent sheaf on a locally Noetherian scheme. Indeed we will study the relation between our definition and the one of A. Grothendieck in EGA I [31] and EGA IV.4 [33]. Moreover it has already been addressed by Kleiman in [43] that the latter contains some errors and has to be modified.
First we are going to define the sheaf of meromorphic functions and study some of its properties. In particular we will generalize a statement from Murfet [59] using the result of Epstein-Yao [21] in order to prove that this sheaf is quasi-coherent if the ring has no embedded associated primes; this will be the aim of Theorem 2.4.19. Finally we also prove in Theorem 2.4.22 that both definitions of the torsion subsheaf are equivalent in the Noetherian case.

2.4.1 Total quotient ring

For our goal we first need the following notion.

Definition 2.4.1. [[11], 10.9.8]
Let R be a ring and S be the set of all NZDs in R. Hence $1 \in S$, $0 \notin S$ and S is multiplicatively closed. The *total quotient ring* of R is defined as the localization $R_{\text{tot}} := S^{-1}R$. Since S does not contain zero-divisors, Lemma A.1.2 gives an injection $i : R \hookrightarrow R_{\text{tot}}$. The idea is to generalize the notion of the quotient field. Indeed we have $R_{\text{tot}} = \text{Quot}(R)$ if R is an integral domain.

Lemma 2.4.2. *Let* $M \in \text{Mod}(R)$ *and consider the morphism*

$$\ell \; : \; M \longrightarrow M \otimes_R R_{\text{tot}} \; : \; m \longmapsto m \otimes \tfrac{1}{1} \; .$$

Then $\ker \ell = \mathcal{T}_R(M) \cong \text{Tor}_1(M, R_{\text{tot}}/R)$.

Proof. Since $M \otimes_R R_{\text{tot}} \cong S^{-1}M$ by Lemma A.2.2, we get

$$\begin{aligned} \ell(m) = 0 \;\Leftrightarrow\; \tfrac{m}{1} = 0 \;\Leftrightarrow\; \exists\, s \in S \text{ such that } s * m = 0 \\ \Leftrightarrow\; \exists\, s \neq 0 \text{ which is a NZD and } s * m = 0 \;\Leftrightarrow\; m \in \mathcal{T}_R(M) \; . \end{aligned}$$

The proof of the isomorphism is exactly the same as the one in Proposition C.4.12 since R_{tot} is flat over R (see Corollary A.2.8). □

Remark 2.4.3. Note that Lemma 2.4.2 is a generalization of Proposition C.4.1 and Proposition C.4.12, which give the corresponding results in the case of integral domains.

Proposition 2.4.4. *The tuple* (R_{tot}, i) *satisfies the following universal property: For any ring homomorphism* $\varphi : R \to T$ *such that* φ *maps* S *to units in* T*, there exists a unique homomorphism of rings* $\phi : R_{\text{tot}} \to T$ *such that* $\phi \circ i = \varphi$*, i.e. the following diagram commutes:*

$$\begin{array}{ccc} R_{\text{tot}} & \overset{\exists!\,\phi}{\dashrightarrow} & T \\ {\scriptstyle i}\uparrow & \nearrow_{\varphi} & \\ S \subset R & & \end{array}$$

Proof. This is a particular case of Proposition A.1.4. □

2.4.2 Definition of $\mathcal{K}_\mathcal{X}$

Now we apply the same idea to coherent sheaves on non-integral schemes. Here all schemes $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ are considered to be locally Noetherian. We follow the ideas developed in [33], [43] and [66].

Definition 2.4.5. [[59], Def.10, p.13] , [[43], p.204] and [[66], p.8]
For all $U \subseteq \mathcal{X}$ open, let

$$S(U) = \left\{ f \in \mathcal{O}_\mathcal{X}(U) \;\middle|\; f \neq 0 \text{ and } [f]_x \text{ is a NZD in } \mathcal{O}_{\mathcal{X},x},\ \forall\, x \in U \right\} .$$

Note that $0 \notin S(U)$, so $S(U)$ is neither a subring, nor a submodule of $\mathcal{O}_\mathcal{X}(U)$. It is just a subset.

Lemma 2.4.6. [[59], Def.10, p.13]
The assignment $U \mapsto S(U)$ with the restrictions of $\mathcal{O}_\mathcal{X}$ defines a sheaf of sets.

Proof. For $V \subseteq U$ open, the restriction $S(U) \to S(V) : f \mapsto f_{|V}$ is well-defined since $[f]_x$ is a NZD, $\forall\, x \in V$ as well. In particular, $f_{|V} \neq 0$.
Let $f, g \in S(U)$ and $U = \bigcup_i U_i$ be an open covering. If $f_{|U_i} = g_{|U_i}$ for all i, then $(f - g)_{|U_i} = 0$ in $\mathcal{O}_\mathcal{X}(U_i)$, $\forall\, i$ and hence $f - g = 0$ since $\mathcal{O}_\mathcal{X}$ is a sheaf, i.e. $f = g$. Note that we cannot assume that $f_{|U_i} = 0$ for all i since $f \neq 0$.
If we have local sections $f_i \in S(U_i)$ for all i that agree on intersections, they glue to a global section $f \in \mathcal{O}_\mathcal{X}(U)$ such that $f_{|U_i} = f_i$, $\forall\, i$. But this f also belongs to $S(U)$. Indeed, $f \neq 0$ since all f_i are non-zero and $[f]_x$ is a NZD in $\mathcal{O}_{\mathcal{X},x}$, $\forall\, x \in U$ since $[f]_x = [f_i]_x$ for some i such that $x \in U_i$. □

Proposition 2.4.7. [[43], p.204] and [[66], p.8]
For all open subset $U \subseteq \mathcal{X}$, $S(U)$ is contained in the set of NZDs of $\mathcal{O}_\mathcal{X}(U)$. Moreover $S(U)$ is multiplicatively closed.

Proof. Let $f \in S(U)$ and assume that $\exists\, g \in \mathcal{O}_\mathcal{X}(U)$ such that $f \cdot g = 0$. Hence $[f \cdot g]_x = [f]_x \cdot [g]_x = 0$, $\forall\, x \in U$. But all $[f]_x$ are NZDs, so $[g]_x = 0$ for all x, implying that $g = 0$ and f is a NZD.
$1 \in S(U)$ is obvious. To see that $S(U)$ is multiplicatively closed, let $f, g \in S(U)$. Then $f \cdot g \neq 0$ since both are NZDs and $[f \cdot g]_x = [f]_x \cdot [g]_x$ is a NZD in $\mathcal{O}_{\mathcal{X},x}$ for all $x \in U$ as well. □

Corollary 2.4.8. [[43], p.204] and [[66], p.8]
If $U \cong \operatorname{Spec} R$ is affine, then $S(U)$ is equal to the set of NZDs of $\mathcal{O}_{\mathcal{X}}(U)$.

Proof. This follows from Proposition 1.3.6 which says that a non-zero element in a ring is a NZD if and only if all its localizations are NZDs. If $U \cong \operatorname{Spec} R$ and $\mathcal{O}_{\mathcal{X}}(U) \cong R$, then

$$\begin{aligned} S(U) &= \left\{ f \in \mathcal{O}_{\mathcal{X}}|_U(U) \;\middle|\; f \neq 0 \text{ and } [f]_x \text{ is a NZD in } \mathcal{O}_{\mathcal{X},x},\ \forall\, x \in U \right\} \\ &\cong \left\{ r \in R \;\middle|\; r \neq 0 \text{ and } \tfrac{r}{1} \text{ is a NZD in } R_P,\ \forall\, P \in \operatorname{Spec} R \right\} \\ &= \left\{ r \in R \;\middle|\; r \text{ is a NZD} \right\} . \end{aligned}$$

□

Definition 2.4.9. [[43], p.204] , [[59], Def.10, p.13] and [[66], p.8]
We define the presheaf of rings $Q : U \mapsto S(U)^{-1}\mathcal{O}_{\mathcal{X}}(U)$ with the restrictions

$$Q(U) \longrightarrow Q(V) \;:\; \frac{f}{g} \longmapsto \frac{f_{|V}}{g_{|V}}$$

for $V \subseteq U$ open. Its sheafification is denoted by $\mathcal{K}_{\mathcal{X}}$ and is called the *sheaf of total quotient rings* or the *sheaf of meromorphic functions* on $\mathcal{X}$.
The idea of defining $\mathcal{K}_{\mathcal{X}}$ is to generalize the concept of the function field of an integral scheme.

Remark 2.4.10. In particular Corollary 2.4.8 implies that $Q(U) = \mathcal{O}_{\mathcal{X}}(U)_{\text{tot}}$ for affine open sets $U \subseteq \mathcal{X}$ and hence $Q(\operatorname{Spec} R) \cong R_{\text{tot}}$. But this equality does not hold in general. Indeed the assignment $U \mapsto \mathcal{O}_{\mathcal{X}}(U)_{\text{tot}}$ does not even define a presheaf as its restriction maps may not be defined[3], e.g. a NZD $r \in \mathcal{O}_{\mathcal{X}}(U)$ may become a zero-divisor in $\mathcal{O}_{\mathcal{X}}(V)$ for $V \subseteq U$, so that the section $\frac{1}{r} \in \mathcal{O}_{\mathcal{X}}(U)_{\text{tot}}$ does not have an image in $\mathcal{O}_{\mathcal{X}}(V)_{\text{tot}}$. An example is given in [[43], p.203].

2.4.3 Properties and quasi-coherence

Now we are going to study some properties of the sheaf $\mathcal{K}_{\mathcal{X}}$. In particular we are interested in the question if and/or under which conditions it is quasi-coherent.

[3] This error in EGA IV.4 [[33], 20.1.1 & 20.1.3, p.226-227] has been redressed in the paper of Kleiman.

Lemma 2.4.11. [[33], 20.1.4, p.227-228] , [[59], L.24, p.13] and [[66], 2.2, p.8]
There is an injective morphism $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{K}_{\mathcal{X}}$, making $\mathcal{K}_{\mathcal{X}}$ an $\mathcal{O}_{\mathcal{X}}$–module.

Proof. Let $U \subseteq \mathcal{X}$ be open. By Proposition 2.4.7 $S(U)$ does not contain zero-divisors, so the natural morphism

$$i_U \;:\; \mathcal{O}_{\mathcal{X}}(U) \longrightarrow Q(U) \;:\; f \longmapsto \tfrac{f}{1}$$

is injective by Lemma A.1.2 and we get a morphism of presheaves $i : \mathcal{O}_{\mathcal{X}} \hookrightarrow Q$. Combining with the sheafification $\theta : Q \to \mathcal{K}_{\mathcal{X}}$, we get a morphism of sheaves $\mathcal{O}_{\mathcal{X}} \to \mathcal{K}_{\mathcal{X}}$, turning $\mathcal{K}_{\mathcal{X}}$ into a sheaf of $\mathcal{O}_{\mathcal{X}}$–modules (see Lemma D.1.2). This morphism is still injective because $i_x : \mathcal{O}_{\mathcal{X},x} \hookrightarrow Q_x$ by exactness of taking stalks and θ_x is an isomorphism, so that $\mathcal{O}_{\mathcal{X},x} \hookrightarrow \mathcal{K}_{\mathcal{X},x}$ for all $x \in \mathcal{X}$. □

Proposition 2.4.12. [[59], Prop.25, p.13]
$\mathcal{K}_{\mathcal{X}}$ satisfies the following universal property: For any morphism $\varphi : \mathcal{O}_{\mathcal{X}} \to \mathcal{A}$ of sheaves of rings on $\mathcal{X}$ such that for every open set $U \subseteq \mathcal{X}$ the ring homomorphism $\varphi_U : \mathcal{O}_{\mathcal{X}}(U) \to \mathcal{A}(U)$ maps $S(U)$ to units in $\mathcal{F}(U)$, there exists a unique morphism of sheaves of rings $\phi : \mathcal{K}_{\mathcal{X}} \to \mathcal{A}$ such that $\phi \circ i = \varphi$.

$$\begin{array}{ccc} \mathcal{K}_{\mathcal{X}} & \overset{\exists!\,\phi}{\dashrightarrow} & \mathcal{A} \\ \uparrow & \nearrow_{\varphi} & \\ \mathcal{O}_{\mathcal{X}} & & \end{array}$$

Proof. Following the same argument as in the proof of Proposition A.1.4, we know that $\forall\, U \subseteq \mathcal{X}$ open there exists a unique morphism $\phi'_U : Q(U) \to \mathcal{A}(U)$ such that

$$\begin{array}{ccc} Q(U) & \overset{\exists!\,\phi'_U}{\dashrightarrow} & \mathcal{A}(U) \\ {\scriptstyle i_U}\uparrow & \nearrow_{\varphi_U} & \\ \mathcal{O}_{\mathcal{X}}(U) & & \end{array}$$

Note that here we don't necessarily have $Q(U) = \mathcal{O}_{\mathcal{X}}(U)_{\mathrm{tot}}$ since U may not be affine. Sheafifying the morphism of presheaves $\phi' : Q \to \mathcal{A}$ then gives the desired morphism of sheaves $\phi : \mathcal{K}_{\mathcal{X}} \to \mathcal{A}$. □

As $\mathcal{K}_\mathcal{X}$ is a sheafification of Q, we have $\mathcal{K}_\mathcal{X}(U) \neq Q(U)$ for $U \subseteq \mathcal{X}$ open in general. But now we will show in Proposition 2.4.14 that it is true over affines. The reader may however skip the proof as it not instructive and very technical.

Lemma 2.4.13. *Let* $r, s \in R$. *Then* $\operatorname{Spec} R = D(r) \cup D(s)$ *if and only if* $1 \in \langle r, s \rangle$. *This gives a condition to see when we have a covering of* $\operatorname{Spec} R$. *More generally,*

$$\operatorname{Spec} R = \bigcup_{i=1}^{n} D(r_i) \;\Leftrightarrow\; 1 \in \langle r_1, \ldots, r_n \rangle \ .$$

Proof. $\underline{\Rightarrow}$: $\operatorname{Spec} R = D(r) \cup D(s)$ means that any prime ideal P satisfies either $r \notin P$ or $s \notin P$. Hence $\langle r, s \rangle = R$, otherwise there is a maximal ideal $\mathfrak{M}$ containing it, which contradicts $\mathfrak{M} \in \operatorname{Spec} R$.
$\underline{\Leftarrow}$: If $1 = ar + bs$ for some $a, b \in R$, then every prime ideal P must satisfy $r \notin P$ or $s \notin P$, otherwise $1 \in P$ and P would not be proper. □

Proposition 2.4.14. [[57], Lecture 9, p.61-62]
If $U \subseteq \mathcal{X}$ *is an affine open subset, then*

$$\mathcal{K}_\mathcal{X}(U) \cong \mathcal{O}_\mathcal{X}(U)_{\text{tot}} = S(U)^{-1}\mathcal{O}_\mathcal{X}(U) \ . \tag{2.8}$$

Proof. We have to show that the assignment $U \mapsto \mathcal{O}_\mathcal{X}(U)_{\text{tot}}$ defines a sheaf when only ranging over affines, i.e. if $U \cong \operatorname{Spec} R$ is affine and covered by (finitely many) affines $U_i = D(r_i) \cong \operatorname{Spec}(R_{r_i})$, then local sections $f_i \in \mathcal{O}_\mathcal{X}(U_i)_{\text{tot}}$ that agree on intersections $U_i \cap U_j$ glue to a section $f \in \mathcal{O}_\mathcal{X}(U)_{\text{tot}}$. Note that

$$\mathcal{O}_\mathcal{X}(U)_{\text{tot}} \cong R_{\text{tot}} \qquad \text{and} \qquad \mathcal{O}_\mathcal{X}(U_i)_{\text{tot}} \cong (R_{r_i})_{\text{tot}} \ .$$

Let $f_i = \frac{s_i}{t_i}$ where $s_i, t_i \in R_{r_i}$ and the t_i are NZDs. Multiplying all of them by r_i^N for large N, we may assume that $s_i = \frac{a_i}{1}$ and $t_i = \frac{b_i}{1}$ for some $a_i, b_i \in R$. Indeed,

$$f_i = \frac{s_i}{t_i} = \frac{s_i \cdot r_i^N/1}{t_i \cdot r_i^N/1} = \frac{s_i'/r_i^{n_i} \cdot r_i^N/1}{t_i'/r_i^{m_i} \cdot r_i^N/1} = \frac{(s_i' \cdot r_i^{N-n_i})/1}{(t_i' \cdot r_i^{N-m_i})/1} = \frac{a_i/1}{b_i/1}$$

for $N > \max_i\{n_i, m_i\}$. Saying that the sections agree on intersections means:

$$\begin{aligned} & f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j} \;\Leftrightarrow\; \tfrac{s_i}{t_i} = \tfrac{s_j}{t_j} \text{ as elements in } \mathcal{O}_{\mathcal{X}}(U_i \cap U_j)_{\text{tot}} \\ & \quad\Leftrightarrow\; \text{there is a NZD } c_{ij} \in \mathcal{O}_{\mathcal{X}}(U_i \cap U_j) \cong R_{r_i r_j} \text{ such that } c_{ij} \cdot (s_i t_j - s_j t_i) = 0 \\ & \quad\Leftrightarrow\; s_i t_j - s_j t_i = 0 \text{ in } R_{r_i r_j} \text{ (since } c_{ij} \text{ is a NZD)} \\ & \quad\Leftrightarrow\; \tfrac{a_i}{1} \cdot \tfrac{b_j}{1} - \tfrac{a_j}{1} \cdot \tfrac{b_i}{1} = 0 \text{ in } R_{r_i r_j} \\ & \quad\Leftrightarrow\; \exists\, d_{ij} \in \mathbb{N} \text{ such that } (r_i r_j)^{d_{ij}} \cdot (a_i b_j - a_j b_i) = 0 \text{ in } R\,. \end{aligned}$$

Taking $D > \max_{i,j}\{d_{ij}\}$, we moreover may represent f_i as a fraction $\frac{a_i/1}{b_i/1}$ such that $a_i b_j = a_j b_i$ in R because

$$f_i = \frac{r_i^D \cdot a_i/1}{r_i^D \cdot b_i/1}$$

with $r_i^D a_i \cdot r_j^D b_j - r_j^D a_j \cdot r_i^D b_i = (r_i r_j)^D \cdot (a_i b_j - a_j b_i) = 0$. Now we have to find $f = \frac{\alpha}{\beta} \in R_{\text{tot}}$ where $\beta \in R$ is a NZD such that $f_{|U_i} = f_i$ for all i, i.e.

$$\tfrac{\alpha}{\beta} = \tfrac{s_i}{t_i} \text{ as elements in } \mathcal{O}_{\mathcal{X}}(U_i)_{\text{tot}} \;\Leftrightarrow\; \tfrac{\alpha}{1} \cdot t_i - s_i \cdot \tfrac{\beta}{1} = 0 \text{ in } R_{r_i}\,.$$

We start by defining the ideal

$$I = \left\{ r \in R \;\middle|\; \tfrac{r}{1} \cdot s_i \in \langle\, t_i \,\rangle \trianglelefteq R_{r_i},\, \forall\, i \right\}.$$

Then $b_j \in I$, $\forall\, j$ because

$$\tfrac{b_j}{1} \cdot s_i = \tfrac{b_j \cdot a_i}{1} = \tfrac{a_j \cdot b_i}{1} = s_j \cdot t_i\,.$$

Next let $c \in \operatorname{Ann}_R(I)$, so that $c \cdot b_j = 0$, $\forall\, j$. Since $t_j = \frac{b_j}{1}$ is a NZD in R_{r_j}, we hence need that $\frac{c}{1} = 0$, i.e. $\forall\, j$, $\exists\, \ell_j$ such that $r_j^{\ell_j} \cdot c = 0$ (note that this does not change if we replace b_j by $r_j^D b_j$). Let $L = \max_j \ell_j$, so that $r_j^L \cdot c = 0$ for all j. By Lemma 2.4.13 we have $1 \in \langle\, r_1, \ldots, r_n \,\rangle$. For all $M \in \mathbb{N}$ we then get

$$c = c \cdot 1 = c \cdot 1^M = c \cdot \Big(\sum_j x_j r_j\Big)^M = c \cdot \sum_{i_1 + \ldots + i_n = M} \lambda_{i_1,\ldots,i_n} \cdot r_1^{i_1} \cdot \ldots \cdot r_n^{i_n}$$

for some $\lambda_{i_1,\ldots,i_n} \in R$. Taking M large enough we can achieve that for all possible values of $i_1, \ldots, i_n$ at least one of them is bigger than L, hence all terms cancel. It follows that $c = 0$. So we have a non-zero ideal I such that $\operatorname{Ann}_R(I) = \{0\}$.

Since R is Noetherian Corollary B.2.26 implies that I must contain a NZD β, which satisfies by definition

$$\tfrac{\beta}{1} \cdot s_i = h_i \cdot t_i \text{ for some } h_i \in R_{r_i},\ \forall\, i\ .$$

In particular $\frac{\beta}{1} \cdot \frac{s_i}{t_i} = h_i$ for all i. But the $\frac{s_i}{t_i}$ are such that they agree on intersections $U_i \cap U_j$, hence so do the $h_i \in R_{r_i} \cong \mathcal{O}_\mathcal{X}(U_i)$. $\mathcal{O}_\mathcal{X}$ being a sheaf, they glue to a global $\alpha \in \mathcal{O}_\mathcal{X}(U) \cong R$ such that $\alpha_{|U_i} = h_i$ for all i, i.e. $\frac{\alpha}{1} = h_i$ in R_{r_i}. Thus we set $f := \frac{\alpha}{\beta} \in \mathcal{O}_\mathcal{X}(U)_{\text{tot}}$ and get $\frac{\beta}{1} \cdot s_i = \frac{\alpha}{1} \cdot t_i$ in R_{r_i} for all i, which is what we wanted. □

Remark 2.4.15. The isomorphism (2.8) may fail if the rings are not Noetherian; it may for example happen that $\mathcal{O}_\mathcal{X}(U)_{\text{tot}} \subsetneq \mathcal{K}_\mathcal{X}(U)$. An example is given in [[43], p.204-205].

Proposition 2.4.16. cf. [[66], 2.1 p.8]
The stalks of $\mathcal{K}_\mathcal{X}$ are the total quotient rings of the stalks of $\mathcal{O}_{\mathcal{X},x}$, i.e.

$$\mathcal{K}_{\mathcal{X},x} \cong (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}\ , \quad \forall\, x \in \mathcal{X}$$

Proof. Since $\mathcal{K}_{\mathcal{X},x} \cong Q_x$ it suffices to prove that $Q_x \cong (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}$, $\forall\, x \in \mathcal{X}$. Fix $x \in \mathcal{X}$; we consider the morphism

$$\varphi_U\ :\ Q(U) = S(U)^{-1}\mathcal{O}_\mathcal{X}(U) \longrightarrow (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}\ :\ \frac{f}{g} \longmapsto \frac{[f]_x}{[g]_x}$$

for some open neighborhood U of x. It is well-defined since $[g]_x$ is a NZD by definition. Moreover if $\frac{f}{g} = \frac{f'}{g'}$, then $\exists\, h \in S(U)$ such that $h \cdot (fg' - f'g) = 0$, hence

$$[h]_x \cdot [fg' - f'g]_x = [h]_x \cdot \big([f]_x \cdot [g']_x - [f']_x \cdot [g]_x\big) = 0\ .$$

Since $[h]_x$ is a NZD, this implies that $[f]_x \cdot [g']_x - [f']_x \cdot [g]_x = 0$ and hence $\frac{[f]_x}{[g]_x} = \frac{[f']_x}{[g']_x}$. Now we get the morphism

$$\varphi_x\ :\ Q_x \to (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}\ :\ \left[\frac{f}{g}\right]_x \longmapsto \frac{[f]_x}{[g]_x}$$

on the inductive limit, which is injective since $\frac{[f]_x}{[g]_x} = 0$ if and only if $[f]_x = 0$ (since elements in the multiplicative subset are NZDs), which means that f is

zero on a neighborhood of x and therefore $\frac{f}{g}$ is zero on a neighborhood of x as well.
To prove surjectivity, let $\frac{[f]_x}{[g]_x} \in (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}$ be given. The germs $[f]_x$ and $[g]_x$ can be represented by $f, g \in \mathcal{O}_{\mathcal{X}}(U)$ for some affine open neighborhood $U \cong \operatorname{Spec} R$ of x with $g \neq 0$ since $[g]_x \neq 0$. Let x correspond to some $P \in \operatorname{Spec} R$, so that we have an element $g \in R$ such that $\frac{g}{1}$ is a NZD in R_P. By Proposition 2.1.15 we can find $r \notin P$ such that $\frac{g}{1}$ is also a NZD in R_r. In other words there is an affine open neighborhood $V = D(r) \subseteq U$ of x such that $g_{|V}$ is a NZD in $\mathcal{O}_{\mathcal{X}}(V)$, hence so are all its germs (Proposition 1.3.6) and

$$h := \frac{f_{|V}}{g_{|V}} \in Q(V)$$

can be chosen as a preimage: $\varphi_x\big([h]_x\big) = \frac{[f]_x}{[g]_x}$. □

Remark 2.4.17. Again this may be false in the non-Noetherian case, see [[43], p.204]. In general we only have an injection $\mathcal{K}_{\mathcal{X},x} = S_x^{-1}\mathcal{O}_{\mathcal{X},x} \hookrightarrow (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}$, where S_x is contained in (but not equal to) the set of NZDs of $\mathcal{O}_{\mathcal{X},x}$.

Proposition 2.4.18. [[33], 20.1.1, p.226] and [[66], 2.2, p.8]
$\mathcal{K}_{\mathcal{X}}$ is a flat $\mathcal{O}_{\mathcal{X}}$–module.

Proof. By exactness of taking stalks, a sheaf is flat if and only if all its stalks are flat. So it follows from $\mathcal{K}_{\mathcal{X},x} \cong (\mathcal{O}_{\mathcal{X},x})_{\text{tot}}, \forall x \in \mathcal{X}$, where $(\mathcal{O}_{\mathcal{X},x})_{\text{tot}}$ is a flat module over $\mathcal{O}_{\mathcal{X},x}$ (Corollary A.2.8). □

Now we are ready to study quasi-coherence of $\mathcal{K}_{\mathcal{X}}$. Unfortunately this is not true in general as it has already been pointed out by Kleiman in [[43], p.205]. We will see this in Example 2.4.24. However it is true in the case where all Noetherian rings defining $\mathcal{X}$ have no embedded primes. For this we will again use the result from Epstein-Yao (Proposition 2.2.10).
The following statement has been proven by Murfet in [59] in the case of integral domains. Our contribution is a modification of its proof so that the result also holds true more generally in a Noetherian ring without embedded primes.

Theorem 2.4.19. cf. [[59], Lemma.28, p.14]
Let $\mathcal{X} = \operatorname{Spec} R$ *where* R *is a Noetherian ring with no embedded primes. Then we have a canonical isomorphism*

$$\widetilde{R_{\mathrm{tot}}} \xrightarrow{\sim} \mathcal{K}_{\mathcal{X}} \ .$$

Proof. We want to show that $\widetilde{R_{\mathrm{tot}}}$ satisfies the universal property of $\mathcal{K}_{\mathcal{X}}$ from Proposition 2.4.12, hence they are canonically isomorphic. Let $\varphi : \mathcal{O}_{\mathcal{X}} \to \mathcal{A}$ be any morphism of sheaves of rings on $\mathcal{X}$ such that φ_U maps elements from $S(U)$ to units in $\mathcal{A}(U)$ for every open set $U \subseteq \mathcal{X}$. In particular $\mathcal{A}$ gets the structure of an $\mathcal{O}_{\mathcal{X}}$–module. By Theorem 1.1.13 we know that the functor $\sim$ is a left adjoint of the functor of global sections $\Gamma(\mathcal{X}, \cdot)$. Thus

$$\varphi \in \operatorname{Hom}(\mathcal{O}_{\mathcal{X}}, \mathcal{A}) = \operatorname{Hom}\big(\widetilde{R}, \mathcal{A}\big) \;\cong\; \operatorname{Hom}_R\big(R, \mathcal{A}(\mathcal{X})\big)$$

and we get a homomorphism of R-modules $f : R \to \mathcal{A}(\mathcal{X})$. Let $S \subset R$ denote the set of NZDs in R. Then $f = \varphi_{\mathcal{X}}$ and f maps elements from S to units in $\mathcal{A}(\mathcal{X})$ because $S \cong S(\mathcal{X})$ under the identification $R \cong \mathcal{O}_{\mathcal{X}}(\mathcal{X})$. Alternatively by using Proposition 1.3.6,

$$\begin{aligned} S(\mathcal{X}) &= \big\{\, t \in \mathcal{O}_{\mathcal{X}}(\mathcal{X}) \;\big|\; t \neq 0,\ [t]_x \text{ is a NZD in } \mathcal{O}_{\mathcal{X},x},\ \forall\, x \in \mathcal{X} \,\big\} \\ &\cong \big\{ r \in R \;\big|\; r \neq 0,\ \tfrac{r}{1} \text{ is a NZD in } R_P,\ \forall\, P \in \operatorname{Spec} R \,\big\} \\ &= \{\, \text{all NZDs in } R \,\} = S \ . \end{aligned}$$

By the universal property of R_{tot}, we now have a unique homomorphism of rings $\psi : R_{\mathrm{tot}} \to \mathcal{A}(\mathcal{X})$ such that

$$\begin{array}{ccc} R_{\mathrm{tot}} & \overset{\psi}{\dashrightarrow} & \mathcal{A}(\mathcal{X}) \\ {\scriptstyle i}\uparrow & \nearrow_{f} & \\ R & & \end{array}$$

Applying $\sim$ and composing with the canonical morphism $\widetilde{\mathcal{A}(\mathcal{X})} \to \mathcal{A}$ induced by (1.4), we obtain

$$\begin{array}{cccc} \widetilde{R_{\mathrm{tot}}} & \overset{\widetilde{\psi}}{\dashrightarrow} & \widetilde{\mathcal{A}(\mathcal{X})} & \longrightarrow \mathcal{A} \\ {\scriptstyle \widetilde{i}}\uparrow & \nearrow_{\widetilde{f}} & & \\ \widetilde{R} & & & \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{R_{\mathrm{tot}}} & \overset{\exists!}{\dashrightarrow} & \mathcal{A} \\ {\scriptstyle j}\uparrow & \nearrow_{\varphi} & \\ \mathcal{O}_{\mathcal{X}} & & \end{array}$$

and the morphism $\widetilde{R_{\text{tot}}} \to \mathcal{A}$ is unique since it comes from $R_{\text{tot}} \to \mathcal{A}(\mathcal{X})$ by adjunction, which is unique. Now it only remains to show that the morphism of sheaves of rings $j = \widetilde{i} : \mathcal{O}_{\mathcal{X}} \to \widetilde{R_{\text{tot}}}$ maps elements from $S(U)$ to units for every open set $U \subseteq \mathcal{X}$. For this it suffices to prove it for every affine open set $D(r)$. Indeed let $U = \bigcup_i U_i$ for some affine U_i and $s \in S(U)$ with restrictions $s_i = s_{|U_i} \in S(U_i)$. If $j_{U_i}(s_i)$ are units for all i, then

$$j_{U_i}(s_i) \cdot t_i = 1 \;\Leftrightarrow\; t_i = \frac{1}{j_{U_i}(s_i)} = \frac{1}{j_U(s)_{|U_i}}\ ,\ \forall\, i$$

for some sections t_i over U_i, then these agree on intersections (since the s_i do) and glue to a global section t such that $j_U(s) \cdot t = 1$, i.e. $j_U(s)$ will be a unit as well.

On the other hand, note that $j_{D(r)} : \mathcal{O}_{\mathcal{X}}(D(r)) \to \widetilde{R_{\text{tot}}}(D(r))$ is nothing but the injection of rings $i_r : R_r \hookrightarrow (R_{\text{tot}})_r$ and we are left to prove that NZDs in R_r are mapped to units in $(R_{\text{tot}})_r$. First assume that R is an integral domain. If $\frac{a}{r^n} \in R_r$ is a NZD, then $a \neq 0$ and a is a NZD in R, hence

$$i_r \ :\ \frac{a}{r^n} \longmapsto \frac{a/1}{r^n/1} \qquad \text{with inverse} \qquad \frac{r^n/a}{1/1} \cdot \frac{a/1}{r^n/1} = \frac{r^n/1}{r^n/1} = 1\ .$$

Now let R be a ring with no embedded primes and $\frac{a}{r^n} \in R_r$ a NZD. In particular $\frac{a}{1}$ is a NZD, so by Proposition 2.2.10 we can find $w = r^m$ and $s \in R$ with $w \cdot s = 0$ such that $wa + s$ is a NZD in R. Hence the inverse of $\frac{a}{r^n}$ in $(R_{\text{tot}})_r$ is

$$\frac{(r^n w)/(wa+s)}{1/1} \cdot \frac{a/1}{r^n/1} = \frac{r^n \cdot \frac{wa}{wa+s}}{r^n \cdot 1/1} = \frac{\frac{wa+s}{wa+s} - \frac{s}{wa+s}}{1/1} = 1 - \frac{\frac{s}{1} \cdot \frac{1}{wa+s}}{1/1} = 1 - 0 = 1$$

because $\frac{w}{1} \cdot \frac{s}{1} = 0$, i.e. $\frac{s}{1}/\frac{1}{1} = 0$ in $(R_{\text{tot}})_r$. Note however that $\frac{s}{1} \in R_{\text{tot}}$ is non-zero. This finishes the proof and shows that $\mathcal{K}_{\mathcal{X}}$ is indeed canonically isomorphic to the sheaf associated to R_{tot}. □

Corollary 2.4.20. *If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a locally Noetherian scheme whose rings that define the local spectra have no embedded primes, then $\mathcal{K}_{\mathcal{X}}$ is quasi-coherent. More precisely, if $\mathcal{X} = \bigcup_i U_i$ is an affine covering with $U_i \cong \operatorname{Spec} R_i$ for some Noetherian rings R_i with no embedded primes, then*

$$\mathcal{K}_{\mathcal{X}}|_{U_i} \;\cong\; \widetilde{(R_i)_{\text{tot}}}\ .$$

In particular, this holds for integral schemes where $(R_i)_{\text{tot}} = \operatorname{Quot}(R_i)$, $\forall\, i$.

2.4.4 Relation with the torsion subsheaf

With the results from the previous section we can now understand the relation between torsion and meromorphic functions.

Definition 2.4.21. [[33], 20.1.5, p.228]
Let $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$. We define the *sheaf of meromorphic sections* on $\mathcal{X}$ by $\mathcal{K}_{\mathcal{X}}(\mathcal{F}) := \mathcal{F} \otimes \mathcal{K}_{\mathcal{X}}$. Combining the identity $\mathcal{F} \to \mathcal{F}$ and the morphism $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{K}_{\mathcal{X}}$ from Lemma 2.4.11, we moreover obtain a canonical morphism $\psi : \mathcal{F} \to \mathcal{K}_{\mathcal{X}}(\mathcal{F})$.

Theorem 2.4.22. cf. [[33], 20.1.5, p.228] and [[66], 2.6, p.10][4]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally Noetherian scheme, $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$ and $\psi : \mathcal{F} \to \mathcal{K}_{\mathcal{X}}(\mathcal{F})$ be the canonical morphism. Then

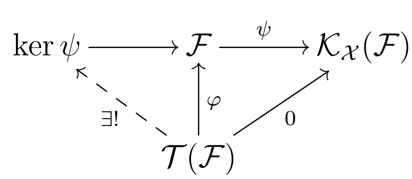
$$\mathcal{T}(\mathcal{F}) \;\cong\; \ker \psi \;. \tag{2.9}$$

In particular, $\mathcal{F}$ is torsion-free if and only if ψ is injective.

Proof. We want to show that there exists a morphism $\mathcal{T}(\mathcal{F}) \to \ker \psi$ such that $\mathcal{T}(\mathcal{F})_x \cong (\ker \psi)_x$ for all $x \in \mathcal{X}$, hence that the 2 sheaves are isomorphic. Since $(\ker \psi)_x \cong \ker(\psi_x)$, we may consider

$$\psi_x \;:\; \mathcal{F}_x \longrightarrow \big(\mathcal{K}_{\mathcal{X}}(\mathcal{F})\big)_x = \big(\mathcal{F} \otimes \mathcal{K}_{\mathcal{X}}\big)_x \;\cong\; \mathcal{F}_x \otimes_{\mathcal{O}_{\mathcal{X},x}} \mathcal{K}_{\mathcal{X},x} \;\cong\; \mathcal{F}_x \otimes_{\mathcal{O}_{\mathcal{X},x}} (\mathcal{O}_{\mathcal{X},x})_{\mathrm{tot}}$$

by Proposition 2.4.16. The same argument as in Lemma 2.4.2 then shows that $\ker(\psi_x) = \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$. Hence $\mathcal{T}(\mathcal{F})$ and $\ker \psi$ have the same stalks. In order to construct $\mathcal{T}(\mathcal{F}) \to \ker \psi$, let $\varphi : \mathcal{T}(\mathcal{F}) \hookrightarrow \mathcal{F}$ be the morphism defining the subsheaf and consider



where $\psi \circ \varphi = 0$ since

$$\varphi_x \;:\; \big(\mathcal{T}(\mathcal{F})\big)_x \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \hookrightarrow \mathcal{F}_x$$

is just the inclusion, hence $\psi_x \circ \varphi_x = (\psi \circ \varphi)_x = 0$, $\forall\, x \in \mathcal{X}$. □

[4] Trautmann [66] states the result for integral schemes, but this fact is not used in the proof.

Remark 2.4.23. (2.9) is consistent with the definition of the torsion subsheaf of Grothendieck in EGA I [[31], 7.4.1, p.163] where he defines it in the case of an integral scheme $\mathcal{X}$ as the kernel of the morphism $\mathcal{F} \to \mathcal{F} \otimes \mathcal{R}(\mathcal{X})$, where $\mathcal{R}(\mathcal{X})$ is the function field of $\mathcal{X}$ (that has to be replaced by $\mathcal{K}_\mathcal{X}$ in the non-integral case).

Example 2.4.24. We know that the torsion subsheaf of $\mathcal{F} = \widetilde{M}$ on $\mathcal{X} = \operatorname{Spec} R$ from Example E.4 is not quasi-coherent. Hence $\mathcal{K}_\mathcal{X}$ cannot be quasi-coherent neither, otherwise $\ker \psi$ would be quasi-coherent. There is also an alternative way to see this: let $\mathcal{X} = \operatorname{Spec} R$ be any affine scheme and $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_R)$ given by $\mathcal{F} \cong \widetilde{M}$ for some $M \in \operatorname{Mod}(R)$. Then $\forall\, r \in \mathcal{O}_\mathcal{X}(\mathcal{X}) \cong R$, the canonical map

$$\big(\mathcal{F}(\mathcal{X})\big)_r \longrightarrow \mathcal{F}\big(D(r)\big) \;:\; \frac{s}{r^n} \longmapsto \frac{1}{r^n} * s_{|D(r)} \;, \tag{2.10}$$

where $\mathcal{F}(\mathcal{X})$ is a module over $\mathcal{O}_\mathcal{X}(\mathcal{X})$, $s_{|D(r)} \in \mathcal{F}(D(r))$ and $\frac{1}{r^n} \in \mathcal{O}_\mathcal{X}(D(r))$, is an isomorphism since quasi-coherent sheaves satisfy

$$\widetilde{M}\big(D(r)\big) \cong M_r \;.$$

We want to show that this is not satisfied for $\mathcal{K}_\mathcal{X}$ in Example E.4. With $r = \bar{Z}$, let us compute

$$\begin{aligned} \big(\mathcal{K}_\mathcal{X}(\mathcal{X})\big)_r &\cong \big(\mathcal{O}_\mathcal{X}(\mathcal{X})_{\mathrm{tot}}\big)_r = \big(R_{\mathrm{tot}}\big)_{\bar{Z}} \;, \\ \mathcal{K}_\mathcal{X}\big(D(r)\big) &\cong \mathcal{K}_\mathcal{X}(\operatorname{Spec} R_r) \cong (R_{\bar{Z}})_{\mathrm{tot}} \end{aligned}$$

by (2.8) since $\mathcal{X}$ and $D(r)$ are affine. The zero-divisors are

$$\mathrm{ZD}(R) = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \qquad , \qquad \mathrm{ZD}(R_{\bar{Z}}) = \langle\, \tfrac{\bar{X}}{\bar{1}} \,\rangle = \{0\}$$

because $\bar{Z} \cdot \bar{X} = \bar{0}$, so $R_{\bar{Z}}$ is an integral domain and we get

$$\begin{aligned} \big(R_{\bar{Z}}\big)_{\mathrm{tot}} = \operatorname{Quot}(R_{\bar{Z}}) &= \left\{ \frac{\bar{f}/\bar{Z}^n}{\bar{g}/\bar{Z}^m} \;\middle|\; \bar{g}/\bar{1} \neq 0 \text{ in } R_{\bar{Z}} \right\} \\ &= \left\{ \frac{\bar{f}/\bar{Z}^n}{\bar{g}/\bar{Z}^m} \;\middle|\; \bar{g} \notin \operatorname{Ann}_R(\bar{Z}) \right\} = \left\{ \frac{\bar{f}/\bar{Z}^n}{\bar{g}/\bar{Z}^m} \;\middle|\; \bar{g} \notin \langle\, \bar{X} \,\rangle \right\} \end{aligned}$$

since $\operatorname{Ann}_R(\bar{Z}) = \langle\, \bar{X} \,\rangle$ is a prime ideal not containing $\bar{Z}$, so $\bar{Z}\bar{g} = \bar{0} \Leftrightarrow \bar{Z}^k\bar{g} = \bar{0}$ for some $k \geq 1$. As the NZDs in R are (classes of) polynomials with non-zero constant term, we also find

$$R_{\mathrm{tot}} = \left\{ \frac{\bar{f}}{\bar{g}} \;\middle|\; \bar{g}(0) \neq 0 \right\} \qquad , \qquad \big(R_{\mathrm{tot}}\big)_{\bar{Z}} = \left\{ \frac{\bar{f}/\bar{g}}{\bar{Z}^n} \;\middle|\; \bar{g}(0) \neq 0 \right\}$$

with R_{tot} seen as an R-module. This gives the canonical morphism from (2.10) as

$$\big(R_{\text{tot}}\big)_{\bar{Z}} \longrightarrow \big(R_{\bar{Z}}\big)_{\text{tot}} \;:\; \frac{\bar{f}/\bar{g}}{\bar{Z}^n} \longmapsto \frac{\bar{1}}{\bar{Z}^n}\cdot\frac{\bar{f}/\bar{g}}{\bar{1}/\bar{1}} = \frac{\bar{f}/\bar{Z}^n}{\bar{g}/\bar{1}}\ ,$$

which is well-defined since $\bar{g}(0) \neq 0$, so $\bar{g} \notin \langle\, \bar{X}\,\rangle$. However this map is not surjective; we can take e.g. $\bar{g} = \bar{Y}$. Then $\bar{Y} \notin \langle\, \bar{X}\,\rangle$, so the denominator in $(R_{\bar{Z}})_{\text{tot}}$ is well-defined, but $\bar{Y}$ it is a zero-divisor in R and vanishes at 0, i.e. $\bar{1}/\bar{Y}$ does not exist in R_{tot}.

Remark 2.4.25. In particular, Example 2.4.24 shows that $S^{-1}(R_{\text{tot}})$ is in general not equal to $(S^{-1}R)_{\text{tot}}$ as an R-module. But this can be achieved by constructing global NZDs when there are no embedded primes. Indeed it is shown in [[21], 4.7, p.11] that for a Noetherian ring R with no embedded primes and a multiplicatively closed subset $S \subset R$, we have the isomorphism of R-modules

$$S^{-1}(R_{\text{tot}}) \;\cong\; (S^{-1}R)_{\text{tot}}\ .$$

2.5 The Grothendieck criterion

Using the properties of the sheaf of meromorphic functions, one can obtain a very powerful criterion to decide whether a quasi-coherent sheaf on a locally Noetherian scheme is torsion-free by only looking at the associated primes of the involved rings and modules. It has been proven by Grothendieck in [33].

Definition 2.5.1. Let M be an R-module and $a \in R$. The *homothety* of M *with respect to a* is the R-module homomorphism

$$h_a \;:\; M \longrightarrow M \;:\; m \longmapsto a * m\ .$$

Lemma 2.5.2. [[6], IV.§1.n°1.Cor.2, p.308-309]
Let R be a Noetherian ring, $M \in \operatorname{Mod}(R)$ and $a \in R$. Then h_a is injective if and only if a does not belong to any prime ideal in $\operatorname{Ass}_R(M)$.

Proof. $\underline{\Rightarrow}$: Assume that $\exists\, P \in \operatorname{Ass}_R(M)$ such that $a \in P$. Proposition B.2.25 gives $P = \operatorname{Ann}_R(x)$ for some $x \in M$, $x \neq 0$ and $a * x = 0$, which implies that h_a is not injective.

$\underline{\Leftarrow}$: If h_a is not injective, then $\exists\, x \in M$, $x \neq 0$ such that $a * x = 0$. Let $N = \langle\, x \,\rangle \leq M$. Since $x \neq 0$, we have $N \neq \{0\}$ and thus $\operatorname{Ass}_R(N) \neq \emptyset$ since R is Noetherian (see Proposition B.3.4). Take $P \in \operatorname{Ass}_R(N)$, i.e. $P = \operatorname{Ann}_R(r * x)$ for some $r \in R$. In particular, $P \in \operatorname{Ass}_R(M)$ by Proposition B.3.5 and $a \in P$ since $a * (r * x) = r * (a * x) = 0$. □

Lemma 2.5.3. *Let M be an R-module and $\ell : M \to M \otimes_R R_{\text{tot}}$ the morphism from Lemma 2.4.2. Then ℓ is injective if and only if h_a is injective for every NZD $a \in R$.*

Proof. Note that ℓ may also be written as $\ell : M \to M_{\text{tot}}$ because of Lemma A.2.2, where the denominators in M_{tot} consist of NZDs in R, i.e. $\ell(x) = \frac{x}{1}$, $\forall\, x \in M$.
$\underline{\Rightarrow}$: Assume that ℓ is injective and let $a \in R$ be a NZD. Then

$$h_a(x) = 0 \;\Leftrightarrow\; a * x = 0 \;\Rightarrow\; \tfrac{x}{1} = \ell(x) = 0 \;\Rightarrow\; x = 0\,.$$

$\underline{\Leftarrow}$: Let $\ell(x) = \frac{x}{1} = 0$, i.e. there is a NZD $a \in R$ such that $a * x = 0$, which implies that $x = 0$ as h_a is injective. □

Corollary 2.5.4. *For every module M over a Noetherian ring R, we have*

$$\begin{aligned} \mathcal{T}_R(M) = \{0\} \;&\Leftrightarrow\; \ell \text{ is injective} \\ &\Leftrightarrow\; \text{for every NZD } a \in R,\; a \notin P, \forall\, P \in \operatorname{Ass}_R(M) \\ &\Leftrightarrow\; \text{no prime ideal in } \operatorname{Ass}_R(M) \text{ contains a NZD}\,. \end{aligned}$$

Proof. Combine Lemma 2.4.2, Lemma 2.5.2 and Lemma 2.5.3. □

Definition 2.5.5. [[32], 3.1.1, p.36]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally Noetherian scheme and $\mathcal{F} \in \mathtt{QCoh}(\mathcal{O}_{\mathcal{X}})$. We say that a point $x \in \mathcal{X}$ is *associated to $\mathcal{F}$* if the maximal ideal $\mathfrak{M}_x$ of the stalk $\mathcal{O}_{\mathcal{X},x}$ is an associated prime of $\mathcal{F}_x$, i.e. if $\mathfrak{M}_x \in \operatorname{Ass}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$. The set of all points that are associated to $\mathcal{F}$ is denoted by $\operatorname{Ass}(\mathcal{F})$.

Remark 2.5.6. The definition already implies that $\operatorname{Ass}(\mathcal{F}) \subseteq \operatorname{supp} \mathcal{F}$ since Proposition B.3.4 gives

$$x \in \operatorname{Ass}(\mathcal{F}) \;\Rightarrow\; \operatorname{Ass}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \neq \emptyset \;\Rightarrow\; \mathcal{F}_x \neq \{0\}\,.$$

Proposition 2.5.7. [[32], 3.1.2, p.36]
Let $\mathcal{X} = \operatorname{Spec} R$ *for a Noetherian ring* R*,* M *an* R*-module and* $\mathcal{F} = \widetilde{M}$*. We identify* $x \in \mathcal{X}$ *with* $P \in \operatorname{Spec} R$*. Then*

$$x \in \operatorname{Ass}(\mathcal{F}) \;\Leftrightarrow\; P \in \operatorname{Ass}_R(M) \;.$$

Proof. follows from the behaviour of associated primes under localization (see Proposition B.3.7). Let $S = R \setminus P$ and denote the set of prime ideals which are contained in P by $\mathcal{P} \subseteq \operatorname{Spec} R$. Then

$$\begin{aligned} \mathfrak{M}_x \in \operatorname{Ass}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \;&\Leftrightarrow\; P_P \in \operatorname{Ass}_{R_P}(M_P) \\ &\Leftrightarrow\; S^{-1}P \in S^{-1}\big(\operatorname{Ass}_R(M) \cap \mathcal{P}\big) \;\Leftrightarrow\; P \in \operatorname{Ass}_R(M) \end{aligned}$$

since $P \in \mathcal{P}$ anyway. □

Theorem 2.5.8 (Grothendieck)**.** [[33], 20.1.6, p.228]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a locally Noetherian scheme and* $\mathcal{F} \in \texttt{QCoh}(\mathcal{O}_{\mathcal{X}})$*. Then* $\mathcal{F}$ *is torsion-free if and only if* $\operatorname{Ass}(\mathcal{F}) \subseteq \operatorname{Ass}(\mathcal{O}_{\mathcal{X}})$*.*

Proof. Since a sheaf is torsion-free if and only if all its stalks are torsion-free, it suffices to check the property on affine schemes. So let $\mathcal{X} = \operatorname{Spec} R$ with associated primes $P_1, \ldots, P_\alpha$ in R and $\mathcal{F} = \widetilde{M}$ for some R-module M.
If $\psi : \mathcal{F} \to \mathcal{K}_{\mathcal{X}}(\mathcal{F})$ denotes the morphism from Section 2.4.4, then Theorem 2.4.22 and Corollary 2.5.4 imply that

$$\begin{aligned} \mathcal{T}(\mathcal{F}) = 0 \;&\Leftrightarrow\; \ker\psi = 0 \;\Leftrightarrow\; \psi_x : \mathcal{F}_x \to (\mathcal{F}_x)_{\text{tot}} \text{ is injective, } \forall\, x \in \mathcal{X} \\ &\Leftrightarrow\; \psi_P : M_P \to (M_P)_{\text{tot}} \text{ is injective, } \forall\, P \in \operatorname{Spec} R \\ &\Leftrightarrow\; \forall\, P \in \operatorname{Spec} R, \text{ no prime ideal in } \operatorname{Ass}_{R_P}(M_P) \text{ contains a NZD} \\ &\Leftrightarrow\; \operatorname{Ass}_R(M) \subseteq \{P_1, \ldots, P_\alpha\} = \operatorname{Ass}_R(R) \;. \end{aligned}$$

It remains to explain the last equivalence. Recall that Proposition B.3.7 gives

$$\operatorname{Ass}_{R_P}(M_P) = \big\{\, Q_P \;\big|\; Q \in \operatorname{Ass}_R(M),\, Q \subseteq P \,\big\} \;.$$

$\underline{\Rightarrow}$: Assume that $\exists\, P \in \operatorname{Ass}_R(M)$ such that $P \neq P_i$, $\forall\, i$. We show that $\operatorname{Ass}_{R_P}(M_P)$ then contains a prime ideal that contains a NZD. Consider all associated primes that are strictly contained in P (there is at least one since every prime ideal contains a minimal prime); denote them by $P_1, \dots, P_\beta$ for some $\beta \leq \alpha$, i.e. $P_i \subsetneq P$, $\forall\, i \leq \beta$. Hence the union is still contained in P and this inclusion is still strict, otherwise Prime Avoidance implies

$$\begin{aligned} \textstyle\bigcup_{i=1}^{\beta} P_i = P \;\Rightarrow\; P \subseteq \bigcup_{i=1}^{\beta} P_i \;\Rightarrow\; P \subseteq P_j \text{ for some } j \leq \beta \\ \Rightarrow\; P \subseteq P_j \subseteq \textstyle\bigcup_{i=1}^{\beta} P_i = P \;\Rightarrow\; P = P_j\ , \end{aligned}$$

which contradicts that $P \notin \{P_1, \dots, P_\alpha\}$. Thus $\exists\, r \in P \setminus \bigcup_{i=1}^{\beta} P_i$. Now we consider the localization R_P. Since $P \in \mathrm{Ass}_R(M)$, we immediately get that $P_P \in \mathrm{Ass}_{R_P}(M_P)$. The zero-divisors in R_P are given by the localizations of those associated primes that are contained in P. But r has been chosen to not belong to any of them, so $\frac{r}{1}$ is a NZD in R_P. Now $\frac{r}{1} \in P_P$ gives the desired statement.
$\underline{\Leftarrow}$: Fix $P \in \operatorname{Spec} R$ and let a be any NZD in R_P. If $\mathrm{Ass}_R(M) \subseteq \{P_1, \dots, P_\alpha\}$, then

$$\mathrm{Ass}_{R_P}(M_P) \subseteq \big\{\, Q_P \;\big|\; Q \in \{P_1, \dots, P_\alpha\},\, Q \subseteq P \,\big\} = \big\{\, (P_i)_P \;\big|\; P_i \subseteq P \,\big\}\ ,$$

where $(P_i)_P \subseteq \mathrm{ZD}(R_P)$, so a cannot belong to a prime ideal in $\mathrm{Ass}_{R_P}(M_P)$. □

Remark 2.5.9. The proof allows to explicitly construct stalks that are not torsion-free if the condition is not satisfied. Indeed if $\exists\, x \in \mathrm{Ass}(\mathcal{F}) \setminus \mathrm{Ass}(\mathcal{O}_\mathcal{X})$, then it is shown that $\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \neq \{0\}$ and thus $\mathcal{T}(\mathcal{F}) \neq 0$.
Vice-versa, if $x \in \mathrm{Ass}(\mathcal{F}) \cap \mathrm{Ass}(\mathcal{O}_\mathcal{X})$, then $\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) = \{0\}$ since ψ_x is injective.

Example 2.5.10. Let us check the criterion of Theorem 2.5.8 on Example E.4, where we know that $\mathcal{F} = \widetilde{M}$ is not torsion-free. By (2.7) we have

$$\begin{aligned} \mathrm{Ass}_R(R) &= \big\{\, \langle\, \bar{X} \,\rangle\ ,\ \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}\ , \\ \mathrm{Ass}_R(M) &= \big\{\, \langle\, \bar{X}, \bar{Z} \,\rangle\ ,\ \langle\, \bar{X}, \bar{Y} \,\rangle\ ,\ \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}\ , \end{aligned}$$

hence $\langle\, \bar{X}, \bar{Z} \,\rangle$ and $\langle\, \bar{X}, \bar{Y} \,\rangle$ are associated primes of M which are not in $\mathrm{Ass}_R(R)$. By Remark 2.5.9 we can thus immediately conclude from Theorem 2.5.8 that the stalks at these two prime ideals are not torsion-free (which is indeed the case as shown in Example 2.3.4).

Remark 2.5.11. Looking at the computations of supp $\mathcal{T}(\mathcal{F})$ in Example 2.3.4, one may ask if there is a relation between torsion in the stalks at prime ideals and torsion in the stalks at the maximal ideals containing them. The answer to this question is Yes, and it will be explained in Proposition 3.3.9.

Remark 2.5.12. Finally let us also point out that Theorem 2.5.8 can be used to give an alternative proof of Corollary 2.2.22. Indeed if a Noetherian ring has no embedded primes, then $P \in \mathrm{Ass}_R(M)$ is contained in some $Q \in \mathrm{Ass}_R(R)$ if and only if $P = Q$ since Q is minimal. Hence we get

$$\begin{aligned} &M \in \mathrm{Mod}(R) \text{ is torsion-free} \\ &\Leftrightarrow\ \forall\, P \in \mathrm{Ass}_R(M),\ \exists\, Q \in \mathrm{Ass}_R(R) \text{ such that } P = Q \\ &\Leftrightarrow\ \mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(R)\ \Leftrightarrow\ \widetilde{M} \in \mathtt{QCoh}(\mathcal{O}_R) \text{ it torsion-free}\ . \end{aligned}$$

2.6 Some more facts

In this final section we want to present an alternative proof of the fact that coherent torsion sheaves have smaller-dimensional support in each component (Proposition 1.4.21 and Theorem 1.4.23). For this we will use several preliminary results, which are also interesting in themselves. In the following we always consider a coherent sheaf $\mathcal{F}$ on a locally Noetherian scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

2.6.1 Preliminaries

Lemma 2.6.1. [[61], Thm.12, p.41-42]
Let M be a finitely generated R-module and $S \subset R$ a multiplicatively closed subset. Then

$$S^{-1}\big(\mathrm{Ann}_R(M)\big) = \mathrm{Ann}_{S^{-1}R}\left(S^{-1}M\right)\ . \tag{2.11}$$

Proof. $\subseteq$: if $\frac{r}{s}$ is such that $r \in \mathrm{Ann}_R(M)$ and $s \in S$, then $\frac{r}{s} * \frac{m}{t} = 0$ for all $\frac{m}{t} \in S^{-1}M$
$\supseteq$: if $\frac{r}{s} * \frac{m}{t} = 0$ for all $\frac{m}{t} \in S^{-1}M$, then $\forall\, m \in M$, $\exists\, b_m \in S$ such that $(b_m \cdot r) * m = 0$. In particular, if $m_1, \ldots, m_n$ are the generators of M, then

$\exists\, b_i \in S$ such that $(b_i \cdot r) * m_i = 0$, $\forall\, i$. Let $b := b_1 \cdot \ldots \cdot b_n \in S$, so $b \cdot r$ annihilates all m_i, i.e. $b \cdot r \in \mathrm{Ann}_R(M)$ and

$$\frac{r}{s} = \frac{b \cdot r}{b \cdot s} \ .$$

□

Proposition 2.6.2. *Let $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$ and $x \in \mathcal{X}$ be fixed. If there is a NZD $f_x \in \mathcal{O}_{\mathcal{X},x}$ such that $f_x * \mathcal{F}_x = \{0\}$, then there exists an affine open neighborhood U of x and a NZD $f \in \mathcal{O}_{\mathcal{X}}(U)$ such that $f * \mathcal{F}|_U = 0$.*

Proof. It suffices to prove the statement locally. Let $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{F} \cong \widetilde{M}$ for some finitely generated R-module M with generators $m_1, \ldots, m_n$ and let x correspond to some $P \in \operatorname{Spec} R$. Hence $\mathcal{F}_x \cong M_P$ is generated by $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$. Since f_x annihilates $\mathcal{F}_x$, we get $f_x * \frac{m_i}{1} = 0$, $\forall\, i$ and all the $m_i^x = \frac{m_i}{1}$ are torsion elements in M_P. By Corollary 2.1.18, we thus can find affine open neighborhoods U_i of x such that each $\frac{m_i}{1}$ is torsion on U_i, i.e.

$$m_i^{U_i} = \tfrac{m_i}{1} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big), \ \forall\, i \ .$$

Let $U \subseteq \bigcap_i U_i$ be an affine open neighborhood of x and restrict all these torsion elements to U. By Remark 2.1.6 the restriction of a torsion element on an affine to a smaller affine is still a torsion element. Write $U = D(r)$, so now we have

$$m_i^U := m_i^{U_i}|_U = \tfrac{m_i}{1} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big) \cong \mathcal{T}_{R_r}(M_r) \ , \ \forall\, i \ ,$$

i.e. for all i, there is a NZD $f_i \in \mathcal{O}_{\mathcal{X}}(U) \cong R_r$ such that $f_i * \frac{m_i}{1} = 0$. Take $f = f_1 \cdot \ldots \cdot f_n$, which is still a NZD in $\mathcal{O}_{\mathcal{X}}(U)$. Then $f * \frac{m_i}{1} = 0$, $\forall\, i$ and $f * \mathcal{F}|_U = 0$ since $\mathcal{F}|_U$ is the sheaf associated to the module M_r, which is generated by $\frac{m_1}{1}, \ldots, \frac{m_n}{1} \in M_r$. □

Remark 2.6.3. In general one cannot expect that $[f]_x = f_x$, i.e. the germ of f at x may not be equal to the given germ f_x.
However they only differ by a unit. Indeed consider the proof of Proposition 2.1.17 where $\frac{a}{1} * \frac{m}{1} = 0$ in M_P for some NZD $\frac{a}{1} \in R_P$, i.e. $\exists\, b \notin P$ such that $ba * m = 0$. We proved that $\frac{m}{1}$ is still torsion in some M_r where it is annihilated by the NZD $\frac{ba}{1}$. Then $\frac{a}{1} \neq \frac{ba}{1}$ as elements in M_P, but we have $\frac{a}{1} = \frac{1}{b} \cdot \frac{ba}{1}$ where $\frac{1}{b}$ is a unit in M_P, so in the stalk the 2 NZDs only differ by a unit.

Corollary 2.6.4. *If $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$ is such that $\mathcal{F}_x$ is a torsion $\mathcal{O}_{\mathcal{X},x}$–module for some $x \in \mathcal{X}$, then there is an affine open neighborhood U of x such that $\mathcal{F}|_U \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}}|_U)$ is a torsion sheaf.*

Proof. If $\mathcal{F}_x$ is a torsion module, then its annihilator contains a NZD f_x by Proposition 1.3.5, i.e. $f_x * \mathcal{F}_x = \{0\}$. So by Proposition 2.6.2 there is an affine open neighborhood U of x and a NZD $f \in \mathcal{O}_{\mathcal{X}}(U)$ such that $f * \mathcal{F}|_U = 0$. Taking stalks for all $y \in U$, we get $f_y * \mathcal{F}_y = \{0\}$ where $f_y \in \mathcal{O}_{\mathcal{X},y}$ is still a NZD (see Proposition 1.3.6). Hence $\mathcal{F}_y$ is a torsion $\mathcal{O}_{\mathcal{X},y}$–module, $\forall\, y \in U$. □

Proposition 2.6.5. *For $x \in \mathcal{X}$, denote the unique maximal ideal of the local ring $\mathcal{O}_{\mathcal{X},x}$ by $\mathfrak{M}_x$. If $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$, then*

$$\operatorname{supp} \mathcal{F} \subseteq \left\{ x \in \mathcal{X} \;\middle|\; \operatorname{Ann}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \subseteq \mathfrak{M}_x \right\} . \tag{2.12}$$

Proof. Let $U \cong \operatorname{Spec} R$ be an affine open subset such that $\mathcal{F} \cong \widetilde{M}$ for finitely generated M. Then

$$\operatorname{supp}(\mathcal{F}|_U) = \left\{ P \in \operatorname{Spec} R \;\middle|\; M_P \neq \{0\} \right\} ,$$
$$M_P \neq \{0\} \;\Leftrightarrow\; \operatorname{Ann}_R(M) \subseteq P \;\Rightarrow\; \big(\operatorname{Ann}_R(M)\big)_P = \operatorname{Ann}_{R_P}(M_P) \subseteq P_P$$

by using (2.11) and Proposition B.3.11, where P_P is the unique maximal ideal of R_P. Repeating the same argument on an affine covering of $\mathcal{X}$, we get the inclusion (2.12). □

Remark 2.6.6. The proof of (2.12) is easy in the case of schemes. However one can show that it actually holds true for any coherent sheaf on a locally ringed space.
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be any non-trivial locally ringed space and $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$. Fix $x \in \mathcal{X}$. Then $\mathcal{F}_x$ is a finitely generated module over $\mathcal{O}_{\mathcal{X},x}$ and $K = \mathcal{O}_{\mathcal{X},x}/\mathfrak{M}_x$ is a field. Define $\mathcal{F}(x) := \mathcal{F}_x/(\mathfrak{M}_x\mathcal{F}_x)$, which is a finite-dimensional vector space over K. By Nakayama's Lemma (Proposition D.1.11), we moreover have

$$\mathcal{F}(x) = \{0\} \;\Leftrightarrow\; \mathcal{F}_x = \{0\}$$

since if $\mathcal{F}(x)$ is generated by $\bar{z}_1, \ldots, \bar{z}_n$, then $\mathcal{F}_x$ is generated by $z_1, \ldots, z_n$. Now let $r_x \in \operatorname{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$ be non-zero and set $\mathcal{E}_x := r_x * \mathcal{F}_x$, so obviously $\mathcal{E}_x = \{0\}$

and hence $\mathcal{E}(x) = \{0\}$. On the other hand we get $\mathcal{E}(x) = \overline{r_x} \cdot \mathcal{F}(x)$ since $\overline{r_x * m} = \overline{r_x} \cdot \overline{m}$, so $\mathcal{E}(x) = \{0\} \Leftrightarrow \overline{r_x} \cdot \mathcal{F}(x) = \{0\}$. Dealing with vector spaces, this means that at least one of them must be zero. If $\overline{r_x} \neq 0$, then $\mathcal{F}(x) = \{0\}$ and hence $\mathcal{F}_x = \{0\}$. Thus if $x \in \operatorname{supp} \mathcal{F}$, we have $\mathcal{F}_x \neq \{0\}$, so $\overline{r_x} = 0$ and $r_x \in \mathfrak{M}_x$, which shows that $\operatorname{Ann}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \subseteq \mathfrak{M}_x$.

2.6.2 Local irreducible decomposition and result

Let $P_1, \ldots, P_\alpha$ be the associated primes of a Noetherian ring R. We know that $\mathcal{X} = \operatorname{Spec} R$ decomposes into the irreducible subschemes $\mathcal{X}_i = V(P_i)$. Now let $U = D(r)$ be an affine open subset. We are interested in the intersection $\mathcal{X}_i \cap U$ (if it is non-empty). More precisely, $U \cong \operatorname{Spec}(R_r)$ also decomposes into irreducible components $\mathcal{X}_i'$, given by the associated primes in R_r. By (1.6) these are given by $P_i' := (P_i)_r$ for $i \in \{1, \ldots, \gamma\}$ such that $r \notin P_i$. Proposition 1.1.1 ensures that

$$\left\{ P \in \operatorname{Spec} R \;\middle|\; P_i \subseteq P \text{ and } r \notin P \right\} \cong \left\{ Q \in \operatorname{Spec}(R_r) \;\middle|\; (P_i)_r \subseteq Q \right\},$$

i.e. $V(P_i) \cap D(r) \cong V'(P_i')$[5] and thus $\mathcal{X}_i \cap U = \mathcal{X}_i'$.

Proposition 2.6.7. *Let $\mathcal{X} = \operatorname{Spec} R$ be affine for a Noetherian ring R and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$. If $\mathcal{F}$ is a torsion sheaf, then* $\dim\big((\operatorname{supp} \mathcal{F}) \cap \mathcal{X}_i\big) < \dim \mathcal{X}_i$, $\forall\, i$.

Proof. Let $P_1, \ldots, P_\alpha$ be the associated primes in R, defining the components $\mathcal{X}_1, \ldots, \mathcal{X}_\alpha$. We fix $i \in \{1, \ldots, \alpha\}$ and $x \in (\operatorname{supp} \mathcal{F}) \cap \mathcal{X}_i$ (if it is non-empty, otherwise the inequality is trivial). Since $\mathcal{F}$ is torsion, its stalks are finitely generated torsion modules and we know that $\operatorname{Ann}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$ contains a NZD f_x. By Proposition 2.6.2 there exists an affine open neighborhood $U = D(r) \subset \operatorname{Spec} R$ of x and a NZD $f \in \mathcal{O}_\mathcal{X}(U)$ such that $f * \mathcal{F}|_U = 0$. Denote $A = R_r$, so that $U \cong \operatorname{Spec} A$. Since $f_y * \mathcal{F}_y = \{0\}$ for all $y \in U$, f_y is in the annihilator of $\mathcal{F}_y$ and (2.12) gives

$$(\operatorname{supp} \mathcal{F}) \cap U \subseteq \left\{ y \in U \;\middle|\; f_y \in \mathfrak{M}_y \right\} \cong \left\{ P \in \operatorname{Spec} A \;\middle|\; \tfrac{f}{1} \in P_P \right\}$$

because $f \in \mathcal{O}_\mathcal{X}(U) \cong A$.

[5] We use the notation $V(\ )$ for closed sets in $\operatorname{Spec} R$ and $V'(\ )$ for closed sets in $\operatorname{Spec} R_r$.

$\frac{f}{1} \in P_P$ also implies that $f \in P$, otherwise $\frac{f}{1}$ would be a unit in A_P. Hence

$$(\operatorname{supp} \mathcal{F}) \cap U \subseteq \left\{ P \in \operatorname{Spec} A \;\middle|\; \tfrac{f}{1} \in P_P \right\} = \left\{ P \in \operatorname{Spec} A \;\middle|\; f \in P \right\} = V'(f) \; .$$

Let $P'_1, \ldots, P'_\gamma$ be the associated primes in A, defining components $\mathcal{X}'_1, \ldots, \mathcal{X}'_\gamma$ in U. Then x belongs to $\mathcal{X}_i \cap U = \mathcal{X}'_i$. As $V'(f)$ is closed in $\operatorname{Spec} A$, the set $V'(f) \cap \mathcal{X}'_i$ is closed in $\mathcal{X}'_i$. However

$$V'(f) \cap \mathcal{X}'_i \subsetneq \mathcal{X}'_i$$

since otherwise $\mathcal{X}'_i \subseteq V'(f) \Leftrightarrow V'(P'_i) \subseteq V'(f)$, implying that $f \in P'_i$, which contradicts that $f \in A$ is a NZD. So we have a closed subset in $\mathcal{X}'_i$, which is irreducible and open in itself, hence the Zariski topology (closed sets have empty interior) implies that the codimension of $V'(f) \cap \mathcal{X}'_i$ in $\mathcal{X}'_i$ is positive. Combining everything we find

$$\begin{aligned} \dim \big((\operatorname{supp} \mathcal{F}) \cap \mathcal{X}_i \big) &= \dim \big((\operatorname{supp} \mathcal{F}) \cap U \cap \mathcal{X}_i \big) \\ &= \dim \big((\operatorname{supp} \mathcal{F}) \cap \mathcal{X}'_i \big) < \dim \mathcal{X}'_i = \dim(\mathcal{X}_i \cap U) = \dim \mathcal{X}_i \; , \end{aligned}$$

where all sets are non-empty (as they contain x) and intersecting with U does not change the dimension because U is open in $\mathcal{X}$. Repeating this argument for each i and a chosen x belonging to $(\operatorname{supp} \mathcal{F}) \cap \mathcal{X}_i$, we get the drop of dimension in each component of $\mathcal{X}$. □

Remark 2.6.8. In the particular case where $\mathcal{X} = \mathbb{A}^n_{\mathbb{K}} = \operatorname{Spec} \big(\mathbb{K}[X_1, \ldots, X_n] \big)$, the proof becomes easier. As above we obtain that $\mathcal{F}$ is locally annihilated by some NZD $f \in \mathcal{O}_{\mathcal{X}}(U)$, i.e. $f * \mathcal{F}|_U = 0$. Localizing this relation at all closed points (i.e. all maximal ideals) in U gives

$$f * \mathcal{F}|_U = 0 \;\Leftrightarrow\; f_{\mathfrak{M}} * \mathcal{F}_{\mathfrak{M}} = \{0\}, \; \forall \, \mathfrak{M} \in U \;\Leftrightarrow\; \overline{f_{\mathfrak{M}}} \cdot \mathcal{F}(\mathfrak{M}) = \{0\}, \; \forall \, \mathfrak{M} \in U \; ,$$

and we get $\overline{f_{\mathfrak{M}}} = 0$ if $\mathcal{F}_{\mathfrak{M}} \neq \{0\}$ (i.e. if $\mathfrak{M} \in \operatorname{supp} \mathcal{F}$). But $\overline{f_{\mathfrak{M}}}$ is nothing but evaluation of (a fraction of) f at the point $m \in \mathbb{K}^n$ defined by the maximal ideal $\mathfrak{M}$. Indeed the residue field is given by

$$K = \mathcal{O}_{X,x} / \mathfrak{M}_x \;\cong\; \mathbb{K}[X_1, \ldots, X_n]_{\mathfrak{M}} / \mathfrak{M}_{\mathfrak{M}} \;\cong\; \mathbb{K} \; .$$

Recall that the correspondence is given by $\mathfrak{M} = \langle\, X_1 - a_1, \dots, X_n - a_n \,\rangle$ if the point has the coordinates $m = (a_1, \dots, a_n)$. A polynomial $s \in \mathbb{K}[X_1, \dots, X_n]$ satisfies $s \in \mathfrak{M} \Leftrightarrow s(m) = 0$, hence dividing out $\mathfrak{M}_{\mathfrak{M}}$ means to evaluate at m. More precisely, if we write $f_{\mathfrak{M}} = \frac{f}{g}$ with $g \notin \mathfrak{M}$, then $g(m) \neq 0$ and

$$\overline{f_{\mathfrak{M}}} = \frac{f(m)}{g(m)} \; .$$

So $\overline{f_{\mathfrak{M}}} = 0 \Leftrightarrow f_{\mathfrak{M}} \in \mathfrak{M}_{\mathfrak{M}}$ means that $f(m) = 0$, i.e. if $\mathfrak{M} \in \operatorname{supp} \mathcal{F}$, then $\mathfrak{M}$ corresponds to some $m \in V(f)$ and we get as before that $(\operatorname{supp} \mathcal{F}) \cap U \subseteq V(f)$, the vanishing set of the polynomial f.

Example 2.6.9. Let us apply the inclusion (2.12) to Example E.3. Recall that the torsion subsheaf is given by $\mathcal{T}(\mathcal{F}) = \widetilde{T}$ for $T = \langle\, [\bar{X}\bar{Z}] \,\rangle \leq M$. Computations in Example 1.4.27 have shown that

$$\operatorname{Ann}_R(T) = \langle\, \bar{X} - 1, \bar{Y} \,\rangle \qquad \text{and} \qquad \operatorname{supp} \mathcal{T}(\mathcal{F}) = V\big(\langle\, \bar{X} - 1, \bar{Y} \,\rangle\big) = \mathcal{Z}_2 \; .$$

T is annihilated by the global NZD $\bar{Y} + \bar{X} - 1$, hence $(\bar{Y} + \bar{X} - 1)_P * T_P = \{0\}$ for all $P \in \mathcal{X}$, so

$$\begin{aligned} \operatorname{supp} \mathcal{F} &\subseteq \big\{\, P \in \operatorname{Spec} R \;\big|\; (\bar{Y} + \bar{X} - 1)_P \in P_P \,\big\} \\ &= \big\{\, P \in \operatorname{Spec} R \;\big|\; \bar{Y} + \bar{X} - 1 \in P \,\big\} = V\big(\bar{Y} + \bar{X} - 1\big) \; , \end{aligned}$$

which is obviously true since $\bar{Y} + \bar{X} - 1 \in \langle\, \bar{X} - 1, \bar{Y} \,\rangle$. Moreover one sees that

$$V\big(\bar{Y} + \bar{X} - 1\big) \cap \mathcal{X}_i \subsetneq \mathcal{X}_i \; , \quad \forall\, i \; .$$

On the other hand one can also look for NZDs that annihilate $\mathcal{T}(\mathcal{F})$ locally. We have for example $\bar{Y}_P * [\bar{X}\bar{Z}]_P = 0$ and $(\bar{X} - 1)_P * [\bar{X}\bar{Z}]_P = 0$. However these relations do not always give torsion since $\bar{Y}_P$ may be a zero-divisor, e.g. for $P = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle$, or e.g. $\bar{Y}_P = 0$ for $P = \langle\, \bar{X}, \bar{Y}, \bar{Z} - 1 \,\rangle$ since $\bar{Y}\bar{Z}(\bar{X} - 1) = \bar{0}$. So we have to look for prime ideals P on which the relations are useful. In Example 2.2.18 we have shown that $\frac{\bar{Y}}{\bar{1}}$ remains a NZD on $U = D(\bar{X})$. Looking for P such that $\bar{X} - 1 \in P_2 \setminus P$, one similarly finds that $\frac{\bar{X} - 1}{\bar{1}}$ is a NZD on $V = D(\bar{X} - 1)$. Now $U \cup V$ is already a covering of $\mathcal{X}$ since any prime $P \in \mathcal{X}$

either satisfies $\bar{X} \notin P$ or $\bar{X}-1 \notin P$, otherwise $\bar{1} \in P$ (see also Lemma 2.4.13). So we get the local annihilations of local NZDs

$$\tfrac{\bar{Y}}{\bar{1}} * \mathcal{F}|_U = 0 \qquad \text{and} \qquad \tfrac{\bar{X}-1}{\bar{1}} * \mathcal{F}|_V = 0 \;,$$

which imply that $\operatorname{supp}(\mathcal{F}|_U) \subseteq V'\big(\tfrac{\bar{Y}}{\bar{1}}\big)$ on $D(\bar{X})$ and $\operatorname{supp}(\mathcal{F}|_V) \subseteq V'\big(\tfrac{\bar{X}-1}{\bar{1}}\big)$ on $D(\bar{X}-1)$. The associated primes on U, resp. V are the localizations of those that do not contain $\bar{X}$, resp. $\bar{X}-1$, hence

$$P_1' = \big\langle\, \tfrac{\bar{Z}}{\bar{1}} \,\big\rangle \quad , \quad P_2' = \big\langle\, \tfrac{\bar{X}-1}{\bar{1}} \,\big\rangle \qquad \text{and} \qquad P_1'' = \big\langle\, \tfrac{\bar{Z}}{\bar{1}} \,\big\rangle \quad , \quad P_2'' = \big\langle\, \tfrac{\bar{X}}{\bar{1}}, \tfrac{\bar{Y}}{\bar{1}} \,\big\rangle \;,$$

which again shows that $V'\big(\tfrac{\bar{Y}}{\bar{1}}\big) \cap \mathcal{X}_i' \subsetneq \mathcal{X}_i'$ and $V'\big(\tfrac{\bar{X}-1}{\bar{1}}\big) \cap \mathcal{X}_i'' \subsetneq \mathcal{X}_i''$ for all i. Note however that if we try to use the (correct) inclusions

$$\operatorname{supp} \mathcal{F} \subseteq V(\bar{Y}) \qquad \text{and} \qquad \operatorname{supp} \mathcal{F} \subseteq V(\bar{X}-1) \;,$$

then $V(\bar{Y}) \cap \mathcal{X}_3 = \mathcal{X}_3$ and $V(\bar{X}-1) \cap \mathcal{X}_2 = \mathcal{X}_2$, so it is important to only consider the local intersections. The reason is that $\bar{Y}$ and $\bar{X}-1$ are (globally) zero-divisors whereas the above proof only works for NZDs.

Chapter 3

Purity and its relations with torsion

The aim of this chapter is to discuss and illustrate the relations, but also the differences between the concepts of purity and torsion-freeness of a sheaf in the non-integral case. First we review the criterion of Huybrechts-Lehn which characterizes pure sheaves by looking at their associated points. Then we show in Theorem 3.1.17 that purity and torsion-freeness are equivalent on schemes where all components have the same dimension, but also give examples to show that they are not equivalent in general.

Our main motivation is to show that every coherent sheaf of pure dimension on a Noetherian scheme is torsion-free as a sheaf on its Fitting support; this is the aim of Theorem 3.5.3. For this we explain how a sheaf can be restricted to its support and that this restriction does not affect purity. After this we give a necessary condition in Proposition 3.2.12 for sheaves to be pure when looking at their annihilator support and see why this condition fails for the Fitting support. Another occuptaion of this chapter is to compare the properties of the annihilator and the Fitting support of a coherent sheaf $\mathcal{F}$. It turns out that there are fundamental differences between them. The annihilator on one hand allows to prove some criteria regarding torsion-freeness and purity, making $\mathcal{Z}_a(\mathcal{F})$ a support that is easier to handle in examples, but $\mathcal{Z}_f(\mathcal{F})$ commutes with pullbacks, making it the preferred subscheme structure to work with.

Finally we present many examples to illustrate there is no clear relation between torsion-freeness on $\mathcal{Z}_a(\mathcal{F})$ and torsion-freeness on $\mathcal{Z}_f(\mathcal{F})$ as soon as one of them has embedded components.

3.1 Characterizations of purity

The goals of this section are to find a criterion for purity by looking at the associated primes of the module (Theorem 3.1.11) and to prove that purity and torsion-freeness of an $\mathcal{O}_{\mathcal{X}}$–module are equivalent under some conditions involving the associated primes of the ring (Theorem 3.1.17).

3.1.1 Definition and examples

Definition 3.1.1. [[38], 1.1.2, p.3]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. The *dimension* of $\mathcal{F}$ is the dimension of its support as a topological space ; we denote

$$\dim \mathcal{F} = \dim(\operatorname{supp} \mathcal{F}) \ .$$

Now let $d \leq \dim \mathcal{X}$. We say that $\mathcal{F}$ is *pure of dimension d* if $\dim \mathcal{F} = d$ and $\mathcal{F}$ has no non-zero proper coherent subsheaves $\mathcal{F}' \subset \mathcal{F}$ such that $\dim \mathcal{F}' < d$.

Remark 3.1.2. If $\mathcal{X} = \operatorname{Spec} R$, the dimension of $\mathcal{X}$ is equal to the maximum of the dimensions of its irreducible components $\mathcal{X}_i$ defined by the associated primes of R, i.e.

$$\dim R = \dim \mathcal{X} = \max_{i=1,\ldots,\alpha} \dim \mathcal{X}_i \ .$$

This is why we need $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ to be Noetherian instead of just locally Noetherian. If $\mathcal{X}$ cannot be covered by finitely many affine schemes, then $\dim \mathcal{X}$ may not be well-defined.
Similarly $\dim \mathcal{F}$ is equal to the maximum of the dimensions of the irreducible components of its support. Note that taking $\mathcal{Z}_a(\mathcal{F})$ or $\mathcal{Z}_f(\mathcal{F})$ as closed subscheme doesn't make a difference as they define the same topological space. Thus purity is a topological condition.

Example 3.1.3. The sheaf $\mathcal{F} = \widetilde{M}$ from Example E.3 is not pure. Indeed its support consists of a plane and two lines, hence $\dim \mathcal{F} = 2$. Rigorously we have

$$\begin{aligned}\dim(\operatorname{supp} \widetilde{M}) &= \dim \big(\operatorname{Spec}(R/\operatorname{Ann}_R(M))\big) = \dim \big(R/\langle\, \bar{Y}\bar{Z}\,\rangle\big) \\ &= \dim \big(\mathbb{K}[X,Y,Z]/\langle\, YZ, XZ(X-1)\,\rangle\big) = 2\end{aligned}$$

by taking the chain of prime ideals $\langle\, \bar{Z}\,\rangle \subsetneq \langle\, \bar{Z}, \bar{X}\,\rangle \subsetneq \langle\, \bar{Z}, \bar{X}, \bar{Y}\,\rangle$. However $\mathcal{T}(\mathcal{F}) = \widetilde{T}$ is a non-zero coherent subsheaf (since $T = \mathcal{T}_R(M)$ is finitely generated) with only 1-dimensional support. From the computations in (1.15), it follows that $\operatorname{Ann}_R(T) = \langle\, \bar{X}-1, \bar{Y}\,\rangle$ and hence

$$\begin{aligned}\dim(\operatorname{supp} \widetilde{T}) &= \dim \big(R/\langle\, \bar{X}-1, \bar{Y}\,\rangle\big) \\ &= \dim \big(\mathbb{K}[X,Y,Z]/\langle\, X-1, Y\,\rangle\big) = \dim \big(\mathbb{K}[Z]\big) = 1\ .\end{aligned}$$

Another example is given by the submodule $N = \langle\, [\bar{Z}]\,\rangle \leq M$. In Example 2.2.18 we computed

$$\begin{gathered}\operatorname{Ann}_R(N) = \operatorname{Ann}_R([\bar{Z}]) = \langle\, \bar{X}(\bar{X}-1), \bar{Y}\,\rangle = \langle\, \bar{X}, \bar{Y}\,\rangle \cap \langle\, \bar{X}-1, \bar{Y}\,\rangle\ , \\ \dim(\operatorname{supp} \widetilde{N}) = \dim \big(R/\langle\, \bar{X}(\bar{X}-1), \bar{Y}\,\rangle\big) = \dim \big(\mathbb{K}[X,Z]/\langle\, X(X-1)\,\rangle\big) = 1\end{gathered}$$

by taking the chain of primes ideals $\langle\, \bar{X}\,\rangle \subsetneq \langle\, \bar{X}, \bar{Z}\,\rangle$. Geometrically this means that $\widetilde{N}$ is supported on the union of the two lines $\mathcal{X}_3$ and $\mathcal{Z}_2$. Thus $\widetilde{T}$ and $\widetilde{N}$ are 2 examples of non-zero proper coherent subsheaves of $\widetilde{M}$ with support in smaller dimension.

Example 3.1.4. The sheaf $\mathcal{F} = \widetilde{M}$ from Example E.4 is not pure neither. Consider for example the submodule $N = \langle\, [\bar{X}]\,\rangle \leq M$. Since

$$\operatorname{Ann}_R\big([\bar{X}]\big) = \langle\, \bar{X}, \bar{Y}, \bar{Z}\,\rangle\ ,$$

we see that $[\bar{X}]_P = 0$ for all $P \in \operatorname{Spec} R \setminus \{\mathfrak{M}\}$. Hence the subsheaf $\widetilde{N}$ has stalks $N_P = \{0\}$, $\forall\, P \neq \mathfrak{M}$ and is supported on one point, i.e. in dimension 0.

Lemma 3.1.5. *Let $\mathcal{X} = \operatorname{Spec} R$ be affine. Then $\mathcal{F}, \mathcal{G} \in \mathtt{Coh}(\mathcal{O}_R)$ are pure of dimension d if and only if $\mathcal{F} \oplus \mathcal{G}$ is pure of dimension d.*

Proof. $\Leftarrow$: Assume for example that $\mathcal{G}$ is not pure and has a non-zero proper coherent subsheaf $0 \neq \mathcal{G}' \subsetneq \mathcal{G}$ with $\dim \mathcal{G}' < d$. Then $0 \oplus \mathcal{G}'$ is a non-zero proper coherent subsheaf of $\mathcal{F} \oplus \mathcal{G}$ whose support has dimension $< d$.
$\Rightarrow$: Write $\mathcal{F} \cong \widetilde{M}$ and $\mathcal{G} \cong \widetilde{N}$ for some finitely generated R-modules M and N, so that $\mathcal{F} \oplus \mathcal{G}$ is given by the sheaf associated to the R-module $M \oplus N$. We shall show that $M \oplus N$ has no submodule that defines a sheaf of dimension $< d$. Let $L \leq M \oplus N$ be any non-zero submodule and $(m, n) \in L$ with $(m, n) \neq (0, 0)$. If $m \neq 0$, consider $\langle\, m \,\rangle \oplus \{0\} \leq L$; if $n \neq 0$, consider $\{0\} \oplus \langle\, n \,\rangle \leq L$. Since $\mathcal{F}$ and $\mathcal{G}$ are pure, $\langle\, m \,\rangle$ and $\langle\, n \,\rangle$ are submodules that define sheaves of dimension d (if they are non-zero), hence L also defines a sheaf with d-dimensional support. □

In order to study other examples later on, let us already mention the following technical lemmata.

Lemma 3.1.6. *Let R be a reduced Noetherian ring and denote the associated primes by $P_1, \ldots, P_\alpha$. Fix i and consider P_i as an R-module. Then $(P_i)_P = \{0\}$ for all $P \in V(P_i) \setminus \bigcup_{j \neq i} V(P_j)$, i.e. for all prime ideals P such that $P_i \subseteq P$ but $P_j \nsubseteq P$, $\forall\, j \neq i$.*

Proof. $P_j \nsubseteq P$ means that $\exists\, a_j \in P_j$ such that $a_j \notin P$ for all $j \neq i$. Define $a := \prod_{j \neq i} a_j$, so that $a \in \bigcap_{j \neq i} P_j$ and $a \notin P$. Thus $a \cdot p \in P_1 \cap \ldots \cap P_\alpha = \mathrm{nil}(R)$, $\forall\, p \in P_i$. As R is reduced, this means that $a \cdot p = 0$. $a \notin P$ then implies that $\frac{p}{1} = 0$ as an element in $(P_i)_P$ for all $p \in P_i$. Hence $(P_i)_P = \{0\}$. □

Remark 3.1.7. The geometric interpretation of this result is that the subsheaf of the structure sheaf $\mathcal{O}_R$ which is defined by the ideal of an irreducible component has zero stalks on the points (primes) that only belong to this component. This agrees with the intuitive interpretation that sections of such a subsheaf are functions that vanish on the given component.
Also note that the condition $P \in V(P_i)$ with $P_j \nsubseteq P$, $\forall\, j \neq i$ implies that P_i is a minimal prime since otherwise there exists $k \neq i$ such that $P_k \subsetneq P_i \subseteq P$.

Remark 3.1.8. The assumption of R being reduced is necessary. Consider e.g. the ring

$$R = \mathbb{K}[X, Y, Z]/\langle\, XZ^2, YZ^2 \,\rangle$$

with the primary decomposition $\langle XZ^2, YZ^2 \rangle = \langle Z^2 \rangle \cap \langle X, Y \rangle$, which gives the associated primes $P_1 = \langle \bar{Z} \rangle$ and $P_2 = \langle \bar{X}, \bar{Y} \rangle$. If we take $P = \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle$, then $P_1 \subseteq P$ and $P_2 \nsubseteq P$, but $\bar{Z}_P \neq 0$ since $\operatorname{Ann}_R(\bar{Z}) = \langle \bar{X}\bar{Z}, \bar{Y}\bar{Z} \rangle \subseteq P$.

However we can change the statement of Lemma 3.1.6 in such a way that it is also holds true in the non-reduced case.

Lemma 3.1.9. *Let R be a Noetherian ring with associated primes $P_1, \ldots, P_\alpha$ and primary decomposition $\{0\} = Q_1 \cap \ldots \cap Q_\alpha$. For fixed i, we get $(Q_i)_P = \{0\}$ for all $P \in V(P_i) \setminus \bigcup_{j \neq i} V(P_j)$.*

Proof. Note that $P_j \nsubseteq P$ implies that $Q_j \nsubseteq P$, otherwise $\operatorname{Rad}(Q_j) \subseteq \operatorname{Rad}(P)$, i.e. $P_j \subseteq P$.
Using the same technique as in Lemma 3.1.6 we can find an element $a \in \bigcap_{j \neq i} Q_j$ such that $a \notin P$. Let $p \in Q_i$ be arbitrary. Then $a \cdot p \in Q_1 \cap \ldots \cap Q_\alpha = \{0\}$, i.e. $\frac{p}{1} = 0$ in $(Q_i)_P$. □

Remark 3.1.10. Now if we consider again the example in Remark 3.1.8, then $Q_1 = \langle \bar{Z}^2 \rangle$ and $\bar{Z}_P^2 = 0$ since $\bar{X} \cdot \bar{Z}^2 = 0$ with $\bar{X} \notin P$.

3.1.2 The criterion of Huybrechts-Lehn

Using the associated primes of M, we can give the following precise characterization of purity.

Theorem 3.1.11 (Huybrechts-Lehn)**.** [[38], 1.1.2, p.3]
Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ with $d = \dim \mathcal{F}$. Then $\mathcal{F}$ is pure of dimension d if and only if all points in $\operatorname{Ass}(\mathcal{F})$ are of dimension d.

Proof. Since $\mathcal{X}$ can be covered by affine open sets, it suffices to prove the statement in the affine case. Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring R and $\mathcal{F} = \widetilde{M}$ be coherent, so that $x \in \operatorname{Ass}(\mathcal{F})$ corresponds to $P \in \operatorname{Ass}_R(M)$, see Proposition 2.5.7. By Corollary B.3.17 and Proposition 1.4.4 we have

$$\begin{aligned} \operatorname{supp} M = V\big(\operatorname{Ann}_R(M)\big) &= V\big(\operatorname{Rad}\big(\operatorname{Ann}_R(M)\big)\big) \\ &= V\big(\textstyle\bigcap_P P\big) = \bigcup_P V(P) \end{aligned} \tag{3.1}$$

as topological spaces. Hence saying that $\dim \mathcal{F} = d$ means that the maximal dimension of a component that prime ideals in $\mathrm{Ass}_R(M)$ can define is d as well, i.e. $\dim V(P) \leq d$, $\forall\, P \in \mathrm{Ass}_R(M)$.

$\underline{\Rightarrow}$: Assume that $P \in \mathrm{Ass}_R(M)$ defines a component $V(P)$ of dimension $< d$. By definition P is given as $P = \mathrm{Ann}_R(x)$ for some $x \in M$, $x \neq 0$. Consider the non-zero submodule $N = \langle\, x\, \rangle \leq M$. Then $\mathrm{supp}\, N \subseteq V(P)$ since $\forall\, Q \in \mathrm{Spec}\, R$,

$$\tfrac{x}{1} = 0 \text{ in } M_Q \;\Leftrightarrow\; \exists\, a \notin Q \text{ such that } a * x = 0 \;\Leftrightarrow\; \exists\, a \in \mathrm{Ann}_R(x) \setminus Q = P \setminus Q \;.$$

Hence $\frac{x}{1} \neq 0$ in M_Q implies that $P \subseteq Q$, i.e. $Q \in V(P)$. But then $N \leq M$ is a submodule such that $\dim(\mathrm{supp}\, N) \leq \dim V(P) < d$, which contradicts purity.

$\underline{\Leftarrow}$: If all prime ideals in $\mathrm{Ass}_R(M)$ define components of dimension d, then in particular they are all minimal (since components defined by embedded primes have smaller dimension, see Lemma 1.4.19). Let $N \leq M$ be any non-zero submodule. Then $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M)$ by Proposition B.3.5, so all primes in $\mathrm{Ass}_R(N)$ also define components of dimension d. But then a similar formula as (3.1) implies that $\dim(\mathrm{supp}\, N) = d$ as well. Thus M defines a sheaf of pure dimension d. □

Remark 3.1.12. [[6], IV.§2.n°3, Thm.1, p.319]
If M is a finitely generated module over a Noetherian ring R, there is a direct way to find the primes in $\mathrm{Ass}_R(M)$. Indeed there exists a decomposition of the submodule $\{0\} \leq M$ of the form

$$\{0\} = \bigcap_{P \in \mathrm{Ass}_R(M)} N(P) \;,$$

where $N(P) \leq M$ are submodules such that $\mathrm{Ass}_R\big(N(P)\big) = \{P\}$. Hence the associated primes of M are exactly those that arise as unique associated primes of the submodules in the decomposition of $\{0\}$. However in practise such a decomposition is not always easy to compute, even with `Singular`, and it is especially hard if M is some abstract module.

This is why we want to look at the associated primes which define the support of the sheaf. Unfortunately there is no general statement as Theorem 3.1.11 in this case. We will prove the corresponding result in Proposition 3.2.12, but already point out that it will in particular depend on the chosen subscheme structure of the support.

With the criterion of Huybrechts-Lehn we can already give a first class of examples of sheaves that are pure. The following example describes under which conditions a structure sheaf is pure.

Example 3.1.13. Let $\mathcal{X} = \operatorname{Spec} R$ be affine and $d = \dim \mathcal{X}$. Then the structure sheaf $\mathcal{O}_R$ is pure of dimension d if and only if R has no associated prime (minimal or embedded) which defines a component of dimension $< d$.

Proof. $\underline{\Rightarrow}$: Assume by contraposition that P_i defines a component such that $\dim V(P_i) < d$ and let $\{0\} = Q_1 \cap \ldots \cap Q_\alpha$ be the primary decomposition of $\{0\}$ in R. Consider the ideal $Q = \bigcap_{j \neq i} Q_j$. Then $Q \neq \{0\}$, otherwise the decomposition is not minimal. We will show that $Q_P = \{0\}$ for all $P \in \operatorname{Spec} R$ such that $P \notin V(P_i)$ and hence that $\operatorname{supp} Q \subseteq V(P_i)$, i.e. Q defines a subsheaf of $\mathcal{O}_R$ whose support is contained in $V(P_i)$ and thus has dimension $< d$. Let $p \in Q$ be arbitrary and assume that $P \notin V(P_i) = V(Q_i)$, i.e. $Q_i \nsubseteq P$. Then $\exists\, a \in Q_i \setminus P$ and as in Lemma 3.1.9 we obtain $a \cdot p \in Q_1 \cap \ldots \cap Q_\alpha = \{0\}$, so that $\frac{p}{1} = 0$ in Q_P.
$\underline{\Leftarrow}$: If all associated primes of R define components of the same dimension, we can apply Theorem 3.1.11 with $\operatorname{Ass}_R(R) = \operatorname{Ass}\big(\{0\}\big)$ and see that $\mathcal{O}_R = \widetilde{R}$ is pure of dimension d. □

Remark 3.1.14. In other words, the structure sheaf of an affine scheme is pure if and only if all irreducible components (minimal and embedded ones) of the scheme have the same dimension.

Example 3.1.15. The pullback of a pure sheaf may not be pure any more. More precisely, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of Noetherian schemes and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{Y})$ is pure of dimension $d = \dim \mathcal{F}$, then $f^*\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ does not need to be pure as well. This can easily be seen on affine schemes.
Let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ be a morphism of schemes for some Noetherian rings R, S and giving rise to a ring homomorphism $\varphi : R \to S$ which turns S into an R-module. If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_R)$ is given by a finitely generated R-module M, then (1.3) implies that $f^*\mathcal{F}$ is given by the S-module $M \otimes_R S$.
Now consider $R = \mathbb{K}[X, Y, Z]$ with $M = R$, i.e. $\mathcal{F} = \mathcal{O}_R$ and $S = R/\langle XZ, YZ \rangle$,

which describes the subscheme of $\mathbb{A}^3_{\mathbb{K}}$ given a plane and a line (see Example E.2), hence $\varphi : R \to S$ is just the projection. Example 3.1.13 implies that $\mathcal{O}_R$ is pure of dimension 3 since $\operatorname{Spec} R$ is an integral scheme, so $\{0\}$ is the only associated prime of R. But $M \otimes_R S \cong S$, so that $f^*\mathcal{F} \cong \mathcal{O}_S$. However S has minimal primes that define components of different dimensions, thus $\mathcal{O}_S$ is not pure by Example 3.1.13.

3.1.3 Relation with torsion-freeness

Definition 3.1.16. Let $\mathcal{X} = \operatorname{Spec} R$ be an affine scheme for a Noetherian ring R and $d = \dim \mathcal{X}$. We say that $\mathcal{X}$ has *equidimensional components* if R has no embedded primes and all minimal primes $P \in \operatorname{Ass}_R(R)$ define components $V(P) \subseteq \mathcal{X}$ of dimension d.

The first important result that relates torsion-freeness and purity of a sheaf is the following.

Theorem 3.1.17 (Leytem)**.** *Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_R)$ with $\dim \mathcal{F} = \dim \mathcal{X} = d$. If $\mathcal{X}$ has equidimensional components, then $\mathcal{F}$ is pure of dimension d if and only if $\mathcal{F}$ is torsion-free on $\mathcal{X}$.*

Proof. $\underline{\Rightarrow}$: Assume that $\mathcal{F} \cong \widetilde{M}$ is pure of dimension d. Since there are no embedded primes, we know by Theorem 2.2.13 that the torsion subsheaf $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is coherent and given by $\mathcal{T}_R(M)$, which is a torsion module. Hence (1.11) in Proposition 1.4.21 gives

$$\dim\big(\operatorname{supp} \mathcal{T}(\mathcal{F})\big) = \dim\big(\operatorname{supp} \widetilde{\mathcal{T}_R(M)}\big) < \dim \mathcal{X} = \dim \mathcal{F} \ ,$$

i.e. $\mathcal{T}(\mathcal{F})$ is a coherent subsheaf of $\mathcal{F}$ whose support has dimension $< d$. In particular it is proper since $\dim \mathcal{F} = d$. $\mathcal{F}$ being pure, we get $\mathcal{T}(\mathcal{F}) = 0$ and thus that $\mathcal{F}$ is torsion-free.

$\underline{\Leftarrow}$: Assume that $\mathcal{F} \cong \widetilde{M}$ is torsion-free and let $N \subset M$ be any non-zero proper submodule. $\mathcal{F}$ being torsion-free, we get that M and hence N are torsion-free as well (see Corollary 1.3.10). In particular, $\operatorname{Ann}_R(N)$ only contains zero-divisors by Proposition 1.3.5 and Prime Avoidance gives $\operatorname{Ann}_R(N) \subseteq P_i$ for some minimal prime P_i (since there are no embedded primes). But then

$$V(P_i) \subseteq V\big(\operatorname{Ann}_R(N)\big) \;\Leftrightarrow\; \mathcal{X}_i \subseteq \operatorname{supp} \widetilde{N}$$
$$\Rightarrow\; \dim \mathcal{X}_i = \dim(\operatorname{supp} \widetilde{N}) = \dim \mathcal{X} = \dim \mathcal{F}$$

because all minimal primes define components of dimension d. Hence every non-zero proper coherent subsheaf of $\mathcal{F}$ has dimension d as well, i.e. $\mathcal{F}$ is pure. □

Remark 3.1.18. The condition about $\mathcal{X}$ having equidimensional components is crucial. Indeed there are examples of schemes with no embedded primes and a sheaf $\mathcal{F}$ with $\dim \mathcal{F} = \dim \mathcal{X}$ which is torsion-free but not pure.

Example 3.1.19. Consider Example E.2 and the structure sheaf $\mathcal{O}_R$ (i.e. we take $M = R$). This is a free sheaf, hence it is torsion-free by Example 2.2.3. The associated primes are both minimal, but define components of different dimensions since $V(P_1)$ is a line and $V(P_2)$ is a plane. In Example 3.1.15 we have seen that $\mathcal{O}_R$ is thus not pure. To see this directly, we look at the subsheaf $\mathcal{F} \subseteq \mathcal{O}_R$ defined by the submodule $P_2 \leq M$. Let $P \in \operatorname{Spec} R$; if $\bar{X} \notin P$ or $\bar{Y} \notin P$, then $\bar{Z}_P = 0$ since $\bar{X} \cdot \bar{Z} = \bar{0}$ or $\bar{Y} \cdot \bar{Z} = \bar{0}$. Hence prime ideals in the support must satisfy $\langle \bar{X}, \bar{Y} \rangle \subseteq P$ and we get $\operatorname{supp} \mathcal{F} \subseteq V(P_1)$ with $\mathcal{F} \neq 0$, so that $\dim \mathcal{F} \leq 1$ while $\dim \mathcal{O}_R = 2$ (actually $\dim \mathcal{F} = 1$).

Remark 3.1.20. The condition $\dim \mathcal{F} = \dim \mathcal{X}$ is important as well, otherwise $\mathcal{F}$ may be a torsion sheaf itself and we get $\mathcal{T}(\mathcal{F}) = \mathcal{F}$ (compare with the proof of Theorem 3.1.17). Note however that $\dim \mathcal{F} = \dim \mathcal{X}$ does not imply that $\operatorname{supp} \mathcal{F} = \mathcal{X}$. It even fails in very easy situations.

Example 3.1.21. If all components have the same dimension:
Consider the "cross" in $\mathbb{A}^2_{\mathbb{K}}$ described by $R = \mathbb{K}[X,Y]/\langle XY \rangle$ and the structure sheaf of a line given by $M = R/\langle \bar{X} \rangle$. Both $\mathcal{X} = \operatorname{Spec} R$ and $\operatorname{supp} M$ are 1-dimensional, but $\operatorname{supp} M \subsetneq \mathcal{X}$.

Example 3.1.22. If there is just 1 associated prime (the scheme is irreducible):
Consider the double line in $\mathbb{A}^2_{\mathbb{K}}$ given by $R = \mathbb{K}[X,Y]/\langle Y^2 \rangle$. The only associated prime is $\langle \bar{Y} \rangle$. But if we consider the structure sheaf of a simple line, then $M = R/\langle \bar{Y} \rangle$ has support $V(\bar{Y})$ while $\mathcal{X} = \operatorname{Spec} R = V(\bar{Y}^2)$, i.e. $\operatorname{supp} M \subsetneq \mathcal{X}$ again and both are 1-dimensional.

Remark 3.1.23. If we want the implication $\dim \mathcal{F} = \dim \mathcal{X} \Rightarrow \operatorname{supp} \mathcal{F} = \mathcal{X}$ to be true, the previous examples show that we need only 1 associated prime whose irreducible component is reduced (see also Lemma 3.1.24). But a reduced and irreducible scheme is integral by Lemma 1.1.5, i.e. it is only true for integral schemes.

Lemma 3.1.24. *Let R be a Noetherian ring and $\{0\} = Q_1 \cap \ldots \cap Q_\alpha$ a primary decomposition with $P_i = \operatorname{Rad}(Q_i)$. If P_i is an associated prime such that $Q_i \subsetneq P_i$ (i.e. the component $\mathcal{X}_i = V(Q_i)$ has a multiple structure), then P_i is given by the annihilator of a nilpotent element (of degree 2).*

Proof. The proof contains elements from the proof of Proposition B.2.25.
Denote $I_i = \bigcap_{j \neq i} Q_j$. We are given that $P_i = \operatorname{Ann}_R(x)$ for some $x \in R$ such that $x \in I_i \cap P_i^{m-1}$, where $m \geq 1$ is minimal such that $P_i^m \subseteq Q_i$. Taking radicals, $x \in I_i$ already implies that $x \in P_j$, $\forall\, j \neq i$, so we are left to show that $x \in P_i$ in order to obtain that x is nilpotent. But this follows from $Q_i \subsetneq P_i$ as we thus have $m \geq 2$. In particular, $x \in P_i$ and $x^2 = 0$. □

In all examples we see that the condition about $\mathcal{X}$ having equidimensional components is quite important, otherwise purity may even fail for very "easy" sheaves (e.g. locally free sheaves), even though they are torsion-free, see Example 3.1.19. Actually this condition also gives the converse of Proposition 1.4.21 in a more general setting.

Corollary 3.1.25. *If $\mathcal{X} = \operatorname{Spec} R$ has equidimensional components, then*

$$\begin{aligned} M \textit{ is a torsion } R\textit{-module} &\Leftrightarrow \widetilde{M} \textit{ is a torsion } \mathcal{O}_R\textit{–module} \\ &\Leftrightarrow \dim(\operatorname{supp} \widetilde{M}) < \dim \mathcal{X}\ . \end{aligned}$$

Proof. The first equivalence is given by Lemma 2.2.6. The second implication, $\Rightarrow$ is always true because of (1.11). For $\Leftarrow$ we use the characterization from Theorem 1.4.23, which says that a module is torsion if and only if the dimension of its support drops in each component. But if all components have the same dimension $d = \dim \mathcal{X}$, this condition is equivalent to a drop of the global dimension. Indeed, if the dimension of the support is $< d$, then it has dropped in each component as they all have dimension d. □

The next result contains the notion of the Hilbert polynomial $P_{\mathcal{F}}$ of a coherent sheaf $\mathcal{F}$ on a projective scheme. We refer to Section 4.1.1 for the precise definition and some properties of $P_{\mathcal{F}}$. Nevertheless we want to include the statement already at this point since it contains information about the torsion of $\mathcal{F}$ and the dimension of its support.

Corollary 3.1.26. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a projective scheme over a Noetherian ring, $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ and assume that the Hilbert polynomial $P_{\mathcal{F}}$ has degree $d < \dim \mathcal{X}$. Then $\mathcal{F}|_U$ is a torsion sheaf on all affine schemes $U \cong \operatorname{Spec} R$ such that U has equidimensional components.*

Proof. Let $\mathcal{F}|_U \cong \widetilde{M}$ for some R-module M. We have $d = \deg P_{\mathcal{F}} = \dim \mathcal{F}$, so if

$$d = \dim \mathcal{F} = \dim(\mathcal{F}|_U) = \dim(\operatorname{supp} \widetilde{M}) < \dim \mathcal{X} = \dim U = \dim(\operatorname{Spec} R) \ ,$$

then M is a torsion module over R by Corollary 3.1.25. From this we conclude that $\mathcal{F}|_U \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}}|_U) \cong \mathtt{Coh}(\mathcal{O}_R)$ is a torsion sheaf on U. □

Example 3.1.27. Let $\mathcal{X} = \mathbb{P}^n_{\mathbb{K}}$ be the projective space, which is covered by $n+1$ copies of the affine space, i.e. $\mathcal{X} = \bigcup_{i=0}^n U_i$ where $U_i \cong \mathbb{A}^n_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[X_1, \ldots, X_n]$. Since polynomial rings over fields are integral domains, the corresponding spectrum has in particular equidimensional components. So if $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ is such that $\deg P_{\mathcal{F}} < n$, then $\mathcal{F}$ is a torsion sheaf on each U_i and hence a torsion sheaf on $\mathcal{X}$ since the stalk $\mathcal{F}_x$ is a torsion module for all $x \in \mathcal{X}$.

Remark 3.1.28. We can also prove the fact from Example 3.1.27 directly. If $\mathcal{Z} = \mathcal{Z}_f(\mathcal{F})$ is the Fitting support of $\mathcal{F}$, we have the exact sequence

$$0 \longrightarrow \operatorname{Fitt}_0(\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

of $\mathcal{O}_{\mathcal{X}}$–modules since the subscheme $\mathcal{Z}$ is defined by the Fitting ideal sheaf (Definition 1.4.3). Fix $x \in \mathcal{X}$; if $x \notin \mathcal{Z}$, then $\mathcal{F}_x = \{0\}$ and this is a torsion module. For $x \in \mathcal{Z}$ we get

$$\begin{aligned} 0 \longrightarrow \big(\operatorname{Fitt}_0(\mathcal{F})\big)_x \longrightarrow \mathcal{O}_{\mathcal{X},x} \longrightarrow \mathcal{O}_{\mathcal{Z},x} \longrightarrow 0 \\ \Leftrightarrow \qquad 0 \longrightarrow I_P \longrightarrow R_P \longrightarrow (R/I)_P \longrightarrow 0 \ , \end{aligned}$$

where $R = \mathbb{K}[X_1, \ldots, X_n]$, $U \cong \operatorname{Spec} R$, $\mathcal{F}|_U \cong \widetilde{M}$, $I = \operatorname{Fitt}_0(M)$ and x corresponds to some $P \in U$. Then $I_P \neq \{0\}$, otherwise $\mathcal{O}_{\mathcal{X},x} \cong \mathcal{O}_{\mathcal{Z},x}$ and by coherence this would imply that $\mathcal{O}_{\mathcal{X}}|_V \cong \mathcal{O}_{\mathcal{Z}}|_V$ for some open neighborhood $V \subseteq U$ of x, which is impossible as $\dim \mathcal{Z} < \dim \mathcal{X}$ and V is dense in U. Hence I_P contains a non-zero element. But

$$I_P = \big(\operatorname{Fitt}_0(M)\big)_P \subseteq \big(\operatorname{Ann}_R(M)\big)_P = \operatorname{Ann}_{R_P}(M_P)$$

by Lemma 1.4.2 and (2.11) since M is finitely generated. So this non-zero element (i.e. a NZD as R_P is an integral domain) belongs to the annihilator of M_P, implying that $M_P \cong \mathcal{F}_x$ is a torsion module by Proposition 1.3.5. Finally we get $\mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) = \mathcal{F}_x$, $\forall\, x \in \mathcal{X}$.

3.1.4 Relation with the torsion filtration

Some authors define the torsion subsheaf to be a maximal subsheaf which is supported in smaller dimension. As we know from Section 1.4.4 that torsion and a drop of the dimension are not always equivalent, we want to study the relation between both definitions. Here we follow the ideas from Huybrechts-Lehn [38], Chapter 1.1 and Bakker [3], Chapter 2.1.

Definition 3.1.29. [[3], 2.1, p.9-10]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Noetherian scheme with $n = \dim \mathcal{X}$. For $d \in \{0, \ldots, n\}$ we denote by $\mathtt{Coh}_d(\mathcal{O}_{\mathcal{X}})$ the category of coherent $\mathcal{O}_{\mathcal{X}}$–modules that are supported in dimension $\leq d$. It is a full subcategory of $\mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ and we have the inclusion functor $i_d : \mathtt{Coh}_d(\mathcal{O}_{\mathcal{X}}) \to \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$.

Proposition 3.1.30. cf. [[3], 2.1, p.10]
The functor i_d admits a right adjoint $T_d : \mathtt{Coh}(\mathcal{O}_{\mathcal{X}}) \to \mathtt{Coh}_d(\mathcal{O}_{\mathcal{X}})$ such that for all $\mathcal{G} \in \mathtt{Coh}_d(\mathcal{O}_{\mathcal{X}})$ and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$,

$$\operatorname{Hom}\big(i_d(\mathcal{G}), \mathcal{F}\big) = \operatorname{Hom}\big(\mathcal{G}, T_d(\mathcal{F})\big)\ .$$

Proof. Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$; we define $T_d(\mathcal{F})$ to be the sheaf of sections of $\mathcal{F}$ that are supported in dimension $\leq d$, i.e. for $U \subseteq \mathcal{X}$ open,

$$T_d(\mathcal{F})(U) = \big\{\, s \in \mathcal{F}(U) \;\big|\; \dim(\operatorname{supp} s) \leq d \,\big\}\ .$$

This defines a submodule of $\mathcal{F}(U)$ as

$$\operatorname{supp}(s+t) \subseteq \operatorname{supp} s \cup \operatorname{supp} t \qquad \text{and} \qquad \operatorname{supp}(f * s) \subseteq \operatorname{supp} s$$

for $s, t \in \mathcal{F}(U)$ and $f \in \mathcal{O}_{\mathcal{X}}(U)$, so $s+t$ and $f * s$ are still sections that are supported in dimension $\leq d$. Moreover $T_d(\mathcal{F})$ is coherent again: if $\mathcal{F}|_U \cong \widetilde{M}$ for some finitely generated module M over a Noetherian ring R, then

$$T_d(\mathcal{F})|_U \cong \widetilde{M_d} \qquad \text{where} \qquad M_d = \big\{\, m \in M \;\big|\; \dim\big(\operatorname{supp}\langle\, m\,\rangle\big) \leq d \,\big\}$$

and M_d is again finitely generated since R is Noetherian. For the adjunction, first note that every morphism $\mathcal{G} \to T_d(\mathcal{F})$ in $\mathtt{Coh}_d(\mathcal{O}_{\mathcal{X}})$ is also a morphism $i_d(\mathcal{G}) \to \mathcal{F}$. Conversely let $\varphi : i_d(\mathcal{G}) \to \mathcal{F}$ be a morphism of coherent sheaves. To prove that $\operatorname{im}\varphi \subseteq T_d(\mathcal{F})$ it suffices to show that for every section $s \in \mathcal{G}(U)$, its image $\varphi(s)$ is also supported in dimension $\leq d$. But this is satisfied because

$$\big[\varphi(s)\big]_x = \varphi_x\big([s]_x\big) \,, \tag{3.2}$$

thus if $[s]_x = 0$, then so is its image. In particular, $\dim\big(\operatorname{supp}\varphi(s)\big) \leq d$. □

Theorem 3.1.31. [[38], 1.1.4, p.4] and [[3], 2.1, p.10]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ with $d = \dim \mathcal{F}$. Then

$$0 \subseteq T_0(\mathcal{F}) \subseteq T_1(\mathcal{F}) \subseteq \ldots \subseteq T_{d-1}(\mathcal{F}) \subseteq T_d(\mathcal{F}) = \mathcal{F} \tag{3.3}$$

is the unique filtration such that $T_i(\mathcal{F})$ is a maximal coherent[1] *subsheaf of $\mathcal{F}$ of dimension $\leq i$.*

Definition 3.1.32. (3.3) is called the *torsion filtration* of $\mathcal{F}$. By definition each one of the quotients $T_i(\mathcal{F})/T_{i-1}(\mathcal{F})$ is a sheaf of pure dimension i, if it is non-zero (since subsheaves whose support has smaller dimension are divided out). In particular we see that $\mathcal{F}$ is pure of dimension d if and only if $T_{d-1}(\mathcal{F}) = 0$.

Now we prove the relation between the torsion subsheaf $\mathcal{T}(\mathcal{F})$ and the term $T_{d-1}(\mathcal{F})$ in the torsion filtration. Indeed it is a priori not clear why (3.3) is called a "torsion" filtration as it only involves the dimension of the sections.

[1]The word "coherent" is not explicitly mentioned in [38], but we have shown in Proposition 3.1.30 that it is indeed the case.

Proposition 3.1.33 (Leytem)**.** *Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring R and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_R)$ with $d = \dim \mathcal{F} = \dim \mathcal{X}$. If $\mathcal{X}$ has equidimensional components, then $\mathcal{T}(\mathcal{F}) = T_{d-1}(\mathcal{F})$.*

Proof. The inclusion $\subseteq$ is true when R has no embedded primes because then $\mathcal{T}(\mathcal{F})$ is coherent and supported in dimension $\leq d-1$ by Theorem 2.2.13 and Proposition 1.4.21. If in addition all minimal primes define components of the same dimension, we know by Corollary 3.1.25 that a coherent sheaf is torsion if and only if it is supported in smaller dimension. Since $T_{d-1}(\mathcal{F})$ satisfies this (and is coherent), it is torsion and thus contained in $\mathcal{T}(\mathcal{F})$ by Lemma 2.2.5. □

Remark 3.1.34. The formula $\mathcal{T}(\mathcal{F}) = T_{d-1}(\mathcal{F})$ can be seen as a generalization of Theorem 3.1.17. Indeed it implies that a sheaf $\mathcal{F}$ with $\dim \mathcal{F} = \dim \mathcal{X}$ is pure of dimension d if and only if it is torsion-free.

Remark 3.1.35. The assumption $\dim \mathcal{F} = \dim \mathcal{X}$ in Proposition 3.1.33 is again necessary. This is compatible with the results from Theorem 3.1.17 and Remark 3.1.20. Assume for example that $d = \dim \mathcal{F} < \dim \mathcal{X}$. Then $\mathcal{F}$ is itself supported in smaller dimension, hence a torsion sheaf on $\mathcal{X}$ by Corollary 3.1.25 (for schemes with equidimensional components) and we get $\mathcal{T}(\mathcal{F}) = \mathcal{F} = T_d(\mathcal{F})$.

Example 3.1.36. In general $\mathcal{T}(\mathcal{F})$ and $T_{d-1}(\mathcal{F})$ may not be related at all. Consider Example E.4 where $\dim \mathcal{X} = 2$ and $d = \dim \mathcal{F} = 1$. $\mathcal{T}(\mathcal{F})$ is not coherent and supported on $\operatorname{supp} \mathcal{F} \setminus \{\mathfrak{M}\}$, i.e. in dimension 1 as well. On the other hand $T_0(\mathcal{F})$ is coherent and given by

$$T_0(\mathcal{F}) = \widetilde{\langle\, [\bar{X}]\, \rangle}\,, \tag{3.4}$$

which is supported on $\{\mathfrak{M}\}$ only. Thus no one can be included in the other one. So we see that two subsheaves of $\mathcal{F}$ that are equal in “nice” situations can be quite different in general.

Proof. Intuitively (3.4) is clear, but let us also prove it rigorously. We are looking for a maximal R-submodule $N \leq M$ such that $\operatorname{supp} N$ is 0-dimensional.
Obviously the support of $\langle\, [\bar{X}]\, \rangle$ is of dimension 0 as it is only supported on $\{\mathfrak{M}\}$. But is it maximal (maybe there are submodules that are supported on finitely

many points, or on double points) ? Since M is torsion-free, so is N and its annihilator is contained in the set of zero-divisors:

$$\mathrm{Ann}_R(N) \subseteq \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle = \mathfrak{M} \qquad \Rightarrow \qquad V(\mathfrak{M}) = \{\mathfrak{M}\} \subseteq \mathrm{supp}\, N \ ,$$

i.e. $\mathfrak{M}$ is in the support of any (non-zero) submodule of M. Assume that N is supported on finitely many points ($\mathfrak{M}$ included). Then its annihilator contains a finite intersection of maximal ideals:

$$\mathfrak{M} \cap \mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_n \subseteq \mathrm{Ann}_R(N) \ ,$$

where each $\mathfrak{M}_i$ is either of the form $\langle\, \bar{X}, \bar{Y} - \lambda, \bar{Z} \,\rangle$ or of the form $\langle\, \bar{X}, \bar{Y}, \bar{Z} - \mu \,\rangle$ for some $\lambda, \mu \neq 0$ (since maximal ideals in the support of M are of this form, see Example 2.3.4). In particular we obtain $\bar{X} \in \mathrm{Ann}_R(N)$. As R is Noetherian, N is again finitely generated; denote its generators by $[\bar{g}_1], \ldots, [\bar{g}_k]$. $\bar{X}$ being in the annihilator implies that no $[\bar{g}_i]$ has a constant term. Now we proceed by induction on n. For $n = 1$ we have

$$\langle\, \bar{X} \,,\, \bar{Y}(\bar{Y} - \lambda) \,,\, \bar{Z} \,\rangle \subseteq \mathrm{Ann}_R(N) \qquad \text{or} \qquad \langle\, \bar{X} \,,\, \bar{Y} \,,\, \bar{Z}(\bar{Z} - \mu) \,\rangle \subseteq \mathrm{Ann}_R(N) \ .$$

Consider e.g. the first one. $\bar{Z} \in \mathrm{Ann}_R(N)$ implies that the $[\bar{g}_i]$ cannot have terms in $[\bar{Z}]$ otherwise these do not vanish since $\bar{Z}$ only annihilates $[\bar{X}]$ and $[\bar{Y}]$. Similarly $\bar{Y}(\bar{Y} - \lambda) \in \mathrm{Ann}_R(N)$ means that there are no terms in $[\bar{Y}]$. But then the $[\bar{g}_i]$ are just polynomials in $[\bar{X}]$ which together with $\bar{X}^2 = \bar{0}$ implies that $N = \langle\, [\bar{X}] \,\rangle$. Similarly in the second case.
Our induction hypothesis is that $\mathfrak{M} \cap \mathfrak{M}_1 \cap \ldots \cap \mathfrak{M}_n$ always contains $\bar{X}$ (clear), a term of the form $\bar{Y} \cdot f(\bar{Y})$ and a term of the form $\bar{Z} \cdot h(\bar{Z})$. By the same argument as before this will show that $N = \langle\, [\bar{X}] \,\rangle$. It is satisfied for $n = 1$. Assume that it is true for n. The following intersection gives

$$\begin{aligned} &\langle\, \bar{X} \,, \ldots ,\, \bar{Y} \cdot f(Y) \,, \ldots ,\, \bar{Z} \cdot h(\bar{Z}) \,\rangle \ \cap \ \langle\, \bar{X}, \bar{Y} - \lambda, \bar{Z} \,\rangle \\ &= \langle\, \bar{X} \,, \ldots ,\, \bar{Y}(\bar{Y} - \lambda) \cdot f(Y) \,, \ldots ,\, \bar{Z} \cdot h(\bar{Z}) \,\rangle \ , \end{aligned}$$

and similarly when intersecting with $\langle\, \bar{X}, \bar{Y}, \bar{Z} - \mu \,\rangle$. Hence the statement holds true for $n + 1$.
Until now we have shown that any $N \leq M$ with 0-dimensional support can

only be supported on $\{\mathfrak{M}\}$, at least topologically (since we were using $\mathrm{Ann}_R(N)$ instead of the Fitting ideal). It remains to show that this support cannot be a double point, which is given by $\langle \bar{Y}, \bar{Z} \rangle$. Assume again that N is generated by some $[\bar{g}_1], \ldots, [\bar{g}_k]$ such that k is minimal. In order to obtain $\bar{Y}$ as a minor in the Fitting ideal we need that all other relations only involve $\bar{1}$. But then all generators with $\bar{1}$ can be omitted since $\bar{1}$ is a unit (just replace them in the relations). In order for k to be minimal we thus need $k = 1$, which implies as before that $N = \langle [\bar{X}] \rangle$ since $\mathrm{Ann}_R(N) = \mathrm{Fitt}_0(N)$ for 1 generator. □

3.2 Sheaves on their support

If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$, the assumption $\dim \mathcal{F} = \dim \mathcal{X}$ is in general not satisfied, so our results from Theorem 3.1.17 and Proposition 3.1.33 cannot be used. The main idea of this section is to consider $\mathcal{F}$ as a sheaf on its support $\mathcal{Z}$ (whose structure has to be specified), in which case $\dim \mathcal{F} = \dim \mathcal{Z}$ is obviously true, so we can again apply our results for $\mathcal{F}$, now seen as a sheaf on the subscheme $\mathcal{Z}$.

3.2.1 Modules over quotients

Let R be a ring, M an R-module and $I \trianglelefteq R$ an ideal such that $I \subseteq \mathrm{Ann}_R(M)$. Thus we have the exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \,.$$

Denote $A := R/I$. We define an A-module structure on M by $\bar{s} * m := s * m$, which is well-defined as $i * m = 0$, $\forall\, i \in I$. On the other hand we can define $N := M \otimes_R A$, where A is an R-module via the projection map, i.e. $r * \bar{s} := \bar{r} \cdot \bar{s}$. By straight-forward computations one then shows

Lemma 3.2.1. *$M \cong N$ as A-modules.*

Thus for any ideal $I \subseteq \mathrm{Ann}_R(M)$, one can define an R/I-module structure on an R-module either directly or by means of a tensor product. One may ask whether the associated primes of M will change under this new module structure, i.e. do they depend on the ring over which we consider M as a module? The following result shows that the answer is No.

Proposition 3.2.2. *If $I \trianglelefteq R$ is such that $I \subseteq \mathrm{Ann}_R(M)$ and $\pi : R \to R/I$, then*

$$\mathrm{Ass}_{R/I}(M) = \pi\big(\,\mathrm{Ass}_R(M)\big)\ .$$

Proof. Recall that

$$\mathrm{Spec}(R/I) \;\cong\; V(I) = \big\{\, P \in \mathrm{Spec}\, R \;\big|\; I \subseteq P \,\big\}$$

since prime ideals in R/I are in 1-to-1 correspondence with prime ideals in R containing I. As the elements in M do not change, neither do their annihilators (up to taking classes of the generators). So it only remains to check that primes in $\mathrm{Ass}_R(M)$ contain I. This follows from Theorem B.3.15, which gives the inclusion $\mathrm{Ass}_R(M) \subseteq \mathrm{supp}\, M$. Indeed if $P \in \mathrm{Ass}_R(M)$ is such that $I \nsubseteq P$, then $\exists\, r \in I \setminus P$ and $M_P = \{0\}$ since $r * M = \{0\}$, so P would not be in the support of M. □

Remark 3.2.3. Even if the sets of associated primes are not exactly the same, we will always write $\mathrm{Ass}_{R/I}(M) = \mathrm{Ass}_R(M)$ in the following, having in mind that one has to take the classes of the generators of the primes because of the module structure $\bar{r} * m = r * m$ for $r \in R$, $m \in M$.

3.2.2 Purity and torsion-freeness on the support

Now we apply the same idea as before to sheaves and the ideal which defines their support.

Definition 3.2.4. Let M be a finitely generated R-module with coherent sheaf $\mathcal{F} \cong \widetilde{M}$ on the affine scheme $\mathcal{X} = \mathrm{Spec}\, R$. We denote $\mathcal{Z} = \mathrm{supp}\, \mathcal{F}$ (the subscheme structure has yet to be specified) and consider the closed immersion of schemes $i : \mathcal{Z} \hookrightarrow \mathcal{X}$. Next we want to "forget" about the structure of $\mathcal{F}$ as an $\mathcal{O}_{\mathcal{X}}$–module and only see it as an $\mathcal{O}_{\mathcal{Z}}$–module on its support. As $\mathcal{Z} = V(I)$ for some $I \subseteq \mathrm{Ann}_R(M)$ (usually the Fitting ideal), M also carries an R/I-module structure via $\bar{r} * m = r * m$ and is thus still finitely generated. By pulling back, i then defines the functor

$$i^* \;:\; \mathtt{Coh}(\mathcal{O}_R) \longrightarrow \mathtt{Coh}(\mathcal{O}_{R/I}) \;\Leftrightarrow\; \mathtt{Coh}(\mathcal{O}_{\mathcal{X}}) \longrightarrow \mathtt{Coh}(\mathcal{O}_{\mathcal{Z}})$$

with $i^*\widetilde{M} \cong \widetilde{M}$ by (1.3) where $M \in \operatorname{Mod}(R)$ on the RHS is considered as an R/I-module as in Lemma 3.2.1. Hence the underlying set of the module did not change under this transformation.

The next result is essential as it shows that purity of a coherent sheaf $\mathcal{F}$ is a notion that is independent of the "ambient space" $\mathcal{X}$ on which $\mathcal{F}$ is considered.

Proposition 3.2.5. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. Then $\mathcal{F}$ is pure of dimension d as an $\mathcal{O}_{\mathcal{X}}$–module if and only if $\mathcal{F}$ is pure of dimension d as an $\mathcal{O}_{\mathcal{Z}}$–module.*

Proof. Let $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{F} \cong \widetilde{M}$ and $I \subseteq \operatorname{Ann}_R(M)$ defining the subscheme structure on $\mathcal{Z} = V(I)$. Let $N \leq M$ be a submodule (unspecified whether as an R-module or an R/I-module). We have the homeomorphism of topological spaces $V(I) \cong \operatorname{Spec}(R/I)$ given by $P \mapsto \pi(P)$ and inverse $Q \mapsto \pi^{-1}(Q)$, where $\pi : R \to R/I$. So

$$\begin{aligned} \left\{ Q \in \operatorname{Spec}(R/I) \;\middle|\; N_Q \neq \{0\} \right\} &\cong \left\{ P \in V(I) \;\middle|\; N_P \neq \{0\} \right\} \\ &= \left\{ P \in \operatorname{Spec} R \;\middle|\; N_P \neq \{0\} \right\} , \end{aligned}$$

where the last equality holds because if $P \notin V(I)$, then $I \nsubseteq P$ and $\exists\, r \in I \setminus P$, so $r \in \operatorname{Ann}_R(M)$ and $r * N = \{0\}$, i.e. $N_P = \{0\}$. Being homeomorphic, the supports of submodules of M on $\mathcal{Z}$ and on $\mathcal{X}$ thus have the same dimension. It follows that $M \in \operatorname{Mod}(R)$ has a submodule with support of dimension $< d$ if and only if $M \in \operatorname{Mod}(R/I)$ has a submodule of dimension $< d$. □

Remark 3.2.6. Note that this result is compatible with Proposition 3.2.2. Indeed by the criterion for purity of Huybrechts-Lehn (Theorem 3.1.11), a sheaf being pure or not is completely determined by the dimension of its associated points. As these do not change under pulling back to the support, neither does the property of being pure.

Remark 3.2.7. On the other hand, the notion of torsion-freeness of a sheaf strongly depends on the space where it is considered.

Example 3.2.8. Let $R = \mathbb{K}[X,Y]$ with $\mathcal{X} = \operatorname{Spec} R = \mathbb{A}^2_{\mathbb{K}}$ be the affine plane and $M = R/I$ with $I = \langle X \rangle$ describe the structure sheaf of a line. Thus $\dim \mathcal{X} = 2$ and $\dim(\operatorname{supp} M) = 1$.
M is a torsion R-module since $X \in \operatorname{Ann}_R(M)$ is a NZD. But it is (obviously) free as an R/I-module. So we get a torsion sheaf on $\mathcal{X}$, but which is torsion-free, and even free, on its support.

Example 3.2.9. Let us also analyze this fact on Example E.4.
If we denote $I = \langle \bar{Y}\bar{Z} \rangle$, then $\operatorname{Ann}_R(M) = \operatorname{Fitt}_0(M) = I$ as M is generated by one element. As rings we have $R/I \cong \mathbb{K}[X,Y,Z]/\langle XY, X^2, XZ, YZ \rangle$ and M is also a module over R/I.
From Example 2.3.1 we know that M is a torsion-free module over R, but there is torsion on the stalks, e.g. for $P = \langle \bar{X}, \bar{Y}, \bar{Z} - 1 \rangle$ we have $\bar{Y}_P * [\bar{Z}]_P = 0$ where $[\bar{Z}]_P \neq 0$ and $\bar{Y}_P \in R_P$ is a NZD. On the other hand we have $M \cong R/I$ as modules over R/I, i.e. M is a free R/I-module and its corresponding sheaf is locally free on $\operatorname{supp} M$ (obvious since it is the structure sheaf of the support). In particular, all its stalks are free and hence torsion-free as well. So what happened to the torsion relation on the support ?
Let us study the relations $\bar{Y}_P * [\bar{Z}]_P = 0$ and $\bar{Z}_P * [\bar{Y}]_P = 0$ over R/I. Since $M \cong R/I$, they reduce to $\bar{Y}_P \cdot \bar{Z}_P = 0$ and $\bar{Z}_P \cdot \bar{Y}_P = 0$ in the localization of this ring (now with P seen as a prime ideal in R/I, compare Remark 3.2.3). So we immediately see that $\bar{Y}_P$ and $\bar{Z}_P$ are zero-divisors as they annihilate each other, i.e. they are no torsion elements. More precisely we even have $\bar{Y}_P = \bar{0}$ since $\bar{Z} \cdot \bar{Y} = \bar{0}$ with $\bar{Z} \notin P$.

Example 3.2.10. Something similar happens in Example E.3, where

$$I = \operatorname{Ann}_R(M) = \langle \bar{Y}\bar{Z} \rangle$$

and the global NZD $\bar{Y} + \bar{X} - 1$, which defines the global torsion element $[\bar{X}\bar{Z}]$, becomes a zero-divisor in the quotient $R/I \cong \mathbb{K}[X,Y,Z]/\langle YZ, XZ(X-1) \rangle$, so the torsion relation reduces to

$$\big(\bar{Y} + \bar{X} - 1\big) \cdot \bar{X}\bar{Z} = \bar{0} \ .$$

Remark 3.2.11. The fact that torsion-freeness of a sheaf depends on the ring can also be seen by using Grothendieck's criterion (Theorem 2.5.8). Indeed the associated primes of M remain the same (Proposition 3.2.2), whereas the associated primes of the ring change, hence the condition $\text{Ass}(\mathcal{F}) \subseteq \text{Ass}(\mathcal{O}_{\mathcal{X}})$ becomes $\text{Ass}(\mathcal{F}) \subseteq \text{Ass}(\mathcal{O}_{\mathcal{Z}})$ and may not be satisfied anymore. The fact that the primes of the ring change, but those of the module do not also illustrates that the notion of torsion depends on the ring on which the sheaf is considered, but the sheaf does not.

Now we are ready to give a criterion which allows to see whether a sheaf is pure by only looking at its support (see Remark 3.1.12).

Proposition 3.2.12 (Leytem)**.** *Let $\mathcal{X} = \text{Spec}\, R$ be affine and $\mathcal{F} \cong \widetilde{M}$ be coherent with $d = \dim \mathcal{F}$. If the annihilator support $\mathcal{Z}_a(\mathcal{F})$ of $\mathcal{F}$ has a component of dimension $< d$, then $\mathcal{F}$ is not pure.*

Proof. Considering the annihilator support means that we have $I = \text{Ann}_R(M)$, so we can consider the R-module M also as a module over R/I. In the following we restrict all computations to the subscheme $\text{Spec}(R/I) \hookrightarrow \mathcal{X}$.
The support having a component of smaller dimension means that the ring R/I has a prime (minimal or embedded) which defines a component of dimension $< d$ in $\text{Spec}\, R/I$; let's denote it by $Q \trianglelefteq R/I$. It is given by the annihilator of a non-zero element, i.e. $Q = \text{Ann}_{R/I}(\bar{x})$ for some $\bar{x} \in R/I$, $\bar{x} \neq \bar{0}$ by Proposition B.2.25. Consider the submodule $N := \bar{x} * M$. Then $N \neq \{0\}$ since $\bar{x} \neq \bar{0}$ means that $x \notin \text{Ann}_R(M)$. In particular, $\text{supp}\, N \neq \emptyset$ as there always exists $P \in \text{Spec}(R/I)$ such that $N_P \neq \{0\}$, see Proposition A.2.11. On the other hand, $N_P = \{0\}$ for all $P \in \text{Spec}(R/I)$ such that $\frac{\bar{x}}{\bar{1}} = 0$ as an element in $(R/I)_P$. But

$$\frac{\bar{x}}{\bar{1}} = 0 \;\Leftrightarrow\; \exists\, \bar{y} \notin P \text{ such that } \bar{x} \cdot \bar{y} = \bar{0} \;\Leftrightarrow\; \exists\, \bar{y} \in \text{Ann}_{R/I}(\bar{x}) \setminus P \;\Leftrightarrow\; \exists\, \bar{y} \in Q \setminus P\,.$$

By contraposition, $\frac{\bar{x}}{\bar{1}} \neq 0 \;\Leftrightarrow\; Q \subseteq P$ and $N_P \neq \{0\}$, hence we obtain $Q \subseteq P$ and $\emptyset \neq \text{supp}\, N \subseteq V(Q)$. This means that all the primes which define non-zero stalks of N lie in the component $V(Q)$. But then $\dim(\text{supp}\, N) \leq \dim V(Q) < d$, i.e. N defines a coherent subsheaf of $\mathcal{F}$ which is supported in dimension $< d$. Hence $\mathcal{F}$ is not pure. $\square$

Remark 3.2.13. Hence for a coherent sheaf $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_R)$ on $\mathcal{X} = \operatorname{Spec} R$, there are only 2 cases:
– If $\mathcal{Z}_a(\mathcal{F})$ has a component of dimension $< d$, then $\mathcal{F}$ is not pure.
– If $\mathcal{Z}_a(\mathcal{F})$ has equidimensional components, then purity and torsion-freeness of $\mathcal{F}$ on $\mathcal{Z}_a(\mathcal{F})$ are equivalent by Theorem 3.1.17.

Corollary 3.2.14. *Let $\mathcal{X} = \operatorname{Spec} R$ be affine, $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$, $\mathcal{Z} = \mathcal{Z}_a(\mathcal{F})$ and assume that $\mathcal{F}$ is pure of dimension d. Then $\mathcal{O}_\mathcal{Z}$ is pure of dimension d.*

Proof. If $\mathcal{F}$ is pure, we know by Proposition 3.2.12 that $\mathcal{Z} = \mathcal{Z}_a(\mathcal{F})$ has equidimensional components. Example 3.1.13 then implies that $\mathcal{O}_\mathcal{Z}$ is pure of dimension $d = \dim \mathcal{Z}$. □

Remark 3.2.15. It is important to consider the annihilator support as the condition $\bar{x} \neq \bar{0}$ is crucial. If we would consider the Fitting support, it is possible that $x \notin \operatorname{Fitt}_0(M)$ but $x \in \operatorname{Ann}_R(M)$, so that $N = \{0\}$. We will encounter this situation in Remark 3.4.19. In general a lot of things can happen for the Fitting support, e.g. two sheaves may have exactly the same scheme-theoretical support, but one of them is pure while the other one is not (see Example 3.4.18). So there cannot exist a criterion to decide whether a sheaf is pure by only looking at the components of its Fitting support.

Remark 3.2.16. Using the criterion of Huybrechts-Lehn we will see an alternative proof of Proposition 3.2.12 in Remark 3.4.8.

Example 3.2.17. The converse of Proposition 3.2.12 is false. Consider for example $\mathcal{X} = \mathbb{A}^2_\mathbb{K} = \operatorname{Spec} \mathbb{K}[X, Y]$ and $\mathcal{F} = \widetilde{M}$ where

$$M = \mathbb{K}[X, Y]/\langle\, X \,\rangle \,\oplus\, \mathbb{K}[X, Y]/\langle\, X, Y \,\rangle = R/I \oplus R/J \ ,$$

i.e. $\mathcal{F}$ is the structure sheaf of a line and a (simple) point lying on that line. Then $\operatorname{Ann}_R(M) = I$, so $\mathcal{Z}_a(\mathcal{F})$ is 1-dimensional and just consists of a line. In particular it has equidimensional components. But $R/J \leq M$ is a non-trivial submodule with 0-dimensional support since $R/J \cong \mathbb{K}$. Thus $\mathcal{F}$ is not pure of dimension 1. On the other hand note that M is a torsion module over R

because $X \in \mathrm{Ann}_R(M)$ is a NZD, hence $\mathcal{F}$ is a torsion sheaf on $\mathcal{X}$. As $\mathcal{F}$ is not pure and $\mathcal{Z}_a(\mathcal{F})$ just has 1 component, Theorem 3.1.17 implies that $\mathcal{F}$ cannot be torsion-free on $\mathcal{Z}_a(\mathcal{F})$ neither. Indeed this is checked by $\bar{Y} * (\bar{0}, \bar{1}) = (\bar{0}, \bar{0})$ where $\bar{Y} \in R/I$ is a NZD.

Remark 3.2.18. Example 3.2.17 also shows that there exist sheaves on integral schemes (which have in particular equidimensional components) that are not pure. However we get $\dim \mathcal{F} < \dim \mathcal{X}$ since $\mathcal{F}$ is a torsion sheaf on $\mathcal{X}$, so $\mathcal{F}$ is not a sheaf of maximal dimension.

Example 3.2.19. Let us find a sheaf $\mathcal{F}$ on a scheme $\mathcal{X}$ with equidimensional components such that $\dim \mathcal{F} = \dim \mathcal{X}$ and $\mathcal{F}$ is not pure. By Theorem 3.1.17 it suffices to give an example of a sheaf of maximal dimension that is not torsion-free. The torsion subsheaf of such a sheaf then gives a non-zero coherent subsheaf (as there are no embedded primes, see Theorem 2.2.13) which is supported in smaller dimension.
Consider e.g. $R = \mathbb{K}[X, Y, Z]/\langle XY \rangle$, so $\operatorname{Spec} R$ describes the union of the 2 planes given by the equations $X = 0$ and $Y = 0$. As a module we take $M = R/\langle \bar{X}\bar{Z} \rangle$, thus $\dim(\operatorname{supp} M) = \dim R$. Moreover $\bar{Z} * [\bar{X}] = [\bar{0}]$ gives a non-trivial torsion element and $\mathrm{Ann}_R([\bar{X}]) = \langle \bar{Y}, \bar{Z} \rangle$ implies that $[\bar{X}] \in \mathcal{T}_R(M)$ is supported on the line $\{Y = Z = 0\}$. Thus the sheaf associated to M is not pure as $\langle [\bar{X}] \rangle \leq M$ defines a non-trivial coherent subsheaf that is supported in smaller dimension.

In Example 3.2.8 we have seen that a module which is torsion-free over R/I for $I \subseteq \mathrm{Ann}_R(M)$ may not be torsion-free over R. To end this section, we want to show that nevertheless the converse is true if R is Noetherian and reduced.

Proposition 3.2.20. *Let R be a reduced Noetherian ring, M an R-module and $I \trianglelefteq R$ an ideal such that $I \subseteq \mathrm{Ann}_R(M)$. If M is torsion-free over R, then M is also torsion-free over R/I.*

Proof. Let $m \in M$ be annihilated by a NZD $\bar{r} \in R/I$. To show that $m = 0$, it suffices to prove that the preimage of $\bar{r}$ under $R \to R/I$ also contains a NZD, i.e. $\exists\, i \in I$ such that $r + i$ is a NZD, because then $(r + i) * m = \bar{r} * m = 0$, so

$m = 0$ by torsion-freeness of M over R.
If the chosen $r \in R$ is a NZD, take $i = 0$. If r is a zero-divisor, it belongs to some associated prime $P_i = \mathrm{Ann}_R(a)$ for some $a \in R$. $r \cdot a = 0$ implies that $a \in I$, otherwise $\bar{r} \cdot \bar{a} = \bar{0}$ and $\bar{r}$ would be a zero-divisor. Then $r + a \notin P_i$ since $r \in P_i$ and $a \notin P_i$ (as R is reduced). Then we continue in the same matter. If $r + a$ is a NZD, take $i = a$. If it is still a zero-divisor, it belongs to another associated prime $P_j = \mathrm{Ann}_R(b)$ for some $b \in R$. Again $b \in I$, otherwise $\bar{r}$ would be a zero-divisor. Moreover $b \notin P_j$ since R is reduced and

$$(r + a) \cdot b = 0 \;\Rightarrow\; a \cdot b = -r \cdot b \in P_i \;\Rightarrow\; b \in P_i \quad \text{since } a \notin P_i \;.$$

Hence we get $r + a + b \notin P_i \cup P_j$. Continuing the same way we obtain the desired statement as R only has finitely many associated primes (keep adding elements from I as long until the sum does not belong to any of the primes any more and is thus a NZD annihilating m). □

Remark 3.2.21. A more elegant proof of Proposition 3.2.20 is given in Proposition D.3.6. This one uses the theory of so-called essential ideals.

Remark 3.2.22. In general we do not have a similar result for sheaves ; if $\mathcal{X}$ is reduced and $\mathcal{F}$ is torsion-free on $\mathcal{X}$, we cannot conclude that $\mathcal{F}$ is torsion-free on its support since torsion-freeness of sheaves and modules are in general not equivalent. However we have

Corollary 3.2.23. *Let $\mathcal{X}$ be a reduced locally Noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf on $\mathcal{X}$. Take any closed subscheme structure $\mathcal{Z}$ on the support of $\mathcal{F}$. If $\mathcal{F}$ is torsion-free on $\mathcal{X}$ and $\mathcal{Z}$ has no embedded components, then $\mathcal{F}$ is torsion-free on $\mathcal{Z}$.*

Proof. It suffices to prove the statement on affines. Let $\mathcal{X} = \operatorname{Spec} R$ for a reduced Noetherian ring R, $\mathcal{F} = \widetilde{M}$ for some R-module M and $I \subseteq \mathrm{Ann}_R(M)$ such that $\mathcal{Z} = V(I)$. If $\mathcal{F}$ is torsion-free on $\mathcal{X}$, then M is a torsion-free module over R by Corollary 1.3.10. Proposition 3.2.20 then implies that M is also torsion-free over R/I. But R/I has no embedded primes since $\mathcal{Z}$ was assumed to have no embedded components, hence $\mathcal{F}$ is torsion-free on $V(I) \cong \operatorname{Spec}(R/I)$ by Corollary 2.2.22. □

Example 3.2.24. Proposition 3.2.20 does not hold true if the ring is not reduced. Consider e.g. the ring $R = \mathbb{K}[X,Y]/\langle\, XY, X^2 \,\rangle$ and the ideal $M = \langle\, \bar{X}, \bar{Y} \,\rangle$ as a module over R. Then M is torsion-free over R (as a submodule of R, which is free) and $I = \operatorname{Ann}_R(M) = \langle\, \bar{X} \,\rangle$, thus $R/I \cong \mathbb{K}[\bar{Y}]$ is an integral domain. However the module structure $R/I \times M \to M$ gives e.g. $\bar{Y} * \bar{X} = \bar{X}\bar{Y} = \bar{0}$, where $\bar{Y} \in R/I$ is a NZD. Hence M is not torsion-free over R/I. Geometrically this means that we have found a coherent subsheaf $\mathcal{F} \subseteq \mathcal{O}_R$ which is torsion-free on $\operatorname{Spec} R$, but not on $\mathcal{Z}_a(\mathcal{F})$. Moreover it is not pure since $\dim(\operatorname{supp} M) = \dim \mathbb{K}[\bar{Y}] = 1$, but $\langle\, \bar{X} \,\rangle \leq M$ is supported in dimension 0.

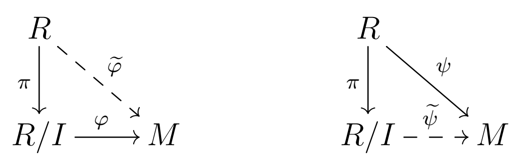
3.2.3 Global sections on the support

Next we are interested in the global sections of a sheaf that has be restricted to its support. It will turn out that these are exactly the same as those on the whole scheme.

Proposition 3.2.25. *Let M be an R-module and $I \trianglelefteq R$. If $I \subseteq \operatorname{Ann}_R(M)$, then*

$$\operatorname{Hom}_R(R/I, M) \cong \operatorname{Hom}_R(R, M) \cong M \ .$$

Proof. The second isomorphism is clear. Let $\pi : R \to R/I$, $\pi(r) = \bar{r}$ be the projection map.
If $\varphi : R/I \to M$, then one defines $\widetilde{\varphi} : R \to M : r \mapsto \varphi(\bar{r})$, i.e. $\widetilde{\varphi} = \varphi \circ \pi$.



If $\psi : R \to M$, we set $\widetilde{\psi} : R/I \to M : \bar{r} \mapsto \psi(r)$. This is well-defined as for $i \in I \subseteq \operatorname{Ann}_R(M)$, we get $\psi(i) = \psi(i \cdot 1) = i * \psi(1) = 0$ where $\psi(1) \in M$. □

Corollary 3.2.26. *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a scheme with $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ and denote its support by $\mathcal{Z}$ (with some subscheme structure). Then*

$$\operatorname{Hom}(\mathcal{O}_\mathcal{Z}, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{O}_\mathcal{X}, \mathcal{F}) \cong \mathcal{F}(\mathcal{X}) \ ,$$

which means that taking global sections of $\mathcal{F}$ on $\mathcal{X}$ is equivalent to taking global sections on $\mathcal{Z}$.

Remark 3.2.27. For $I \subseteq \mathrm{Ann}_R(M)$, consider the short exact sequence of R-modules

$$0 \longrightarrow I \xrightarrow{j} R \xrightarrow{\pi} R/I \longrightarrow 0\ .$$

Now we apply the left exact contravariant functor $\mathrm{Hom}_R(\,\cdot\,, M)$. This gives

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(R/I, M) &\xrightarrow{\circ\pi} \mathrm{Hom}_R(R, M) \\ &\xrightarrow{\circ j} \mathrm{Hom}_R(I, M) \longrightarrow \mathrm{Ext}^1(R/I, M) \longrightarrow \dots\ . \end{aligned}$$

In Proposition 3.2.25 we showed that $\circ\pi$ is an isomorphism. But exactness of the sequence implies that $\mathrm{im}(\circ\pi) = \ker(\circ j)$. And indeed for every $\rho \in \mathrm{Hom}_R(R, M)$ we get $\rho \circ j = 0$ because $\rho(i) = 0$, $\forall\, i \in I$. So $\circ j$ is the zero map. On the other hand this does not imply that $\mathrm{Hom}_R(I, M) = \{0\}$.

Example 3.2.28. Consider $R = \mathbb{K}[X]$ and $M = \mathbb{K} \cong \mathbb{K}[X]/\langle\, X\,\rangle$, so we get $I = \mathrm{Ann}_R(M) = \langle\, X\,\rangle$. Then we can set

$$\begin{aligned} f \ :\ \langle\, X\,\rangle \longrightarrow \mathbb{K}\ :\ X &\longmapsto \alpha \\ X \cdot g &\longmapsto \alpha \cdot g(0) \end{aligned}$$

for some $\alpha \in \mathbb{K}^*$ and this is a well-defined R-module homomorphism $I \to M$. Another example is the sequence of $\mathbb{Z}$-modules

$$\mathbb{Z} \xrightarrow{p} \mathbb{Z}/n\mathbb{Z} \xrightarrow{*n} \mathbb{Z}/n\mathbb{Z}\ ,$$

where p is surjective and $\mathrm{im}\, p = \ker(*n)$, but $\mathbb{Z}/n\mathbb{Z} \neq \{0\}$ even if $*n$ is actually the zero morphism.

Remark 3.2.29. So we illustrated that if M is an R-module and $I \subseteq \mathrm{Ann}_R(M)$ is an ideal, there may exist non-zero R-module homomorphisms $I \to M$.

3.3 Examples and torsion components

In the following we illustrate the concepts of purity and torsion-freeness on some examples of coherent sheaves $\mathcal{F} = \widetilde{M}$ on $\mathcal{X} = \mathrm{Spec}\, R$ and on $\mathcal{Z} = \mathcal{Z}_f(\mathcal{F}) = V(I)$ for $I = \mathrm{Fitt}_0(M)$; in this section all modules will be generated by 1 element, so

$\mathrm{Ann}_R(M) = \mathrm{Fitt}_0(M)$. In particular we are interested in answering the following questions. Is M torsion-free over R and/or over R/I ? Is $\mathcal{F}$ torsion-free on $\mathcal{X}$ and/or on $\mathcal{Z}$? Is $\mathcal{F}$ of pure dimension ? For these we will use the criteria from Proposition 1.3.3, Theorem 2.5.8 and Theorem 3.1.11.
Moreover we use Grothendieck's criterion to see where the torsion appears in the case when there exists $P \in \mathrm{Ass}_R(M) \setminus \mathrm{Ass}_R(R)$. Its proof allows to find a NZD in R_P that creates the torsion ; it is given by $\frac{r}{1}$ for $r \in P$ that does not belong to any prime P_i of R that is strictly contained in P. A neatly arranged summary of all the results is given in Appendix E.

3.3.1 Example E.3

The ring $R = \mathbb{K}[X,Y,Z]/\langle YZ(X-1), XZ(X-1)\rangle$ defines the components $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$, so there are no embedded primes and the torsion subsheaf of any coherent sheaf is coherent.

Example 3.3.1. For $M = R/\langle \bar{Y}\bar{Z}\rangle$, we know from Example 1.4.27 that the support $\mathcal{Z}$ consists of the plane $\mathcal{X}_1$ and the two parallel lines $\mathcal{Z}_2$ and $\mathcal{Z}_3 = \mathcal{X}_3$. Hence $\dim \mathcal{F} = 2$. The torsion submodule is $\mathcal{T}_R(M) = \langle [\bar{X}\bar{Z}]\rangle$, so M is not torsion-free over R. Since torsion remains after localization, $\mathcal{F}$ is not torsion-free on $\mathcal{X}$ neither. If we take $P = P_2'$, then $P_2 \subsetneq P$, so $\bar{Y}_P$ is a NZD such that $\bar{Y}_P * [\bar{X}\bar{Z}]_P = 0$. The global torsion element $[\bar{X}\bar{Z}]$ is supported on $V(P)$, so all stalks $M_{\mathfrak{M}}$ for $P \subseteq \mathfrak{M}$ have torsion too (see Example 1.4.29). Finally we saw in Example 1.4.27 that $\mathcal{T}(\mathcal{F})$ is only supported on the line $\mathcal{Z}_2$ and thus defines a coherent subsheaf of $\mathcal{F}$ with 1-dimensional support, i.e. $\mathcal{F}$ is not pure of dimension 2. Now what happens as a sheaf on the support ?
Let $I = \mathrm{Ann}_R(M) = \langle \bar{Y}\bar{Z}\rangle$; then $M \cong R/I$, so M is a free R/I-module (obvious as $\mathcal{F}$ is the structure sheaf of $\mathcal{Z}$). In particular, M is torsion-free over R/I and $\mathcal{F}$ is torsion-free on $\mathcal{Z}$. But it is not pure on $\mathcal{Z}$ as it is not pure on $\mathcal{X}$. Hence torsion-freeness on $\mathcal{Z}$ and purity are not equivalent. This is because not all minimal primes in R/I define components of the same dimension. One could also see this by the fact that $\mathcal{F} = \mathcal{O}_{R/I}$ and $\mathrm{Spec}(R/I)$ has components of dimension < 2, so Example 3.1.13 tells that $\mathcal{O}_{R/I}$ is not of pure dimension 2.
Note that a subsheaf of $\mathcal{F}$ with smaller dimension on $\mathcal{Z}$ can be given by the sub-

sheaf of $\mathcal{F}$ which has smaller dimension on $\mathcal{X}$, i.e. by $\mathcal{T}(\mathcal{F})$. In Example 3.2.10 we already illustrated that the torsion relation $(\bar{Y}+\bar{X}-1)*[\bar{X}\bar{Z}]=[\bar{0}]$ on R is not longer torsion in R/I since $\bar{Y}+\bar{X}-1\in R/I$ is a zero-divisor. On $\mathcal{Z}$, the torsion subsheaf $\mathcal{T}(\mathcal{F})\subset\mathcal{F}=\mathcal{O}_{R/I}$ is described by the ideal $\langle\,\bar{X}\bar{Z}\,\rangle\subset R/I$. Moreover it satisfies $(\bar{X}-1)\cdot\bar{X}\bar{Z}=\bar{0}$ and $\bar{Y}\cdot\bar{X}\bar{Z}=\bar{0}$, so the support is indeed included in $\mathcal{Z}_2$.

3.3.2 Example E.4 and Example E.5

The ring $R=\mathbb{K}[X,Y,Z]/\langle\,XY,X^2,XZ\,\rangle$ with components $\mathcal{X}_1,\mathcal{X}_2$ has an embedded prime which defines an embedded double point on the plane $\mathcal{X}_1$.

Example 3.3.2. We saw in Example 2.3.1 that the R-module $M=R/\langle\,\bar{Y}\bar{Z}\,\rangle$ is torsion-free and that its corresponding sheaf is supported on $\mathcal{Z}=\mathcal{Z}_1\cup\mathcal{Z}_2\cup\mathcal{Z}_3$, hence $\dim\mathcal{F}=1$. $\mathcal{F}$ is however not torsion-free and its torsion subsheaf $\mathcal{T}(\mathcal{F})$, which is not coherent, is supported on $\mathcal{Z}\setminus\{\mathfrak{M}\}$, where $\mathfrak{M}=\langle\,\bar{X},\bar{Y},\bar{Z}\,\rangle$ is a the maximal ideal, see Example 2.3.4. Here we can take $\bar{Z}\in P=P_1'$ and $\bar{Y}\in Q=P_2'$ with $P_1\subsetneq P$ and $P_1\subsetneq Q$, so we find the torsion relations $\bar{Z}_P*[\bar{Y}]_P=0$ and $\bar{Y}_Q*[\bar{Z}]_Q=0$. Since however $P_3'=P_3$, there is no torsion on the embedded component. Moreover Example 3.1.4 shows that it is not pure of dimension 1 as $\langle\,[\bar{X}]\,\rangle\leq M$ defines a subsheaf with 0-dimensional support.
But again $\mathcal{F}$ is free and hence torsion-free as a sheaf on its support because $I=\mathrm{Ann}_R(M)=\langle\,\bar{Y}\bar{Z}\,\rangle$, $M\cong R/I$ and $\mathcal{F}=\mathcal{O}_{R/I}$. Being not pure on $\mathcal{X}$, it is not pure on $\mathcal{Z}$ neither. This can also be seen by the fact that $\mathcal{Z}_a(\mathcal{F})$ has an embedded prime, so according to the constructive proof of Proposition 3.2.12, we can explicitly find a subsheaf with smaller dimension using its annihilator. We have $\langle\,\bar{X},\bar{Y},\bar{Z}\,\rangle=\mathrm{Ann}_R(\bar{X})$ where $\bar{X}\in R/I$ is nilpotent and $\bar{X}\notin\mathrm{Ann}_R(M)$, so the wanted subsheaf is $N:=\bar{X}*M=\bar{X}*\langle\,[\bar{1}]\,\rangle=\langle\,[\bar{X}]\,\rangle$.

Example 3.3.3. Now let us take a look at what may happen for a sheaf whose support no longer has embedded components, i.e. we shall divide out the nilpotent element $\bar{X}$. Consider $M=R/\langle\,\bar{X},\bar{Y}\bar{Z}\,\rangle$, which we now will call Example E.5. The primary decomposition becomes

$$\langle\,XY,X^2,XZ,X,YZ\,\rangle=\langle\,X,YZ\,\rangle=\langle\,X,Y\,\rangle\cap\langle\,X,Z\,\rangle\;,$$

so the support of $\mathcal{F}$ is $\mathcal{Z}_1 \cup \mathcal{Z}_2$ and $\dim \mathcal{F} = 1$. Let us first analyze what happens on the support. As $I = \mathrm{Ann}_R(M) = \langle \bar{X}, \bar{Y}\bar{Z} \rangle$, $M \cong R/I$ says that M is free over R/I and that $\mathcal{F} = \mathcal{O}_{R/I}$ is free, hence torsion-free on $\mathcal{Z}$. This also implies that $\mathcal{F} = \mathcal{O}_{\mathcal{Z}}$ is pure of dimension 1 as $\mathcal{Z}$ has equidimensional components (see Example 3.1.13). Alternatively this can be seen by the criterion of Huybrechts-Lehn since the associated primes $P_1' = \langle X, Y \rangle$ and $P_2' = \langle X, Z \rangle$ of M define components of the same dimension. However the situation on $\mathcal{X}$ is quite different. First we obtain that M is also torsion-free over R since P_1' and P_2' are both contained in P_2. But there is torsion locally since they do not belong to $\mathrm{Ass}_R(R)$. Consider for example the relation $\bar{Y}_P * [\bar{Z}]_P = [\bar{0}]$ at $P = \langle \bar{X}, \bar{Y}, \bar{Z} - 1 \rangle$. $\bar{Y}_P \in R_P$ is a NZD and $[\bar{Z}]_P \neq 0$ since

$$\bar{f} * [\bar{Z}] = [\bar{0}] \;\Leftrightarrow\; \bar{f} \cdot \bar{Z} \in \langle \bar{X}, \bar{Y}\bar{Z} \rangle = \langle \bar{X}, \bar{Y} \rangle \cap \langle \bar{X}, \bar{Z} \rangle \;\Rightarrow\; \bar{f} \in \langle \bar{X}, \bar{Y} \rangle \subseteq P ,$$

thus $[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$. So again we have a module which is torsion-free but where certain stalks are not. Indeed one exactly shows as in Example 2.3.4 that $\mathcal{T}_{R_P}(M_P) \neq \{0\}$ for all $P \in \mathcal{Z}$, $P \neq \mathfrak{M}$ since the topological spaces are the same. Alternatively this also follows from Grothendieck's criterion since $P_1', P_2' \in \mathrm{Ass}_R(M) \setminus \mathrm{Ass}_R(R)$. So the torsion subsheaf $\mathcal{T}(\mathcal{F})$ is not coherent on $\mathcal{X}$ as its support is not closed. The most interesting fact however is that $\mathcal{F}$ is of pure dimension 1 without being torsion-free on $\mathcal{X}$. This is due to the embedded component.

3.3.3 Examples of coherent torsion with embedded primes

Next we consider the ring

$$R = \mathbb{K}[X, Y, Z] / \langle XZ, YZ^2 \rangle .$$

The primary decomposition

$$\langle XZ, YZ^2 \rangle = \langle Z \rangle \cap \langle X, Y \rangle \cap \langle X, Z^2 \rangle$$

gives the associated primes

$$P_1 = \langle \bar{Z} \rangle = \mathrm{Ann}_R(\bar{X}) \quad , \quad P_2 = \langle \bar{X}, \bar{Y} \rangle = \mathrm{Ann}_R(\bar{Z}^2) ,$$
$$P_3 = \langle \bar{X}, \bar{Z} \rangle = \mathrm{Ann}_R(\bar{Y}\bar{Z}) ,$$

where $\bar{Y}\bar{Z} \in R$ is nilpotent of degree 2. The set of zero-divisors in R is given by $\mathrm{ZD}(R) = \langle \bar{X}, \bar{Y} \rangle \cup \langle \bar{X}, \bar{Z} \rangle$. $\mathcal{X}$ describes the union of a plane, a perpendicular line and a double line which is embedded into the plane. We denote them by $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ respectively. As expected we have $\dim \mathcal{X} = 2$ since $\mathbb{K}[X,Y,Z] \twoheadrightarrow R$ and we have the chain of primes $\langle \bar{Z} \rangle \subsetneq \langle \bar{X}, Z \rangle \subsetneq \langle \bar{X}, Y, Z \rangle$.

Example 3.3.4. We take the module $M = R/\langle \bar{X} \rangle$, which is generated by $[\bar{1}]$, and call it Example E.6. It describes the structure sheaf of the line and the double line since $\mathcal{X} \cap V(\bar{X}) = \mathcal{X}_2 \cup \mathcal{X}_3$. As $\mathrm{Ann}_R(M) = \langle \bar{X} \rangle$, this can also be seen on the primary decomposition

$$\langle XZ, YZ^2, X \rangle = \langle X, YZ^2 \rangle = \langle X, Y \rangle \cap \langle X, Z^2 \rangle \ .$$

As rings we have $R/\langle \bar{X} \rangle \cong \mathbb{K}[Y,Z]/\langle YZ^2 \rangle$, so $\dim \mathcal{F} = 1$. By definition $I = \mathrm{Ann}_R(M) = \langle \bar{X} \rangle$, we see again that $M \cong R/I$ and $\mathcal{F} = \mathcal{O}_{R/I}$, so M is torsion-free over R/I and $\mathcal{F}$ is torsion-free on $\mathcal{Z}$. Moreover $\mathcal{F} = \mathcal{O}_{\mathcal{Z}}$ is pure of dimension 1 as R/I has no embedded primes and $\mathcal{Z}$ has equidimensional components (Example 3.1.13). To check torsion-freeness on R and $\mathcal{X}$, denote $P'_1 = \langle \bar{X}, \bar{Y} \rangle$ and $P'_2 = \langle \bar{X}, \bar{Z} \rangle$. Then we see that

$$\mathrm{Ass}_R(M) = \{ P'_1 \, , \, P'_2 \} \subseteq \{ P_1 \, , \, P_2 \, , \, P_3 \} = \mathrm{Ass}_R(R) \ ,$$

hence M is torsion-free over R and $\mathcal{F}$ is torsion-free on $\mathcal{X}$, i.e. $\mathcal{T}(\mathcal{F}) = 0$. In particular $\mathcal{T}(\mathcal{F})$ is coherent even though R has embedded primes. This gives an example of a pure and torsion-free sheaf on a scheme that has an embedded component.

Example 3.3.5. On the same ring we now consider $M = R/\langle \bar{X}\bar{Y} \rangle$, which gives the structure sheaf of $\mathcal{X} \cap \{XY = 0\}$. We will call it Example E.7. Intuitively this gives the union of the line $\mathcal{X}_2$, the double line $\mathcal{X}_3$ and another line. Indeed the primary decomposition gives

$$\begin{aligned} &\langle XZ, YZ^2, XY \rangle = \langle X, Y \rangle \cap \langle Y, Z \rangle \cap \langle X, Z^2 \rangle \\ \Rightarrow\ &\mathrm{Ass}_R(M) = \left\{ P'_1 = \langle \bar{X}, \bar{Y} \rangle \ , \ P'_2 = \langle \bar{Y}, \bar{Z} \rangle \ , \ P'_3 = \langle \bar{X}, \bar{Z} \rangle \right\} , \end{aligned}$$

so that $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$ with $\mathcal{Z}_1 = \mathcal{X}_2$ and $\mathcal{Z}_3 = \mathcal{X}_3$. Hence $\dim \mathcal{F} = 1$ and as before we see that $\mathcal{F}$ is free and torsion-free on $\mathcal{Z}$. Moreover it is pure of dimension 1 since $\mathcal{Z}$ has equidimensional components. More interesting things happen however on $\mathcal{X}$.

Concerning torsion of M over R, we have $P_2' \nsubseteq P_i$ for all i, hence M is not torsion-free over R. In order to find the torsion submodule, let $\bar{f} \in R$ be a NZD and $[\bar{g}] \in M$ such that $\bar{f} * [\bar{g}] = [\bar{0}]$. Then

$$\begin{aligned}\bar{f} \cdot \bar{g} \in \langle\, \bar{X}\bar{Y} \,\rangle = \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{Y}, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Z}^2 \,\rangle \;\subseteq\; P_2 \cap \langle\, \bar{Y}, \bar{Z} \,\rangle \cap P_3 \\ \Rightarrow\; \bar{g} \in P_2 \cap P_3 = \langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle \;,\end{aligned}$$

so that $[\bar{X}]$ and $[\bar{Y}\bar{Z}]$ are candidates for torsion elements. But $\big(\bar{Y} + \bar{Z}\big) * [\bar{X}] = [\bar{0}]$ where $\bar{Y} + \bar{Z}$ is a NZD, hence we already get $[\bar{X}] \in \mathcal{T}_R(M)$. To show that there are no other torsion elements, assume that $\exists\, \bar{f}_1, \bar{f}_2 \in R$ such that

$$\bar{f}_1 * [\bar{X}] + \bar{f}_2 * [\bar{Y}\bar{Z}] \in \mathcal{T}_R(M) \;.$$

Then $[\bar{f}_2\bar{Y}\bar{Z}] \in \mathcal{T}_R(M)$ and by the relations in R, we may assume that $\bar{f}_2$ only depends on $\bar{Y}$. But this is not possible as all elements that annihilate $[\bar{Y}\bar{Z}]$ are contained in P_3. So finally we get $\mathcal{T}_R(M) = \langle\, [\bar{X}] \,\rangle$. This torsion also remains on all localizations such that $[\bar{X}]_P \neq 0$. As $\bar{Z} * [\bar{X}] = 0$ and $\bar{Y} * [\bar{X}] = 0$ are all relations annihilating $[\bar{X}]$, we get that $[\bar{X}]_P \neq 0$ for all P on the line $\mathcal{Z}_2$, i.e. $\mathcal{T}_{R_P}(M_P) \neq \{0\}$, $\forall\, P \in \mathcal{Z}_2$. So $\mathcal{F}$ is not torsion-free on $\mathcal{X}$, although it is pure of dimension 1.

Let us find all $P \in \mathcal{X}$ on which the localization is not torsion-free (i.e. let us determine the support of $\mathcal{T}(\mathcal{F})$). We already know that $\mathcal{Z}_2 \subseteq \operatorname{supp} \mathcal{T}(\mathcal{F})$. Note that the relation may simplify, e.g. for $P = \langle\, \bar{X} - \lambda, \bar{Y}, \bar{Z} \,\rangle$ for $\lambda \neq 0$ we get $\bar{Z}_P = 0$, so that $\bar{X}_P$ and $\bar{Y}_P$ are NZDs and $\bar{Y}_P * [\bar{X}]_P = 0$. This is nothing but the global relation $\big(\bar{Y} + \bar{Z}\big) * [\bar{X}] = [\bar{0}]$ since $\bar{Z}$ vanishes in the localization. Also note that $[\bar{Y}]_P = 0$ since $\bar{X} \notin P$, so multiples of $[\bar{X}]_P$ are the only torsion elements for such primes. Now consider $P = \langle\, \bar{X}, \bar{Y}, \bar{Z} - \lambda \,\rangle$ for $\lambda \neq 0$. As $\bar{Z} \notin P$, we get $\bar{X}_P = \bar{Y}_P = 0$. So there is no torsion since the remaining element $[\bar{Z}]_P$ cannot be annihilated at all. Finally we look at $P = \langle\, \bar{X}, \bar{Y} - \lambda, \bar{Z} \,\rangle$ for $\lambda \neq 0$. Here we obtain that $\bar{Y}_P$ is a NZD together with the relations $\bar{Z}_P^2 = 0$, $\bar{X}_P\bar{Z}_P = 0$

and $[\bar{X}]_P = 0$. One sees that there is no possible combination to obtain torsion. Finally we have $\operatorname{supp} \mathcal{T}(\mathcal{F}) = \mathcal{Z}_2$. This implies in particular that the torsion subsheaf is coherent since every local torsion element comes from a global one (all local ones are multiples of $[\bar{X}]_P$, so they all come from $[\bar{X}]$) and we get

$$\mathcal{T}(\mathcal{F}) = \widetilde{\langle\, [\bar{X}] \,\rangle}\ .$$

Hence on $\mathcal{X} = \operatorname{Spec} R$ we have a coherent sheaf $\mathcal{F}$ which is pure but not torsion-free and whose torsion subsheaf is coherent even though R has embedded primes.

Example 3.3.6. Consider the ring $R = \mathbb{K}[X,Y,Z]/\langle\, XZ, X^2 \,\rangle$ with the module $M = R/\langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle$ and the primary decompositions

$$\langle\, XZ, X^2 \,\rangle = \langle\, X \,\rangle \cap \langle\, X^2, Z \,\rangle \quad , \quad \langle\, XZ, X^2, X, YZ \,\rangle = \langle\, X, Y \,\rangle \cap \langle\, X, Z \,\rangle\ .$$

Hence $\{\bar{0}\} = \langle\, \bar{X} \,\rangle \cap \langle\, \bar{X}^2, \bar{Z} \,\rangle = Q_1 \cap Q_2$ with

$$\operatorname{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{X} \,\rangle\ ,\ P_2 = \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\}\ ,$$
$$\operatorname{Ass}_R(M) = \big\{\, P_1' = \langle\, \bar{X}, \bar{Y} \,\rangle\ ,\ P_2' = \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\}\ .$$

If $\mathcal{X}_1 = V(P_1)$, $\mathcal{X}_2 = V(Q_2)$, $\mathcal{Z}_1 = V(P_1')$ and $\mathcal{Z}_2 = V(P_2')$, then $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ is a plane with an embedded double line and $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$ gives the (simple) "cross" inside of that plane. In particular $\dim \mathcal{X} = 2$, $\dim \mathcal{F} = 1$ and $\mathcal{F}$ is torsion-free on $\mathcal{Z}$ and pure of dimension 1.
But since P_1' is not included in any of the primes from $\operatorname{Ass}_R(R)$, we see that M is not torsion-free over R and that $\mathcal{F}$ is not torsion-free over $\mathcal{X}$. As $\bar{Y} \in R$ is a NZD with $\bar{Y} * [\bar{Z}] = [\bar{0}]$, we obtain that $[\bar{Z}] \in M$ is a global torsion element. One finds that $\mathcal{T}_R(M) = \langle\, [\bar{Z}] \,\rangle$. The torsion remains in all localizations such that $[\bar{Z}]_P \neq 0$, so that $\mathcal{T}_{R_P}(M_P) \neq \{0\}$ for all $P \in \mathcal{Z}_1$. The question is whether there are other stalks on which torsion may appear. By Remark 2.5.9 we know that the answer is No (if an associated prime $P \in \operatorname{Ass}_R(M)$ is also an associated prime of R, then M_P is torsion-free). Hence $\mathcal{T}_{R_Q}(M_Q) = \{0\}$ for all primes of the form $Q = \langle\, \bar{X}, \bar{Y} - \lambda, \bar{Z} \,\rangle \in \mathcal{Z}_2$ with $\lambda \neq 0$. Finally we have

$$\mathcal{T}(\mathcal{F}) = \widetilde{\langle\, [\bar{Z}] \,\rangle} \qquad \text{with} \qquad \operatorname{supp} \mathcal{T}(\mathcal{F}) = \mathcal{Z}_1\ .$$

3.3.4 Torsion in a given component

We noticed that all examples in Section 3.3 have a property in common: for each module M, torsion in the stalk M_P only appeared at prime ideals P which belong to a component of $\mathcal{Z}$ that is not a component of $\mathcal{X}$; independent of whether M was torsion-free or not. This is a consequence of Grothendieck's criterion. Indeed we have

Corollary 3.3.7. *Let R be a Noetherian ring and M a module over R.*
1) If $P \in \mathrm{Ass}_R(M) \setminus \mathrm{Ass}_R(R)$, then $\mathcal{T}_{R_P}(M_P) \neq \{0\}$.
2) Vice-versa, if $M \in \mathrm{Ass}_R(M) \cap \mathrm{Ass}_R(R)$, then $\mathcal{T}_{R_P}(M_P) = \{0\}$.

Proof. follows from Theorem 2.5.8 and Remark 2.5.9. □

In other words, if $V(P)$ is an irreducible component of $\mathcal{Z}$ that is not a component of $\mathcal{X}$, then there is torsion in M_P and if $V(P) \subseteq \mathcal{Z}$ is also a component of $\mathcal{X}$, there is none.
On the other hand we cannot say that there is torsion in $M_{\mathfrak{M}}$ for all maximal ideals $\mathfrak{M}$ containing P, i.e. we cannot say what happens at a closed point by just looking at the component(s) it belongs to. Similarly if $P \in \mathrm{Ass}_R(M)$ belongs to $\mathrm{Ass}_R(R)$, there still may exist points on the component $V(P)$ at which the stalks are not torsion-free.

Example 3.3.8. Consider Example E.4 at the maximal ideal $\mathfrak{M} = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$. $\mathfrak{M}$ belongs to the lines $\mathcal{Z}_1 = V(P_1')$ and $\mathcal{Z}_2 = V(P_2')$, which are not components of $\mathcal{X}$ itself. However we have shown in Example 2.3.4 that $\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) = \{0\}$, even if $M_{P_1'}$ and $M_{P_2'}$ are not torsion-free. Torsion-freeness at $\mathfrak{M}$ is due to the fact that the component $V(\mathfrak{M}) \subseteq \mathcal{Z}$ is also a component of $\mathcal{X}$.

If $\mathfrak{M}$ is a maximal ideal containing P, we cannot conclude:

$$\mathcal{T}_{R_P}(M_P) \neq \{0\} \quad \not\Rightarrow \quad \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\} \ ,$$
$$\mathcal{T}_{R_P}(M_P) = \{0\} \quad \not\Rightarrow \quad \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) = \{0\} \ .$$

Nevertheless we now show that if P defines a "component with torsion", then generically there is also torsion on $M_{\mathfrak{M}}$. The converse (if P gives a "torsion-free component", then there is generically no torsion in $M_{\mathfrak{M}}$) is true as well, but the exact statement is slightly different.

Proposition 3.3.9 (Leytem)**.** *Let $\mathcal{X} = \operatorname{Spec} R$ for a Noetherian ring R, $\mathcal{F} \cong \widetilde{M}$ be quasi-coherent and fix $P \in \mathcal{X}$.*
1) If $\mathcal{T}_{R_P}(M_P) \neq \{0\}$, then $\exists\, r \notin P$ such that

$$\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\} \ \text{ for all } \mathfrak{M} \in V(P) \cap D(r) \ .$$

2) If $\mathcal{T}_{R_P}(M_P) = \{0\}$ and $\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\}$ for some $\mathfrak{M} \in V(P)$, then

$$\exists\, r \notin P \ \text{ such that } \mathfrak{M} \in V(r) \ .$$

Proof. 1) Let $\frac{m}{1} \in \mathcal{T}_{R_P}(M_P)$ be a non-zero torsion element. By Corollary 2.1.18 we know that $\frac{m}{1} \in \mathcal{T}_{R_r}(M_r)$ for some $r \notin P$ where $\frac{m}{1} \in M_r$ is still non-zero and $\frac{m}{1} \in \mathcal{T}_{R_Q}(M_Q), \forall\, Q \in D(r)$. Moreover $\frac{m}{1} \neq 0$ in $M_{\mathfrak{M}}$ for all $\mathfrak{M}$ containing P since $\operatorname{Ann}_R(m) \subseteq P \subseteq \mathfrak{M}$. Finally

$$0 \neq \tfrac{m}{1} \in \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \ , \quad \forall\, \mathfrak{M} \in V(P) \cap D(r) \ .$$

2) Let $\frac{a}{1} * \frac{m}{1} = 0$ where $\frac{a}{1} \in R_{\mathfrak{M}}$ is a NZD and $\frac{m}{1} \in M_{\mathfrak{M}}$ is non-zero, i.e. $\exists\, b \notin \mathfrak{M}$ such that $ba*m = 0$. Since $P \subseteq \mathfrak{M}$, every associated prime of R that is contained in P is also contained in $\mathfrak{M}$. In particular, all zero-divisors in R_P remain zero-divisors in $R_{\mathfrak{M}}$. Hence $\frac{a}{1}$ is also a NZD in R_P. But then $\frac{m}{1} = 0$ in M_P since $ba*m = 0$ with $b \notin P$ and M_P is torsion-free. From this we get that $\frac{m}{1} = 0$ on an open neighborhood $D(r)$ of P (since $\mathcal{F}$ is a sheaf), i.e. $\frac{m}{1} = 0$ in M_r with $r \notin P$. But as $\frac{m}{1} \neq 0$ in $M_{\mathfrak{M}}$, we hence obtain $\mathfrak{M} \notin D(r)$, i.e. $\mathfrak{M} \in V(r) = \mathcal{X} \setminus D(r)$. □

Remark 3.3.10. The first statement says that if there is torsion in M_P, there is a dense open neighborhood of P in $V(P)$ on which we have torsion too. The second one says that if M_P is torsion-free, then torsion in $V(P)$ can only appear in a set with empty interior. So in both cases we obtain a generic situation.

Example 3.3.11. Consider Example E.4, where e.g. $P = P_2' \in \operatorname{Ass}_R(M)$ is not an associated prime of R, so we know that $\mathcal{T}_{R_P}(M_P) \neq \{0\}$. A torsion relation is given by $\bar{Y}_P * [\bar{Z}]_P = 0$. Taking $r = \bar{Z} \in P_2 \setminus P$, we find that $[\bar{Z}]/\bar{1}$ is still torsion

over $D(\bar{Z})$, see Example 2.3.1. The intersection $V(P) \cap D(\bar{Z})$ is then equal to the line $\mathcal{Z}_2$ with the origin removed.
Similarly we have $P_1' \in \operatorname{Ass}_R(M) \setminus \operatorname{Ass}_R(R)$, hence by taking $r = \bar{Y} \in P_2 \setminus P_1'$, one finds that $\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\}$ for all $\mathfrak{M} \in V(P_2') \cap D(\bar{Y})$, which is equal to the line $\mathcal{Z}_1$ with the origin removed. So we have torsion on all points of the "cross", except the origin (compare Example 2.3.4).

Example 3.3.12. Now consider Example E.3. Here $P = \langle \bar{Z} \rangle \in \operatorname{Ass}_R(M)$ is an associated prime of R too, so $\mathcal{T}_{R_P}(M_P) = \{0\}$. At $\mathfrak{M} = \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle \in V(P)$ we however have torsion given by $\bar{Y}_{\mathfrak{M}} * [\bar{Z}]_{\mathfrak{M}} = 0$. But $[\bar{Z}]_P = 0$ since $\bar{Y} \notin P$, so according to the proof of Proposition 3.3.9 we see that $\mathfrak{M} \in V(\bar{Y})$, which means that $\mathfrak{M}$ can only belong to the line

$$V(P) \cap V(\bar{Y}) = V\big(P + \langle \bar{Y} \rangle\big) = V\big(\langle \bar{Y}, \bar{Z} \rangle\big) ,$$

and this one has empty interior in the plane $V(P)$. Now we can continue the same way by looking at the prime $L = \langle \bar{Y}, \bar{Z} \rangle$ that defines the line. Since $\bar{X} - 1 \notin L$, the global torsion element $[\bar{X}\bar{Z}]$ vanishes and we have $\mathcal{T}_{R_L}(M_L) = \{0\}$. But still $\mathfrak{M} \in V(L)$, so we get $[\bar{Z}]_L = 0$ as well with $\bar{X}(\bar{X} - 1) \notin L$, hence $\mathfrak{M} \in V(\bar{X}(\bar{X} - 1))$. So now

$$\begin{aligned} \mathfrak{M} \in V(L) \cap V\big(\bar{X}(\bar{X} - 1)\big) &= V\big(\langle \bar{X}(\bar{X} - 1), \bar{Y}, \bar{Z} \rangle\big) \\ &= \big\{ \langle \bar{X}, \bar{Y}, \bar{Z} \rangle \, , \, \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle \big\} , \end{aligned}$$

which means that these 2 maximal ideals are the only ones in $V(P)$ on which torsion may appear.

Remark 3.3.13. By repeating the above argument with $[\bar{Z}]_{\mathfrak{M}} = [\bar{X}\bar{Z}]_{\mathfrak{M}}$ (see Remark 2.2.19) one can omit $\bar{X}$ and immediately finds $\mathfrak{M} = \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle$ as only solution.

Example 3.3.14. Let $P = \langle \bar{Z} \rangle$ in Example E.7; as $P \notin \operatorname{supp} M$, we trivially get $\mathcal{T}_{R_P}(M_P) = \{0\}$. However for $\mathfrak{M} = \langle \bar{X} - 1, \bar{Y}, \bar{Z} \rangle \in V(P)$ we have the torsion relation $\bar{Y}_{\mathfrak{M}} * [\bar{X}]_{\mathfrak{M}} = 0$ where $\bar{Y}_{\mathfrak{M}}$ is a NZD, see Example 3.3.5. Since $\bar{Y} \notin P$ we obtain $[\bar{X}]_P = 0$, hence $\mathfrak{M} \in V(\bar{Y})$ and we get

$$\mathfrak{M} \in V(P) \cap V(\bar{Y}) = V\big(\langle \bar{Y}, \bar{Z} \rangle\big) .$$

Now take $L = \langle \bar{Y}, \bar{Z} \rangle$; this is an associated prime of M but not of R, so $\mathcal{T}_{R_L}(M_L) \neq \{0\}$ with $\bar{Y}_L * [\bar{X}]_L = 0$ as well. Then Proposition 3.3.9 says that there is generically torsion at the maximal ideals in the line $\mathcal{Z}_2 = V(L)$. And indeed we know that all stalks in $\mathcal{Z}_2$ have torsion.

Remark 3.3.15. Example 3.3.12 and Example 3.3.14 show that there is in general no information about the codimension of the subspace where torsion can appear in a "torsion-free component". More precisely, for $P \in \operatorname{Spec} R$ denote

$$TP := \left\{ \mathfrak{M} \in V(P) \;\middle|\; \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\} \right\} .$$

If $\mathcal{T}_{R_P}(M_P) = \{0\}$, we have proved in Proposition 3.3.9 that $\operatorname{codim}_{V(P)}(TP) \geq 1$. But it is also possible that we have an equality, as well as a strict inequality.

In the case where the torsion subsheaf is quasi-coherent, we can even say something more.

Proposition 3.3.16. *Let $\mathcal{X} = \operatorname{Spec} R$ for a Noetherian ring R, $\mathcal{F} \cong \widetilde{M}$ be quasi-coherent and assume that $\mathcal{T}(\mathcal{F})$ is quasi-coherent too. If $\mathcal{T}_{R_P}(M_P) \neq \{0\}$ for some $P \in \mathcal{X}$, then $\mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}}) \neq \{0\}$ for all $\mathfrak{M} \in V(P)$.*

Proof. Theorem 2.2.8 says that $\mathcal{T}(\mathcal{F})$ being coherent means that

$$\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P) \;, \quad \forall\, P \in \mathcal{X} \;.$$

Hence if M_P has torsion, then the local (non-zero) torsion element in $\mathcal{T}_{R_P}(M_P)$ comes from a global torsion element $m \in \mathcal{T}_R(M)$ with $\operatorname{Ann}_R(m) \subseteq P$ and m remains torsion in all localizations where it does not vanish. But $\operatorname{Ann}_R(m) \subseteq \mathfrak{M}$ as well, so $0 \neq \frac{m}{1} \in \mathcal{T}_{R_{\mathfrak{M}}}(M_{\mathfrak{M}})$ for all $\mathfrak{M} \in V(P)$. □

3.4 Annihilator vs. Fitting support

In most of the examples we encountered until now, the module was generated by 1 element and hence the annihilator support coincided with the Fitting support. But in general the properties of a sheaf $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ restricted to $\mathcal{Z}_a(\mathcal{F})$ or to $\mathcal{Z}_f(\mathcal{F})$ can be quite different. In this section we are going to analyze the relations and differences between both supports. Moreover we want to understand which one of them is a "better" choice.

3.4.1 Associated primes of M, $\mathrm{Ann}_R(M)$ and $\mathrm{Fitt}_0(M)$

We start with the following observation.

Lemma 3.4.1. *Let I, I' be ideals in a Noetherian ring R with $\mathrm{Rad}(I) = \mathrm{Rad}(I')$, so that $V(I) = V(I')$ as topological spaces. Then I and I' have the same minimal associated primes, i.e. the minimal primes in $\mathrm{Ass}(I)$ and $\mathrm{Ass}(I')$ are the same.*

Proof. The argument will be symmetric with respect to I and I', so we only show that every minimal associated prime of I is a minimal associated prime of I'. Let $I = Q_1 \cap \ldots \cap Q_\alpha$ and $I' = Q'_1 \cap \ldots \cap Q'_\beta$ be minimal primary decompositions. This gives

$$P_1 \cap \ldots \cap P_\alpha = \mathrm{Rad}(I) = \mathrm{Rad}(I') = P'_1 \cap \ldots \cap P'_\beta \ .$$

We may assume that all primes in these intersections are minimal, otherwise they can be omitted. Let P_i be a fixed minimal prime of I. By Prime Avoidance the inclusion $P'_1 \cap \ldots \cap P'_\beta = \mathrm{Rad}(I) \subseteq P_i$ implies that there is a minimal prime P'_j of I' such that $P'_j \subseteq P_i$. Assume that $P'_j \subsetneq P_i$, i.e. $\exists\, r \in P_i \setminus P'_j$. Since P_i is minimal, we get $P_k \nsubseteq P_i$ for all $k \neq i$ and $\exists\, r_k \in P_k \setminus P_i$. In particular, $r_k \notin P'_j$, $\forall\, k \neq i$. But then

$$r \cdot \prod_{k \neq i} r_k \in \big(P_1 \cap \ldots \cap P_\alpha\big) \setminus P'_j \ ,$$

which contradicts that $\mathrm{Rad}(I) = \mathrm{Rad}(I')$. Hence $P_i = P'_j$ and P_i is also a minimal prime for I'. □

Corollary 3.4.2. *Let M be a finitely generated module over a Noetherian ring. Then the minimal primes in $\mathrm{Ass}\big(\mathrm{Ann}_R(M)\big)$ and $\mathrm{Ass}\big(\mathrm{Fitt}_0(M)\big)$ are the same.*

Remark 3.4.3. Actually we can say that an even stronger statement holds true: let $I = \mathrm{Ann}_R(M)$, $I' = \mathrm{Fitt}_0(M)$ with minimal primary decompositions $I = \bigcap_i Q_i$ and $I' = \bigcap_j Q'_j$ as above. Since $V(I)$ is a subscheme of $V(I')$, we get

$$\mathrm{Spec}(R/I) \text{ is a subscheme of } \mathrm{Spec}(R/I') \;\Rightarrow\; \forall\, i,\, \exists\, j \text{ such that } Q'_j \subseteq Q_i \ .$$

It means that every component of $\mathrm{Spec}(R/I)$ is contained in a component of $\mathrm{Spec}(R/I')$, which is a much stronger condition as it also gives information about embedded components and components with non-reduced structures.

By Proposition 3.2.5 and Proposition 3.2.2 we know that purity of a sheaf $\mathcal{F}$ is independent of the "ambient space" and that the associated primes do not depend on the ring over which $\mathcal{F}$ is considered, so we do not need to study purity of $\mathcal{F}$ on $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$.
But Grothendieck's criterion shows that torsion-freeness of $\mathcal{F}$ heavily depends on the ring over which the sheaf is considered. So we are interested in the primes in $\operatorname{Ass}_R\big(R/\operatorname{Ann}_R(M)\big) = \operatorname{Ass}\big(\operatorname{Ann}_R(M)\big)$ and

$$\operatorname{Ass}_R\big(R/\operatorname{Fitt}_0(M)\big) = \operatorname{Ass}\big(\operatorname{Fitt}_0(M)\big)\ ,$$

as well as in their relation to the primes in $\operatorname{Ass}_R(M)$. Unfortunately these are in general not the same. But we have the following result.

Proposition 3.4.4. [[6], IV.§1.n°4.Thm.2, p.313] and [[55], 6.5, p.39]
Let R be a Noetherian ring and M a finitely generated R-module. Then the minimal primes in

$$\operatorname{Ass}_R(M) \quad , \quad \operatorname{supp} M \quad , \quad \operatorname{Ass}\big(\operatorname{Ann}_R(M)\big) \quad , \quad \operatorname{Ass}\big(\operatorname{Fitt}_0(M)\big)$$

are the same. Hence we always obtain the same (topological) decomposition

$$\operatorname{supp} M = \bigcup_{P \text{ minimal}} V(P)\ .$$

Remark 3.4.5. In particular for any $\mathcal{X} = \operatorname{Spec} R$, as soon as a minimal prime P of the annihilator or the Fitting support is not contained in the set $\operatorname{Ass}_R(R)$, then M_P is not torsion-free.

We can partially deal with the embedded primes by the following result.

Proposition 3.4.6. [[11], 10.66.4]
Let M be a finitely generated module over a Noetherian ring R and $I = \operatorname{Ann}_R(M)$. If $\operatorname{Ass}_R(M)$ does not contain embedded primes, then $\operatorname{Ass}(I)$ has no embedded primes neither.

Proof. Let $P \in \operatorname{Ass}(I)$ be arbitrary; it is given by the annihilator of an element $\bar{x} \in R/I$, $\bar{x} \neq \bar{0}$. So the submodule $N = x * M$ is non-zero as $x \notin \operatorname{Ann}_R(M)$. In particular we have $P \subseteq \operatorname{Ann}_R(N)$ because $P = \operatorname{Ann}_R(\bar{x})$, i.e. elements that annihilate $\bar{x}$ also annihilate N. Then any associated prime $Q \in \mathrm{Ass}_R(N)$ satisfies $P \subseteq Q$ because $\forall\, m \in M$,

$$\begin{aligned} r \in P = \mathrm{Ann}_R(\bar{x}) \;&\Rightarrow\; r \cdot x \in I \;\Rightarrow\; r * (x * m) = (r \cdot x) * m = 0 \\ &\Rightarrow\; \mathrm{Ann}_R(\bar{x}) \subseteq \mathrm{Ann}_R(x * m) \;. \end{aligned}$$

Moreover $Q \in \mathrm{Ass}_R(M)$ by Proposition B.3.5 since $N \leq M$. Now assume that P is embedded, i.e. there is a minimal prime $P' \in \mathrm{Ass}(I)$ such that $P' \subsetneq P$. P' being minimal, we know that $P' \in \mathrm{Ass}_R(M)$ as well. Now we take any prime $Q \in \mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M)$, which exists by Proposition B.3.4 since $N \neq \{0\}$. Then $P' \subsetneq P \subseteq Q$ and Q would be an embedded prime in $\mathrm{Ass}_R(M)$. □

Remark 3.4.7. By contraposition: if $\mathrm{Ass}\big(\mathrm{Ann}_R(M)\big)$ has embedded primes, then so has $\mathrm{Ass}_R(M)$. Similarly as in Proposition 3.2.12 we see that the argument does not work for $I = \mathrm{Fitt}_0(M)$ since N could be zero if $x \in \mathrm{Ann}_R(M)\backslash\mathrm{Fitt}_0(M)$.

Remark 3.4.8. As mentioned in Remark 3.2.16, Proposition 3.4.6 and Theorem 3.1.11 now allow to give an alternative proof of Proposition 3.2.12.
Let $P \in \mathrm{Ass}\big(\mathrm{Ann}_R(M)\big)$ be a prime ideal which defines a component of dimension $< d$. If P is minimal, then $P \in \mathrm{Ass}_R(M)$ by Proposition 3.4.4 and the criterion of Huybrechts-Lehn implies that $\mathcal{F}$ is not pure. If P is embedded, then $\mathrm{Ass}_R(M)$ also contains an embedded prime by Remark 3.4.7 and again $\mathcal{F}$ is not pure because of Theorem 3.1.11.

3.4.2 Properties of $\mathrm{Ann}_R(M)$ and $\mathrm{Fitt}_0(M)$

We mainly want to discuss 2 aspects of the annihilator and the Fitting ideal. Since the criteria of Grothendieck and Huybrechts-Lehn are based on the associated primes of M, we first want to know which one of them is "closer" to $\mathrm{Ass}_R(M)$. Secondly we are interested in knowing which one behaves "better" with respect to the scheme structures.

The advantages of $\mathrm{Fitt}_0(M)$ are that it defines a richer structure on the support and that it encodes information about the finite presentation of the module (i.e. the locally free resolution of the sheaf). On the other hand $\mathrm{Ann}_R(M)$

can easier deal with embedded primes (Proposition 3.2.12 and Proposition 3.4.6) which allows conclusions about purity.

Lemma 3.4.9. [[18], V-11, p.221] and [[30], 7.2.7, p.346]
Let R be a (not nec. Noetherian) ring and M, N two R-modules of finite presentation. Then $\mathrm{Ann}_R(M \oplus N) = \mathrm{Ann}_R(M) \cap \mathrm{Ann}_R(N)$ *and*

$$\mathrm{Fitt}_0(M \oplus N) = \mathrm{Fitt}_0(M) \cdot \mathrm{Fitt}_0(N) \ .$$

Proof. The formula for the annihilator follows from $r * (m, n) = 0 \Leftrightarrow r * m = 0$ and $r * n = 0$. For the Fitting ideal, consider the finite presentations

$$R^{m_1} \xrightarrow{\varphi} R^{m_0} \longrightarrow M \longrightarrow 0 \qquad \text{and} \qquad R^{n_1} \xrightarrow{\psi} R^{n_0} \longrightarrow N \longrightarrow 0 \ ,$$

where φ is of type $m_1 \times m_0$ and ψ is of the type $n_1 \times n_0$, so that the direct sum is given by

$$R^{m_1+n_1} \xrightarrow{A} R^{m_0+n_0} \longrightarrow M \oplus N \longrightarrow 0 \ ,$$

where

$$A = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}$$

is a matrix of type $(m_1 + n_1) \times (m_0 + n_0)$. If we compute its minors of order $m_0 + n_0$, the blocks of zeros imply that these will just consist of products of the minors of φ and ψ. □

Corollary 3.4.10. [[30], 7.2.8, p.346]
Let $I_1, \ldots, I_k \trianglelefteq R$ be ideals in a Noetherian ring and consider the R-modules $M_i = R/I_i$. Then $\mathrm{Ann}_R\big(M_1 \oplus \ldots \oplus M_k\big) = I_1 \cap \ldots \cap I_k$ *and*

$$\mathrm{Fitt}_0\big(M_1 \oplus \ldots \oplus M_k\big) = I_1 \cdot \ldots \cdot I_k \ .$$

Proof. Since each R-module R/I_i is generated by $\bar{1}$, we obtain that

$$\mathrm{Ann}_R(R/I_i) = \mathrm{Fitt}_0(R/I_i) = I_i$$

for all $i \in \{1, \ldots, k\}$ and use the formulas of Lemma 3.4.9 by induction. □

Remark 3.4.11. Hence comparing the annihilator and Fitting ideals is also related to comparing the ideals $I \cap J$ and $I \cdot J$ as in Section 1.2.4. So similarly as in that discussion, the Fitting support can cause problems as it may become too big. Consider e.g. Example 3.4.23 and Example 3.4.27 below; in these we will define M in order to obtain the structure sheaf of a certain component, but taking the Fitting ideal of $M \oplus M$ gives a support that is equal to all of the spectrum, so we lose the (smaller) subscheme that we actually wanted to study.

Remark 3.4.12. We obtain a particular case for $M = R/I \oplus R/I$ for some ideal $I \trianglelefteq R$. Indeed

$$\mathrm{Ass}_R(R/I) = \mathrm{Ass}(I) = \mathrm{Ass}\big(\mathrm{Ann}_R(R/I)\big) = \mathrm{Ass}\big(\mathrm{Fitt}_0(R/I)\big)$$

since R/I is generated by one element. But $\mathrm{Ann}_R(M) = I$, $\mathrm{Fitt}_0(M) = I^2$ and

$$\begin{aligned}\mathrm{Ass}_R(M) = \mathrm{Ass}_R\big(R/I \oplus R/I\big) &= \mathrm{Ass}_R(R/I) \cup \mathrm{Ass}_R(R/I)\\ &= \mathrm{Ass}(I) = \mathrm{Ass}\big(\mathrm{Ann}_R(M)\big)\end{aligned}$$

by Corollary B.3.6. Hence for modules of this type, the annihilator is closer related to $\mathrm{Ass}_R(M)$ than the Fitting ideal. This illustrates again the impression that $\mathrm{Ann}_R(M)$ gives in general easier criteria for torsion-freeness and purity.

However the Fitting ideal has the following important property: it commutes with pullbacks. This is very useful in the theory of moduli spaces.

Proposition 3.4.13. [[1], p.179-180]
Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of schemes and $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_\mathcal{Y})$. Then $\mathrm{Fitt}_0(f^*\mathcal{F})$ is generated by $\mathrm{Fitt}_0(\mathcal{F})$ as an $\mathcal{O}_\mathcal{X}$–module.

This follows from right exactness of the functor $f^* : \mathtt{Mod}(\mathcal{O}_\mathcal{Y}) \to \mathtt{Mod}(\mathcal{O}_\mathcal{X})$. Indeed if

$$\mathcal{O}_\mathcal{Y}^m \longrightarrow \mathcal{O}_\mathcal{Y}^n \longrightarrow \mathcal{F} \longrightarrow 0$$

is a locally free resolution of $\mathcal{F}$ on $\mathcal{Y}$, then

$$f^*\mathcal{O}_\mathcal{Y}^m \longrightarrow f^*\mathcal{O}_\mathcal{Y}^n \longrightarrow f^*\mathcal{F} \longrightarrow 0 \qquad \Leftrightarrow \qquad \mathcal{O}_\mathcal{X}^m \longrightarrow \mathcal{O}_\mathcal{X}^n \longrightarrow f^*\mathcal{F} \longrightarrow 0$$

is a locally free resolution of $f^*\mathcal{F}$ on $\mathcal{X}$. We will prove the corresponding statement for modules.

Let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ be a morphism of affine schemes and $\mathcal{F} \cong \widetilde{M}$ a quasi-coherent sheaf on $\operatorname{Spec} R$. So we can pull back $\mathcal{F}$ on $\operatorname{Spec} S$ via $f^*\mathcal{F}$. On the level of modules this corresponds to a homomorphism of rings $\varphi : R \to S$, which turns S into an R-module via $r * s = \varphi(r) \cdot s$. Then

Lemma 3.4.14. [[16], 20.5, p.494] and [[52], 34621]
The Fitting support commutes with pullbacks, in the sense that

$$\operatorname{Fitt}_0\big(M \otimes_R S\big) \;\cong\; \operatorname{Fitt}_0(M) \otimes_R S\ . \tag{3.5}$$

Proof. Let

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

be a finite presentation of M over R with $A = (a_{ij})_{ij}$ for some $a_{ij} \in R$. Tensoring by S we get

$$S^m \xrightarrow{B} S^n \longrightarrow M \otimes_R S \longrightarrow 0\ ,$$

where $B = \varphi(A)$. This is because the tensor product transforms an R-module homomorphism $R \to R : r \mapsto a \cdot r$ for some $a \in R$ into $S \to S : s \mapsto a * s$ (still as R-modules). As a morphism of S-modules this gives $S \to S : s \mapsto \varphi(a) \cdot s$. Now $\operatorname{Fitt}_0(M \otimes_R S)$ is an ideal in S that is generated by subdeterminants of $B = \varphi(A)$. Since φ is a ring homomorphism we get $\operatorname{sdet}(\varphi(A)) = \varphi(\operatorname{sdet} A)$, i.e. the Fitting ideal on S is generated by the generators of the Fitting ideal on R that we see as elements in S via φ. This corresponds to $\operatorname{Fitt}_0(M) \otimes_R S$. ☐

Remark 3.4.15. The idea behind these operations is that we can first pull back to S and then take the ideal over S or first take the ideal over R and then pull back to S. This gives the S-modules

$$\operatorname{Fitt}_0\big(M \otimes_R S\big) \quad , \quad \operatorname{Ann}_S\big(M \otimes_R S\big) \quad , \quad \operatorname{Fitt}_0(M) \otimes_R S \quad , \quad \operatorname{Ann}_R(M) \otimes_R S\ .$$

Example 3.4.16. In general a formula as in (3.5) does not hold true for the annihilator. Indeed we always have a morphism of S-modules

$$\operatorname{Ann}_R(M) \otimes_R S \longrightarrow \operatorname{Ann}_S\big(M \otimes_R S\big) \ : \ r \otimes s \longmapsto r * s\ , \tag{3.6}$$

which is well-defined since $(r*s)*(m\otimes t)=m\otimes(r*st)=(r*m)\otimes(s\cdot t)=0$, $\forall\, m\in M,\ t\in S$. But it may not be injective: consider e.g. $M=R/I$ and $S=R/J$ for some ideals $I,J\trianglelefteq R$ such that $\{0\}\neq I\cdot J\subsetneq I\cap J$. Then Corollary D.2.6 gives

$$\mathrm{Ann}_R(M)\otimes_R S=\mathrm{Ann}_R(R/I)\otimes_R R/J=I\otimes_R R/J\ \cong\ I/(I\cdot J)\ ,$$
$$\mathrm{Ann}_S\big(M\otimes_R S\big)=\mathrm{Ann}_{R/J}\big(R/I\otimes_R R/J\big)\ \cong\ \mathrm{Ann}_{R/J}\big(R/(I+J)\big)\ .$$

Taking an element $r\in(I\cap J)\setminus I\cdot J$, we find that $\bar r\neq\bar 0$ but it is mapped to zero since $r\in J$. Alternatively we can see this $\bar r$ as an element $r\otimes\bar 1\in I\otimes_R R/J$ which is non-zero because

$$r\otimes\bar 1=(r\cdot 1)\otimes\bar 1\neq 1\otimes(r*\bar 1)=0\qquad\text{and}\qquad r\otimes\bar 1\neq(i\cdot j)\otimes\bar 1=i\otimes\bar j=0$$

as $1\notin I$ and r cannot be written as a product $i\cdot j$ with $i\in I$ and $j\in J$. Taking $I=J$ such that $\{0\}\neq I^2\subsetneq I$, we even get a stronger counter-example since then $R/(I+I)=R/I$ and $\mathrm{Ann}_{R/I}(R/I)=\{0\}$, so (3.6) is nothing but the morphism $I/I^2\to\{0\}$.

Remark 3.4.17. The fact that $\mathrm{Fitt}_0(M)$ commutes with pullbacks is however the crucial aspect why one prefers in general to consider the Fitting support of a sheaf instead of its annihilator support. This way one ensures that the support behaves functorially with respect to morphisms of schemes and sheaves and that the scheme structure is respected under pulling back.
On the other hand this gives several disadvantages regarding criteria for torsion-freeness and purity, whom we are now forced to deal with. The following examples in Section 3.4.3 show that the Fitting support may have a lot of unexpected and unpleasant properties.

3.4.3 Examples: creation and disappearance of embedded primes

Example 3.4.18. Inspired from Example E.4 and Example E.5 let us consider the non-reduced ring $R=\mathbb{K}[X,Y,Z]/\langle\, XY,X^2,XZ\,\rangle$ together with the modules

$M = R/\langle \bar{X}, \bar{Y}\bar{Z} \rangle$ and $N = R/\langle \bar{Y}\bar{Z} \rangle$. We set

$$F = M \oplus M \qquad , \qquad G = N \oplus N$$

and look at the sheaves $\mathcal{F} = \widetilde{F}$ and $\mathcal{G} = \widetilde{G}$. In Example 3.3.3 and Example 3.3.2 we have seen that $\widetilde{M}$ is of pure dimension, but $\widetilde{N}$ is not. By Lemma 3.1.5 we hence obtain that $\mathcal{F}$ is pure and $\mathcal{G}$ is not. Alternatively one can see this by looking at the annihilator supports. We denote

$$I = \text{Ann}_R(F) = \text{Ann}_R(M) = \langle \bar{X}, \bar{Y}\bar{Z} \rangle \ ,$$
$$J = \text{Ann}_R(G) = \text{Ann}_R(N) = \langle \bar{Y}\bar{Z} \rangle \ ,$$

so that we get the free modules $F \cong (R/I)^2$ and $G \cong (R/J)^2$. Since $\mathcal{Z}_a(\mathcal{F})$ has equidimensional components, torsion-freeness of $\mathcal{F} = \mathcal{O}_{R/I} \oplus \mathcal{O}_{R/I}$ implies that $\mathcal{F}$ is of pure dimension. $\mathcal{Z}_a(\mathcal{G})$ having an embedded component, we conclude on the other hand from Proposition 3.2.12 that $\mathcal{G}$ is not pure. As in Remark 3.4.12 we get

$$\text{Ass}_R(F) = \text{Ass}\big(\text{Ann}_R(F)\big) = \big\{ \langle \bar{X}, \bar{Y} \rangle \ , \ \langle \bar{X}, \bar{Z} \rangle \big\} \ ,$$
$$\text{Ass}_R(G) = \text{Ass}\big(\text{Ann}_R(G)\big) = \big\{ \langle \bar{X}, \bar{Y} \rangle \ , \ \langle \bar{X}, \bar{Z} \rangle \ , \ \langle \bar{X}, \bar{Y}, \bar{Z} \rangle \big\} \ .$$

Now we want to compute $\mathcal{Z}_f(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{G})$. For this we have to look at the relations of the generators of F and G. To simplify notations, let's denote them by $(1,0)$ and $(0,1)$. For G we have

$$\begin{matrix} \bar{Y}\bar{Z} * (1,0) + \bar{0} * (0,1) = (0,0) \\ \bar{0} * (1,0) + \bar{Y}\bar{Z} * (0,1) = (0,0) \end{matrix} \qquad \longrightarrow \qquad \begin{pmatrix} \bar{Y}\bar{Z} & \bar{0} \\ \bar{0} & \bar{Y}\bar{Z} \end{pmatrix} \ .$$

$\text{Fitt}_0(G)$ is generated by all minors of order 2 (the number of generators), i.e. $\text{Fitt}_0(G) = \langle (\bar{Y}\bar{Z})^2 \rangle$. The primary decomposition gives

$$\langle XY, X^2, XZ, (YZ)^2 \rangle = \langle X, Y^2 \rangle \cap \langle X, Z^2 \rangle \cap \langle Z, X^2, XY, Y^2 \rangle \ ,$$

so the associated primes in R are

$$\langle \bar{X}, \bar{Y} \rangle = \text{Ann}_R(\bar{Y}\bar{Z}^2) \qquad , \qquad \langle \bar{X}, \bar{Z} \rangle = \text{Ann}_R(\bar{Y}^2\bar{Z}) \ ,$$
$$\langle \bar{X}, \bar{Y}, \bar{Z} \rangle = \text{Ann}_R(\bar{X}) \ .$$

This means that the Fitting support of $\mathcal{G}$ consists of 2 double lines and a triple point at their intersection (already a lot more complicated than 2 lines and a double point at the intersection). Note that all primes are given by annihilators of nilpotent elements. The triple point is moreover an embedded component. For F we get

$$\begin{array}{c} \bar{Y}\bar{Z} * (1,0) + \bar{0} * (0,1) = (0,0) \\ \bar{0} * (1,0) + \bar{Y}\bar{Z} * (0,1) = (0,0) \\ \bar{X} * (1,0) + \bar{0} * (0,1) = (0,0) \\ \bar{0} * (1,0) + \bar{X} * (0,1) = (0,0) \end{array} \qquad \longrightarrow \qquad \begin{pmatrix} \bar{Y}\bar{Z} & \bar{0} \\ \bar{0} & \bar{Y}\bar{Z} \\ \bar{X} & \bar{0} \\ \bar{0} & \bar{X} \end{pmatrix}.$$

Again $\mathrm{Fitt}_0(F)$ is generated by all minors of order 2, i.e. $(\bar{Y}\bar{Z})^2$, $\bar{X}^2 = \bar{0}$ and $\bar{X}\bar{Y}\bar{Z} = \bar{0}$, so $\mathrm{Fitt}_0(F) = \langle\, (\bar{Y}\bar{Z})^2 \,\rangle$ as well. Thus $\mathcal{Z}_f(\mathcal{F}) = \mathcal{Z}_f(\mathcal{G})$, although $\mathcal{F}$ is pure and $\mathcal{G}$ is not. This shows that there cannot exist a criterion to decide whether a sheaf is pure by only looking at the components of its Fitting support (compare Remark 3.2.15). Finally we have

$$\mathrm{Ass}\big(\mathrm{Fitt}_0(F)\big) = \mathrm{Ass}\big(\mathrm{Fitt}_0(G)\big) = \big\{\, \langle\, \bar{X}, \bar{Y} \,\rangle \,,\ \langle\, \bar{X}, \bar{Z} \,\rangle \,,\ \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}$$

and Grothendieck's criterion allows to see that $\mathcal{F}$ and $\mathcal{G}$ are torsion-free on both of their supports.

Remark 3.4.19. The idea of the proof of Proposition 3.2.12 does not work for the Fitting support here (compare Remark 3.2.15). Indeed consider $\mathcal{F}$ and the embedded triple point of $\mathcal{Z}_f(\mathcal{F})$, whose associated prime is given by $\mathrm{Ann}_R(\bar{X})$. We have $\bar{X} \notin \mathrm{Fitt}_0(F)$ but $\bar{X} \in \mathrm{Ann}_R(F)$, so defining the submodule $\bar{X} * F = \{0\}$ does not help. On the other hand it works for $\mathcal{G}$ since $\bar{X} \notin \mathrm{Ann}_R(G)$.

Remark 3.4.20. Example 3.4.18 also illustrates that the Fitting support of a pure sheaf may have embedded components (in contrast to its annihilator support, see Proposition 3.2.12).

Indeed we know from Corollary 3.2.14 that if $\mathcal{F}$ is pure of dimension d, then $\mathcal{O}_{\mathcal{Z}_a(\mathcal{F})}$ is also pure of dimension d, but $\mathcal{O}_{\mathcal{Z}_f(\mathcal{F})}$ does not need to. Actually the only thing that can happen for the Fitting support is a creation of embedded primes since the minimal primes of $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$ are the same (Corollary 3.4.2), so if all components of $\mathcal{Z}_a(\mathcal{F})$ are of dimension d, then all minimal primes in $\mathcal{Z}_f(\mathcal{F})$ also define components of dimension d.

Example 3.4.21. Creation of new embedded primes in the Fitting support may even happen for integral schemes ; consider Example 3.2.17 with $M = R/I \oplus R/J$. We have seen that the annihilator support is a line and has only one component, but its Fitting support is given by

$$\mathrm{Fitt}_0(M) = I \cdot J = \langle\, X^2, XY \,\rangle = \langle\, X \,\rangle \cap \langle\, X^2, Y \,\rangle \ ,$$

thus defines a line with an embedded double point. Note however that the sheaf $\widetilde{M}$ is not pure.

Example 3.4.22. We can also illustrate another aspect why Fitting supports are difficult to handle. Consider again Example 3.4.18 ; if $I = \mathrm{Ann}_R(F) = \langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle$ and $I' = \mathrm{Fitt}_0(F) = \langle\, (\bar{Y}\bar{Z})^2 \,\rangle$, then $I' \subsetneq I$ and $\varphi :\ R/I' \twoheadrightarrow R/I$. $F \cong (R/I)^2$ implies that F is a free module over R/I, generated by $(1,0)$ and $(0,1)$. However it is not longer free over R/I', e.g. we have

$$\bar{X} * (1,0) = \varphi(\bar{X}) \cdot \big([\bar{1}], [\bar{0}]\big) = [\bar{X}] \cdot \big([\bar{1}], [\bar{0}]\big) = [\bar{0}] \cdot \big([\bar{1}], [\bar{0}]\big) = (0,0) \ ,$$

where $\bar{X} \in R/I'$ is non-zero, but $[\bar{X}] \in R/I$ is zero, so $(1,0)$ is not linearly independent over R/I'. However R/I is still torsion-free since R/I' has more zero-divisors:

$$\mathrm{ZD}(R/I) = \langle\, \bar{X}, \bar{Y} \,\rangle \cup \langle\, \bar{X}, \bar{Z} \,\rangle \qquad , \qquad \mathrm{ZD}(R/I') = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \ .$$

If there is a NZD $\bar{r} \in R/I'$ annihilating some $m \in M$, then $\bar{r}$ is also a NZD in R/I annihilating m and M would have torsion over R/I. The same argument shows that every torsion-free module over R/I is also torsion-free over R/I'. Unfortunately this result does not hold true in general.

Example 3.4.23. Consider $R = \mathbb{K}[X,Y,Z]/\langle\, XZ, X^2 \,\rangle$ and the quotient module $M = R/\langle\, \bar{X}\bar{Y} \,\rangle$. We have the primary decompositions

$$\langle\, XZ, X^2 \,\rangle = \langle\, X \,\rangle \cap \langle\, Z, X^2 \,\rangle \qquad , \qquad \langle\, XZ, X^2, XY \,\rangle = \langle\, X \,\rangle \cap \langle\, X^2, Y, Z \,\rangle \ ,$$

so $\mathcal{X}$ consists of a plane with an embedded double line and M describes the structure sheaf of that plane with an embedded double point at the origin. Let

$N = M \oplus M$, so $I = \mathrm{Ann}_R(N) = \langle\, \bar{X}\bar{Y} \,\rangle$ and $I' = \mathrm{Fitt}_0(N) = \langle\, (\bar{X}\bar{Y})^2 \,\rangle$ where $\bar{X}^2\bar{Y}^2 = \bar{0}$, i.e. $\mathcal{Z}_f(\widetilde{N}) = \operatorname{Spec} R$. N is free of rank 2 over R/I, generated by $(1,0)$ and $(0,1)$, and hence torsion-free. But it is neither free, nor torsion-free over R/I' because

$$\mathrm{ZD}(R/I) = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \qquad , \qquad \mathrm{ZD}(R/I') = \langle\, \bar{X}, \bar{Z} \,\rangle \ .$$

The generator $(1,0)$ is e.g. annihilated by the non-zero element $\bar{X}\bar{Y} \in R/I'$ (while $\bar{X}\bar{Y}$ is zero in R/I), so the generating set is not linearly independent. Moreover $\bar{Y}$ is a NZD in R/I' with $\bar{Y} * \big([\bar{X}], [\bar{0}]\big) = 0$ in N, thus $\mathcal{T}_{R/I'}(N) \neq \{0\}$. Let us also analyze this using the associated primes.

$$\begin{gathered} \mathrm{Ass}_R(R) = \big\{\, \langle\, \bar{X} \,\rangle \ , \ \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\} \ , \\ \mathrm{Ass}_R(M) = \big\{\, \langle\, \bar{X} \,\rangle \ , \ \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\} = \mathrm{Ass}_R(N) \ , \\ \mathrm{Ass}\,\big(\, \mathrm{Ann}_R(N)\big) = \mathrm{Ass}_R(N) \qquad , \qquad \mathrm{Ass}\,\big(\, \mathrm{Fitt}_0(N)\big) = \mathrm{Ass}_R(R) \ . \end{gathered}$$

The inclusion $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}\,\big(\, \mathrm{Ann}_R(N)\big)$ shows that the sheaf $\widetilde{N}$ is torsion-free on $\mathcal{Z}_a(\widetilde{N})$. But $\mathrm{Ass}_R(N) \nsubseteq \mathrm{Ass}\,\big(\, \mathrm{Fitt}_0(N)\big)$, so it is not torsion-free on its Fitting support.

Remark 3.4.24. This gives yet another illustration of the difficulty of Fitting supports: the module $N = M \oplus M$ defines a sheaf whose direct summands are (torsion-)free on their Fitting supports since $\mathrm{Fitt}_0(M) = I$, but it is not torsion-free itself. In other words, $\widetilde{M}$ is torsion-free on $\mathcal{Z}_f(\widetilde{M})$, but $\widetilde{N} = \widetilde{M} \oplus \widetilde{M}$ is not torsion-free on $\mathcal{Z}_f(\widetilde{N})$.

Example 3.4.25. Something similar also happens in Example 3.2.24. Let us compute the Fitting support of the subsheaf $\mathcal{F} \subseteq \mathcal{O}_R$. M is generated by $\bar{X}$ and $\bar{Y}$, which give the relations

$$\begin{pmatrix} -\bar{Y} & \bar{X} \\ \bar{Y} & \bar{0} \\ \bar{0} & \bar{X} \\ \bar{X} & \bar{0} \end{pmatrix} ,$$

i.e. $\mathrm{Fitt}_0(M) = \langle\, XY, X^2 \,\rangle$ and thus $\mathcal{Z}_f(\mathcal{F}) = \operatorname{Spec} R$. In particular we see that $\mathcal{Z}_a(\mathcal{F})$ is just a simple line since $\mathrm{Ann}_R(M) = \langle\, \bar{X} \,\rangle$, whereas $\mathcal{Z}_f(\mathcal{F})$ is a line with an embedded double point, i.e. $\mathcal{Z}_a(\mathcal{F}) \subsetneq \mathcal{Z}_f(\mathcal{F})$. Hence $\mathcal{F}$ is an example of a sheaf which is torsion-free on its Fitting support (since it is equal to $\operatorname{Spec} R$), but not on its annihilator support. Thus we have exactly the opposite situation as in Example 3.4.23.

Remark 3.4.26. Intuitively one could think that taking the Fitting support can only create more zero-divisors (i.e. more embedded components) since it contains $\mathcal{Z}_a(\mathcal{F})$ as a closed subscheme, see Remark 3.4.3. Example 3.4.23 however shows that this is not the case; in general $Z_f(\mathcal{F})$ does not have more embedded components than $Z_a(\mathcal{F})$. Indeed it is true that their minimal primes are the same since both define the same topological space, but an embedded component of $\mathcal{Z}_a(\mathcal{F})$ can e.g. disappear in $\mathcal{Z}_f(\mathcal{F})$ to become part of a bigger component.
In general nothing can be said for embedded primes of $\mathcal{Z}_a(\mathcal{F})$. They may remain, but can also disappear in another component of $\mathcal{Z}_f(\mathcal{F})$, which can either be embedded (Example 3.4.23) or minimal (Example 3.4.27). This is due to the embedded component in the ring R. Indeed if R has embedded components, these may be divided out in the module so that they are not "seen" by the annihilator support, but they "come back" as soon as the Fitting support is strictly bigger. Hence $\mathcal{Z}_f(\mathcal{F})$ also takes care of the structure of the ring and "remembers" where the module came from.

Example 3.4.27. We want to illustrate that embedded primes of $\mathcal{Z}_a(\mathcal{F})$ can disappear in $\mathcal{Z}_f(\mathcal{F})$ and become part of a minimal prime whose component was given a non-reduced structure.
Let $R = \mathbb{K}[X,Y,Z]/\langle X^2 \rangle$ represent a double plane and $M = R/\langle \bar{X}\bar{Z}, \bar{X}\bar{Y} \rangle$ define the structure sheaf of a simple plane and an embedded double point. Take $N = M \oplus M$, so that $I = \operatorname{Ann}_R(N) = \langle \bar{X}\bar{Z}, \bar{X}\bar{Y} \rangle$ and

$$I' = \operatorname{Fitt}_0(N) = \langle (\bar{X}\bar{Z})^2, \bar{X}\bar{Z}\bar{X}\bar{Y}, (\bar{X}\bar{Y})^2 \rangle = \{\bar{0}\}$$

since $\bar{X}^2 = \bar{0}$. Hence we get the primary decompositions

$$\langle X^2, XZ, XY \rangle = Q_1 \cap Q_2 = \langle X \rangle \cap \langle X^2, Y, Z \rangle \quad , \quad \langle X^2 \rangle = Q_1' = \langle X^2 \rangle$$

with associated primes

$$\mathrm{Ass}_R(R) = \left\{ P_1' = \langle \bar{X} \rangle \right\} ,$$
$$\mathrm{Ass}_R(M) = \left\{ P_1 = \langle \bar{X} \rangle \ , \ P_2 = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle \right\} = \mathrm{Ass}_R(N) \ ,$$
$$\mathrm{Ass}\left(\mathrm{Ann}_R(N)\right) = \mathrm{Ass}_R(N) \qquad , \qquad \mathrm{Ass}\left(\mathrm{Fitt}_0(N)\right) = \mathrm{Ass}_R(R) \ .$$

So we see that the minimal primes of R/I and R/I' are the same while the embedded prime P_2 has disappeared. Indeed it is characterized by $P_1 \subsetneq P_2$ with $Q_1 \nsubseteq Q_2$ (otherwise the primary decomposition is not minimal). But we have $Q_1' \subseteq Q_2$, which means that the plane $V(P_1)$ has been "doubled" to become $V(Q_1')$ and now contains the double point, that was not contained in the simple plane. Looking at the associated primes, we again conclude that the sheaf defined by N is torsion-free on its annihilator support, but not on its Fitting support.

Example 3.4.28. Consider $R = \mathbb{K}[X,Y,Z]$ with the modules $M = R/\langle X, YZ \rangle$, $N = R/\langle Z, XY \rangle$ and $L = M \oplus N$, i.e. the support of L consists of the union of two "crosses". When considering the Fitting support, a certain multiple structure will be put on their intersection. Lemma 3.4.9 gives

$$\begin{aligned}
\mathrm{Ann}_R(L) = \langle X, YZ \rangle \cap \langle Z, XY \rangle &= \langle XY, XZ, YZ \rangle \\
&= \langle X, Y \rangle \cap \langle X, Z \rangle \cap \langle Y, Z \rangle \ , \\
\mathrm{Fitt}_0(L) = \langle X, YZ \rangle \cdot \langle Z, XY \rangle &= \langle XZ, X^2Y, YZ^2, XY^2Z \rangle \\
&= \langle X, Y \rangle \cap \langle X^2, XZ, Z^2 \rangle \cap \langle Y, Z \rangle \ , \\
\mathrm{Ass}_R(L) = \mathrm{Ass}_R(M) \cup \mathrm{Ass}_R(N) &= \left\{ \langle X, Y \rangle \ , \ \langle X, Z \rangle \ , \ \langle Y, Z \rangle \right\} \ .
\end{aligned}$$

If $\mathcal{L} = \widetilde{L}$, let us denote the components of $\mathcal{Z}_a(\mathcal{L})$ by $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ and those of $\mathcal{Z}_f(\mathcal{L})$ by $\mathcal{Z}_1', \mathcal{Z}_2', \mathcal{Z}_3'$. Topologically we have $\mathcal{Z}_i = \mathcal{Z}_i'$ for all i, but $\mathcal{Z}_2'$ has a richer structure than $\mathcal{Z}_2$. Although $\mathcal{Z}_f(\mathcal{L})$ transformed $\mathcal{Z}_2$ into $\mathcal{Z}_2'$, it did not create any embedded components.

As M and N are just quotients, their corresponding sheaves are (torsion-)free on $\mathrm{supp}\, M$ and $\mathrm{supp}\, N$, hence pure of dimension 1 by Theorem 3.1.17 as all components (the lines) have the same dimension. Thus $\mathcal{L}$ is pure as well (Lemma 3.1.5) and therefore torsion-free on $\mathcal{Z}_a(\mathcal{L})$ and $\mathcal{Z}_f(\mathcal{L})$ because both have equidimensional components. This is again checked by observing that

$$\mathrm{Ass}_R(L) = \mathrm{Ass}\left(\mathrm{Ann}_R(L)\right) = \mathrm{Ass}\left(\mathrm{Fitt}_0(L)\right) \ .$$

Example 3.4.29. Let $R = \mathbb{K}[X,Y,Z]$ and $\mathcal{F} = \widetilde{M}$, where $M = M_1 \oplus M_2 \oplus M_3$ with

$$M_1 = R/\langle\, X \,\rangle \quad , \quad M_2 = R/\langle\, Y^2 \,\rangle \quad , \quad M_3 = R/\langle\, X^3, Z \,\rangle \ ,$$

i.e. $\mathcal{F}$ is the sum of the structure sheaves of a plane, a double plane and a triple line. Corollary 3.4.2 then allows to compute

$$\begin{aligned} \mathrm{Ann}_R(M) &= \langle\, X \,\rangle \cap \langle\, Y^2 \,\rangle \cap \langle\, X^3, Z \,\rangle = \langle\, XY^2Z \,,\, X^3Y^2 \,\rangle \ , \\ \mathrm{Fitt}_0(M) &= \langle\, X \,\rangle \cdot \langle\, Y^2 \,\rangle \cdot \langle\, X^3, Z \,\rangle = \langle\, XY^2Z \,,\, X^4Y^2 \,\rangle \\ &= \langle\, X \,\rangle \cap \langle\, Y^2 \,\rangle \cap \langle\, X^4, Z \,\rangle \ , \\ \mathrm{Ass}_R(M) = \mathrm{Ass}_R(M_1) \cup \mathrm{Ass}_R(M_2) \cup \mathrm{Ass}_R(M_3) &= \big\{ \langle\, X \,\rangle \ , \ \langle\, Y \,\rangle \ , \ \langle\, X, Z \,\rangle \big\} \ . \end{aligned}$$

As in Example 3.4.28 the sheaves associated to M_1, M_2, M_3 are torsion-free on their support and hence pure (but not of the same dimension). However $\mathcal{F}$ is not pure anymore because $M_3 \leq M$ defines a subsheaf with smaller-dimensional support. On the other hand $\mathcal{F}$ is still torsion-free on $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$, even if these have embedded components.

Example 3.4.30. Consider $R = \mathbb{K}[X,Y,Z]/\langle\, X^2 \,\rangle$ with $M = R/\langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle$ and $N = M \oplus M$, so that M describes the structure sheaf of a simple “cross” in a double plane. Computing the Fitting ideal, we find

$$\mathrm{Fitt}_0(N) = \mathrm{Ann}_R(N) \cdot \mathrm{Ann}_R(N) = \langle\, \bar{X}\bar{Y}\bar{Z}, \bar{Y}^2\bar{Z}^2 \,\rangle$$

and hence the primary decomposition

$$\langle\, X^2, XYZ, (YZ)^2 \,\rangle = \langle\, X^2, XY, Y^2 \,\rangle \cap \langle\, X^2, XZ, Z^2 \,\rangle$$

shows that $\mathcal{Z}_f(\widetilde{N})$ “recovers” the structure of $\mathrm{Spec}\, R$ and gives 2 triple lines. But it does not create a new embedded component. The sheaf is still pure of dimension 1 and torsion-free on its support since

$$\mathrm{Ass}_R(N) = \mathrm{Ass}\big(\mathrm{Ann}_R(N)\big) = \mathrm{Ass}\big(\mathrm{Fitt}_0(N)\big) = \big\{ \langle\, X, Y \,\rangle \ , \ \langle\, X, Z \,\rangle \big\} \ .$$

3.4.4 Remark: projective varieties in the literature

In the literature the condition "let $\mathcal{X}$ be a variety" often means that $\mathcal{X}$ is irreducible, reduced and sometimes even smooth. In particular the general convention is that projective varieties are locally given by integral domains. If this is the case we know that purity and torsion-freeness of a sheaf on $\mathcal{X}$ are equivalent and e.g. the following statements hold true:

[[62], Section 2, p.278]
"We work with coherent sheaves on an algebraic variety X [...] If $\dim \operatorname{supp} A = \dim X$ then A has pure support iff A has no torsion, because $\dim \operatorname{supp} B < \dim X$ is equivalent to B being a torsion sheaf."

[[45], Section 3.2.3, p.36]
"In case X is a projective curve, i.e. $d = 1$ [...] note that a sheaf $\mathcal{F}$ is pure of dimension 1 if and only if has no torsion subsheaf, i.e. $T_0(\mathcal{F}) = 0$."

We want to point out that with our definition of a variety, where is in general non-integral, non-reduced and/or not equidimensional, these assertions do not hold true anymore.
Consider Example E.4 with $\mathcal{F} = \widetilde{M}$ seen as a sheaf on its support $\mathcal{Z} = \operatorname{supp} \mathcal{F}$ (i.e. take $\mathcal{X} = \mathcal{Z}$, so that $\dim \mathcal{F} = \dim \mathcal{X} = 1$). $\mathcal{F}$ is coherent and (torsion-)free on $\mathcal{Z}$, but not pure of dimension 1. Moreover the sheaf $T_0(\mathcal{F})$ is non-zero (see Example 3.1.36) and supported in dimension 0, but torsion-free on $\mathcal{Z}$ as it is a subsheaf of the structure sheaf $\mathcal{O}_{\mathcal{Z}} = \mathcal{F}$. Alternatively we have

$$\operatorname{Ass}_R\big(\langle\, [\bar{X}]\,\rangle\big) = \big\{\, \langle\, \bar{X}, \bar{Y}, \bar{Z}\,\rangle\,\big\}\ ,$$
$$\operatorname{Ass}\big(\operatorname{Ann}_R(M)\big) = \big\{\, \langle\, \bar{X}, \bar{Y}\,\rangle\ ,\ \langle\, \bar{X}, \bar{Z}\,\rangle\ ,\ \langle\, \bar{X}, \bar{Y}, \bar{Z}\,\rangle\,\big\}\ .$$

Remark 3.4.31. Note that there exist integral curves that are not smooth, e.g. the nodal curve in $\mathbb{A}_2$ which is given as the vanishing set of the irreducible polynomial $f(X,Y) = X^3 + Y^3 + XY$, so the coordinate ring $\mathbb{K}[X,Y]/\langle\, f\,\rangle$ is an integral domain (hence so are all stalks), but the curve is singular because it intersects itself.

3.5 Final result and open questions

We are given the following question: $(*)$
Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring R and M a finitely generated R-module. Assume that the coherent $\mathcal{O}_{\mathcal{X}}$–module

$$\mathcal{F} = \widetilde{M}$$

is pure on $\mathcal{X}$. We denote $I = \operatorname{Fitt}_0(M)$ and $R' = R/I$, so that M can also be seen as a module over R' and the Fitting support of $\mathcal{F}$ is $\mathcal{Z} = V(I) \cong \operatorname{Spec} R'$. Is $\mathcal{F}$ torsion-free as an $\mathcal{O}_{\mathcal{Z}}$–module?

3.5.1 Torsion-freeness on different supports

In order to answer this question, we first need to compare torsion-freeness of a sheaf on different supports. Example 3.4.23 and Example 3.4.25 showed that torsion-freeness of $\mathcal{F}$ on $\mathcal{Z}_a(\mathcal{F})$ does not imply torsion-freeness on $\mathcal{Z}_f(\mathcal{F})$, and neither vice-versa. This is always due to the existence of embedded primes. Nevertheless such an implication exists if the support on which $\mathcal{F}$ is torsion-free does not have embedded components.

Proposition 3.5.1 (Leytem)**.** *Let $\mathcal{F} = \widetilde{M}$ for some finitely generated module M over a Noetherian ring R and $I, I' \subseteq \operatorname{Ann}_R(M)$ be two ideals defining different subscheme structures on $\operatorname{supp} \mathcal{F}$. Assume that $\mathcal{F}$ is torsion-free on $V(I)$ and that $\operatorname{Ass}(I)$ has no embedded primes. Then $\mathcal{F}$ is also torsion-free on $V(I')$.*

Proof. We will use Grothendieck's criterion and Proposition 3.2.2. As $\mathcal{F}$ is torsion-free on $V(I)$, we have

$$\operatorname{Ass}_{R/I}(M) \subseteq \operatorname{Ass}_R(R/I) = \operatorname{Ass}(I) \ .$$

Since $\operatorname{Ass}(I)$ has no embedded primes and the minimal primes in $\operatorname{Ass}(I)$ and $\operatorname{Ass}(I')$ are the same (Lemma 3.4.1), the set of associated primes $\operatorname{Ass}(I')$ can only be bigger (i.e. contain some embedded primes that are not associated primes of I). As the associated primes of M are moreover independent of the ring, we obtain

$$\operatorname{Ass}_{R/I'}(M) = \operatorname{Ass}_{R/I}(M) \subseteq \operatorname{Ass}(I) \subseteq \operatorname{Ass}(I') = \operatorname{Ass}_R(R/I') \ ,$$

and hence that $\mathcal{F} = \widetilde{M}$ is torsion-free on $V(I')$. □

Remark 3.5.2. This can e.g. be observed in Example 3.4.22. The ring R/I defining $\mathcal{Z}_a(\mathcal{F})$ does not have embedded primes and $\mathcal{F}$ is torsion-free on $\mathcal{Z}_a(\mathcal{F})$, thus $\mathcal{F}$ is also torsion-free on $\mathcal{Z}_f(\mathcal{F})$.

If we would consider the question $(*)$ for $\mathcal{Z}_a(\mathcal{F})$, the answer is immediately Yes since the annihilator support of a pure sheaf has equidimensional components (Proposition 3.2.12), hence purity implies that it is torsion-free (Theorem 3.1.17). In order to obtain the corresponding result for the Fitting support, we have to include Proposition 3.5.1 into our reasoning.

Theorem 3.5.3 (Leytem)**.** *The answer to the question $(*)$ is Yes, i.e. purity of a coherent sheaf implies torsion-freeness on its Fitting support.*

Proof. Let $I = \mathrm{Fitt}_0(M)$ and $J = \mathrm{Ann}_R(M)$. Since $\mathcal{F}$ is of pure dimension, its annihilator support $\mathcal{Z}_a(\mathcal{F})$ has equidimensional components (Proposition 3.2.12). Purity and Theorem 3.1.17 then imply that $\mathcal{F}$ is torsion-free on $V(J)$. Using Grothendieck's criterion and Proposition 3.5.1 this means

$$\mathrm{Ass}_{R/I}(M) = \mathrm{Ass}_{R/J}(M) \subseteq \mathrm{Ass}(J) \subseteq \mathrm{Ass}(I) = \mathrm{Ass}_R(R/I)$$

since $\mathcal{Z}_a(\mathcal{F}) = V(J)$ has no embedded primes and $\mathrm{Rad}(I) = \mathrm{Rad}(J)$. It follows that

$$\mathrm{Ass}_{R/I}(M) \subseteq \mathrm{Ass}_R(R/I) = \mathrm{Ass}\left(\mathrm{Fitt}_0(M)\right),$$

and hence $\mathcal{F}$ is torsion-free on $\mathcal{Z}_f(\mathcal{F}) = V(I)$ as well. □

Remark 3.5.4. As in the proof of Proposition 3.5.1 we see that the result does not just hold true for $I = \mathrm{Fitt}_0(M)$, but for any ideal $I \subseteq \mathrm{Ann}_R(M)$ such that $V(I) = \operatorname{supp} M$ as topological spaces (since all we need is that the minimal primes are the same).

Remark 3.5.5. The statement of Theorem 3.5.3 is rather obvious in the integral case. Our achievement was to prove that the result holds true for **any** Noetherian ring.

3.5.2 Message and open questions

Our message of Part I of this thesis is the following:

The behaviour of torsion and purity can be very counter-intuitive when there are embedded primes. But Purity always implies torsion-freeness of a sheaf on its support.

Open questions and first attempts

1) Is it possible that the Fitting support of a pure sheaf on a reduced scheme has embedded components ?

A priori it does not seem to be possible since an embedded component in $\mathcal{Z}_f(\mathcal{F})$ can only be created either by an existing embedded component of the scheme (see e.g. Example 3.4.18), which is not possible as it is reduced, or by a subsheaf of $\mathcal{F}$ which is supported in smaller dimension (as in Example 3.4.21), but there is none since $\mathcal{F}$ is pure.

2) Assume that $\mathcal{X}$ is reduced and $\mathcal{F}$ is torsion-free on $\mathcal{X}$. Is $\mathcal{F}$ torsion-free on its support ?

We already proved in Proposition 3.2.20 that if R is a reduced ring and a module M is torsion-free over R, then M is also torsion-free over R/I for $I \subseteq \mathrm{Ann}_R(M)$. However we are interested in knowing whether the corresponding sheaf is torsion-free as well. From Corollary 3.2.23 we know for example that this holds true if the ring R/I has no embedded primes.
Proposition 1.3.3 and Theorem 2.5.8 already imply that $\mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(R)$ since R is reduced and thus has no embedded primes. Moreover torsion-freeness of M over R/I says that every $P \in \mathrm{Ass}_R(M)$ is contained in some $Q \in \mathrm{Ass}(I)$. Torsion on $\mathcal{Z} = V(I)$ can only appear if such a P is strictly contained in an embedded prime Q. So the question reduces to:
If R is reduced, is it possible to create embedded primes in $\mathrm{Ass}(I) = \mathrm{Ass}_R(R/I)$ without creating them in $\mathrm{Ass}_R(M)$? We conjecture that the answer is negative.

In the case one tries to construct a counter-example note that we have the following constraint:
Denote $\mathcal{F} = \widetilde{M}$. If $\mathcal{X} = \operatorname{Spec} R$ has equidimensional components, then torsion-freeness of $\mathcal{F}$ on $\mathcal{X}$ implies that $\mathcal{F}$ is pure (Theorem 3.1.17 and Corollary 3.1.25) and hence that it is torsion-free on $\mathcal{Z}$ by Remark 3.5.4. So in order to obtain a counter-example (if there is one), one needs a reduced scheme $\mathcal{X}$ whose components have different dimensions.

3) Is torsion-freeness and torsionlessness (see Sections C.2 and C.4) of a finitely generated module equivalent when there are no embedded primes, or at least if the ring is reduced ?

Part II

Singular sheaves in the fine Simpson moduli spaces of one-dimensional sheaves on $\mathbb{P}_2$

Chapter 4

Construction and examples of the Simpson moduli spaces

This chapter is a reminder of the construction and properties of the moduli spaces $M_P(\mathcal{X})$ of semistable sheaves on a projective variety $\mathcal{X}$. They have initially been introduced by Gieseker and Maruyama in 1977 and generalized by Simpson in 1994. A short historical note is given in Section 4.2.5. We are especially interested in semistable sheaves on $\mathbb{P}_2$ with linear Hilbert polynomial. For this we are repeating classical results from Simpson [65], Maican [48], Le Potier [47], Freiermuth [23] and Trautmann [25]. While doing so we also include a short initiation to moduli spaces and representability of moduli functors.
From this point of view we are giving a basic introduction to the theory of Simpson moduli spaces; the chapter collects already known results and examples which should be understood in order to deal with the problems that are discussed in Chapter 5 later on. In particular we review the situation of $M_{am+b}(\mathbb{P}_2)$ for $a \leq 3$, where we have $M_{m+1} \cong \mathbb{P}_2$, $M_{2m+1} \cong \mathbb{P}_5$ and $M_{3m+1} \cong \mathfrak{U}(3)$, the universal cubic curve on $\mathbb{P}_2$. The reader who is familiar with the definition and construction of the moduli spaces $M_P(\mathcal{X})$, as well as with singularities of 1-dimensional sheaves with "small" Hilbert polynomial, may only consider Section 4.4; in this one we define the notion of a singular sheaf as in [47] and use the results of Part I to explain why "almost all" stable sheaves in $M_{am+b}(\mathbb{P}_2)$ are vector bundles on a smooth curve of degree a (Corollary 4.4.21).

We include this chapter at this point rather than in a separate appendix so that the reader can get a better idea of the objects we are going to study. Beside introducing moduli spaces, we also want to illustrate some notions and objects in easy situations which we will introduce more generally in Section 5.1, such as parameter spaces and geometric quotients by non-reductive groups. Moreover we explain some tools in Section 4.5 which allow to compute locally free resolutions of coherent sheaves on $\mathbb{P}_2$, such as syzygies and Koszul resolutions. Finally we study the case of M_{3m+1} in more detail and reprove that the subvariety of singular sheaves in M_{3m+1} is given by the universal singular locus of $\mathfrak{U}(3)$, which is smooth, irreducible and of codimension 2.

4.1 Preliminaries

We start with some preliminary results. Let us first recall the following facts.

Definition 4.1.1. A *projective scheme* over $\mathbb{K}$ is a scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ that is locally of finite type over $\operatorname{Spec}\mathbb{K}$ and which can be embedded as a closed subscheme into some projective space $\mathbb{P}^n_{\mathbb{K}}$. In particular this implies that $\mathcal{X}$ is Noetherian as it is compact and any point has an affine open neighborhood $U \subseteq \mathcal{X}$ such that $\mathcal{O}_{\mathcal{X}}(U)$ is a finitely generated $\mathbb{K}$-algebra (i.e. isomorphic to a quotient of a polynomial ring). The closed embedding $\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathbb{K}}$ moreover gives rise to a very ample invertible sheaf $\mathcal{O}(1)$ on $\mathcal{X}$ by pulling back the twisting sheaf $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(1)$ of Serre.

Theorem 4.1.2 (Grothendieck's Vanishing Theorem)**.** [[35], III, Th. 2.7, p.208]
Let X be a Noetherian topological space of finite dimension n.
Then $H^i(X, \mathcal{F}) = \{0\}$ for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X.

Theorem 4.1.3 (Serre's Theorem A)**.** [[64], p.36]
Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a projective scheme over $\mathbb{K}$ with very ample sheaf $\mathcal{O}(1)$.
If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$, then there exists an integer $n_0 \in \mathbb{Z}$ such that $\mathcal{F}(n)$ is generated by global sections for all $n \geq n_0$.

Theorem 4.1.4 (Serre's Theorem B)**.** [[35], Th. 5.2, p.228] and [[64], Th. p.36] *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a projective scheme over $\mathbb{K}$ with very ample sheaf $\mathcal{O}(1)$ and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. Then the cohomology spaces $H^i(\mathcal{X}, \mathcal{F})$ are finite-dimensional vector spaces over $\mathbb{K}$ for all $i \geq 0$. Moreover there exists an integer $m_0 \in \mathbb{Z}$ such that $\mathcal{F}(m)$ is acyclic[1] for all $m \geq m_0$.*

Remark 4.1.5. In analytic geometry Serre's Theorem B is also called the Finiteness Theorem of Cartan-Serre, see e.g. [[22], Th. 9.1.1, p.257]

4.1.1 Hilbert polynomials

Definition 4.1.6. Let $\mathcal{X}$ be a projective scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. The *Euler characteristic* of $\mathcal{F}$, denoted by $\chi(\mathcal{X}, \mathcal{F})$, is the integer defined as

$$\chi(\mathcal{X}, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \cdot h^i(\mathcal{F}) \ , \tag{4.1}$$

where $h^i(\mathcal{F}) = \dim_{\mathbb{K}} \big(H^i(\mathcal{X}, \mathcal{F}) \big)$. Theorem 4.1.2 and Theorem 4.1.4 ensure that this is well-defined.

Definition 4.1.7. Let $\mathcal{X}$ be a projective scheme over $\mathbb{K}$ and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$, $\mathcal{F} \neq 0$. We denote $d = \dim \mathcal{F}$ the dimension of the support of $\mathcal{F}$ as a closed topological subspace of $\mathcal{X}$. It can be shown, see e.g. [[68], 18.6.1, p.483] or [[64], Prop. p.38], that the Euler characteristic of the twisted sheaf

$$\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}(m) = \mathcal{F} \otimes \mathcal{O}(1)^{\otimes m}$$

is a polynomial expression in m of degree d. Thus we can define the *Hilbert polynomial* $P_{\mathcal{F}}$ of $\mathcal{F}$ by the formula

$$P_{\mathcal{F}}(m) = \chi\big(\mathcal{X}, \mathcal{F}(m)\big) \in \mathbb{Q}[m]$$

for $m \in \mathbb{Z}$. This is a numerical polynomial (i.e. a polynomial with rational coefficients that takes integer values on integers) with $\deg P_{\mathcal{F}} = d$. In particular, $P_{\mathcal{F}} = 0$ if and only if $\mathcal{F} = 0$ (in which case $d = -1$).

[1] We recall that the definition of an acyclic sheaf is given in Section 1.1.2.

Remark 4.1.8. Whenever we mention the Hilbert polynomial of an $\mathcal{O}_{\mathcal{X}}$–module on a projective scheme, the sheaf is assumed to be coherent otherwise its Hilbert polynomial may not exist because of infinite-dimensional vector spaces occurring in (4.1).

Serre's Theorem B and [[11], 32.33.15] imply that

$$P_{\mathcal{F}}(m) = h^0\big(\mathcal{F}(m)\big) = \dim_{\mathbb{K}} \Gamma\big(\mathcal{X}, \mathcal{F}(m)\big) \quad \text{for} \quad m >> 0 \ .$$

In particular it shows that the leading coefficient of $P_{\mathcal{F}}$ is always > 0 as it is non-zero by definition (if $\mathcal{F} \neq 0$) and dimensions of vector spaces are non-negative. The Euler characteristic is moreover additive in exact sequences, hence so are Hilbert polynomials (since the sheaf $\mathcal{O}(m)$ is invertible and thus flat, see Proposition C.3.12). More generally:

Proposition 4.1.9. [[68], 18.4.A, p.472]
If

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \ldots \longrightarrow \mathcal{F}_n \longrightarrow 0$$

is an exact sequence of coherent $\mathcal{O}_{\mathcal{X}}$–modules, then $\sum_{i=1}^{n} (-1)^i \cdot P_{\mathcal{F}_i} = 0$. In particular we have

$$P_{\mathcal{F} \oplus \mathcal{G}} = P_{\mathcal{F}} + P_{\mathcal{G}} \quad , \quad P_{\mathcal{F}/\mathcal{F}'} = P_{\mathcal{F}} - P_{\mathcal{F}'} \quad , \quad P_{\mathcal{F}(k)}(m) = P_{\mathcal{F}}(m+k) \qquad (4.2)$$

for a coherent subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ and for all $k \in \mathbb{Z}$.

Example 4.1.10. [[11], 32.33.14] and [[68], p.484]
Let $\mathcal{X} = \mathbb{P}^n_{\mathbb{K}}$ be the n-dimensional projective space over $\mathbb{K}$. Then we have $P_{\mathcal{O}_{\mathcal{X}}}(m) = \binom{m+n}{n}$ and for all $k \in \mathbb{Z}$,

$$P_{\mathcal{O}(k)}(m) = \chi\big(\mathbb{P}^n_{\mathbb{K}}, \mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(k)(m)\big) = \binom{m+k+n}{n} \ . \qquad (4.3)$$

We finish the preliminaries by the following important lemma, which states that a twist $\mathcal{F} \mapsto \mathcal{F}(k)$ for some $k \in \mathbb{Z}$ does not change sheaves which are supported in dimension 0.

Lemma 4.1.11. *If* $\mathcal{G} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ *has* 0*-dimensional support, then*

$$\mathcal{G}(k) \cong \mathcal{G} \ , \quad \forall\, k \in \mathbb{Z} \ .$$

Proof. Denote $Z = \operatorname{supp} \mathcal{G}$; as this is closed of dimension 0, we can rewrite $\mathcal{G}$ as a finite sum of skyscrapers

$$\mathcal{G} = \bigoplus_{x \in Z} \operatorname{Sky}_x(\mathcal{G}_x) \ .$$

Since the tensor product commutes with direct sums, it thus suffices to show that twisting does not change a skyscraper sheaf. Let $\mathcal{G} = \operatorname{Sky}_x(M)$ for some (closed) point $x \in \mathcal{X}$ and an $\mathcal{O}_{\mathcal{X},x}$–module M. By definition this is equal to $\mathcal{G} = i_* M$ where $i : \{x\} \hookrightarrow \mathcal{X}$ is the inclusion. We want to show that $\mathcal{G} \otimes \mathcal{O}_\mathcal{X}(k) \cong \mathcal{G}$. Both sheaves already have the same stalks. Hence it remains to show that there exists a morphism between them. For this we use that there is a canonical morphism

$$\begin{aligned} \mathcal{G} \otimes \mathcal{O}_\mathcal{X}(k) = (i_* M) \otimes \mathcal{O}_\mathcal{X}(k) \xrightarrow{\exists} i_*\big(M \otimes_{\mathcal{O}_{\mathcal{X},x}} i^* \mathcal{O}_\mathcal{X}(k)\big) \ &\cong \ i_*\big(M \otimes_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_\mathcal{X}(k)_x\big) \\ &\cong \ i_*\big(M \otimes_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X},x}\big) \ \cong \ i_* M = \mathcal{G} \ . \end{aligned}$$

□

4.1.2 Semistability and s-equivalence

Consider again a projective scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ over $\mathbb{K}$ and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$, $\mathcal{F} \neq 0$ with $d = \dim \mathcal{F} \geq 0$.

Lemma 4.1.12. [[38], 1.2.1, p.10] and [[65], p.55]
The Hilbert polynomial $P_\mathcal{F}$ *can uniquely be written in the form*

$$P_\mathcal{F}(m) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \cdot \frac{m^i}{i!} \tag{4.4}$$

for some rational coefficients $\alpha_i(\mathcal{F}) \in \mathbb{Q}$*,* $i \in \{0, \ldots, d\}$ *such that* $\alpha_d(\mathcal{F}) \in \mathbb{N}$ *and* $\alpha_0(\mathcal{F}) = \chi(\mathcal{X}, \mathcal{F})$*. If* $\mathcal{F}' \subseteq \mathcal{F}$ *is a coherent subsheaf and* $\mathcal{G} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ *with* $d = \dim \mathcal{G} = \dim \mathcal{F}'$*, additivity in exact sequences shows that*

$$\alpha_i(\mathcal{F} \oplus \mathcal{G}) = \alpha_i(\mathcal{F}) + \alpha_i(\mathcal{G}) \qquad \textit{and} \qquad \alpha_i(\mathcal{F}/\mathcal{F}') = \alpha_i(\mathcal{F}) - \alpha_i(\mathcal{F}') \ .$$

Definition 4.1.13. [[38], 1.2.1, p.10] and [[65], p.55]
The leading coefficient $\alpha_d(\mathcal{F})$ of $P_{\mathcal{F}}$ is called the *multiplicity* of $\mathcal{F}$. We also define the *slope* as the quotient $\mu(\mathcal{F}) = \frac{\alpha_{d-1}(\mathcal{F})}{\alpha_d(\mathcal{F})}$ and the *reduced Hilbert polynomial* of $\mathcal{F}$ by

$$p_{\mathcal{F}} = \frac{P_{\mathcal{F}}}{\alpha_d(\mathcal{F})} \,.$$

Notation 4.1.14. If f and g are polynomials we write $f < g$, resp. $f \leq g$ if there exists an integer $m_0 \in \mathbb{N}$ such that $f(m) < g(m)$, resp. $f(m) \leq g(m)$ for all $m \geq m_0$, i.e. if the inequality is satisfied for sufficiently large m.

Definition 4.1.15. [[38], 1.2.4, p.11] , [[48], 2.1, p.5-6] and [[65], p.55]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ with $d = \dim \mathcal{F}$.
$\mathcal{F}$ is called *stable*, resp. *semistable* of dimension d if
1) $\mathcal{F}$ is of pure dimension d, i.e. $\dim \mathcal{F}' = d$ for any proper non-zero coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$.
2) Any proper non-zero coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ satisfies $p_{\mathcal{F}'} < p_{\mathcal{F}}$, resp. $p_{\mathcal{F}'} \leq p_{\mathcal{F}}$.[2]
Note that the notion of (semi)stability depends on the very ample sheaf $\mathcal{O}(1)$, i.e. on the chosen embedding $\mathcal{X} \hookrightarrow \mathbb{P}^n_{\mathbb{K}}$.

Remark 4.1.16. cf. [[38], 1.2.2, p.11 & 1.2.11, p.14]
The definition of the slope is taken from [65] and differs from the one in [38], where it is given by

$$\begin{aligned}\widetilde{\mu}(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} &= \frac{\alpha_d(\mathcal{O}_{\mathcal{X}})}{\alpha_d(\mathcal{F})} \cdot \Big(\alpha_{d-1}(\mathcal{F}) - \frac{\alpha_d(\mathcal{F})}{\alpha_d(\mathcal{O}_{\mathcal{X}})} \cdot \alpha_{d-1}(\mathcal{O}_{\mathcal{X}}) \Big) \\ &= \alpha_d(\mathcal{O}_{\mathcal{X}}) \cdot \mu(\mathcal{F}) - \alpha_{d-1}(\mathcal{O}_{\mathcal{X}})\end{aligned}$$

if $d = \dim \mathcal{F} = \dim \mathcal{X}$, otherwise the rank of $\mathcal{F}$ is zero. But both give equivalent notions for μ-(semi)stability ; one says that a sheaf is *μ-(semi)stable* if it satisfies certain torsion conditions[3] and $\mu(\mathcal{F}')\,(\leq)\,\mu(\mathcal{F})$ for[4] any proper non-zero coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$. As $\alpha_d(\mathcal{O}_{\mathcal{X}}) > 0$, both notions give the same definition.

[2] This is the notion of G-stability in the sense of Giesecker. Here we do not consider the related notion of μ-stability (sometimes also called M-stability in the sense of Mumford), but we will briefly see in Section 4.3.1 that they are equivalent in the case of linear Hilbert polynomials.

[3] that are not of our interest right now

[4] The notation $(\leq)$ means that we take $\leq$ for semistability and $<$ in the stable case.

Remark 4.1.17. [[38], 1.2.5, p.11]
An equivalent definition for (semi)stability would have been to say that a sheaf $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ is (semi)stable if and only if

$$\alpha_d(\mathcal{F}) \cdot P_{\mathcal{F}'} \,(\leq)\, \alpha_d(\mathcal{F}') \cdot P_{\mathcal{F}} \tag{4.5}$$

for all proper coherent subsheaves $\mathcal{F}' \subsetneq \mathcal{F}$. Condition 2) in Definition 4.1.15 is then satisfied. Applying the inequality to the coherent subsheaf $T_{d-1}(\mathcal{F}) \subset \mathcal{F}$, which satisfies $\alpha_d\big(T_{d-1}(\mathcal{F})\big) = 0$ then gives $P_{T_{d-1}(\mathcal{F})} \leq 0$, hence $T_{d-1}(\mathcal{F}) = 0$. But this exactly means that $\mathcal{F}$ is pure of dimension d, so (4.5) includes purity for free.

Definition 4.1.18. [[38], 1.5.1, p.23] , [[67], 12.3, p.78-79] and [[48], 2.4, p.7]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ be semistable with $d = \dim \mathcal{F}$. A *Jordan-Hölder filtration* (sometimes also called a *stable filtration*) of $\mathcal{F}$ is a filtration

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_k = \mathcal{F} \tag{4.6}$$

for some $k \in \mathbb{N}$ by coherent subsheaves $\mathcal{F}_i \subset \mathcal{F}$ such that all $\mathcal{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ with $i \geq 1$ are stable sheaves of dimension d and have the same reduced Hilbert polynomial as $\mathcal{F}$, i.e. $p_{\mathcal{Q}_i} = p_{\mathcal{F}}$ for all $i \in \{1, \ldots, k\}$.

Remark 4.1.19. cf. [[38], 1.5.1, p.23]
This implies that $\mathcal{F}_1 = \mathcal{Q}_1$ is stable, $\mathcal{F}_i$ with $i \geq 2$ is semistable (otherwise $\mathcal{F}$ would not be) and all $\mathcal{F}_i$ have reduced Hilbert polynomial $p_{\mathcal{F}}$. Indeed we have $p_{\mathcal{F}_1} = p_{\mathcal{F}}$ and $P_{\mathcal{Q}_i} = P_{\mathcal{F}_i} - P_{\mathcal{F}_{i-1}}$, hence by induction

$$\alpha_d(\mathcal{Q}_i) \cdot p_{\mathcal{Q}_i} = \alpha_d(\mathcal{F}_i) \cdot p_{\mathcal{F}_i} - \alpha_d(\mathcal{F}_{i-1}) \cdot p_{\mathcal{F}_{i-1}} \quad \Rightarrow \quad p_{\mathcal{F}_i} = \frac{\alpha_d(\mathcal{Q}_i) + \alpha_d(\mathcal{F}_{i-1})}{\alpha_d(\mathcal{F}_i)} \cdot p_{\mathcal{F}}$$

with $p_{\mathcal{Q}_i} = p_{\mathcal{F}}$ and $\alpha_d(\mathcal{Q}_i) = \alpha_d(\mathcal{F}_i) - \alpha_d(\mathcal{F}_{i-1})$, thus $p_{\mathcal{F}_i} = p_{\mathcal{F}}$ for all $i \geq 2$.

Definition 4.1.20. [[38], 1.5.2, p.23] and [[67], 12.3.1, p.79]
For a semistable sheaf $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ with a Jordan-Hölder filtration as in (4.6) we define the *graded sheaf of $\mathcal{F}$* by

$$gr(\mathcal{F}) := \bigoplus_{i=1}^{k} \big(\mathcal{F}_i/\mathcal{F}_{i-1}\big) \, .$$

Proposition 4.1.21. [[38], 1.5.2, p.23-24] and [[67], 12.3.1, p.79-80]
1) Jordan-Hölder filtrations always exist, but don't need to be unique.
2) All quotients $\mathcal{F}/\mathcal{F}_i$ with $i \in \{1, \dots, k-1\}$ in (4.6) are semistable of dimension d with reduced Hilbert polynomial $p_{\mathcal{F}}$.
3) The graded sheaf $gr(\mathcal{F})$ is independent of the chosen filtration of $\mathcal{F}$ (in the sense that different filtrations will give isomorphic graded sheaves).

Thus the following equivalence relation is well-defined.

Definition 4.1.22. [[38], 1.5.3, p.24] , [[67], 12.3.2, p.80] and [[49], p.4]
Two semistable sheaves $\mathcal{F}, \mathcal{G} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ with the same reduced Hilbert polynomial are called *s-equivalent* (*stably equivalent*) if their graded sheaves are isomorphic, i.e.

$$\mathcal{F} \sim \mathcal{G} \;\Leftrightarrow\; gr(\mathcal{F}) \cong gr(\mathcal{G}) \;.$$

This means that there exists a permutation σ of the index set $\{1, \dots, k\}$ such that $\mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathcal{G}_{\sigma(i)}/\mathcal{G}_{\sigma(i)-1}$, $\forall\, i$.

Example 4.1.23. [[67], 12.3.2, p.80]
Obviously two isomorphic sheaves are s-equivalent (since they have isomorphic filtrations). The converse is false ; an illustration of this fact is e.g. the following. Assume that a semistable sheaf $\mathcal{F}$ is given by an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \;, \tag{4.7}$$

where $\mathcal{F}', \mathcal{G} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ are stable. Then $0 \subsetneq \mathcal{F}' \subsetneq \mathcal{F}$ is a Jordan-Hölder filtration of $\mathcal{F}$ since $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}' \cong \mathcal{G}$ are stable and we get

$$gr(\mathcal{F}) \cong \mathcal{F}' \oplus \mathcal{G} = gr(\mathcal{F}' \oplus \mathcal{G}) \;.$$

Thus any extension $\mathcal{F}$ as in (4.7) is s-equivalent to $\mathcal{F}' \oplus \mathcal{G}$. But there are a lot of such extensions that do not split (i.e. which are not isomorphic to the direct sum) ; a concrete example will be given in Example F.1.9.

Remark 4.1.24. However if all sheaves are stable, then being s-equivalent means being isomorphic since stable sheaves have the trivial Jordan-Hölder filtration $0 \subsetneq \mathcal{F}$, and hence $gr(\mathcal{F}) = \mathcal{F}$.

4.1.3 Fibers and flatness

Definition 4.1.25. We denote by $\texttt{Sch}(\mathbb{K})$ the category of Noetherian schemes that are of finite type over $\operatorname{Spec}\mathbb{K}$. Hence its objects are schemes that can be covered by finitely many affine schemes which are spectra of Noetherian rings that are finitely generated $\mathbb{K}$-algebras.

Let $(\mathcal{X},\mathcal{O}_\mathcal{X})$ be a projective scheme over $\mathbb{K}$, $S\in\texttt{Sch}(\mathbb{K})$ and assume that a morphism $f:\mathcal{X}\to S$ is given (one also says that $\mathcal{X}$ is a *scheme over S*). For every closed point $s\in S$, let $\kappa(s)=\mathcal{O}_{S,s}/\mathfrak{M}_s$ be its residue field. The importance of s being closed is made clear by the results below. In particular, closed points are respected by morphisms.

Lemma 4.1.26. [[29], 3.33, p.79-80]
Let $(\mathcal{Y},\mathcal{O}_\mathcal{Y})$ be a scheme that is locally of finite type over some field $\mathbb{K}$ and $y\in\mathcal{Y}$ with residue field $\kappa(y)$. Then y is a closed point if and only if the field extension $\mathbb{K}\hookrightarrow\kappa(y)$ is finite.
In particular if $\mathbb{K}$ is algebraically closed, then it has no non-trivial finite algebraic extensions, thus y is closed if and only if $\kappa(y)\cong\mathbb{K}$.

Lemma 4.1.27. [[53], 516766]
Let $(\mathcal{Y},\mathcal{O}_\mathcal{Y})$ and $(\mathcal{Z},\mathcal{O}_\mathcal{Z})$ be schemes that are locally of finite type over some field $\mathbb{K}$ and $f:\mathcal{Y}\to\mathcal{Z}$ a morphism of $\mathbb{K}$-schemes. If $y\in\mathcal{Y}$ is a closed point, then $f(y)\in\mathcal{Z}$ is closed as well.

Proof. The morphism $f:\mathcal{Y}\to\mathcal{Z}$ induces a homomorphism of local rings

$$f_y^\# \;:\; \mathcal{O}_{\mathcal{Z},f(y)}\to\mathcal{O}_{\mathcal{Y},y}$$

that preserves the maximal ideals, i.e. $f_y^\#\big(\mathfrak{M}_{f(y)}\big)\subseteq\mathfrak{M}_y$. Hence it induces a morphism of fields

$$\mathcal{O}_{\mathcal{Z},f(y)}/\mathfrak{M}_{f(y)}\longrightarrow\mathcal{O}_{\mathcal{Y},y}/\mathfrak{M}_y \quad\Leftrightarrow\quad \kappa\big(f(y)\big)\longrightarrow\kappa(y)$$

which is thus injective. Since y is closed, the extension $\mathbb{K}\hookrightarrow\kappa(y)$ is finite, hence so is $\mathbb{K}\hookrightarrow\kappa\big(f(y)\big)$. If $\mathbb{K}$ is algebraically closed, this can be shown easier by $\mathbb{K}\hookrightarrow\kappa\big(f(y)\big)\hookrightarrow\kappa(y)\cong\mathbb{K}$, thus $f(y)$ is closed as well. □

Remark 4.1.28. So for a projective scheme $\mathcal{X}$ over S, we have $\kappa(s) \cong \mathbb{K}$ for all closed point $s \in S$ and $\{s\} = \operatorname{Spec} \kappa(s)$ defines a closed subscheme of S with inclusion morphism $i_s : \{s\} \hookrightarrow S$.

Definition 4.1.29. [[35], II, p.89] and [[38], 2.1, p.34]
Let $f : \mathcal{X} \to S$ be a morphism. The *fiber* of f over s is defined as the fiber product $\mathcal{X}_s := \mathcal{X} \times_S \operatorname{Spec} \kappa(s)$, which is a scheme over $\kappa(s)$.

$$\begin{array}{ccccc} \mathcal{X}_s & \xrightarrow{\pi_s} & \mathcal{X} & & \\ \downarrow & & \downarrow{\scriptstyle f} & \searrow & \\ \{s\} & \xrightarrow{i_s} & S & \longrightarrow & \operatorname{Spec} \mathbb{K} \end{array} \tag{4.8}$$

For $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ we denote the *restriction* of $\mathcal{F}$ to the fiber $\mathcal{X}_s$ by[5]

$$\mathcal{F}|_s := \pi_s^* \mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}_s})$$

Proposition 4.1.30. [[35], II, Ex. 3.10, p.92]
There is a homeomorphism of topological spaces $\mathcal{X}_s \cong f^{-1}(\{s\})$. In particular $\mathcal{X}_s$ can be seen as a closed subscheme of $\mathcal{X}$, hence it is also a projective scheme over $\mathbb{K}$.

Example 4.1.31. Consider the trivial product $\mathcal{X}_S := \mathcal{X} \times_{\mathbb{K}} S$, which is a scheme over S. Then the fiber over s for a closed point $s \in S$ is $(\mathcal{X}_S)_s \cong \mathcal{X}$ since

$$\begin{array}{ccccc} (\mathcal{X} \times_{\mathbb{K}} S)_s & \longrightarrow & \mathcal{X} \times_{\mathbb{K}} S & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \{s\} & \longrightarrow & S & \longrightarrow & \operatorname{Spec} \mathbb{K} \end{array}$$

where $(\mathcal{X} \times_{\mathbb{K}} S)_s = (\mathcal{X} \times_{\mathbb{K}} S) \times_S \operatorname{Spec} \kappa(s) \cong \mathcal{X} \times_{\mathbb{K}} \operatorname{Spec} \mathbb{K} \cong \mathcal{X}$.

Definition 4.1.32. [[67], 2.3, p.10] and [[38], 2.1.1, p.34-35]
The morphism $f : \mathcal{X} \to S$ gives rise to a homomorphism $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{\mathcal{X},x}$ of local rings for all $x \in \mathcal{X}$, turning every module over $\mathcal{O}_{\mathcal{X},x}$ also into a module over $\mathcal{O}_{S,f(x)}$. A sheaf $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ is called *flat over S* (or *S-flat*) if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$–module, $\forall\, x \in \mathcal{X}$.

[5] Here we do not use the standard notation $\mathcal{F}_s$ of the literature in order to avoid confusions with the stalks of $\mathcal{F}$.

Flatness is a property which ensures that the fibers and the restrictions $\mathcal{F}|_s$ behave in some sense "continuously". More precisely we have the following important results.

Proposition 4.1.33. [[67], 2.3, p.10 & 17.6, p.98]
Let $T \in \mathtt{Sch}(\mathbb{K})$ with a morphism $f : T \to S$ and consider the fiber product

$$\begin{array}{ccc} \mathcal{X} \times_S T & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

1) If $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ is S-flat on $\mathcal{X}$, then $\pi^*\mathcal{F}$ is T-flat on $\mathcal{X} \times_S T$.
2) Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_{\mathcal{X}}$–modules and assume that $\mathcal{H}$ is S-flat. Then the sequence of pullbacks

$$0 \longrightarrow \pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{G} \longrightarrow \pi^*\mathcal{H} \longrightarrow 0$$

is an exact sequence of sheaves on $\mathcal{X} \times_S T$. In particular in the setting of (4.8), the sequence of restrictions $0 \to \mathcal{F}|_s \to \mathcal{G}|_s \to \mathcal{H}|_s \to 0$ is exact on $\mathcal{X}_s$ for all closed points $s \in S$.

Theorem 4.1.34. [[35], III, Thm. 9.9, p.261-263] , [[67], 2.5 & 2.5.1, p.11-12]
Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_{\mathbb{K}} \times_{\mathbb{K}} S$, considered as a scheme over S, and assume that $\mathcal{F}$ is flat over S. Then the function

$$s \longmapsto P_{\mathcal{F}|_s} = \chi\big(\mathbb{P}^n_{\mathbb{K}}, \mathcal{F}|_s(m)\big)$$

with values in numerical polynomials is locally constant on the closed points of S.[6] *If S is moreover connected, the Hilbert polynomial of the restriction $\mathcal{F}|_s$ on $\mathbb{P}^n_{\mathbb{K}}$ is independent of $s \in S$ closed.*[7]

[6] It is important to only consider the closed points, otherwise the fiber over s may not be a projective scheme.

[7] Hartshorne [35] only proves the statement in the case where S is an integral scheme, so it is irreducible and hence connected; Trautmann generalized the proof in the non-reduced case. On the other hand [[38], 2.1.2, p.35] states this result for any morphism $f : \mathcal{X} \to S$, but only gives a reference to Hartshorne where it is not proven in such a generality.

Proposition 4.1.35. [[67], 2.5.2, p.12]
The converse of Theorem 4.1.34 holds true if S is a reduced scheme, i.e. if S is reduced and $s \mapsto P_{\mathcal{F}|_s}$ is a constant function on the closed points of S, then $\mathcal{F}$ is flat over S.

4.2 The Simpson moduli functor

Now we are able to define the main objects that we will be working with in the following. We denote by $\mathtt{Sch}_c(\mathbb{K})$ the full subcategory of connected Noetherian schemes of finite type over $\operatorname{Spec}\mathbb{K}$. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a projective scheme over $\mathbb{K}$, $S \in \mathtt{Sch}_c(\mathbb{K})$ and denote $\mathcal{X}_S = \mathcal{X} \times_{\mathbb{K}} S$ with the projection $\pi_S : \mathcal{X}_S \to S$.

4.2.1 Definition and properties

Definition 4.2.1. cf. [[65], p.58][8]
Fix a polynomial $P \in \mathbb{Q}[m]$ of degree d. We say that $\mathcal{E}$ is a *semistable sheaf on $\mathcal{X}_S/S$ with Hilbert polynomial P* if $\mathcal{E}$ is a coherent sheaf on $\mathcal{X}_S$ that is flat over S and such that for each closed point $s \in S$, the restriction $\mathcal{E}|_s$ on the fiber $(\mathcal{X}_S)_s \cong \mathcal{X}$ is a semistable sheaf of pure dimension d and Hilbert polynomial P. This is well-defined as we know from Theorem 4.1.34 that Hilbert polynomials in the fibers are independent of the closed point s if S is connected.

Definition 4.2.2. [[65], p.65] , [[67], 13.1, p.83-84] and [[38], 4.1, p.90]
For $S \in \mathtt{Sch}_c(\mathbb{K})$ and a numerical polynomial $P \in \mathbb{Q}[m]$ of degree d we define

$$\mathcal{M}_P(S) := \Big\{ [\mathcal{E}] \;\Big|\; \mathcal{E} \text{ is a semistable sheaf on } \mathcal{X}_S/\,S \\ \text{of pure dimension } d \text{ and Hilbert polynomial } P \Big\}\,, \qquad (4.9)$$

where the equivalence class $[\mathcal{E}]$ is defined as follows: two S-flat sheaves $\mathcal{E}$ and $\mathcal{F}$ on $\mathcal{X}_S$ are equivalent, $\mathcal{E} \sim \mathcal{F}$, if there exists a line bundle (i.e. an invertible sheaf) $\mathcal{L}$ on S such that $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^* \mathcal{L}$.

[8] The definition in [65] is more general as it considers the case of arbitrary projective schemes $\mathcal{X} \to S$. We restrict ourselves to the case of $\mathcal{X}_S \to S$. The upcoming definition of $\mathcal{M}_P(S)$ however is exactly the same as in [65].

So in other words $[\mathcal{E}] \in \mathcal{M}_P(S)$ is a class of a coherent sheaf on $\mathcal{X}_S$ that is flat over S and such that for every closed point $s \in S$, the restriction $\mathcal{E}|_s$ on $\mathcal{X}$ is a semistable sheaf of pure dimension d and Hilbert polynomial P. Elements in $\mathcal{M}_P(S)$ are also called *families over S*.

Lemma 4.2.3. [[67], p.84] and [[38], 4.1, p.90]
$\sim$ is an equivalence relation on $\mathtt{Coh}(\mathcal{O}_{\mathcal{X}_S})$ and if $\mathcal{E} \sim \mathcal{F}$, then $\mathcal{E}|_s \cong \mathcal{F}|_s$ for all closed points $s \in S$.

Proof. We use that pullbacks of invertible sheaves are invertible and commute with tensor products.
− reflexive: $\mathcal{E} \sim \mathcal{E}$ because $\mathcal{E} \otimes \pi_S^*\mathcal{O}_S \cong \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_S} \cong \mathcal{E}$.
− symmetric: $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^*\mathcal{L}$
$\Rightarrow\ \mathcal{E} \otimes \pi_S^*(\mathcal{L}^*) \cong \mathcal{F} \otimes \pi_S^*\mathcal{L} \otimes \pi_S^*(\mathcal{L}^*) \cong \mathcal{F} \otimes \pi_S^*(\mathcal{L} \otimes \mathcal{L}^*) \cong \mathcal{F} \otimes \pi_S^*\mathcal{O}_S \cong \mathcal{F}$.
− transitive: $\mathcal{E} \sim \mathcal{F}$ and $\mathcal{F} \sim \mathcal{G}$
$\Rightarrow\ \mathcal{E} \cong \mathcal{G} \otimes \pi_S^*\mathcal{L}' \otimes \pi_S^*\mathcal{L} \cong \mathcal{F} \otimes \pi_S^*(\mathcal{L}' \otimes \mathcal{L})\ \Rightarrow\ \mathcal{E} \sim \mathcal{G}$.
These properties holds since $\mathrm{Pic}(\mathcal{X}_S)$ is an abelian group, see Proposition 1.1.10.
For the restrictions consider the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi_s} & \mathcal{X}_S \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi_S} \\ \{s\} & \xrightarrow{i_s} & S \end{array}$$

If $\mathcal{E} \sim \mathcal{F}$, then

$$\mathcal{E}|_s = \pi_s^*\mathcal{E} \cong \pi_s^*(\mathcal{F} \otimes \pi_S^*\mathcal{L}) \cong \pi_s^*\mathcal{F} \otimes \pi_s^*\pi_S^*\mathcal{L} \cong \pi_s^*\mathcal{F} \otimes q^*i_s^*\mathcal{L} \cong \pi_s^*\mathcal{F} = \mathcal{F}|_s$$

since $i_s^*\mathcal{L}$ is a line bundle over one point and hence trivial (of rank 1), so that $i_s^*\mathcal{L} \cong \mathcal{O}_{\mathbb{K}}$ and $q^*i_s^*\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}$. □

Proposition 4.2.4. [[67], 13.1, p.84] , [[38], 4.1, p.90] and [[48], p.7]
The assignment $S \mapsto \mathcal{M}_P(S)$ is functorial and contravariant. It defines the Simpson moduli functor

$$\mathcal{M}_P\ :\ \mathtt{Sch_c}(\mathbb{K})^{\mathrm{op}} \longrightarrow \mathtt{Set}\ :\ S \longmapsto \mathcal{M}_P(S)\ .$$

Proof. Let a morphism $f : T \to S$ in $\mathtt{Sch_c}(\mathbb{K})$ be given ; we are intended to construct a pullback map $f^* : \mathcal{M}_P(S) \to \mathcal{M}_P(T)$. For $[\mathcal{E}] \in \mathcal{M}_P(S)$ we define $f^*[\mathcal{E}] = [p^*\mathcal{E}]$, where $\mathcal{X}_S \times_S T \cong \mathcal{X}_T$ so that $p \simeq \mathrm{id}_{\mathcal{X}} \times f$ and

$$\begin{array}{ccccccc} \mathcal{X} & \xrightarrow{\pi_t} & \mathcal{X}_T & \xrightarrow{p} & \mathcal{X}_S & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_S} & & \downarrow \\ \{t\} & \longrightarrow & T & \xrightarrow{f} & S & \longrightarrow & \operatorname{Spec}\mathbb{K} \end{array}$$

We have to show that this is independent of the class $[\mathcal{E}]$ and that $[p^*\mathcal{E}]$ indeed defines an element in $\mathcal{M}_P(T)$. If $\mathcal{E} \sim \mathcal{F}$ on $\mathcal{X}_S$ with $\mathcal{E} \cong \mathcal{F} \otimes \pi_S^*\mathcal{L}$, then $p^*\mathcal{E} \cong p^*\mathcal{F} \otimes p^*\pi_S^*\mathcal{L} \cong p^*\mathcal{F} \otimes \pi_T^*(f^*\mathcal{L})$, i.e. $p^*\mathcal{E} \sim p^*\mathcal{F}$ on $\mathcal{X}_T$. Moreover $p^*\mathcal{E}$ is still coherent and flat over T as a pullback.

It remains to show the properties on the fibers. For this note that for any closed point $t \in T$ we have $(p^*\mathcal{E})|_t = \pi_t^* p^*\mathcal{E} \cong (p \circ \pi_t)^*\mathcal{E} \cong \pi_{f(t)}^*\mathcal{E} = \mathcal{E}|_{f(t)}$ because

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\pi_t} & \mathcal{X}_T & \xrightarrow{p} & \mathcal{X}_S \\ \downarrow & & \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \pi_S} \\ \{t\} & \xrightarrow{i_t} & T & \xrightarrow{f} & S \end{array} \qquad \cong \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi_{f(t)}} & \mathcal{X}_S \\ \downarrow & & \downarrow{\scriptstyle \pi_S} \\ \{f(t)\} & \xrightarrow{i_{f(t)}} & S \end{array}$$

where $f \circ i_t = i_{f(t)}$ and $f(t)$ is a closed point in S. Now since $\mathcal{E}|_{f(t)}$ is semistable of pure dimension d with Hilbert polynomial P, so is $(p^*\mathcal{E})|_t$. □

Remark 4.2.5. Functoriality is the actual motivation of Definition 4.2.2, which is rather abstract. Indeed one cannot define $\mathcal{M}_P(S)$ to be the space of semistable sheaves on $\mathcal{X}_S$ since e.g. purity is not preserved under pullbacks, see Example 3.1.15. Thus semistability has to be defined fiberwise.

Definition 4.2.6. [[67], 4.2, p.19-20] and [[48], 2.7 & 2.8, p.8]
A scheme $M \in \mathtt{Sch_c}(\mathbb{K})$ is called a *fine moduli space* of semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P if there exists an isomorphism of functors

$$\mathcal{M}_P \xrightarrow{\sim} \operatorname{Hom}(\,\cdot\,, M)\ , \tag{4.10}$$

where Hom is taken in the category $\mathtt{Sch_c}(\mathbb{K})$. One also says that M *represents* the functor $\mathcal{M}_P$. If this is the case, the class $[\mathcal{U}] \in \mathcal{M}_P(M)$ that corresponds to the identity $\mathrm{id}_M \in \operatorname{Hom}(M, M)$ under this isomorphism is called the *universal* *family*. By definition $\mathcal{U}$ is a family over M, i.e. a semistable coherent sheaf on $\mathcal{X}_M/M$ that is flat over M.

Remark 4.2.7. $\mathcal{U}$ has the following important property: For any $S \in \mathtt{Sch}_\mathtt{c}(\mathbb{K})$ and $[\mathcal{E}] \in \mathcal{M}_P(S)$, there exists a unique morphism $f : S \to M$, given by (4.10), such that $[\mathcal{E}] = f^*[\mathcal{U}]$, i.e. there is a line bundle $\mathcal{L}$ on S such that $\mathcal{E} \cong F^*\mathcal{U} \otimes \pi_S^*\mathcal{L}$.

$$\begin{array}{ccc} \mathcal{X}_S & \xrightarrow{F} & \mathcal{X}_M \\ \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_M} \\ S & \xrightarrow{f} & M \end{array}$$

This means that every family over S is a unique pullback of the universal sheaf $\mathcal{U}$ on $\mathcal{X}_M/M$ (in the sense defined above, i.e. up to some twist by a line bundle). In particular, M is unique up to canonical isomorphism. We also have a set bijection

$$M \cong \mathrm{Hom}\big(\{\mathrm{pt}\}, M\big) \cong \mathcal{M}_P(\mathrm{pt}) \ , \tag{4.11}$$

where $\{\mathrm{pt}\} = \mathrm{Spec}\,\mathbb{K}$ is a closed point. Hence the closed points of M are in 1-to-1 correspondence with elements in $\mathcal{M}_P(\mathrm{Spec}\,\mathbb{K})$. The latter are classes of coherent sheaves on $\mathcal{X}_{\mathrm{Spec}\,\mathbb{K}}$ that are flat over $\mathrm{Spec}\,\mathbb{K}$ and such that the restriction to the fiber over closed points is a semistable sheaf of pure dimension d and Hilbert polynomial P. Here we have $\mathcal{X}_{\mathrm{Spec}\,\mathbb{K}} = \mathcal{X} \times_{\mathbb{K}} \mathrm{Spec}\,\mathbb{K} \cong \mathcal{X}$ and pt is the only closed point in $\mathrm{Spec}\,\mathbb{K}$. Flatness is always satisfied since $\mathcal{O}_{\mathbb{K},\mathrm{pt}} \cong \mathbb{K}$ and modules over $\mathbb{K}$ are always free, hence flat. It remains to check what happens to the equivalence relation ; since every line bundle $\mathcal{L}$ over $\mathrm{Spec}\,\mathbb{K}$ is trivial, we get $\mathcal{E} \otimes \pi^*_{\mathrm{Spec}\,\mathbb{K}}\mathcal{L} \cong \mathcal{E}$ and the equivalence classes become isomorphism classes. Summarizing we obtain

$$\big\{ \text{closed points of } M \big\} \overset{1:1}{\longleftrightarrow} \mathcal{M}_P(\mathrm{Spec}\,\mathbb{K}) \cong \big\{ \text{isomorphism classes of semistable sheaves on } \mathcal{X} \text{ of pure dimension } d \text{ and Hilbert polynomial } P \big\} \, .$$

Remark 4.2.8. We will see in Section 4.2.3 that the closed points of fine moduli spaces must necessarily correspond to isomorphism classes of stable sheaves, otherwise one may construct examples that ruin continuity conditions.

Corollary 4.2.9. *Assume that $M \in \mathtt{Sch}_\mathtt{c}(\mathbb{K})$ is a fine moduli space and let $[\mathcal{F}]$ be an isomorphism class in M which is closed, i.e. $[\mathcal{F}]$ corresponds to a closed point $m \in M$. Then the restriction $\mathcal{U}|_m$ on the fiber $\mathcal{X}$ is isomorphic to $\mathcal{F}$.*

Proof. $[\mathcal{F}] \in M \cong \mathcal{M}_P(\operatorname{Spec}\mathbb{K})$, so representabiliy of M implies that there exists a unique morphism $f : \operatorname{Spec}\mathbb{K} \to M$ such that $[\mathcal{F}] = [F^*\mathcal{U}]$ which, since all line bundles on $\operatorname{Spec}\mathbb{K}$ are trivial, means that $\mathcal{F} \cong F^*\mathcal{U}$.

$$\begin{array}{ccc} \mathcal{X}_{\operatorname{Spec}\mathbb{K}} & \xrightarrow{F} & \mathcal{X}_M \\ \downarrow & & \downarrow{\scriptstyle \pi_M} \\ \operatorname{Spec}\mathbb{K} & \xrightarrow{f} & M \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi_m} & \mathcal{X}_M \\ \downarrow & & \downarrow{\scriptstyle \pi_M} \\ \{m\} & \xrightarrow{i_m} & M \end{array}$$

But uniqueness of f together with $\mathcal{X}_{\operatorname{Spec}\mathbb{K}} \cong \mathcal{X}$ and $\{m\} \cong \operatorname{Spec}\mathbb{K}$ imply that f is nothing but the inclusion i_m, hence these two diagrams are isomorphic. In particular $\mathcal{U}|_m = \pi_m^*\mathcal{U} \cong F^*\mathcal{U} \cong \mathcal{F}$. □

Definition 4.2.10. [[38], 2.2.1, p.40]
A scheme $M \in \mathtt{Sch}_\mathtt{c}(\mathbb{K})$ is said to be a *coarse moduli space* of semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P (or that it *corepresents* the Simpson moduli functor $\mathcal{M}_P$) if there is a natural transformation of functors

$$\alpha \ : \ \mathcal{M}_P \longrightarrow \operatorname{Hom}(\,\cdot\,, M)$$

which satisfies the following universal property: for any other connected scheme $N \in \mathtt{Sch}_\mathtt{c}(\mathbb{K})$ with a natural transformation $\beta : \mathcal{M}_P \to \operatorname{Hom}(\,\cdot\,, N)$, there exists a unique morphism of schemes $h : M \to N$ over $\mathbb{K}$ such that $\beta = (h\circ) \circ \alpha$, i.e. we have the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_P & \xrightarrow{\quad\alpha\quad} & \operatorname{Hom}(\,\cdot\,, M) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle h\circ} & \\ & \operatorname{Hom}(\,\cdot\,, N) & \end{array}$$

where $\beta(S) = h \circ \alpha(S)$, $\forall\, S \in \mathtt{Sch}_\mathtt{c}(\mathbb{K})$. This again ensures that M is uniquely given up to canonical isomorphism.

Remark 4.2.11. cf. [[67], 13.1, p.84] and [[48], 2.5, p.7]
Some authors, such as Maican [48], Trautmann [67] and Harris-Morrison [34], also require that the natural transformation $\alpha : \mathcal{M}_P \to \operatorname{Hom}(\,\cdot\,, M)$ of a coarse moduli space must be a set bijection on closed points, i.e.

$$\alpha(\operatorname{Spec}\mathbb{K}) \ : \ \mathcal{M}_P(\operatorname{Spec}\mathbb{K}) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{Spec}\mathbb{K}, M) \cong M \ .$$

As in Remark 4.2.7 this implies again that closed points in M are in 1-to-1 correspondence with isomorphism classes of stable sheaves on $\mathcal{X}$ of pure dimension d and Hilbert polynomial P.
However we will explain in Section 4.2.3 why this additional condition is in general not a good choice in the case of the Simpson moduli functor $\mathcal{M}_P$.

Lemma 4.2.12. [[67], 13.1, p.84]
If the natural transformation $\alpha : \mathcal{M}_P \to \mathrm{Hom}(\,\cdot\,, M)$ is a bijection on closed points, then it is given by

$$\alpha(S) \ : \ \mathcal{M}_P(S) \longrightarrow \mathrm{Hom}(S, M) \ : \ [\mathcal{E}] \longmapsto \big(S \to M \ : \ s \mapsto [\mathcal{E}|_s] \big) \tag{4.12}$$

for all $S \in \mathtt{Sch_c}(\mathbb{K})$ and $s \in S$ closed. Here $[\mathcal{E}|_s]$ denotes the isomorphism class.

Proof. $\alpha(S)$ is well-defined since $[\mathcal{E}] \in \mathcal{M}_P(S)$ means that its restriction $\mathcal{E}|_s$ is semistable on $\mathcal{X}$ of pure dimension d and Hilbert polynomial P. Lemma 4.2.3 moreover ensures that its isomorphism class is independent of the representative of the equivalence class $[\mathcal{E}]$. Fix $s \in S$ closed and consider the morphism of schemes $i_s : \mathrm{Spec}\,\mathbb{K} \to S : \mathrm{pt} \mapsto s$. Since α is a natural transformation we get the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_P(S) & \xrightarrow{\alpha(S)} & \mathrm{Hom}(S, M) \\ {\scriptstyle i_s^*}\downarrow & & \downarrow{\scriptstyle \circ\, i_s} \\ \mathcal{M}_P(\mathrm{Spec}\,\mathbb{K}) & \xrightarrow{\alpha(\mathrm{Spec}\,\mathbb{K})} & M \end{array}$$

because $M \cong \mathrm{Hom}(\mathrm{Spec}\,\mathbb{K}, M)$ via $m \mapsto (\mathrm{pt} \mapsto m)$. To find the value of $\alpha(S)[\mathcal{E}](s)$, we compute

$$\begin{aligned} \alpha(S)[\mathcal{E}](s) = \big(\alpha(S)[\mathcal{E}] \circ i_s\big)(\mathrm{pt}) &= \alpha(\mathrm{Spec}\,\mathbb{K}) i_s^*[\mathcal{E}](\mathrm{pt}) \\ &= \alpha(\mathrm{Spec}\,\mathbb{K})[\mathcal{E}|_s](\mathrm{pt}) = [\mathcal{E}|_s] \end{aligned}$$

since $\alpha(\mathrm{Spec}\,\mathbb{K})$ and (4.11) give a bijection between $\mathcal{M}_P(\mathrm{Spec}\,\mathbb{K})$ and M when evaluating at pt. □

Remark 4.2.13. If $\alpha(\mathrm{Spec}\,\mathbb{K})$ is not a bijection, then a similar formula as in Lemma 4.2.12 holds true. In that case we obtain the same expression than (4.12),

but where $[\mathcal{E}|_s]$ means s-equivalence class. This is still well-defined since isomorphic sheaves are also s-equivalent.
The reason for this is explained in Theorem 4.2.14 below. Indeed elements of $\mathcal{M}_P(\operatorname{Spec}\mathbb{K})$ are still isomorphism classes of semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P. The only difference is that there is no longer a 1-to-1 correspondence since M may have less closed points.

4.2.2 Theorem of Simpson

The following deep result was proven in 1994 by Carlos T. Simpson in [65], Theorem 1.21, p.71-73.

Theorem 4.2.14 (Simpson)**.** *Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a projective scheme over $\mathbb{K}$ and $P \in \mathbb{Q}[m]$ a fixed numerical polynomial of degree d.*

1) *There exists a moduli space $M_P(\mathcal{X}) \in \mathtt{Sch_c}(\mathbb{K})$ which universally corepresents*[9] *the Simpson moduli functor $\mathcal{M}_P$.*

2) *$M_P(\mathcal{X})$ is also a projective scheme over $\mathbb{K}$.*

3) *The closed points of $M_P(\mathcal{X})$ are in bijection with s-equivalence classes of coherent semistable sheaves on $\mathcal{X}$ of pure dimension d and Hilbert polynomial P.*

4) *There is a dense open subscheme $M_P^s(\mathcal{X}) \subseteq M_P(\mathcal{X})$ whose closed points parametrize isomorphism classes of stable sheaves on $\mathcal{X}$ with Hilbert polynomial P (for which isomorphism and s-equivalence classes coincide).*

Sketch of proof. We briefly describe how $M_P(\mathcal{X})$ is constructed ; more information can be found in [[65], p.65-66] and [[47], p.4-5].
For a numerical polynomial $P \in \mathbb{Q}[m]$ and a coherent sheaf $\mathcal{G} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$, the Hilbert scheme $\operatorname{Hilb}_P(\mathcal{G})$ classifies (equivalence classes of) quotients $\mathcal{G} \twoheadrightarrow \mathcal{Q}$ with

[9]Denote $h_M = \operatorname{Hom}(\,\cdot\,, M)$. For our purposes we do not need the definition of a universal corepresentation, but let us nevertheless mention it for completion ; it means that there is a natural transformation $\alpha : \mathcal{M}_P \to h_M$ such that for any morphism $\phi : h_N \to h_M$, the fiber product $h_N \times_{h_M} \mathcal{M}_P$ is corepresented by h_N. So in particular $M_P(\mathcal{X})$ is also a coarse moduli space by choosing the identity $h_M \to h_M$.

given Hilbert polynomial $P_{\mathcal{Q}} = P$. We fix $M \in \mathbb{N}$ large enough such that the twisted sheaf $\mathcal{F}(M)$ is generated by global sections and $H^i(\mathcal{F}(M)) = \{0\}, \forall\, i \geq 1$ for all semistable sheaves $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ with Hilbert polynomial P. This is possible because of Serre's Theorems A and B and since the set of semistable sheaves with fixed Hilbert polynomial is bounded, see [[65], 1.1, p.56-57], which ensures that M can be chosen independent of $\mathcal{F}$. Now let $V \cong \mathbb{K}^{P(M)}$ be a vector space of dimension $P(M)$ and define the invertible sheaf $\mathcal{G} = V \otimes \mathcal{O}(-M)$, where V is seen as the trivial vector bundle with fiber V. Then consider the Hilbert scheme $\mathrm{Hilb}_P(\mathcal{G})$, on which the group $\mathrm{SL}(V)$ acts by composition

$$V \otimes \mathcal{O}(-M) \overset{\mathrm{SL}(V)}{\longrightarrow} V \otimes \mathcal{O}(-M) \longrightarrow \mathcal{Q}\,.$$

The open subset $\Omega \subset \mathrm{Hilb}_P(\mathcal{G})$ of points that are semistable under the action of $\mathrm{SL}(V)$ (in the sense of Geometric Invariant Theory, see Appendix D.4 for more information) then describes semistable quotients $\mathcal{Q}$, so we take

$$M_P(\mathcal{X}) := \Omega / \mathrm{SL}(V)\,, \tag{4.13}$$

which can shown to be a projective scheme by using GIT.

Remark 4.2.15. In some particular cases we will see more precise descriptions of Ω and the action of $\mathrm{SL}(V)$ in terms of affine spaces and matrices acting on exact sequences. This is e.g. done in Remark 4.6.23 and Remark 5.1.43.

Remark 4.2.16. The definitions of a coarse moduli space in [[48], 2.5, p.7-8] and [[23], 3.6, p.28] contain the condition about $\alpha(\mathrm{Spec}\,\mathbb{K})$ being a bijection, but still claim that $M_P(\mathcal{X})$ is a coarse moduli space. This is not the case since α being a bijection on closed points does not imply that the closed points of M are in 1-to-1 correspondence with s-equivalence classes of semistable sheaves as stated in Theorem 4.2.14 (compare Remark 4.2.7 and Remark 4.2.11).

4.2.3 Representability and properly semistable sheaves

In general it is not possible to obtain a fine moduli space for the functor $\mathcal{M}_P$ as soon as there are semistable sheaves that are not stable (such sheaves are called *properly semistable*). Similarly having a set bijection between closed points of the moduli space and $\mathcal{M}_P(\operatorname{Spec}\mathbb{K})$ is not possible since such a construction would not be continuous. The reasons for this are the following.

Lemma 4.2.17. [[38], 4.1.2, p.91]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ be semistable with Hilbert polynomial P and assume that we have a corepresentation $\alpha : \mathcal{M}_P \to \operatorname{Hom}(\,\cdot\,, M)$ for some $M \in \mathtt{Sch}_{\mathtt{c}}(\mathbb{K})$. If there exists a non-split exact sequence of semistable sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{4.14}$$

with the same reduced Hilbert polynomial $p_{\mathcal{F}}$, then the closed points of M cannot be in 1-to-1 correspondence with elements from $\mathcal{M}_P(\operatorname{Spec}\mathbb{K})$, i.e. α cannot be a bijection on closed points.

Proof. If (4.14) holds one can construct a coherent sheaf $\mathcal{E}$ on $\mathcal{X}_{\mathbb{A}^1} = \mathcal{X} \times_{\mathbb{K}} \mathbb{A}^1_{\mathbb{K}}$ which is flat over $\mathbb{A}^1_{\mathbb{K}}$ such that

$$\mathcal{E}|_0 \cong \mathcal{F}' \oplus \mathcal{G} \qquad , \qquad \mathcal{E}|_t \cong \mathcal{F}\,,\ \forall\, t \neq 0\,,$$

where we identify closed points in $\mathbb{A}^1_{\mathbb{K}}$ via $t \leftrightarrow \langle\, X - t\,\rangle$. Hence $[\mathcal{E}] \in \mathcal{M}_P(\mathbb{A}^1_{\mathbb{K}})$ since its restrictions to the fibers at closed points are semistable sheaves on $\mathcal{X}$ with Hilbert polynomial P. If $\alpha(\operatorname{Spec}\mathbb{K})$ would be a bijection, then Lemma 4.2.12 gives a morphism $f : \mathbb{A}^1_{\mathbb{K}} \to M : t \mapsto [\mathcal{E}|_t]$. But this is not continuous at 0 since $\mathcal{F} \ncong \mathcal{F}' \oplus \mathcal{G}$, so f is not a morphism of schemes. Hence α cannot be a bijection on closed points. ☐

Remark 4.2.18. Intuitively one could consider the two curves $\mathbb{K} \to M$ (where $\mathbb{K}$ is endowed with some topology) defined by $t \mapsto [\mathcal{F}]$ and $t \mapsto [\mathcal{E}|_t]$. For $t \neq 0$ the sheaves $\mathcal{F}$ and $\mathcal{E}|_t$ are isomorphic and the curves agree ; one could say that both are "constant". But for $t \to 0$ we have $\mathcal{E}|_0 \ncong \mathcal{F}$, so the "constant" sequence $t \mapsto [\mathcal{E}|_t]$ has a "limit" at 0 with a different value. It is clear that situations like this should not happen if we want a "good" moduli space M to parametrize our sheaves.
The way out of this issue is to consider s-equivalence classes instead of isomorphism classes. Indeed if $\mathcal{F}'$ and $\mathcal{G}$ in (4.14) are stable, then $\mathcal{F}$ is s-equivalent to $\mathcal{F}' \oplus \mathcal{G}$ by Example 4.1.23 and $\mathcal{E}|_0 \sim \mathcal{E}|_t$ for $t \neq 0$, so the "limit" of the sequence

$t \mapsto [\mathcal{E}|_t]$ (where [ ] now means s-equivalence class) will still have the same value. This illustrates that closed points of a moduli space must at least correspond to s-equivalence classes of semistable sheaves with Hilbert polynomial P, otherwise the (classes of) sheaves in M do not vary "continuously".
More generally there is no hope of having a bijection on closed points when exact sequences like (4.14) exist, which e.g. occurs in the following case.

Corollary 4.2.19. *If there exists a properly semistable sheaf on $\mathcal{X}$ with Hilbert polynomial P, then $\alpha : \mathcal{M}_P \to \mathrm{Hom}(\,\cdot\,, M)$ is not a bijection on closed points.*

Proof. It suffices to consider the Jordan-Hölder filtration (4.6) of $\mathcal{F}$. As $\mathcal{F}$ is not stable, the filtration is non-trivial and we have $0 \neq \mathcal{F}_1 \subsetneq \mathcal{F}$. Hence we can choose

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_1 \longrightarrow 0\ .$$

This is because $p_{\mathcal{F}_1} = p_{\mathcal{F}}$ by Remark 4.1.19, so we have

$$P_{\mathcal{F}/\mathcal{F}_1} = \big(\alpha_d(\mathcal{F}) - \alpha_d(\mathcal{F}_1)\big) \cdot p_{\mathcal{F}/\mathcal{F}_1}\ ,$$
$$P_{\mathcal{F}/\mathcal{F}_1} = P_{\mathcal{F}} - P_{\mathcal{F}_1} = \alpha_d(\mathcal{F}) \cdot p_{\mathcal{F}} - \alpha_d(\mathcal{F}_1) \cdot p_{\mathcal{F}_1} = \big(\alpha_d(\mathcal{F}) - \alpha_d(\mathcal{F}_1)\big) \cdot p_{\mathcal{F}}$$

and obtain $p_{\mathcal{F}/\mathcal{F}_1} = p_{\mathcal{F}}$ since $\alpha_d(\mathcal{F}) - \alpha_d(\mathcal{F}_1)$ is non-zero (otherwise $\mathcal{F}/\mathcal{F}_1 = 0$). Semistability of $\mathcal{F}/\mathcal{F}_1$ follows from Proposition 4.1.21. Moreover this sequence does not split. ☐

4.2.4 Representability in the stable case

The following result gives a condition for representabiliy in the case of stable sheaves.

Definition 4.2.20. [[38], 4.1, p.90] and [[48], p.8]
Let $P \in \mathbb{Q}[m]$ be a numerical polynomial of degree d. Similarly as for the Simpson moduli functor $\mathcal{M}_P$ we can define a functor $\mathcal{M}_P^s : \mathtt{Sch_c}(\mathbb{K})^{\mathrm{op}} \to \mathtt{Set}$ that should classify stable sheaves by

$$\mathcal{M}_P^s(S) := \Big\{ [\mathcal{E}] \ \Big|\ \mathcal{E} \text{ is a stable sheaf on } \mathcal{X}_S/\,S \text{ of pure dimension } d \text{ and Hilbert polynomial } P \Big\}\ ,$$

where $[\mathcal{E}]$ is the same equivalence class as in (4.9).

Theorem 4.2.21. [[38], 4.6.5 & 4.6.6, p.119-120] and [[48], 2.9, p.9]
Let $P \in \mathbb{Q}[m]$ be a numerical polynomial of degree $d \leq \dim \mathcal{X}$ and write it of the form

$$P(m) = \sum_{i=0}^{d} \beta_i \cdot \binom{m+i-1}{i} \tag{4.15}$$

for some integral coefficients $\beta_i \in \mathbb{Z}$. If $\gcd(\beta_0, \dots, \beta_d) = 1$, then the open subscheme $M_P^s(\mathcal{X})$ from Theorem 4.2.14 is a fine moduli space for the functor $\mathcal{M}_P^s$. In particular there exists a universal family on $\mathcal{X} \times_{\mathbb{K}} M_P^s(\mathcal{X})$.

4.2.5 Some historical remarks

The initial motivation for studying moduli spaces of semistable sheaves is that there is no moduli space which classifies all coherent sheaves on a projective scheme. The way out of this problem is to add the semistability condition. The first achievements on this topic have been done by D. Giesecker and M. Maruyama in 1977. However they defined semistable sheaves to be torsion-free instead of pure. On integral projective schemes this e.g. implies that the moduli spaces are empty if the degree of the fixed Hilbert polynomial is strictly less than the dimension of the scheme (as sheaves on "nice" schemes which are supported in smaller dimension are torsion).
Maruyama proved existence of the moduli space of stable sheaves on a smooth projective variety $\mathcal{X}$ with a fixed Hilbert polynomial in [50] by using Mumford's Geometric Invariant Theory. The semistable case presented technical difficulties which first had been solved by Gieseker in [27] in case $\mathcal{X}$ is a surface. For higher-dimensional $\mathcal{X}$ existence of the moduli space of semistable sheaves is proven in [51] one year later.
In 1994 Simpson generalized the definition of (semi)stability by replacing the condition on torsion-freeness by purity in [65]. This way he also managed to prove existence of non-trivial moduli spaces of semistable sheaves with fixed Hilbert polynomial of degree $d < \dim \mathcal{X}$. This is why the spaces $M_P(\mathcal{X})$ are called Simpson moduli spaces, even though Giesecker and Maruyama introduced the concept almost 20 years before. Showing boundedness of the set of semistable sheaves with fixed Hilbert polynomial is by the way an essential part of the proof, in the

initial one of Maruyama as well as in the one of Simpson.
From a pedagogical point of view one however prefers to speak about the statement of Simpson because of its bigger generality. For a better understanding it is also useful to consider textbooks such as [38] which summarize all important results before studying the original article(s).

4.3 Simpson moduli spaces on $\mathbb{P}_2$

From now on we always consider the (classical) projective plane $\mathbb{P}_2 = \mathbb{P}(\mathbb{K}^3)$ with structure sheaf $\mathcal{O}_{\mathbb{P}_2}$ given by regular functions and fixed homogeneous coordinates $(x_0 : x_1 : x_2)$, resp. the corresponding projective scheme

$$\mathcal{X} = \mathbb{P}^2_{\mathbb{K}} = \operatorname{Proj} \mathbb{K}[X_0, X_1, X_2]$$

and Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}_2}(1)$ as very ample line bundle.

Example 4.3.1. Hence by (4.3) the Hilbert polynomials of the twisted sheaves $\mathcal{O}_{\mathbb{P}_2}(k)$, $k \in \mathbb{Z}$ are

$$\begin{aligned} P_{\mathcal{O}_{\mathbb{P}_2}(k)}(m) &= \binom{m+k+2}{2} = \frac{(m+k+2)(m+k+1)}{2} \\ &= \frac{1}{2} \cdot m^2 + \frac{2k+3}{2} \cdot m + \frac{k^2+3k+2}{2} \; . \end{aligned}$$

Lemma 4.3.2. *If $\mathcal{F}, \mathcal{G} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$, then we have for all $n \in \mathbb{Z}$*

$$\operatorname{Hom}\big(\mathcal{F}(n), \mathcal{G}\big) \;\cong\; \operatorname{Hom}\big(\mathcal{F}, \mathcal{G}(-n)\big) \; .$$

Proof. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}_2}(n)$, so that $\mathcal{E}^* \cong \mathcal{O}_{\mathbb{P}_2}(-n)$. Definition 1.1.9 and (1.2) give

$$\begin{aligned} \operatorname{Hom}\big(\mathcal{F}(n), \mathcal{G}\big) &= \operatorname{Hom}\big(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}\big) \;\cong\; \operatorname{Hom}\big(\mathcal{F}\,,\, \mathcal{H}\mathtt{om}(\mathcal{E}, \mathcal{G})\big) \\ &\cong\; \operatorname{Hom}\big(\mathcal{F}\,,\, \mathcal{E}^* \otimes \mathcal{G}\big) \;\cong\; \operatorname{Hom}\big(\mathcal{F}, \mathcal{G}(-n)\big) \; . \end{aligned}$$

□

4.3.1 Sheaves with linear Hilbert polynomial

We are particularly interested in semistable sheaves on $\mathbb{P}_2$ whose Hilbert polynomial is linear, i.e. sheaves that are supported in dimension $d = 1$. More precisely, we consider $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ with a Hilbert polynomial

$$P_{\mathcal{F}}(m) = \alpha_1(\mathcal{F}) \cdot m + \alpha_0(\mathcal{F}) \ ,$$

where $\alpha_0(\mathcal{F}) \in \mathbb{Z}$ and $\alpha_1(\mathcal{F}) \in \mathbb{N}$.[10]

Remark 4.3.3. For such sheaves the conditions for being (semi)stable become
1) $\mathcal{F}$ is of pure dimension 1, i.e. $\mathcal{F}$ has no non-zero proper coherent subsheaves with 0-dimensional support.
2) Every proper non-zero coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ satisfies $\mu(\mathcal{F}') < \mu(\mathcal{F})$, resp. $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. The second property follows from

$$p_{\mathcal{F}}(m) = \frac{\alpha_1(\mathcal{F}) \cdot m + \alpha_0(\mathcal{F})}{\alpha_1(\mathcal{F})} = m + \frac{\alpha_0(\mathcal{F})}{\alpha_1(\mathcal{F})} = m + \mu(\mathcal{F}) \ ,$$

so that $p_{\mathcal{F}'} \,(\leq)\, p_{\mathcal{F}} \ \Leftrightarrow\ \mu(\mathcal{F}')\,(\leq)\,\mu(\mathcal{F})$. In particular this proves that the conditions of (semi)stability and μ-(semi)stability are equivalent for linear Hilbert polynomials, see also [[23], 3.3, p.20].

Notation 4.3.4. In the following we always write the Hilbert polynomial as $P_{\mathcal{F}}(m) = am + b$, where $a \geq 1$ is the multiplicity of $\mathcal{F}$ and b is the Euler characteristic

$$b = \chi(\mathbb{P}_2, \mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) \ . \tag{4.16}$$

Lemma 4.3.5. cf. [[23], 3.1, p.23-24]
If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ has Hilbert polynomial $P_{\mathcal{F}}(m) = am + b$, any proper subsheaf $\mathcal{F}' \subsetneq \mathcal{F}$ has Hilbert polynomial $\mu m + r$ with $\mu \leq a$ and if $\mu = a$, then $r < b$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0 \ ,$$

where the quotient $\mathcal{F}/\mathcal{F}'$ has Hilbert polynomial $(a-\mu)m + (b-r)$ by (4.2). Hence $a \geq \mu$ since the leading coefficient must be non-negative. If $a = \mu$, the same argument gives $b \geq r$. But the constant terms cannot be equal, otherwise $\mathcal{F}/\mathcal{F}' = 0$ since it has zero Hilbert polynomial and $\mathcal{F} = \mathcal{F}'$ by Definition 4.1.7, which contradicts that $\mathcal{F}'$ is proper. □

[10] If $\alpha_1(\mathcal{F}) = 0$, the Hilbert polynomial of $\mathcal{F}$ is constant, which means that $\mathcal{F}$ is supported on finitely many points (with multiplicities) and thus equal to a direct sum of skyscraper sheaves.

Proposition 4.3.6. [[23], 3.1, p.23-24] and [[48], 2.3, p.6]
If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ is a semistable sheaf with Hilbert polynomial $P_{\mathcal{F}}(m) = am + b$ where a and b are coprime, i.e. $\gcd(a, b) = 1$, then $\mathcal{F}$ is stable.

Proof. Let $\mathcal{F}' \subseteq \mathcal{F}$ be a non-zero subsheaf of $\mathcal{F}$ with Hilbert polynomial given by $P_{\mathcal{F}'}(m) = \mu m + r$. If $\mathcal{F}'$ has the same reduced Hilbert polynomial $p_{\mathcal{F}}$, then

$$p_{\mathcal{F}'} = p_{\mathcal{F}} \;\Leftrightarrow\; m + \tfrac{r}{\mu} = m + \tfrac{b}{a} \;\Leftrightarrow\; \tfrac{r}{\mu} = \tfrac{b}{a} \;\Leftrightarrow\; a \cdot r = \mu \cdot b \,.$$

Since a divides the product $\mu \cdot b$ with $\gcd(a, b) = 1$, we need that a divides μ. Together with $\mu \leq a$ from Lemma 4.3.5 this implies that $\mu = a$, thus $r = b$ and $P_{\mathcal{F}'} = P_{\mathcal{F}}$, which means that $\mathcal{F}'$ is not proper. We have shown that any subsheaf of $\mathcal{F}$ with the same reduced Hilbert polynomial is equal to $\mathcal{F}$ itself. Hence the reduced Hilbert polynomial of a proper non-zero subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$ always satisfies $p_{\mathcal{F}'} < p_{\mathcal{F}}$, i.e. $\mathcal{F}$ is stable. □

Corollary 4.3.7. *If $\gcd(a, b) = 1$, then all semistable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $am + b$ are stable. Hence the moduli functors $\mathcal{M}^s_{am+b}$ and $\mathcal{M}_{am+b}$ are equal and $M^s_{am+b}(\mathbb{P}_2) = M_{am+b}(\mathbb{P}_2)$.*

Corollary 4.3.8. [[47], 3.19, p.20] and [[48], 2.10, p.9]
If $\gcd(a, b) = 1$, then $M_{am+b}(\mathbb{P}_2)$ is a fine moduli space for the Simpson moduli functor $\mathcal{M}_{am+b}$. In particular its closed points correspond to isomorphism classes of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $am + b$.

Proof. Note that for linear Hilbert polynomials the coefficients from (4.4) and (4.15) coincide[11]:

$$\alpha_0 \cdot \frac{m^0}{0!} + \alpha_1 \cdot \frac{m^1}{1!} = \alpha_0 + \alpha_1 \cdot m \quad , \quad \beta_0 \cdot \binom{m-1}{0} + \beta_1 \cdot \binom{m}{1} = \beta_0 + \beta_1 \cdot m \,.$$

The coefficients $\beta_0 = a$ and $\beta_1 = b$ being coprime we know from Theorem 4.2.21 that the functor $\mathcal{M}^s_{am+b}$ is represented by $M^s_{am+b}(\mathbb{P}_2)$. But again a and b are coprime, so the functors $\mathcal{M}^s_{am+b}$ and $\mathcal{M}_{am+b}$ are equal by Corollary 4.3.7 and also their moduli spaces coincide. Thus $\mathcal{M}_{am+b}$ is represented by $M_{am+b}(\mathbb{P}_2)$. □

[11] This is no longer true for higher degrees, e.g. $\frac{m^2}{2!} \neq \binom{m+1}{2}$.

Proposition 4.3.9. cf. [[48], p.6]
Let $f \in \mathbb{K}[X_0, X_1, X_2]$ be a homogeneous polynomial of degree $d \geq 1$ and denote its vanishing set in $\mathbb{P}_2$ by

$$Z(f) = \left\{ (x_0 : x_1 : x_2) \in \mathbb{P}_2 \;\middle|\; f(x_0, x_1, x_2) = 0 \right\} .$$

1) The structure sheaf $\mathcal{O}_C$ of the 1-dimensional curve $C = Z(f)$ has Hilbert polynomial

$$P_{\mathcal{O}_C}(m) = d \cdot m + \tfrac{3d-d^2}{2} . \tag{4.17}$$

2) $\mathcal{O}_C$ is stable.

Proof. 1) C being the vanishing set of the polynomial f, we get the presentation

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_C \longrightarrow 0 , \tag{4.18}$$

which allows to compute the Hilbert polynomial of $\mathcal{O}_C$ by using Example 4.3.1:

$$P_{\mathcal{O}_C}(m) = \frac{(m+2)(m+1)}{2} - \frac{(m-d+2)(m-d+1)}{2} = d \cdot m + \frac{3d-d^2}{2} .$$

2) First we have to show that $\mathcal{O}_C$ is pure of dimension 1. This is true because the subscheme $C \subset \mathbb{P}_2$ is a curve, so it has no components of dimension 0 (see Example 2.2.16). Indeed on an affine open set $U_i \subset \mathbb{P}_2$, it is given by

$$C_{|U_i} = \operatorname{Spec}\left(\mathbb{K}[X_j, X_k]/\langle f_i \rangle\right) ,$$

where f_i is obtained from f by replacing $X_i = 1$. This is a (possibly non-reduced) ring without embedded primes.
To prove stability let $0 \neq \mathcal{I} \subsetneq \mathcal{O}_C$ be a proper coherent subsheaf. It is shown in [[48], 6.8, p.31-32] that there exists a homogeneous polynomial $g \in \mathbb{K}[X_0, X_1, X_2]$ dividing f and the ideal sheaf $\mathcal{J}$ of the curve $C' = Z(g)$ satisfies $\mathcal{I} \subset \mathcal{J} \subset \mathcal{O}_C$ where $\mathcal{J}/\mathcal{I}$ is supported on finitely many points. So we have a subscheme $C' \subsetneq C$ and exact sequences

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C'} \longrightarrow 0 \qquad , \qquad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I} \longrightarrow 0 .$$

$\mathcal{J}/\mathcal{I}$ being supported on finitely many points means that it has constant Hilbert polynomial $c \geq 0$. In particular $P_{\mathcal{I}}$ and $P_{\mathcal{J}}$ only differ by the constant c. Moreover C' is a curve of degree $r \leq d$ (since g divides f), so $\mathcal{O}_{C'}$ has a similar Hilbert polynomial than $\mathcal{O}_C$. Proposition 4.1.9 and $\mathcal{O}_{C'} \cong \mathcal{O}_C/\mathcal{J} \cong (\mathcal{O}_C/\mathcal{I})/(\mathcal{J}/\mathcal{I})$ then imply that

$$P_{O_{C'}} = \mathcal{P}_{O_C} - \mathcal{P}_{\mathcal{I}} - \mathcal{P}_{\mathcal{J}/\mathcal{I}} \;\Rightarrow\; \mathcal{P}_{\mathcal{I}}(m) = d \cdot m + \tfrac{3d-d^2}{2} - r \cdot m - \tfrac{3r-r^2}{2} - c$$
$$\Rightarrow\; \mu(I) = \frac{3d-3r-d^2+r^2-2c}{2(d-r)} = \frac{3-d-r}{2} - \frac{c}{d-r}\,.$$

Note that $d - r \neq 0$, otherwise $\mathcal{I}$ would be supported on finitely many points, which contradicts that $\mathcal{O}_C$ is of pure dimension 1. The slope of $\mathcal{O}_C$ is given by $\mu(\mathcal{O}_C) = \frac{3d-d^2}{2d} = \frac{3-d}{2}$ and this is clearly $> \mu(\mathcal{I})$. Hence $\mathcal{O}_C$ is stable. □

4.3.2 Theorem of Le Potier

Let $P(m) = am + b \in \mathbb{Z}[m]$ be fixed with $a \geq 1$. The following results give some information and properties of the moduli space M_{am+b} of semistable sheaves on $\mathbb{P}_2$ with linear Hilbert polynomial $am+b$. They have first been stated and proven by Joseph Le Potier in [47], Theorem 1.1, p.1.

Theorem 4.3.10 (Le Potier)**.**

1) M_{am+b} is an irreducible projective variety of dimension a^2+1.
[[47], Thm. 3.1, p.10-11]

2) If $a \geq 3$, M_{am+b} is also locally factorial.
[[47], Thm. 3.5, p.12 & p.19]

3) The open subvariety M^s_{am+b} of stable sheaves on $\mathbb{P}_2$ is smooth.
[[47], Prop. 2.3, p.5]

4) In particular if $\gcd(a,b) = 1$, then M_{am+b} is smooth itself.
[[23], Thm. 4.5, p.46]

We also have a precise result about non-representability of the moduli functors.

Theorem 4.3.11. [[47], 3.4, p.11 & 3.19, p.20-22], [[24], p.15], [[48], 2.11, p.9] *Assume that $\gcd(a,b) \neq 1$. Then the closed subscheme $M_{am+b} \backslash M^s_{am+b}$ of properly semistable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $am+b$ has codimension $\geq 2a-3$ and for any open subset $U \subseteq M_{am+b}$ there does not exist a universal family on*

$\mathbb{P}_2 \times U$. In particular there are no fine moduli spaces for the functors $\mathcal{M}_{am+b}$ and $\mathcal{M}^s_{am+b}$.

4.3.3 The Duality Theorem of Maican

Fix $a \geq 1$. A priori one may think that it is necessary to study the moduli spaces M_{am+b} for all $b \in \mathbb{Z}$. But the following results from Maican and Drézet that have been proven in [49] and [15] show that it suffices to restrict the values of β to a (relatively small) finite set.

Proposition 4.3.12. [[15], p.1]
The map

$$M_{am+b} \longrightarrow M_{am+a+b} \ : \ \mathcal{F} \longmapsto \mathcal{F}(1) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2}(1) \tag{4.19}$$

defines an isomorphism of projective varieties with inverse map $\mathcal{G} \longmapsto \mathcal{G}(-1)$.

Proof. First recall that tensoring by $\mathcal{O}_{\mathbb{P}_2}(1)$ is exact since the sheaf is invertible. The map (4.19) is well-defined as

$$P_{\mathcal{F}(1)}(m) = P_{\mathcal{F}}(m+1) = a(m+1) + b = am + a + b$$

and preserves coherence. It also preserves purity, semistability and s-equivalence classes. Indeed if $\mathcal{F}' \subsetneq \mathcal{F}(1)$ is a proper non-zero subsheaf with 0-dimensional support, then $\mathcal{F}'(-1) \subsetneq \mathcal{F}$ would be a 0-dimensional subsheaf of $\mathcal{F}$ (here we use exactness and Lemma 4.1.11). Similarly if $\mathcal{F}'$ is such that $p_{\mathcal{F}'} > p_{\mathcal{F}(1)}$, then $\mathcal{F}'(-1)$ would destabilize $\mathcal{F}$: writing $P_{\mathcal{F}'}(m) = \mu m + r$, we get

$$\tfrac{r}{\mu} = \mu(\mathcal{F}') > \mu(\mathcal{F}(1)) = 1 + \tfrac{b}{a} \quad \Rightarrow \quad -1 + \tfrac{r}{\mu} = \mu(\mathcal{F}'(-1)) > \mu(\mathcal{F}) = \tfrac{b}{a} \ .$$

To see that s-equivalence classes are preserved as well, note that if $\mathcal{F}_0 \subsetneq \ldots \subsetneq \mathcal{F}_k$ is a Jordan-Hölder filtration of $\mathcal{F}$, then $\mathcal{F}_0(1) \subsetneq \ldots \subsetneq \mathcal{F}_k(1)$ is a filtration of $\mathcal{F}(1)$. Here we again use exactness. As before all stability conditions are still satisfied and each $\mathcal{F}_i(1)$ has reduced Hilbert polynomial $p_{\mathcal{F}} + 1$, hence so do the quotients $\mathcal{F}_i(1)/\mathcal{F}_{i-1}(1)$. Finally we admit that the map is also a morphism of projective varieties. □

Remark 4.3.13. In particular the isomorphism $M_{am+b} \cong M_{am+a+b}$ implies that it is enough to consider $b \in \{1, 2, \dots, a\}$. But one can even do a better estimate using another deep result.

Definition 4.3.14. [[49], p.2] , [[38], 1.1.7, p.6] and [[23], 4.3, p.44]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ with $\dim \mathcal{F} = 1$ (i.e. $\mathcal{F}$ has linear Hilbert polynomial). We define the *dual sheaf* of $\mathcal{F}$ by

$$\mathcal{F}^{\mathrm{D}} := \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}_2}) ,$$

where $\omega_{\mathbb{P}_2}$ is the canonical sheaf on $\mathbb{P}_2$ and $\mathcal{E}xt^1(\mathcal{F}, \cdot\,)$ is the first right derived functor of $\mathcal{H}\mathfrak{om}(\mathcal{F}, \cdot\,)$ (so $\mathcal{F}^{\mathrm{D}}$ is indeed a sheaf).[12] From [[35], II, 8.20.1, p.182] we know that $\omega_{\mathbb{P}_2} \cong \mathcal{O}_{\mathbb{P}_2}(-3)$.

Remark 4.3.15. This definition is motivated by the fact that the (usual) dual $\mathcal{F}^*$ of a 1-dimensional sheaf on a smooth surface is zero. Indeed if $\mathcal{X}$ is a smooth projective variety, then the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is pure (see Example 3.1.13) and we get

$$\mathcal{F}^*(U) = \mathrm{Hom}\left(\mathcal{F}|_U, \mathcal{O}_{\mathcal{X}}|_U\right) = \{0\}$$

for every open subset $U \subseteq \mathbb{P}_2$ because $\dim \mathcal{F} < \dim \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}}$ is pure (compare with the proof of Proposition 3.1.30). Hence it makes sense to consider the derived functor of $\mathcal{H}\mathfrak{om}$.

Proposition 4.3.16. [[49], p.3-5] and [[48], p.51-52]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be of pure dimension 1 with Hilbert polynomial $am + b$. Then
1) $\mathcal{E}xt^i(\mathcal{F}, \omega_{\mathcal{X}}) = 0$ for $i > 1$ and $h^i(\mathcal{F}) = h^{1-i}(\mathcal{F}^{\mathrm{D}})$. In particular the Euler characteristic is $\chi(\mathbb{P}_2, \mathcal{F}^{\mathrm{D}}) = -\chi(\mathbb{P}_2, \mathcal{F})$.
2) The Hilbert polynomial of the dual $\mathcal{F}^{\mathrm{D}}$ is[13]

$$P_{\mathcal{F}^{\mathrm{D}}}(m) = (-1)^{\dim \mathcal{F}} \cdot P_{\mathcal{F}}(-m) = -(-am + b) = am - b \ .$$

[12] Freiermuth [23] and Maican in [48] define the dual sheaf by $\mathcal{F}^{\nabla} = \mathcal{E}xt^1(\mathcal{F}, \omega_{\mathbb{P}_2})(1)$, but Proposition 4.3.12 ensures that the additional twist does not change the result as the corresponding moduli spaces will be isomorphic.

[13] The Hilbert polynomial of $\mathcal{F}^{\nabla}$ would be $am + a - b$.

3) $\mathcal{F}$ is (semi)stable if and only if $\mathcal{F}^{\mathrm{D}}$ is (semi)stable.
4) Let $\mathcal{F}, \mathcal{G} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be semistable of pure dimension 1 with the same Hilbert polynomial. Then $\mathcal{F}$ and $\mathcal{G}$ are s-equivalent if and only if $\mathcal{F}^{\mathrm{D}}$ and $\mathcal{G}^{\mathrm{D}}$ are s-equivalent.

Theorem 4.3.17. [[49], p.7] , [[48], 9.4, p.52-53] and [[23], 4.4, p.44-45][14]
For all integers $n \geq 2$ and $a \geq 1$, there is an isomorphism of projective varieties

$$M_{am+b}(\mathbb{P}_n) \xrightarrow{\sim} M_{am-b}(\mathbb{P}_n) \ : \ [\mathcal{F}] \longmapsto [\mathcal{F}^{\mathrm{D}}] \ ,$$

where [ ] denotes the s-equivalence class of semistable sheaves on $\mathbb{P}_n$.

Remark 4.3.18. Hence in order to study the moduli spaces M_{am+b}, for fixed $a \geq 1$ it suffices to consider the values $b \in \{1, 2, \ldots, \lfloor \frac{a}{2} \rfloor, a\}$.

4.4 Support and singular sheaves

Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be a semistable with linear Hilbert polynomial $am+b$ where $0 < b \leq a$ and consider its Fitting support $C := \mathcal{Z}_f(\mathcal{F})$.[15] In particular $\mathcal{F}$ also inherits the structure of a coherent $\mathcal{O}_C$–module (see Section 3.2). C is a projective curve in $\mathbb{P}_2$, but in general it is neither integral, nor reduced. However this is where we can apply our results from Part I and say the following.

Proposition 4.4.1. cf. [[23], 3.1, p.23-25] and [[48], 2.3, p.6][16]
1) $\mathcal{F}$ is a torsion sheaf on $\mathbb{P}_2$.
2) The annihilator support $\mathcal{Z}_a(\mathcal{F})$ has no embedded components.
3) $\mathcal{F}$ is torsion-free as a sheaf on $\mathcal{Z}_a(\mathcal{F})$ and on $\mathcal{Z}_f(\mathcal{F})$, i.e. $\mathcal{F}$ is a torsion-free $\mathcal{O}_C$–module.

Proof. 1) As $\dim \mathcal{F} < \dim \mathbb{P}_2$, this has already been proven in Example 3.1.27 and Remark 3.1.28.
2) $\mathcal{F}$ being pure, this follows from Proposition 3.2.12.
3) follows from Proposition 3.5.1, Theorem 3.5.3 and Remark 3.5.4. □

[14] [48] and [23] only prove the theorem for $\mathbb{P}_2$ and in the case where $\gcd(\alpha, \beta) = 1$.
[15] We refer to Definition 1.4.5 for a reminder of the Fitting support of a coherent sheaf.
[16] Maican does not specify which support he meant, but gives [23] as a reference, which uses the annihilator support.

The aim of this section is to prove that "almost all" sheaves in M^s_{am+b} are not just torsion-free, but actually locally free on their support and can be seen as vector bundles on a 1-dimensional variety.

4.4.1 Properties of the support

Before continuing we briefly introduce the following tool.

Definition 4.4.2. [[38], 1.1.11 & 1.1.12, p.7-8]
Let $\mathcal{X}$ be any projective scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. A section $s \in \Gamma(\mathcal{X}, \mathcal{O}(1))$ is called *$\mathcal{F}$-regular* if the morphism

$$\mathcal{F}(-1) \xrightarrow{s*} \mathcal{F}$$

which is induced by $\mathcal{O}(-1) \xrightarrow{s\cdot} \mathcal{O}_{\mathcal{X}}$ after tensoring by $\mathcal{F}$ is injective. One can show that $\mathcal{F}$-regular sections always exist (in some sense they are even dense). We only give a description which shall illustrate this fact.

Proposition 4.4.3. cf. [[38], 1.1.11, p.9]
A section $s \in \Gamma(\mathcal{X}, \mathcal{O}(1))$ is $\mathcal{F}$-regular if and only if its zero set $H = V(s)$ contains no associated points of $\mathcal{F}$.[17]

Proof. It suffices to prove the statement on affines. Let $U \cong \operatorname{Spec} R$, so that $\mathcal{F}|_U \cong \widetilde{M}$ for some R-module M and $s \in \mathcal{F}(U) \cong M$. Then

$$\begin{aligned} \mathcal{F}(-1) \xrightarrow{s} \mathcal{F} \text{ is injective } &\Leftrightarrow\ \mathcal{F}_x \xrightarrow{s_x} \mathcal{F}_x \text{ is injective, } \forall\, x \\ &\Leftrightarrow\ M_P \xrightarrow{s_P} M_P \text{ is injective, } \forall\, P \\ &\Leftrightarrow\ M \xrightarrow{s} M \text{ is injective} \end{aligned}$$

by Proposition A.2.13. Lemma 2.5.2 moreover implies that this homothety is injective if and only if s does not belong to any associated prime of M. On the other hand if $P \in \operatorname{Ass}_R(M)$, then

$$s \in P \ \Leftrightarrow\ P \in V(s) = H\ .$$

Therefore s not belonging to an associated prime of M means that no associated point of $\mathcal{F}$ belongs to H. □

[17] We recall that the set $\operatorname{Ass}(\mathcal{F})$ is introduced in Definition 2.5.5.

Remark 4.4.4. This allows to see that regular sections always exist in the case of semistable sheaves $\mathcal{F}$ on $\mathbb{P}_2$ with linear Hilbert polynomial. Indeed we know that $C = \mathcal{Z}_f(\mathcal{F})$ is a curve, $s \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathbb{P}_2}(1))$ is a homogeneous polynomial of degree 1 (also called a *linear form*) and the associated primes describe the irreducible components of C. Since there are only finitely many it suffices to choose a non-constant s which does not belong to any of them, which is possible as Prime Avoidance (see Lemma B.1.3) implies that a union of finitely many prime ideals cannot be equal to the whole ring.

Proposition 4.4.5. cf. [[23], p.24]
If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ has linear Hilbert polynomial $am + b$, then its Fitting support is a projective curve of degree a (i.e. the vanishing set of a homogeneous polynomial of degree a).

Proof. Choose an $\mathcal{F}$-regular section s with vanishing set $H = V(s)$. This gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{s} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_H \longrightarrow 0 \ .$$

Tensoring by $\mathcal{F}$ and using that s is $\mathcal{F}$-regular we get

$$0 \longrightarrow \mathcal{F}(-1) \xrightarrow{s} \mathcal{F} \longrightarrow \mathcal{F}|_H \longrightarrow 0 \ , \tag{4.20}$$

where $\mathcal{F}|_H = \mathcal{F} \otimes \mathcal{O}_H$ is the restriction of $\mathcal{F}$ to the line H. So the Hilbert polynomial of $\mathcal{F}|_H$ is

$$P_{\mathcal{F}|_H}(m) = P_{\mathcal{F}}(m) - P_{\mathcal{F}}(m-1) = am + b - \big(a(m-1) + b\big) = a \ ,$$

i.e. $\mathcal{F}|_H$ has constant Hilbert polynomial and is supported on finitely many points. But (B.11) implies that $\mathrm{supp}(\mathcal{F}|_H) = C \cap H$, so the intersection of C and the line H consists of a points (with multiplicities, but generically all points are simple), meaning that C is a curve of degree a. □

Notation 4.4.6. [[67], p.14]
Fix an ordering of the monomials in the variables X_0, X_1, X_2, for example the

lexicographical order with $X_0 > X_1 > X_2$. Every homogeneous polynomial $f \in \mathbb{K}[X_0, X_1, X_2]$ of degree $d \geq 1$ can be uniquely written as a sum

$$f = \sum_{|i|=d} a_{i_0 i_1 i_2} \cdot X_0^{i_0} X_1^{i_1} X_2^{i_2} \ ,$$

where $|i| = i_0 + i_1 + i_2$. For $N = \binom{d+2}{2} = \frac{(d+2)(d+1)}{2}$ we denote by $\langle f \rangle \in \mathbb{K}^N$ the vector of coefficients $a_{i_0 i_1 i_2}$, e.g. $f = a_0 X_0 + a_1 X_1 + a_2 X_2$ gives the vector $\langle f \rangle = (a_0, a_1, a_2)$. In particular $\langle f \rangle \neq 0$ means that f is not the zero polynomial. Via this identification the space of all such homogeneous polynomials can be given the structure of the affine space $\mathbb{A}_N$ endowed with the Zariski topology.

Remark 4.4.7. We also use this notation when considering the vanishing set in $\mathbb{P}_2$ defined by f, i.e. the curve $Z(f) \subset \mathbb{P}_2$ may be identified with an element $\langle f \rangle \in \mathbb{P}_{N-1}$ as multiplication by constants does not change the vanishing set. For example the line defined by $f = a_0 X_0 + a_1 X_1 + a_2 X_2$ is represented by $(a_0 : a_1 : a_2)$, a quadric is given by a point in $\mathbb{P}_5$ and a cubic can be described by elements of $\mathbb{P}_9$.

Definition 4.4.8. Let $\mathcal{C}_d(\mathbb{P}_2)$ denote the Hilbert scheme of all curves in $\mathbb{P}_2$ of degree d. By the notation in Remark 4.4.7 it may be identified with the projective space $\mathbb{P}_{N-1}$ for $N = \binom{d+2}{2}$. Indeed one can show that $\mathbb{P}_{N-1}$ satisfies the properties of being a fine moduli space for the Quot-functor[18] represented by $\mathcal{C}_d(\mathbb{P}_2)$.

Proposition 4.4.9. cf. [[47], p.6]
There is a morphism of projective varieties $\sigma : M_{am+b} \longrightarrow \mathcal{C}_a(\mathbb{P}_2)$, given by $[\mathcal{F}] \longmapsto \mathcal{Z}_f(\mathcal{F})$.

Proof. In order to prove this we use the fact that M_{am+b} is corepresenting the Simpson moduli functor. Let $N = \binom{a+2}{2}$. Since $\mathbb{P}_{N-1}$ is connected we have $\mathcal{C}_a(\mathbb{P}_2) \in \mathtt{Sch}_c(\mathbb{K})$.
Consider the natural transformation $\beta : \mathcal{M}_{am+b} \to \mathrm{Hom}\left(\,\cdot\,, \mathcal{C}_a(\mathbb{P}_2)\right)$ given by

$$\beta(S) \ : \ \mathcal{M}_{am+b}(S) \longrightarrow \mathrm{Hom}\left(S, \mathcal{C}_a(\mathbb{P}_2)\right) \ : \ [\mathcal{E}] \longmapsto \left(s \mapsto \mathcal{Z}_f(\mathcal{E}|_s) \right) \ ,$$

[18] We refer to [[67], Chapter 8, p.42-50] for more information about Quot-functors and Hilbert schemes.

which is well-defined since the fibers $\mathcal{E}|_s$ are independent of the equivalence class in $\mathcal{M}_{am+b}(S)$ by Lemma 4.2.3 and have Hilbert polynomial $am+b$, so Proposition 4.4.5 implies that their Fitting support is a curve of degree a. The corepresentation property from Definition 4.2.10 thus implies existence of a morphism $\sigma : M_{am+b} \to \mathcal{C}_a(\mathbb{P}_2)$ such that $\beta(S) = \sigma \circ \alpha(S)$, $\forall\, S \in \mathtt{Sch}_c(\mathbb{K})$. Formula (4.12) from Lemma 4.2.12 and Remark 4.2.13 now allow to compute

$$\mathcal{Z}_f(\mathcal{E}|_s) = \beta(S)[\mathcal{E}](s) = \big(\sigma \circ \alpha(S)\big)[\mathcal{E}](s) = \sigma\big(\alpha(S)[\mathcal{E}](s)\big) = \sigma\big([\mathcal{E}|_s]\big)\ .$$

Hence we see that $\sigma : [\mathcal{F}] \mapsto \mathcal{Z}_f(\mathcal{F})$. □

4.4.2 Singular sheaves

If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ has linear Hilbert polynomial it cannot be locally free as an $\mathcal{O}_{\mathbb{P}_2}$–module since $\dim \mathcal{F} < 2$. But as it is torsion-free on C (Proposition 4.4.1) we may ask whether $\mathcal{F}$ is locally free on its support, i.e. we can consider C as a projective variety $(C, \mathcal{O}_C)$ and ask if $\mathcal{F}$ is a locally free $\mathcal{O}_C$–module ?
By coherence it suffices to check that the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{C,x}$–module for all $x \in C$.

Definition 4.4.10. [[47], 2.7, p.6]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ with linear Hilbert polynomial $P_{\mathcal{F}}(m) = am+b$. We say that $\mathcal{F}$ is a *non-singular* sheaf if it is locally free on its support. If this is not the case, $\mathcal{F}$ is called *singular*.

Remark 4.4.11. The definition of being (non-)singular does not make sense for (classes of) sheaves in M_{am+b} since non-isomorphic sheaves of the same s-equivalence class may have different stalks ; an illustration is given in Example F.1.7. So one cannot speak of singular sheaves in the Simpson moduli space as soon as there exist properly semistable sheaves. In particular there is no "subvariety" of singular sheaves in M_{am+b}.[19]
On the other hand the notions are well-defined for isomorphism classes, hence being (non-)singular makes sense in the open dense subvariety M^s_{am+b} of stable sheaves.

[19] However there is a way of defining (non-)singular sheaves for s-equivalence classes as well by choosing a specific representative of the class. We briefly explain this idea in Section F.2.

The first important result about singular sheaves is the following.

Proposition 4.4.12. [[47], 3.2, p.10]
The closed subset Ω^{sing} *of* Ω *from* (4.13) *of points that parametrize singular sheaves is of codimension at least* 2*, i.e.*

$$\mathrm{codim}_{\Omega}(\Omega^{sing}) \geq 2 \ .$$

Our next goal is to reprove Freiermuths result which states that sheaves in M^s_{am+b} are non-singular if their support defines a smooth curve in $\mathbb{P}_2$, hence that singular sheaves can only appear if C has singular points.

Remark 4.4.13. We first refer to Proposition D.1.17, which says that a point x of an irreducible curve $V \subset \mathbb{P}_2$ is smooth if and only if the local ring $\mathcal{O}_{V,x}$ is a principal ideal domain. By Lemma D.1.18, this equivalence also holds true for curves defined by reducible polynomials as one can replace the coordinate ring at a smooth point by the coordinate ring of the irreducible component (since smooth points do not lie on an intersection of two components).
We also recall the Structure Theorem of finitely generated modules over PIDs (Theorem D.1.13).

Definition 4.4.14. The (formal) partial derivatives of a homogeneous polynomial $f \in \mathbb{K}[X_0, X_1, X_2]$ of degree $d \geq 1$ are homogeneous polynomials of degree $d-1$ and we denote them by $\partial_i f := \frac{\partial f}{\partial X_i}$.

Lemma 4.4.15. *Let* f *be a non-constant homogeneous polynomial.*
1) If the curve $Z(f) \subset \mathbb{P}_2$ *is smooth, then* f *is irreducible.*
2) The set of singular points is closed in $Z(f)$*.*
3) In particular if $Z(f)$ *is irreducible, the set of its smooth points is open and dense (if non-empty).*

Proof. 1) Assume that f is reducible and writes as a product $f = f_1 \cdot f_2$ for some non-constant homogeneous polynomials f_1, f_2. By Bézout's Theorem on $\mathbb{P}_2$ (Theorem D.1.14), the curves $Z(f_1)$ and $Z(f_2)$ intersect in at least 1 point, so $Z(f)$ will not be smooth at that point.

2) The set of singular points in $Z(f)$ can be described as

$$\begin{aligned} S(f) &= \left\{ x \in Z(f) \;\middle|\; \partial_0 f(x) = \partial_1 f(x) = \partial_2 f(x) = 0 \right\} \\ &= Z(f) \cap Z(\partial_0 f) \cap Z(\partial_1 f) \cap Z(\partial_2 f) \ . \end{aligned}$$

Thus $S(f)$ is closed in $Z(f)$.
3) then follows since non-empty open sets in irreducible spaces are dense. □

Proposition 4.4.16. cf. [[23], 3.1, p.23-25] and [[48], p.6]
Let $\mathcal{F} \in M^s_{am+b}$.
1) If $C = \mathcal{Z}_f(\mathcal{F})$ *is a smooth curve, then* $\mathcal{F}$ *is a locally free* $\mathcal{O}_C$*–module. Thus every stable sheaf with smooth* 1*-dimensional support is non-singular.*
2) More generally, $\mathcal{F}$ *is locally free on the smooth part of* C.

Proof. 1) The stalks $\mathcal{O}_{C,x}$ correspond to localizations of the coordinate ring of the curve C. Thus if C is smooth, $\mathcal{O}_{C,x}$ is a principal ideal domain for all $x \in C$ (see Remark 4.4.13). By Proposition 4.4.1 we also know that $\mathcal{F}$ is a torsion-free $\mathcal{O}_C$–module, i.e. $\mathcal{F}_x$ is torsion-free over $\mathcal{O}_{C,x}$ for all $x \in C$ (alternatively this can be seen by the fact that smooth curves do not have embedded components). But freeness and torsion-freeness over PIDs are equivalent. Hence $\mathcal{F}_x$ is a free $\mathcal{O}_{C,x}$–module, $\forall\, x \in C$. Coherence of $\mathcal{F}$ then implies that it is locally free of finite rank.
2) As the set of singular points is closed in C by Lemma 4.4.15, we can also consider $\mathcal{F}$ as a sheaf on $U = C \setminus S(f)$. But since the stalks $\mathcal{F}_x$ and $\mathcal{O}_{C,x}$ for $x \in U$ do not change, the same argument gives that $\mathcal{F}|_U$ is a locally free $\mathcal{O}_C|_U$–module. □

Remark 4.4.17. In [[45], 4.14, p.49] we are given an example of a semistable sheaf that is torsion-free but not locally free on its support. So it is important to check whether the support of the sheaf is indeed smooth or not.

Next we are interested in knowing “how many” of the sheaves in M^s_{am+b} are singular. By Proposition 4.4.16 this question is already related to the one of how many curves in $\mathbb{P}_2$ are singular.

Proposition 4.4.18. *The set of smooth curves of degree d in $\mathbb{P}_2$ is open and dense in the Hilbert scheme $\mathcal{C}_d(\mathbb{P}_2) \cong \mathbb{P}_{N-1}$ of all curves of degree d.*

Proof. The singular curves can be described by the set of coefficients

$$S = \Big\{ \langle f \rangle \in \mathbb{P}_{N-1} \;\Big|\; \exists\, x \in \mathbb{P}_2 \text{ such that } f(x) = \partial_0 f(x) = \partial_1 f(x) = \partial_2 f(x) = 0 \Big\} . \tag{4.21}$$

It is non-empty and proper. We shall show that it is closed.
If we write $x = (x_0 : x_1 : x_2)$, then each $\partial_i f(x)$ defines a homogeneous polynomial equation in the x_i and the coefficients of f. Hence the set

$$S' = \Big\{ \big(\langle f \rangle \,,\, x\big) \in \mathbb{P}_{N-1} \times \mathbb{P}_2 \;\Big|\; f(x) = \partial_0 f(x) = \partial_1 f(x) = \partial_2 f(x) = 0 \Big\}$$

is closed in the projective variety $\mathbb{P}_{N-1} \times \mathbb{P}_2$ (here one uses the Segre embedding to see that the image of S' is closed in $\mathbb{P}_{3N-1}$). We denote the first projection by $\pi : \mathbb{P}_{N-1} \times \mathbb{P}_2 \to \mathbb{P}_{N-1}$. Since $\mathbb{P}_2$ is a complete variety (see Proposition D.1.16), we know that π is a closed map. Hence $\pi(S') = S$ is closed as well. □

Remark 4.4.19. This result is a particular case of Bertini's Theorem, which similarly states that the set of smooth hypersurfaces of degree d in $\mathbb{P}_n$ is open and dense in $\mathbb{P}_{N-1}$ for $N = \binom{n+d}{n}$.

Definition 4.4.20. We say that a set in an irreducible topological space is *generic* if it contains a non-empty open set. Hence generic sets are in particular dense. The reason why we want to consider such sets is that they are "very big", so we are "not losing too much information". Elements from generic sets are called *generic elements*.

Corollary 4.4.21. cf. [[48], p.6] and [[45], 3.8, p.36]
The singular sheaves in M^s_{am+b} are all contained in a closed subset. Hence the set of non-singular sheaves is generic and a generic $\mathcal{F} \in M^s_{am+b}$ is a vector bundle on a smooth curve of degree a.

Proof. Consider the morphism of projective varieties $\sigma : M_{am+b} \to \mathcal{C}_a(\mathbb{P}_2)$ from Proposition 4.4.9, restricted to the open subvariety M^s_{am+b}. Since stable sheaves with smooth support are non-singular, the singular sheaves must lie in the preimage of the set of singular curves S, which is closed. σ being continuous, all singular sheaves thus lie in the closed set $\sigma^{-1}(S)$, which is also proper as there exist stable sheaves with smooth support (e.g. structure sheaves of lines). So the set of non-singular sheaves contains the non-empty open set $M^s_{am+b} \setminus \sigma^{-1}(S)$ and is generic. Choosing some $\mathcal{F}$ from this generic set, we know that it has smooth support and is therefore a locally free $\mathcal{O}_C$–module, i.e. a vector bundle on its support C, which is a smooth curve of degree a. □

Remark 4.4.22. Using [[47], 2.8, p.6-8] one can show that the subset of singular sheaves in M^s_{am+b} is even closed itself. In Corollary 5.1.39 later on we will give a proof in some particular case.

Remark 4.4.23. We will see in Proposition 4.5.14 that a generic $\mathcal{F} \in M^s_{am+b}$ is even a line bundle over a smooth curve of degree a, i.e. that $\mathcal{F}_x$ is a free $\mathcal{O}_{C,x}$–module of rank 1 for all $x \in C$.

Definition 4.4.24. The set of singular sheaves being closed we already conclude that "almost all" sheaves in M^s_{am+b} are non-singular. Denote the closed subvariety of singular sheaves in M^s_{am+b} by M'_{am+b}. Now we want some more precise information about "how many" sheaves in M^s_{am+b} are singular, i.e. we are interested in the codimension

$$\operatorname{codim}_{M^s_{am+b}}\left(M'_{am+b}\right) .$$

Computing this codimension for certain values of a and b will be our aim for the rest of the thesis.

4.5 Finite resolutions and first examples

Locally free resolutions of coherent sheaves are very useful as they allow to give a concrete description. In this section we briefly explain some methods of constructing them. Moreover we give conditions under which a morphism between direct sums of line bundles in injective (Proposition 4.5.8), resp. surjective (Proposition 4.5.9). Then we apply these methods in Proposition 4.5.14 to show that a generic sheaf in M^s_{am+b} is a line bundle on a smooth curve of degree a.

4.5.1 Syzygies

Definition 4.5.1. [[69], p.456]
Let R be a ring and M a finitely generated R-module with a set of generators $\{m_1, \ldots, m_n\}$. A *syzygy* of M is an element $(r_1, \ldots, r_n) \in R^n$ such that

$$r_1 * m_1 + \ldots + r_n * m_n = 0 \ .$$

Hence syzygies encode the relations between the generators. The set of all syzygies is a submodule of R^n, called the *module of syzygies*. It is equal to the kernel of the R-module homomorphism

$$\varepsilon \ : \ R^n \longrightarrow M \ : \ e_i \longmapsto m_i \ . \tag{4.22}$$

Syzygies can be used recurrently to compute free resolutions of finitely generated R-modules.

Notation 4.5.2. An R-module homomorphism $f : R \to R$ is uniquely determined by its value $f(1)$ and may thus be identified with some $\lambda \in R$. We extend this concept to morphisms of the type $R^n \to R^m$, which may thus be seen as $n \times m$–matrices with entries in R. From now one we will always write such a morphism as

$$R^n \xrightarrow{A} R^m \ .$$

Here we point out that, contrary to the usual convention, we consider elements of R^n as rows vectors and multiply them by the matrix A from the right to get a row vector in R^m.

Example 4.5.3. [[69], p.456]
Let $R = \mathbb{K}[X, Y]$, $\mathfrak{M} = \langle X, Y \rangle$ and $M = R/\mathfrak{M}$. M is generated by $\bar{1}$, which satisfies the relations $X * \bar{1} = 0$ and $Y * \bar{1} = 0$. Hence the module of syzygies of M is $\mathfrak{M} \leq R^1$. Now we consider $\mathfrak{M}$ as an R-module. Since this is not free, we can again compute the corresponding module of syzygies. $\mathfrak{M}$ is generated by X, Y, which satisfy the relation $(-Y) * X + X * Y = 0$. Hence the module of syzygies of $\mathfrak{M}$ is generated by $(-Y, X) \in R^2$. This element does no longer

satisfy a non-trivial relation, hence the module is free of rank 1. Combining all the previous steps, we obtain an exact sequence

$$0 \longrightarrow R \xrightarrow{\varphi} R^2 \xrightarrow{A} R \xrightarrow{\varepsilon} R/\mathfrak{M} \longrightarrow 0 \ ,$$

where $\varphi = (-Y, X)$ and

$$A = \begin{pmatrix} X \\ Y \end{pmatrix} \ .$$

This gives a finite free resolution of the R-module $R/\mathfrak{M}$.

Theorem 4.5.4 (Hilbert's Syzygy Theorem)**.** [[17], 1.1, p.6] and [[69], p.456] *Let $\mathbb{K}$ be a field and $R = \mathbb{K}[X_1, \ldots, X_n]$. Then every finitely generated module over R has a free resolution of length at most n (the length of a resolution is one less than the number of free modules in the resolution). More precisely, the procedure described in Example 4.5.3 ends in $n+1$ steps and gives an exact sequence*

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \ , \tag{4.23}$$

where each F_i is a free R-module of finite rank.

Remark 4.5.5. 1) Hilbert's Syzygy Theorem indeed only holds true for polynomial rings. It is e.g. no longer true for quotients of polynomial rings. Consider $R = \mathbb{K}[X, Y]/\langle XY \rangle$ and the module $M = R/\langle \bar{X} \rangle \cong \mathbb{K}[\bar{Y}]$, which is generated by $\bar{1}$. Then the free resolution of M obtained by the procedure described in Example 4.5.3 is

$$\ldots \longrightarrow R \xrightarrow{\cdot \bar{Y}} R \xrightarrow{\cdot \bar{X}} R \xrightarrow{\cdot \bar{Y}} R \xrightarrow{\cdot \bar{X}} R \longrightarrow M \longrightarrow 0 \ .$$

2) Theorem 4.5.4 does not mean that every resolution is of length at most n. Consider e.g. $R = \mathbb{K}[X]$ and the trivial resolution

$$0 \longrightarrow R \xrightarrow{\psi} R^2 \xrightarrow{A} R^2 \xrightarrow{\varphi} R \longrightarrow 0$$

with

$$\psi = (0, 1) \qquad , \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad , \qquad \varphi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ .$$

On the other hand this sequence is not of the type (4.23) since the surjective morphism on the right must always be ε, hence it does not contradict Theorem 4.5.4.

4.5.2 Global resolutions

Proposition 4.5.6. *Every $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ has a global resolution of the form*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \ , \tag{4.24}$$

where $\mathcal{E}, \mathcal{E}_0, \mathcal{E}_1$ are locally free and $\mathcal{E}_0, \mathcal{E}_1$ are finite direct sums of twisted structure sheaves, i.e. if $r_i = \operatorname{rk} \mathcal{E}_i$, then $\exists\, a_{ij} \in \mathbb{Z}$ such that

$$\mathcal{E}_i = \bigoplus_{j=1}^{r_i} \mathcal{O}_{\mathbb{P}_2}(a_{ij}) \ .$$

Proof. Serre's Theorem A (Theorem 4.1.3) says that $\exists\, n_0 \in \mathbb{N}$ such that $\mathcal{F}(n_0)$ is generated by its global sections, i.e. the sequence

$$\mathcal{O}_{\mathbb{P}_2}^{N_0} \longrightarrow \mathcal{F}(n_0) \longrightarrow 0$$

is exact with $N_0 = h^0(\mathcal{F})$. Twisting back we obtain

$$N_0\, \mathcal{O}_{\mathbb{P}_2}(-n_0) \stackrel{\varphi}{\longrightarrow} \mathcal{F} \longrightarrow 0 \ .$$

Let $\mathcal{K} = \ker \varphi$; the same argument then gives $n_1 \in \mathbb{N}$ such that $\mathcal{K}(n_1)$ is generated by its global sections and

$$N_1\, \mathcal{O}_{\mathbb{P}_2}(-n_1) \longrightarrow \mathcal{K} \longrightarrow 0 \ ,$$

hence

$$N_1\, \mathcal{O}_{\mathbb{P}_2}(-n_1) \stackrel{\psi}{\longrightarrow} N_0\, \mathcal{O}_{\mathbb{P}_2}(-n_0) \longrightarrow \mathcal{F} \longrightarrow 0 \ .$$

So if $\mathcal{K}' = \ker \psi$, we finally obtain the resolution

$$0 \longrightarrow \mathcal{K}' \longrightarrow N_1\, \mathcal{O}_{\mathbb{P}_2}(-n_1) \longrightarrow N_0\, \mathcal{O}_{\mathbb{P}_2}(-n_0) \longrightarrow \mathcal{F} \longrightarrow 0 \ . \tag{4.25}$$

Restricting (4.25) to an affine open set $U \subset \mathbb{P}_2$ such that $\mathcal{F}|_U \cong \widetilde{M}$, we get the exact sequence of finitely generated R-modules

$$0 \longrightarrow K' \longrightarrow R^{N_1} \longrightarrow R^{N_0} \longrightarrow M \longrightarrow 0 \ ,$$

where $R = \mathbb{K}[X, Y]$. As explained in (4.22) the sequence (4.25) is obtained by the same procedure as a resolution using syzygies since finding generators of a kernel exactly corresponds to finding syzygies. Hence Theorem 4.5.4 implies that K' is a free R-module as such resolutions have length at most 2. Repeating this argument on all affines, we obtain that $\mathcal{K}'$ is locally free. Thus (4.24) now immediately follows from the global sequence (4.25). □

Example 4.5.7. In general, the locally free sheaf $\mathcal{E}$ in (4.24) cannot be written as a direct sum of line bundles. Consider the following example on $\mathbb{P}_3$. We recall from [[35], II, 8.20.1, p.182] that the *tangent sheaf* on $\mathbb{P}_n$ is locally free and given by the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_n} \xrightarrow{\varphi} (n+1)\,\mathcal{O}_{\mathbb{P}_n}(1) \longrightarrow \mathcal{T}_{\mathbb{P}_n} \longrightarrow 0 \;,$$

where $\varphi = (X_0, \ldots, X_n)$. Moreover it is indecomposable, i.e. $\mathcal{T}_{\mathbb{P}_n}$ cannot be written be as a direct sum of non-zero locally free sheaves. This can be shown by computing its cocycles. In [[56], 2.2, p.3] it is shown that for $R = \mathbb{K}[X_0, X_1, X_2, X_3]$ and the homogenous ideal

$$I = \langle\, X_0^2, X_3^2, X_1X_3 + X_0X_2 \,\rangle$$

the subscheme $\mathcal{Z} \subset \mathbb{P}_3$ described by R/I (which is contained in a quadruple line) has a minimal resolution given by

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}_3}(-6) \xrightarrow{A_3} 4\,\mathcal{O}_{\mathbb{P}_3}(-5) \xrightarrow{A_2} 5\,\mathcal{O}_{\mathbb{P}_3}(-4) \\ \xrightarrow{A_1} 3\,\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A_0} \mathcal{O}_{\mathbb{P}_3} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \;, \end{aligned}$$

where $A_3 = (-X_2, X_1, -X_0, X_3)$. Hence the resolution is of length 4 and one could think that this contradicts Hilbert's Syzygy Theorem since locally there are only 3 variables (setting $X_i = 1$ on affines). However we can consider

$$0 \longrightarrow \ker A_1 \longrightarrow 5\,\mathcal{O}_{\mathbb{P}_3}(-4) \xrightarrow{A_1} 3\,\mathcal{O}_{\mathbb{P}_3}(-2) \xrightarrow{A_0} \mathcal{O}_{\mathbb{P}_3} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \qquad (4.26)$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_3}(-6) \xrightarrow{A_3} 4\,\mathcal{O}_{\mathbb{P}_3}(-5) \xrightarrow{A_2} \operatorname{coker} A_2 \longrightarrow 0 \qquad (4.27)$$

with $\ker A_1 = \operatorname{coker} A_2$ by exactness. Theorem 4.5.4 implies that $\ker A_1$ is locally free since (4.26) is a resolution of maximal length, hence the corresponding module is free on affines. On the other hand (4.27) and the form of A_3 (up to sign and permutations) say that $\operatorname{coker} A_2 \cong \mathcal{T}_{\mathbb{P}_3}(-6)$. Hence $\ker A_1$ is locally free, but indecomposable. Finally the resolution of $\mathcal{O}_{\mathcal{Z}}$ corresponding to the construction as in (4.23) is

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \xrightarrow{A_1} \mathcal{E}_1 \xrightarrow{A_0} \mathcal{E}_0 \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 ,$$

where all $\mathcal{E}_i$ are decomposable, but $\mathcal{E} = \ker A_1$ is not. Thus it does not contradict Theorem 4.5.4.

4.5.3 Injective and surjective morphisms

As in (4.24), let a sheaf $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be given as the cokernel of a morphism of direct sums of twisted sheaves, i.e.

$$\bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}_2}(n_i) \xrightarrow{A} \bigoplus_{j=1}^{l} \mathcal{O}_{\mathbb{P}_2}(m_j) \longrightarrow \mathcal{F} \longrightarrow 0 . \tag{4.28}$$

Thus A is an $k \times l$–matrix of homogeneous polynomials. In particular this allows to compute the Fitting ideal $\mathrm{Fitt}_0(\mathcal{F}) \subseteq \mathcal{O}_{\mathbb{P}_2}$: it is the sheaf generated by all $l \times l$–minors of A (since l is the number of generators). Again we consider the vectors as rows and multiply them by A from the right:

$$v = (v_1, \dots, v_k) \overset{A}{\longmapsto} v \cdot A = (w_1, \dots, w_l) .$$

Proposition 4.5.8. *A is injective if and only if $l \geq k$ and at least one of the $k \times k$–minors is non-zero. In particular if $k = l$, then A is injective if and only if* $\det A$ *is not the zero polynomial.*

Proof. If we want the morphism defined by A to be injective, we clearly need that $l \geq k$. And if this is the case, the rank-nullity theorem implies that we need A to be of rank k, i.e. there is at least one non-zero $k \times k$–minor since the rank is equal to the order of its largest non-zero minor. □

Proposition 4.5.9. *Let* $\mathcal{F} = \operatorname{coker} A$ *be as in* (4.28). *If $k < l$, then* $\operatorname{supp} \mathcal{F} = \mathbb{P}_2$. *If $k \geq l$, then* $\operatorname{supp} \mathcal{F}$ *is given by all points in $\mathbb{P}_2$ which vanish on all $l \times l$–minors of A. Thus A is surjective if and only if this vanishing set is empty. In particular if $k = l$, then* $\operatorname{supp} \mathcal{F} = Z(\det A)$.

Proof. Let us compute the support of $\mathcal{F}$. Taking stalks at $x \in \mathbb{P}_2$ we get

$$k\,\mathcal{O}_{\mathbb{P}_2,x} \xrightarrow{A_x} l\,\mathcal{O}_{\mathbb{P}_2,x} \longrightarrow \mathcal{F}_x \longrightarrow 0 \ .$$

As $\mathcal{F}_x = \operatorname{coker}(A_x)$, $\operatorname{supp}\mathcal{F}$ is given by all points for which A_x is not surjective, i.e. has rank $< l$. If $k < l$ this is always true, hence $\operatorname{supp}\mathcal{F} = \mathbb{P}_2$ in this case.
So let $k \geq l$. If we want that A_x has rank $< l$ in order not to be surjective (otherwise $\mathcal{F}_x = \{0\}$), we need all $l \times l$–minors in A_x to vanish. Hence $\operatorname{supp}\mathcal{F}$ is given by the points in $\mathbb{P}_2$ on which all $l \times l$–minors of A (which are homogeneous polynomials) vanish. In particular, if $k = l$ and A is a square-matrix, then $\operatorname{supp}\mathcal{F}$ is the set of all points $x \in \mathbb{P}_2$ such that $(\det A)(x) = 0$. □

4.5.4 Applications

Example 4.5.10. [[17], p.6]
Consider the morphism of sheaves $\varphi : 3\,\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}$ given by

$$\varphi = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} \ .$$

φ is surjective since the support of $\mathcal{F} = \operatorname{coker}\varphi$ is the common vanishing set of X_0, X_1, X_2, which is empty. To find a locally free resolution we have to compute the syzygies of φ. We have the relations

$$(-X_2)\cdot X_1 + X_1 \cdot X_2 = 0 \qquad , \qquad X_2 \cdot X_0 + (-X_0)\cdot X_2 = 0 \ ,$$
$$(-X_1)\cdot X_0 + X_0 \cdot X_1 = 0 \ ,$$

so the vectors $v_1 = (0, -X_2, X_1)$, $v_2 = (X_2, 0, -X_0)$ and $v_3 = (-X_1, X_0, 0)$ generate the module of syzygies. But this is still not free since

$$X_0 \cdot v_1 + X_1 \cdot v_2 + X_2 \cdot v_3 = (0,0,0)$$

and we get the vector (X_0, X_1, X_2). This one has no more non-trivial relations, hence we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\psi} 3\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} 3\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow 0 \ , \qquad (4.29)$$

where

$$\psi = (X_0\,,\,X_1\,,\,X_2) \qquad \text{and} \qquad A = \begin{pmatrix} 0 & -X_2 & X_1 \\ X_2 & 0 & -X_0 \\ -X_1 & X_0 & 0 \end{pmatrix}\,.$$

Definition 4.5.11. The exact sequence (4.29) is called the *Koszul resolution on* $\mathbb{P}_2$. Similar resolutions also hold true on $\mathbb{P}_n$.

Definition 4.5.12. Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ and $p \in \mathbb{P}_2$. As $\mathcal{F}_p$ is a module over $\mathcal{O}_{\mathbb{P}_2,p}$, we can consider the quotient $\mathcal{F}(p) := \mathcal{F}_p/\mathfrak{M}_p\mathcal{F}_p$, where $\mathfrak{M}_p \trianglelefteq \mathcal{O}_{\mathbb{P}_2,p}$ is the unique maximal ideal of the local ring. $\mathcal{F}(p)$ is a vector space over the field $\kappa(p) = \mathcal{O}_{\mathbb{P}_2,p}/\mathfrak{M}_p \cong \mathbb{K}$ since p is a closed point (see Lemma 4.1.26).

Lemma 4.5.13. *If*

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{4.30}$$

is a short exact sequence of $\mathcal{O}_{\mathcal{X}}$*–modules, then the sequence of* $\mathbb{K}$*-vector spaces*

$$\mathcal{F}(p) \longrightarrow \mathcal{G}(p) \longrightarrow \mathcal{H}(p) \longrightarrow 0$$

is exact as well.

Proof. By Lemma D.2.5, we know that $\mathcal{F}(p) = \mathcal{F}_p/\mathfrak{M}_p\mathcal{F}_p \cong \mathcal{F}_p \otimes \mathcal{O}_{\mathbb{P}_2,p}/\mathfrak{M}_p$, hence taking the stalk of (4.30) at p and tensoring by $\mathcal{O}_{\mathbb{P}_2,p}/\mathfrak{M}_p$, which is right exact, gives the desired statement. □

Now we are ready to prove the statement we already mentioned in Remark 4.4.23, namely that a generic sheaf in M^s_{am+b} is a line bundle over a smooth curve.

Proposition 4.5.14. cf. [[48], p.6]
Let $\mathcal{F} \in M^s_{am+b}$ *be such that its support* $C = \mathcal{Z}_f(\mathcal{F})$ *is a smooth curve. Then* $\mathcal{F}$ *is a locally free* $\mathcal{O}_C$*–module of rank* 1.

Proof. By Proposition 4.5.6 we know that $\mathcal{F}$ has a locally free resolution of the type

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0\,,$$

where $\mathcal{E}_i = \bigoplus_{j=1}^{r_i} \mathcal{O}_{\mathbb{P}_2}(a_{ij})$ and r_i is the rank of $\mathcal{E}_i$. Since $\mathcal{F}$ has linear Hilbert polynomial, we have $\operatorname{supp}\mathcal{F} \neq \mathbb{P}_2$ and hence $r_1 \geq r_0$ by Proposition 4.5.9. Moreover we need $r_1 = r_0$, i.e. A must be a square-matrix, otherwise its support would be given by the common vanishing set of all minors of order r_0. This is either not 1-dimensional or would contain isolated points, which contradicts purity (an example is given in Remark 4.5.16 below). A being a square-matrix also implies that $\det A$ is not the zero polynomial, otherwise $\operatorname{supp}\mathcal{F} = \mathbb{P}_2$. But then A is injective by Proposition 4.5.8 and we are left with

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 , \tag{4.31}$$

where $r_1 = r_0$. By Corollary 4.4.21 we know that a generic $\mathcal{F} \in M^s_{am+b}$ is a vector bundle on its smooth support C, i.e. $\mathcal{F}_p$ is a free $\mathcal{O}_{C,p}$–module of finite rank, $\forall\, p \in C$. Suppose that it is of rank $n \geq 2$. Using Lemma 4.5.13 we then consider the corresponding exact sequence of vector spaces

$$\mathbb{K}^{r_0} \xrightarrow{A(p)} \mathbb{K}^{r_0} \longrightarrow \mathcal{F}(p) \longrightarrow 0$$

since $\mathcal{E}_{i,p} \cong r_0\, \mathcal{O}_{\mathbb{P}_2,p}$, hence $\mathcal{E}_i(p) \cong \mathbb{K}^{r_0}$. If $\mathcal{F}_p$ is of rank n, then Nakayama's Lemma (Theorem D.1.11) implies that $\dim_{\mathbb{K}} \mathcal{F}(p) = n$. But $\mathcal{F}(p)$ is the cokernel of $A(p)$, hence the evaluated matrix $A(p)$ must have rank $r_0 - n < r_0 - 1$. So in particular all its minors of order $r_0 - 1$ must vanish. This holds for every $p \in C$. Hence the homogeneous polynomials given by the submaximal minors of A have to vanish at infinitely many points, i.e. they are all zero, implying that $\det A$ is zero as well. This contradiction shows that $n = 1$, i.e. that $\mathcal{F}$ is a line bundle on C. □

The fact that 1-dimensional sheaves have a resolution of the type (4.31) is also compatible with the following result.

Proposition 4.5.15. [[70], 2.2, p.4]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be a sheaf of rank 0. Then $\mathcal{F}$ is pure of dimension 1 if and only if there exists an exact sequence

$$0 \longrightarrow \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow 0 , \tag{4.32}$$

where $\mathcal{G}$ is a finite direct sum of line bundles (twisted structure sheaves) on $\mathbb{P}_2$.

Remark 4.5.16. Purity is indeed essential in Proposition 4.5.15. Consider e.g. the structure sheaf of a line and a point in $\mathbb{P}_2$. We may assume that the subscheme $\mathcal{Z} \subset \mathbb{P}_2$ is described by the homogeneous ideal

$$I = \langle\, X_0X_1, X_0X_2 \,\rangle$$

since its vanishing set is $Z(X_0) \cup \{(1:0:0)\}$. We have the relation

$$X_2 \cdot X_0X_1 - X_1 \cdot X_0X_2 = 0 \ ,$$

but none on the generator $(X_2, -X_1)$, hence computing syzygies gives the locally free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\psi} 2\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \ ,$$

where

$$\psi = (X_2\,,\, -X_1) \qquad \text{and} \qquad A = \begin{pmatrix} X_0X_1 \\ X_0X_2 \end{pmatrix} \ .$$

This is minimal by the Theorem 4.5.4, but not of the form (4.32).

4.5.5 Hilbert polynomial $m+1$

Now we are ready to study the first examples. We start with the most simple case M_{m+1}. As $\gcd(1,1) = 1$, Corollary 4.3.7 and Corollary 4.3.8 imply that M_{m+1} is a fine moduli space and we get $M_{m+1} = M^s_{m+1}$ as all involved sheaves are stable. By Theorem 4.3.10 we also know that M_{m+1} is smooth, irreducible and of dimension 2.

Lemma 4.5.17. *Let $z \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ be a non-zero linear form, i.e.*

$$z = aX_0 + bX_1 + cX_2$$

with $a, b, c \in \mathbb{K}$, and consider the line $L = Z(z)$. Then $\mathcal{O}_L \in M_{m+1}$.

Proof. By Proposition 4.3.9 the structure sheaf $\mathcal{O}_L$ is stable and (4.17) implies that it has Hilbert polynomial $m + \frac{3-1}{2} = m+1$. Thus $\mathcal{O}_L \in M_{m+1}$. □

In particular (4.18) also gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\cdot z} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_L \longrightarrow 0\ . \tag{4.33}$$

If we consider this sequence on some affine set $U_i \subset \mathbb{P}_2$, we get

$$0 \longrightarrow \mathbb{K}[X,Y] \xrightarrow{\cdot z_i} \mathbb{K}[X,Y] \longrightarrow \mathbb{K}[X,Y]/\langle\, z_i\,\rangle \longrightarrow 0\ ,$$

where z_i is obtained by replacing $X_i = 1$ in z and $\mathbb{K}[X,Y]/\langle\, z_i\,\rangle$ is the coordinate ring $\mathbb{K}[L_i]$ of the curve $L_i = Z(z_i) \subset \mathbb{A}_2$. This is nothing but the local description of the structure sheaf $\mathcal{O}_L$.
The following result now states that all sheaves in M_{m+1} are actually given by such a resolution.

Proposition 4.5.18. [[15], p.2]
Isomorphism classes of sheaves $\mathcal{F} \in M_{m+1}$ are exactly those that are given by a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\cdot z} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.34}$$

for some $z \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ with $\langle z\rangle \neq 0$.

(4.34) and Proposition 4.5.9 imply that $C = \mathcal{Z}_f(\mathcal{F})$ is given by all points in $\mathbb{P}_2$ on which the linear form z vanishes, i.e. $C = Z(z)$ is a line. But this is the same sitaution as in (4.33) and so we get $\mathcal{F} \cong \mathcal{O}_C$. In particular $\mathcal{F}$ is a (locally) free $\mathcal{O}_C$–module, hence non-singular and there are no singular sheaves in M_{m+1}.

Corollary 4.5.19. [[47], p.36]
Isomorphism classes of stable sheaves in M_{m+1} are exactly the structure sheaves $\mathcal{O}_L$ of lines $L \subset \mathbb{P}_2$. Hence M_{m+1} may be identified with the space of lines in $\mathbb{P}_2$; we get $M'_{m+1} = \emptyset$ and $M_{m+1} \cong \mathcal{C}_1(\mathbb{P}_2) \cong \mathbb{P}_{\binom{3}{2}-1} = \mathbb{P}_2$.

4.5.6 Hilbert polynomial $2m+1$

Similarly $\gcd(2,1) = 1$, so Theorem 4.3.10 implies that M_{2m+1} is a fine moduli space which is smooth, irreducible and of dimension 5. By Proposition 4.3.9 we know again by that the structure sheaf of a quadratic curve (conic) in $\mathbb{P}_2$ is stable and has Hilbert polynomial

$$2 \cdot m + \tfrac{3\cdot 2 - 2^2}{2} = 2m+1\ .$$

Proposition 4.5.20. [[15], p.2]
The isomorphism classes of stable sheaves $\mathcal{F} \in M_{2m+1}$ are exactly those given by a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\cdot q} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0 ,$$

where $q \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2)\big)$ is a quadratic form with $\langle q \rangle \neq 0$.

Exactly the same argument as in Section 4.5.5 shows that $\mathcal{F} \cong \mathcal{O}_C$ for $C = Z(q)$, so again there are no singular sheaves.

Remark 4.5.21. Note here that C does not need to be smooth, e.g. $q = X_0^2$ would give a double line. Hence we get examples of non-singular sheaves with singular support. In particular this shows that the converse of Proposition 4.4.16 is false.

Corollary 4.5.22. [[47], p.36]
The fine moduli space M_{2m+1} is isomorphic to the space of conics in $\mathbb{P}_2$, so we get $M'_{2m+1} = \emptyset$ and $M_{2m+1} \cong \mathcal{C}_2(\mathbb{P}_2) \cong \mathbb{P}_{\binom{4}{2}-1} = \mathbb{P}_5$.

4.6 Hilbert polynomial $3m + 1$

The first non-trivial example is given by M_{3m+1}. Again $\gcd(3,1) = 1$ implies that the moduli space is fine, smooth, irreducible and of dimension 10. Moreover it only contains isomorphism classes of stable sheaves.

Remark 4.6.1. Note that structure sheaves of cubic curves in $\mathbb{P}_2$ (curves of degree 3) do not define elements in M_{3m+1} since by (4.17) they have Hilbert polynomial

$$3 \cdot m + \tfrac{3 \cdot 3 - 3^2}{2} = 3m .$$

If $C = Z(f)$ is the vanishing set of a homogeneous polynomial f of degree 3, then $\mathcal{O}_C \in M_{3m}$ and $\mathcal{O}_C(1) \in M_{3m+3}$. Hence for M_{3m+1} we cannot use the same methods as for $m + 1$ and $2m + 1$.

4.6.1 Description of sheaves in M_{3m+1}

Proposition 4.6.2. [[25], 3.2, p.6] and [[47], p.37]
Let $\mathcal{F} \in M_{3m+1}$ and $C = \mathcal{Z}_f(\mathcal{F})$ be its Fitting support. Then there exists a point $p \in C$ and a non-split extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \text{Sky}_p(\mathbb{K}) \longrightarrow 0 \,. \tag{4.35}$$

Proof. By Proposition 4.4.5 and Remark 4.6.1 we know that $C \subset \mathbb{P}_2$ is a curve of degree 3 and its structure sheaf $\mathcal{O}_C$ has Hilbert polynomial $3m$. Let $C = Z(f)$ be given by the vanishing set of some homogeneous polynomial $f \in \mathbb{K}[X_0, X_1, X_2]$ of degree 3. We obtain an injection $\mathcal{O}_C \hookrightarrow \mathcal{F}$ as follows.
Since $P_{\mathcal{F}}(0) = 1 = h^0(\mathcal{F}) - h^1(\mathcal{F})$ by (4.16), the sheaf $\mathcal{F}$ has a non-zero global section $s \in H^0(\mathbb{P}_2, \mathcal{F})$. Let $U_i \subset \mathbb{P}_2$ be an affine open set and denote by $f_i(X, Y)$ the polynomial in 2 variables obtained by replacing $X_i = 1$ in f. If $\mathcal{F}|_{U_i}$ corresponds to some module M over $\mathbb{K}[X, Y]$, we define

$$\mathbb{K}[X, Y]/\langle\, f_i \,\rangle \longrightarrow M \ : \ \bar{g} \longmapsto g * s_{|U_i} \ ,$$

which is well-defined since $f_i \in \text{Fitt}_0(M) \subseteq \text{Ann}_{\mathbb{K}[X,Y]}(M)$. Under $\sim$ this gives an injective morphism $\mathcal{O}_C \to \mathcal{F}$ since $\mathcal{F}$ is a torsion-free $\mathcal{O}_C$–module (Proposition 4.4.1). The quotient

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{O}_C \longrightarrow 0$$

has Hilbert polynomial $P_{\mathcal{F}}(m) - P_{\mathcal{O}_C}(m) = (3m+1) - 3m = 1$ and is therefore a skyscraper sheaf supported on a single point $p \in \mathbb{P}_2$. This point must belong to C, otherwise $\mathcal{F}_p = \{0\}$. Finally the extension is also non-split because $\mathcal{O}_C \oplus \text{Sky}_p(\mathbb{K})$ is not pure of dimension 1, thus not stable. □

Remark 4.6.3. There exists an equivalent characterization for sheaves in M_{3m+1} by means of resolutions, which we will see in (4.36). Using this description one can also write down explicitly the coordinates of the point $p \in C$ from (4.35). This will be the aim of Lemma 4.6.9.

Proposition 4.6.4. [[25], 3.3, p.6-7 & 6.1, p.15] and [[15], p.2]
The non-trivial extension (4.35) *of* $\mathcal{F} \in M_{3m+1}$ *is equivalent to a resolution of* $\mathcal{O}_{\mathbb{P}_2}$*–modules*

$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{4.36}$$

with A *a matrix of the form*

$$A = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} ,$$

where $z_1, z_2 \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ *are linear forms and* $q_1, q_2 \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2)\big)$ *are quadratic forms such that* $\langle \det A \rangle \neq 0$ *and the vectors* $\langle z_1 \rangle, \langle z_2 \rangle \in \mathbb{K}^3$ *are linearly independent over* $\mathbb{K}$.

Remark 4.6.5. Note that the exact sequence (4.36) is compatible with the formula of Example 4.3.1 since

$$P_{\mathcal{F}}(m) = \frac{m(m+1)}{2} + \frac{(m+2)(m+1)}{2} - 2 \cdot \frac{m(m-1)}{2} = 3m+1\ .$$

Remark 4.6.6. By Proposition 4.5.8 we need $\langle \det A \rangle \neq 0$ for injectivity in (4.36). But why do we need the condition on linear independence ?
Assume that $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are linearly dependent, i.e. $\lambda\, z_1 + \mu\, z_2 = 0$ for some $(\lambda, \mu) \neq (0,0)$. Then one can always perform finitely many row transformations to obtain

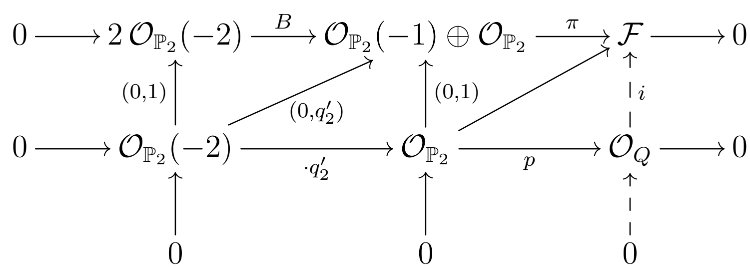
$$A \sim \begin{pmatrix} z_1' & q_1' \\ 0 & q_2' \end{pmatrix} = B$$

for some linear, resp. quadratic forms z_1', q_1', q_2' with $\langle z_1' \rangle \neq 0$ and $\langle q_2' \rangle \neq 0$. Let $g \in \mathrm{GL}_2(\mathbb{K})$ describe the linear transformations of the rows, i.e. $B = g \cdot A$. As g is invertible, the cokernel $\mathcal{F}_B$ of the morphism defined by B will be isomorphic[20] to the one defined by A, i.e.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{A} & \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F}_A & \longrightarrow & 0 \\ & & {\scriptstyle g}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \cong}\uparrow & & \\ 0 & \longrightarrow & 2\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{B} & \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F}_B & \longrightarrow & 0 \end{array}$$

[20] We will give a rigorous proof of this fact more generally in Lemma 5.1.11.

Now consider



where $\mathcal{O}_Q$ is the structure sheaf of the conic $Q = Z(q_2')$. As

$$(\pi \circ (0,1)) \circ (\cdot q_2') = \pi \circ B \circ (0,1) = 0 \; ,$$

$\pi \circ (0,1)$ factorizes through the cokernel and we get a morphism $i : \mathcal{O}_Q \to \mathcal{F}$. This i is even injective: (with an abuse of notation) let $x \in \mathcal{O}_Q$ such that $i(x) = 0$.

$$\begin{gathered} p \text{ surjective } \;\Rightarrow\; \exists\, y \in \mathcal{O}_{\mathbb{P}^2} \text{ such that } x = p(y) \\ 0 = i(x) = i\big(p(y)\big) = \big(\pi \circ (0,1)\big)(y) = \pi(0,y) \\ \Rightarrow\;\; (0,y) \in \operatorname{im} B \;\Rightarrow\; \exists\, a,b \in \mathcal{O}_{\mathbb{P}^2}(-2) \text{ such that } (0,y) = (a,b)\cdot B \\ (a,b)\cdot B = (az_1' \,,\, aq_1' + bq_2') \\ \Rightarrow\;\; a = 0 \text{ since } \langle z_1' \rangle \neq 0 \;\Rightarrow\; (0,y) = (0,b)\cdot B = (0, bq_2') \\ \Rightarrow\;\; y = b \cdot q_2' \;\Rightarrow\; y \in \operatorname{im}(\cdot q_2') = \ker p \;\Rightarrow\; x = p(y) = 0 \end{gathered}$$

But this means that $\mathcal{O}_Q$ is a non-zero proper coherent subsheaf of $\mathcal{F}$. Since Q is a curve of degree 2 we know from (4.17) that $\mathcal{O}_Q$ has Hilbert polynomial $2m+1$, hence its reduced Hilbert polynomial is $m + \frac{1}{2}$. But $P_{\mathcal{F}}(m) = 3m+1$ and $p_{\mathcal{F}}(m) = m + \frac{1}{3} < m + \frac{1}{2}$, which contradicts stability of $\mathcal{F}$. So $\langle z_1 \rangle$ and $\langle z_2 \rangle$ must be linearly independent.

4.6.2 Parameter space and criterion for singularity

We shall determine in which case the matrix A from (4.36) defines a singular sheaf via $\mathcal{F} = \operatorname{coker} A$. For this, we introduce the following notations.

Notation 4.6.7. We write the polynomials z_1, z_2, q_1, q_2 in A as

$$\begin{gathered} z_1 = a_0\, X_0 + a_1\, X_1 + a_2\, X_2 \qquad , \qquad z_2 = b_0\, X_0 + b_1\, X_1 + b_2\, X_2 \\ q_1 = A_{00}\, X_0^2 + A_{01}\, X_0X_1 + A_{02}\, X_0X_2 + A_{11}\, X_1^2 + A_{12}\, X_1X_2 + A_{22}\, X_2^2 \\ q_2 = B_{00}\, X_0^2 + B_{01}\, X_0X_1 + B_{02}\, X_0X_2 + B_{11}\, X_1^2 + B_{12}\, X_1X_2 + B_{22}\, X_2^2 \end{gathered} \tag{4.37}$$

with coefficients $a_i, b_i, A_{ij}, B_{ij} \in \mathbb{K}$, so the space of all such matrices may be identified with the affine variety $\mathbb{A}_{18}$. Now consider

$$X := \left\{ \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \;\middle|\; \langle z_1 \rangle, \langle z_2 \rangle \text{ are lineary independent and } \langle z_1q_2 - z_2q_1 \rangle \neq 0 \right\} ,$$

which parametrizes all the matrices from Proposition 4.6.4.

Lemma 4.6.8. [[41], p.3-4]
X is an open subset of $\mathbb{A}_{18}$, hence a quasi-affine variety. In particular, $X \subset \mathbb{A}_{18}$ is dense.

Proof. Since we consider $\mathbb{A}_{18}$ as an affine variety, closed subsets are given by vanishing sets of polynomials in 18 variables. We shall show that $\mathbb{A}_{18} \setminus X$ is closed. Note that the coefficients of the polynomial $\det A = z_1q_2 - z_2q_1$ are such polynomials in the 18 variables from (4.37). As it is a homogeneous polynomial of degree 3, we get $\binom{5}{2} = 10$ closed conditions, i.e. the condition $\langle z_1q_2 - z_2q_1 \rangle \neq 0$ is open. Saying that $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are linearly independent means that at least one of the 2×2–minors of

$$D = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{4.38}$$

must be non-zero, so in the complement we get that all minors must vanish, i.e. again 3 closed conditions. Hence $X \subset \mathbb{A}_{18}$ is open. □

Lemma 4.6.9. [[41], 1.1, p.5]
The intersection point of the projective lines $Z(z_1)$ and $Z(z_2)$ in $\mathbb{P}_2$ is given by $p = (d_0 : d_1 : d_2) \in \mathbb{P}_2$, where

$$d_0 = a_1b_2 - a_2b_1 \qquad , \qquad d_1 = -(a_0b_2 - a_2b_0) \qquad , \qquad d_2 = a_0b_1 - a_1b_0$$

are the 2×2–minors of the matrix D in (4.38) defined by $\langle z_1 \rangle$ and $\langle z_2 \rangle$.

Proof. As $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are linearly independent, we know that at least one of the minors is non-zero, hence $p \in \mathbb{P}_2$ is well-defined. Moreover the corresponding projective lines meet in exactly 1 point by Theorem D.1.14. Now consider

$$\begin{pmatrix} a_0 & a_0 & b_0 \\ a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \end{pmatrix} \quad , \quad \begin{pmatrix} b_0 & a_0 & b_0 \\ b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \end{pmatrix} .$$

These matrices have determinant zero. Expanding them along the first column gives the equations

$$a_0 d_0 + a_1 d_1 + a_2 d_2 = 0 \qquad \text{and} \qquad b_0 d_0 + b_1 d_1 + b_2 d_2 = 0 ,$$

which exactly means that $(d_0 : d_1 : d_2)$ is the point where both z_1 and z_2 vanish. □

Now we have the following criterion for a sheaf in M_{3m+1} that is parametrized by some $A \in X$ to be singular.

Proposition 4.6.10. [[25], 6.2, p.15] and [[41], 1.2, p.5]
Let $A \in X$ and $p \in \mathbb{P}_2$ be as in Lemma 4.6.9. Then the following statements are equivalent:
1) The sheaf $\mathcal{F} := \operatorname{coker} A$ is singular.
2) The quadratic forms q_1 and q_2 both vanish at p.
3) p is a singular point of the curve $C = Z(\det A) = \mathcal{Z}_f(\mathcal{F})$.

Proof. As $\mathcal{F}$ is coherent, we have that $\mathcal{F}$ is a locally free $\mathcal{O}_C$–module if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_{C,x}$–module, $\forall\, x \in \mathbb{P}_2$. We first show that $\mathcal{F}_x \cong \mathcal{O}_{C,x}$, $\forall\, x \in \mathbb{P}_2 \setminus \{p\}$, i.e. all these stalks are free $\mathcal{O}_{C,x}$–modules of rank 1. Hence whether $\mathcal{F}$ is singular or not only depends on the stalk $\mathcal{F}_p$.
If $x \neq p$, at least one of the linear forms z_1, z_2 does not vanish at x, say e.g. $z_1(x) \neq 0$. Taking stalks at x of the exact sequence (4.36) gives

$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2,x} \xrightarrow{A_x} 2\,\mathcal{O}_{\mathbb{P}_2,x} \longrightarrow \mathcal{F}_x \longrightarrow 0 , \tag{4.39}$$

where the entries of A_x are those of A considered as elements in $\mathcal{O}_{\mathbb{P}_2,x}$. As $z_1(x) \neq 0$, it is invertible in this local ring with inverse $\frac{1}{z_1} \in \mathcal{O}_{\mathbb{P}_2,x}$. But then

$$A_x = \begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{q_1}{z_1} \\ \frac{z_2}{z_1} & \frac{q_2}{z_1} \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{q_1}{z_1} \\ 0 & \frac{q_2}{z_1} - \frac{z_2}{z_1} \cdot \frac{q_1}{z_1} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z_1^2} \cdot (z_1 q_2 - z_2 q_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z_1^2} \cdot \det A \end{pmatrix} ,$$

where $\frac{1}{z_1^2}$ is a unit, hence

$$\mathcal{F}_x \cong \operatorname{coker} A_x = 2\,\mathcal{O}_{\mathbb{P}_2,x} / \operatorname{im} A_x \;\cong\; \mathcal{O}_{\mathbb{P}_2,x} / \operatorname{im}(\cdot \det A) = \mathcal{O}_{\mathbb{P}_2,x} / \langle \det A \rangle = \mathcal{O}_{C,x}$$

since A_x is an isomorphism in the first component.

1) $\Rightarrow$ 2) : For p, we already know that $z_1(p) = z_2(p) = 0$. If now $q_i(p) \neq 0$ for some $i = 1, 2$, then one proceeds as above to obtain

$$A_p \sim \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{q_i^2} \cdot \det A \end{pmatrix} ,$$

thus $\mathcal{F}_p \cong \mathcal{O}_{C,p}$ is free and $\mathcal{F}$ is a locally free $\mathcal{O}_C$–module. An alternative proof goes as follows:
From (4.39) and Lemma 4.5.13 we get the exact sequence of $\mathbb{K}$-vector spaces

$$\mathbb{K}^2 \xrightarrow{A(p)} \mathbb{K}^2 \longrightarrow \mathcal{F}(p) \longrightarrow 0 \,, \tag{4.40}$$

where

$$A(p) = \begin{pmatrix} z_1(p) & q_1(p) \\ z_2(p) & q_2(p) \end{pmatrix} \sim \begin{pmatrix} 0 & q_i(p) \\ 0 & * \end{pmatrix}$$

is the evaluated matrix at p. Since $q_i(p) \neq 0$, $A(p)$ is of rank 1 and the vector space $\mathcal{F}(p) = \operatorname{coker} A(p)$ has dimension 1. Nakayama's Lemma then implies that $\mathcal{F}_p$ is generated by 1 element over $\mathcal{O}_{\mathbb{P}_2,p}$ (but not free since it is torsion). The module structure defined by $\mathcal{O}_{\mathbb{P}_2,p} \twoheadrightarrow \mathcal{O}_{C,p}$ implies that $\mathcal{F}_p$ is also generated by 1 element over $\mathcal{O}_{C,p}$. Hence the annihilator and the Fitting ideal of $\mathcal{F}_p$ coincide (Lemma 1.4.8). But then no non-zero element from $\mathcal{O}_{C,p}$ can annihilate $\mathcal{F}_p$ (since all those that annihilate are divided out). Hence the generator of $\mathcal{F}_p$ has no relations over $\mathcal{O}_{C,p}$ and we conclude that $\mathcal{F}_p \cong \mathcal{O}_{C,p}$ is free.

2) $\Rightarrow$ 1) : Let $q_1(p) = q_2(p) = 0$. We shall show that $\mathcal{F}_p$ is not a free $\mathcal{O}_{C,p}$–module, hence that $\mathcal{F}$ is singular. Assume that $\mathcal{F}_p$ is free. From Proposition 4.5.14 we know that $\mathcal{F}_p$ must be free of rank 1. Hence we get

$$\mathcal{F}(p) = \mathcal{F}_p/\mathfrak{M}_p\mathcal{F}_p \cong \mathcal{O}_{C,p}/\mathfrak{M}_p\mathcal{O}_{C,p} = \mathcal{O}_{C,p}/\mathfrak{M}_{C,p} \cong \mathbb{K} \ , \tag{4.41}$$

where $\mathfrak{M}_{C,p} \trianglelefteq \mathcal{O}_{C,p}$ is the unique maximal ideal. We will prove formula (4.41) afterwards. On the other hand, we have

$$\mathbb{K}^2 \xrightarrow{A(p)} \mathbb{K}^2 \longrightarrow \mathcal{F}(p) \longrightarrow 0$$

with $z_1(p) = z_2(p) = q_1(p) = q_2(p) = 0$, thus $A(p) = 0$ is the zero map and $\mathcal{F}(p) \cong \mathbb{K}^2$. This contradiction shows that $\mathcal{F}_p$ cannot be free.
Note : in general, the evaluated matrix $A(x)$ is not well-defined if one of the entries of A does not vanish at x. But

$$\mathrm{rk}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \mathrm{rk}\begin{pmatrix} \lambda\,\alpha & \lambda^2\,\beta \\ \lambda\,\gamma & \lambda^2\,\delta \end{pmatrix} \ , \ \forall\,\lambda \neq 0 \ ,$$

hence its image and cokernel (as vector spaces) are independent of the chosen coordinates for $x = (x_0 : x_1 : x_2) \in \mathbb{P}_2$.

2) $\Leftrightarrow$ 3) : The vectors $\langle X_0\rangle = (1,0,0)$, $\langle X_1\rangle = (0,1,0)$ and $\langle X_2\rangle = (0,0,1)$ form a basis of $\mathbb{K}^3$. Moreover we know that $\langle z_1\rangle$ and $\langle z_2\rangle$ are linearly independent, hence there exists a third vector $v \in \mathbb{K}^3$ such that $\langle z_1\rangle, \langle z_2\rangle, v$ is another basis of $\mathbb{K}^3$. Write $v = (a,b,c)$ and let z_0 be the linear form defined by $\langle z_0\rangle = v$. So

$$\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & b & c \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = T \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = T^{-1} \cdot \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}$$

and this defines a change of variables as the matrix T is invertible. In particular, $X_i = X_i(z_0, z_1, z_2)$.
Denote $f = \det A$, which is a homogeneous polynomial of degree 3. Now p is a singular point of the curve C if and only if

$$\frac{\partial f}{\partial X_0}(p) = \frac{\partial f}{\partial X_1}(p) = \frac{\partial f}{\partial X_2}(p) = 0 \ . \tag{4.42}$$

By the chain rule, we have

$$\frac{\partial f}{\partial z_i} = \frac{\partial f}{\partial X_0} \cdot \frac{\partial X_0}{\partial z_i} + \frac{\partial f}{\partial X_1} \cdot \frac{\partial X_1}{\partial z_i} + \frac{\partial f}{\partial X_2} \cdot \frac{\partial X_2}{\partial z_i} \ ,$$
$$\frac{\partial f}{\partial X_i} = \frac{\partial f}{\partial z_0} \cdot \frac{\partial z_0}{\partial X_i} + \frac{\partial f}{\partial z_1} \cdot \frac{\partial z_1}{\partial X_i} + \frac{\partial f}{\partial z_2} \cdot \frac{\partial z_2}{\partial X_i} \ ,$$

hence (4.42) is equivalent to

$$\frac{\partial f}{\partial z_0}(p) = \frac{\partial f}{\partial z_1}(p) = \frac{\partial f}{\partial z_2}(p) = 0 \ ,$$

where each $\frac{\partial f}{\partial z_i}$ has to be seen as a function in z_0, z_1, z_2. Recall Euler's formula, which states that

$$z_0 \cdot \frac{\partial f}{\partial z_0} + z_1 \cdot \frac{\partial f}{\partial z_1} + z_2 \cdot \frac{\partial f}{\partial z_2} = 3f \ .$$

Note that $f(p) = z_1(p) = z_2(p) = 0$. Expanding the invertible matrix T along the first line, we find

$$0 \neq \det T = a \cdot d_0 + b \cdot d_1 + c \cdot d_2 = z_0(p) \ ,$$

hence $\frac{\partial f}{\partial z_0}(p) = 0$ is always true. As $f = z_1 q_2 - z_2 q_1$, we moreover have

$$\frac{\partial f}{\partial z_1} = q_2 + z_1 \cdot \frac{\partial q_2}{\partial z_1} - z_2 \cdot \frac{\partial q_1}{\partial z_1} \quad \Rightarrow \quad \frac{\partial f}{\partial z_1}(p) = q_2(p) \ ,$$
$$\frac{\partial f}{\partial z_2} = z_1 \cdot \frac{\partial q_2}{\partial z_2} - q_1 - z_2 \cdot \frac{\partial f}{\partial z_2} \quad \Rightarrow \quad \frac{\partial f}{\partial z_2}(p) = -q_1(p) \ .$$

Summarizing all the above, we obtain

$$p \text{ is a singular point of } C \ \Leftrightarrow \ (4.42) \ \Leftrightarrow \ \frac{\partial f}{\partial z_1}(p) = \frac{\partial f}{\partial z_2}(p) = 0$$
$$\Leftrightarrow \ q_1(p) = q_2(p) = 0 \ ,$$

which finally proves the equivalence. □

Remark 4.6.11. Here we provide the proof of (4.41). As $c : C \hookrightarrow \mathbb{P}_2$ is a closed immersion, the induced surjective map $\mathcal{O}_{\mathbb{P}_2,p} \twoheadrightarrow \mathcal{O}_{C,p}$ is a local homomorphism. So we may reduce the situation to the case of a quotient of local rings

$$R \xrightarrow{\pi} S \longrightarrow 0$$

with unique maximal ideals $\mathfrak{M} \trianglelefteq R$ and $\mathfrak{N} \trianglelefteq S$ such that $\pi(\mathfrak{M}) \subseteq \mathfrak{N}$, or equivalently $\mathfrak{M} = \pi^{-1}(\mathfrak{N})$. Now we have to show that $S/\mathfrak{M}S = S/\mathfrak{N}$, i.e. that $\pi^{-1}(\mathfrak{N}) * S = \mathfrak{N}$. As S is a quotient of R, we may write $S \cong R/I$ for some ideal $I \trianglelefteq R$ and π is just the canonical quotient map.
$\subseteq$: let $r \in R$ such that $\pi(r) = \bar{r} \in \mathfrak{N} \;\Rightarrow\; r * \bar{s} = \bar{r} \cdot \bar{s} \in \mathfrak{N}, \forall\, \bar{s} \in S$
$\supseteq$: let $\bar{r} \in \mathfrak{N}$ and choose $r \in R$ such that $\pi(r) = \bar{r} \;\Rightarrow\; r * \bar{1} = \bar{r}$ with $r \in \pi^{-1}(\mathfrak{N})$

4.6.3 Codimension of the singular sheaves

Proposition 4.6.12. [[41], p.6-9]
Let $X' \subset X$ be the subset of all matrices which define singular sheaves. Then X' is a smooth closed subvariety of X and $\operatorname{codim}_X X' = 2$.

Proof. Let $A \in X$ with the notations as in (4.37). By Proposition 4.6.10, we know that $A \in X'$ if and only if $q_1(p) = q_2(p) = 0$. Writing out the equations, this gives

$$\begin{aligned} F_1 \;&:\; q_1(p) = A_{00}\, d_0^2 + A_{01}\, d_0 d_1 + A_{02}\, d_0 d_2 + A_{11}\, d_1^2 + A_{12}\, d_1 d_2 + A_{22}\, d_2^2 = 0 \\ F_2 \;&:\; q_2(p) = B_{00}\, d_0^2 + B_{01}\, d_0 d_1 + B_{02}\, d_0 d_2 + B_{11}\, d_1^2 + B_{12}\, d_1 d_2 + B_{22}\, d_2^2 = 0 \end{aligned}$$

with $d_0 = a_1 b_2 - a_2 b_1$, $d_1 = a_2 b_0 - a_0 b_2$ and $d_2 = a_0 b_1 - a_1 b_0$. Hence F_1 and F_2 define polynomials in 18 variables. These 2 equations completely determine X', which is hence given by their common vanishing set intersected with X, i.e.

$$X' = X \cap Z(F_1, F_2) \subset \mathbb{A}_{18}\,. \tag{4.43}$$

In particular, this shows that X' is an (affine) algebraic subvariety of codimension 2 in X.

$$X' \xhookrightarrow[\text{codim}=2]{\text{closed}} X \xhookrightarrow{\text{open}} \mathbb{A}_{18}$$

The subvariety $X' \subset X$ is smooth since the Jacobian matrix $J(F_1, F_2)$ has maximal rank at all points in X:

$$\begin{pmatrix} * & \dots & * & d_0^2 & d_0 d_1 & d_0 d_2 & d_1^2 & d_1 d_2 & d_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \dots & * & 0 & 0 & 0 & 0 & 0 & 0 & d_0^2 & d_0 d_1 & d_0 d_2 & d_1^2 & d_1 d_2 & d_2^2 \end{pmatrix}$$

This matrix has always rank 2 since at least one of the d_i is non-zero. $\square$

4.6.4 Group action and quotient map

Until now we only found a criterion to decide whether a matrix from X defines a singular sheaf in M_{3m+1} and that the subvariety $X' \subset X$ giving singular sheaves is of codimension 2. But this doesn't say anything about the codimension of the subvariety of singular sheaves in M_{3m+1} yet. We have to find a description of M'_{3m+1} itself.
For this first note that there is no 1-to-1 correspondence between M_{3m+1} and the parameter space X. More precisely, two different matrices $A, B \in X$ may define isomorphic sheaves $\mathcal{F}_A$ and $\mathcal{F}_B$ via their cokernels; we already saw such an example in Remark 4.6.6. Hence we have to divide out a certain action on X in order to obtain a bijection with M_{3m+1}. This action is given by the group of automorphisms of the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 2\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{A} & \mathcal{O}_{\mathbb{P}_2}(-1)\oplus\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F}_A & \longrightarrow & 0 \\ & & {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & {\scriptstyle \cong}\uparrow & & \\ 0 & \longrightarrow & 2\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{B} & \mathcal{O}_{\mathbb{P}_2}(-1)\oplus\mathcal{O}_{\mathbb{P}^2} & \longrightarrow & \mathcal{F}_B & \longrightarrow & 0 \end{array}$$

Notation 4.6.13. Consider the group

$$G := \operatorname{Aut}\big(2\,\mathcal{O}_{\mathbb{P}_2}(-2)\big) \times \operatorname{Aut}\big(\mathcal{O}_{\mathbb{P}_2}(-1)\oplus\mathcal{O}_{\mathbb{P}_2}\big)$$

The first factor is nothing but $\mathrm{GL}_2(\mathbb{K})$. We denote the second one by H and it consists of matrices of the form

$$H = \left\{ \begin{pmatrix} \lambda & z \\ 0 & \mu \end{pmatrix} \;\middle|\; \lambda,\mu\in\mathbb{K},\ \lambda\mu\neq 0,\ z\in\Gamma\big(\mathbb{P}_2,\mathcal{O}_{\mathbb{P}_2}(1)\big) \right\}. \tag{4.44}$$

As a variety H is isomorphic to $\mathbb{K}^*\times\mathbb{K}^*\times\mathbb{K}^3$. The group $G = \mathrm{GL}_2(\mathbb{K})\times H$ then acts on X by the rule $(g,h)\,.\,A := g\cdot A\cdot h^{-1}$ and this action corresponds to isomorphisms of exact sequences. Hence the orbits of G are in 1-to-1 correspondence with the isomorphism classes of sheaves defined as cokernels by these exact sequences, i.e. with points in M_{3m+1}.

Remark 4.6.14. One is tempted to expect that $X/G \cong M_{3m+1}$. However this is not completely true. The reason for this is that points in X have a non-trivial stabilizer, so the action of G on X is not free. Indeed for all $A \in X$, we have

$$\Gamma := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \times \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subseteq \mathrm{Stab}_G(A) .$$

In Corollary 5.2.37 we will show more generally that Γ is indeed equal to the stabilizer of any $A \in X$. Being independent of A we can hence divide it out and obtain a new group $\mathbb{P}G := G/\Gamma$, which now acts freely on X and whose orbits still correspond bijectively to the points in M_{3m+1}. So we get (at least) a set bijection $X/\mathbb{P}G \cong M_{3m+1}$, which is also compatible with dimensions. Indeed $\mathrm{GL}_2(\mathbb{K})$ may be identified with an open subset of $\mathbb{A}_4$ and H is open in $\mathbb{A}_5$, hence $\dim G = 9$ and $\Gamma \cong \mathbb{K}^*$ implies that $X/\mathbb{P}G$ is of dimension $18-(9-1)=10$, the same as M_{3m+1}. We denote the corresponding quotient map by

$$\nu \;:\; X \longrightarrow X/\mathbb{P}G \cong M_{3m+1} .$$

According to all previous constructions it is given by $[A] \mapsto [\operatorname{coker} A]$. Now we even have

Theorem 4.6.15. [[25], 6.3, p.16] and [[41], p.4]

1) M_{3m+1} is a geometric quotient of X by the action of the (non-reductive) group G.[21] *In particular, the bijection $X/\mathbb{P}G \cong M_{3m+1}$ is an isomorphism of projective varieties.*

2) The quotient map $\nu : X \to M_{3m+1}$ is a morphism of projective varieties and defines a principal bundle over M_{3m+1} with fiber $\mathbb{P}G$.

Proof. We only give certain ideas of how to prove these statements; they both follow from a similar version of the Theorem of Gleason from differential geometry. X is a variety on which the group $\mathbb{P}G$ acts freely, so the quotient $X/\mathbb{P}G$ can be endowed with a structure of a projective variety with quotient topology given by ν. Moreover the projection $\nu : X \to X/\mathbb{P}G$ defines a principal bundle with structure group $\mathbb{P}G$.
Showing that ν is a morphism of projective varieties is done by using that M_{3m+1}

[21] The notions of "geometric quotient" and "(non-)reductive group" are explained more precisely in Appendix D.4.

is a fine moduli space. Indeed, we want to construct an element $[\mathcal{E}] \in \mathcal{M}_{3m+1}(X)$ (this is a sheaf on $\mathbb{P}_2 \times X$) in order to get a unique morphism $X \to M_{3m+1}$ such that $[\mathcal{E}]$ is the pullback of the universal family $[\mathcal{U}] \in \mathcal{M}_{3m+1}(M_{3m+1})$. Since $X \subset \mathbb{A}_{18}$ is a quasi-affine variety, it may also be seen as a quasi-projective variety $(X, \mathcal{O}_X)$. For $n \in \mathbb{Z}$, define

$$\mathcal{O}_{\mathbb{P}_2 \times X}(n) := \mathcal{O}_{\mathbb{P}_2}(n) \boxtimes \mathcal{O}_X \ .$$

Using the Künneth formula, one can show that

$$\Gamma\big(\mathbb{P}_2 \times X \ , \ \mathcal{O}_{\mathbb{P}_2 \times X}(n)\big) \ \cong \ \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(n)\big) \otimes \Gamma(X, \mathcal{O}_X) \ ,$$

i.e. global sections of $\mathcal{O}_{\mathbb{P}_2 \times X}(n)$ for $n \geq 0$ are homogeneous polynomials of degree n with coefficients in $\mathcal{O}_X(X)$. Then we define $\mathcal{E}$ as the cokernel

$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2 \times X}(-2) \xrightarrow{B} \mathcal{O}_{\mathbb{P}_2 \times X}(-1) \oplus \mathcal{O}_{\mathbb{P}_2 \times X} \longrightarrow \mathcal{E} \longrightarrow 0 \ , \tag{4.45}$$

where B is a matrix like x such that $\det B$ is non-zero and its linear entries are linearly independent. Fix $x \in X$ and consider the Cartesian diagram

$$\begin{array}{ccccc} \mathbb{P}_2 & \xrightarrow{\pi_x} & \mathbb{P}_2 \times X & \xrightarrow{p_1} & \mathbb{P}_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_X} & & \\ \{x\} & \longrightarrow & X & & \end{array}$$

In order to compute $\mathcal{E}|_x$, note that

$$\begin{aligned} \mathcal{O}_{\mathbb{P}_2 \times X}(n)|_x &= \pi_x^*\big(\mathcal{O}_{\mathbb{P}_2 \times X}(n)\big) = \pi_x^*\big(\mathcal{O}_{\mathbb{P}_2}(n) \boxtimes \mathcal{O}_X\big) = \pi_x^*\Big(p_1^*\big(\mathcal{O}_{\mathbb{P}_2}(n)\big) \otimes \pi_X^*\mathcal{O}_X\Big) \\ &\cong \ (p_1 \circ \pi_x)^*\big(\mathcal{O}_{\mathbb{P}_2}(n)\big) \otimes (\pi_X \circ \pi_x)^*\mathcal{O}_X \ \cong \ \mathrm{id}^*\left(\mathcal{O}_{\mathbb{P}_2}(n)\right) \otimes \mathcal{O}_{\mathbb{P}_2} \ \cong \ \mathcal{O}_{\mathbb{P}_2}(n) \ . \end{aligned}$$

When applying the right exact functor π_x^* to (4.45), we thus get

$$2\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{x} \mathcal{O}_{\mathbb{P}_2}(-1) \oplus \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{E}|_x \longrightarrow 0 \ ,$$

so that $\mathcal{E}|_x \cong \operatorname{coker} x$, which has Hilbert polynomial $3m+1$ since $x \in X$. Thus $[\mathcal{E}] \in \mathcal{M}_{3m+1}(X)$. By the universal property of fine moduli spaces, we obtain a unique morphism

$$\nu \ : \ X \to M_{3m+1} \ : \ A \mapsto [\operatorname{coker} A]$$

such that $[\mathcal{E}] = \nu^*[\mathcal{U}]$, where

$$\begin{array}{ccc} \mathbb{P}_2 \times X & & \mathbb{P}_2 \times M_{3m+1} \\ \downarrow \pi_X & & \downarrow \pi_M \\ X & \xrightarrow{\nu} & M_{3m+1} \end{array}$$

To see that $\nu : X \to M_{3m+1}$ defines a locally trivial fibration (i.e. a bundle), recall that $\nu(\mathbb{P}G.x) = x$, $\forall\, x \in X$. We want to construct a section $s : M_{3m+1}|_U \to X$ which should be a local inverse of ν. Let $y = [\mathcal{F}] \in M_{3m+1}$ and choose a matrix $A \in X$ such that $\mathcal{F} \cong \operatorname{coker} A$. We set $s(y) := A$. This is locally well-defined since $X/\mathbb{P}G \cong M_{3m+1}$, so there exists an open neighborhood $V \subseteq X$ of A which does not contain any other element from the $\mathbb{P}G$-orbit of A. Hence $s : M_{3m+1}|_U \to V \subseteq X$ with $U := \nu(V)$ is well-defined and we have a local isomorphism

$$\begin{array}{ccc} X_{|V} & \xleftarrow{\ \sim\ } & \mathbb{P}G \times M_{3m+1} \\ & {\scriptstyle\nu}\searrow \quad \swarrow & \\ & M_{3m+1|U} & \end{array}$$

given by $(\bar{g}, y) \mapsto g.s(y)$, which is well-defined since $\nu(g.s(y)) = \nu(s(y)) = y$. □

4.6.5 Universal cubic curve and codimension in M_{3m+1}

Definition 4.6.16. Fix $d \geq 1$ and let $N = \binom{d+2}{2}$. The *universal curve of degree d* on $\mathbb{P}_2$ is defined as

$$\begin{aligned} \mathfrak{U}(d) &= \left\{ (C, x) \in \mathcal{C}_d(\mathbb{P}_2) \times \mathbb{P}_2 \;\middle|\; x \in C \right\} \\ &\cong \left\{ \big(\langle f \rangle, x\big) \in \mathbb{P}_{N-1} \times \mathbb{P}_2 \;\middle|\; f(x) = 0 \right\} . \end{aligned}$$

By projection it defines a projective bundle $\mathfrak{U}(d) \to \mathbb{P}_2$ (a fiber bundle whose fibers are projective spaces). In particular for $d = 3$ we obtain the *universal cubic curve* $\mathfrak{U}(3)$. In coordinates, this is

$$\mathfrak{U}(3) = \Bigg\{ \Big((C_{00} : C_{10} : \ldots : C_{03}) \,,\, (x_0 : x_1 : x_2) \Big) \in \mathbb{P}_9 \times \mathbb{P}_2 \text{ such that } \sum_{i+j+k=3} C_{jk}\, x_0^i x_1^j x_2^k = 0 \Bigg\} . \quad (4.46)$$

The relation between $\mathfrak{U}(3)$ and the moduli space M_{3m+1} is the following.

Theorem 4.6.17. [[47], 5.1, p.36-37] and [[23], 5.4, p.72-73]
There is an isomorphism of projective varieties $M_{3m+1} \cong \mathfrak{U}(3)$ *given by*

$$M_{3m+1} \xrightarrow{\sim} \mathfrak{U}(3) \ : \ [\mathcal{F}] \longmapsto \big(\mathcal{Z}_f(\mathcal{F}) \, , \, p \big) \ ,$$

where $p \in \mathcal{Z}_f(\mathcal{F})$ *is as in the extension* (4.35).
In terms of the parameter space X *where* $\mathcal{F} \cong \operatorname{coker} A$ *for some* $A \in X$ *it can be rewritten as*

$$X/\mathbb{P}G \xrightarrow{\sim} \mathfrak{U}(3) \ : \ [A] \longmapsto \big(\langle \det A \rangle \, , \, p \big) \in \mathbb{P}_9 \times \mathbb{P}_2 \ ,$$

where $p \in Z(z_1) \cap Z(z_2)$ *is as in Lemma 4.6.9.*

Remark 4.6.18. This is well-defined since $\det A = z_1 q_2 - z_2 q_1$ is a homogenous polynomial of degree 3 that vanishes at p. $\langle \det A \rangle$ is also independent of the class of A since

$$\det \big((g,h) \, . \, A \big) = \det \big(g \cdot A \cdot h^{-1} \big) = \det g \cdot \det A \cdot \det h^{-1} = \lambda \cdot \det A$$

for some $\lambda \in \mathbb{K}^*$, hence $\langle \det A \rangle = \langle \lambda \ \det A \rangle$ in $\mathbb{P}_9$.

Remark 4.6.19. The inverse morphism is defined as follows: given a cubic curve $C \in \mathcal{C}_3(\mathbb{P}_2)$ and a point $x \in C$, consider the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{Sky}_x(\mathbb{K}) \longrightarrow 0 \ ,$$

where $\mathcal{O}_C \to \mathrm{Sky}_x(\mathbb{K})$ is the morphism defining the closed subscheme $\{x\} \hookrightarrow C$. Its kernel $\mathcal{G}$ can be shown to be stable and has Hilbert polynomial $3m-1$, so the dual has Hilbert polynomial $3m+1$ by Proposition 4.3.16 and it suffices to take $\mathcal{F} := \mathcal{G}^{\mathrm{D}}$ (since $\mathcal{G}$ is stable if and only if $\mathcal{G}^{\mathrm{D}}$ is stable). In particular the point $p = (d_0 : d_1 : d_2)$ from Lemma 4.6.9 is exactly the one occurring in (4.35). Hence Theorem 4.6.17 says that giving a sheaf in M_{3m+1} is equivalent to giving a cubic curve and a point lying on that curve.

Corollary 4.6.20. [[41], 1.4, p.5]
Let $\mathcal{F} \in M_{3m+1}$ be given by $\mathcal{F} \cong \operatorname{coker} A$ and represented by $\big(\langle \det A\rangle\,,\,p\big) \in \mathfrak{U}(3)$. Then $[\mathcal{F}]$ is the isomorphism class of a singular sheaf if and only if p is a singular point of the curve $Z(\det A) \subset \mathbb{P}_2$.

Proof. Combine Theorem 4.6.17 and Proposition 4.6.10. □

Proposition 4.6.21. [[41], p.6]
The subvariety M'_{3m+1} of (isomorphism classes of) singular sheaves in M_{3m+1} is a smooth closed subvariety of codimension 2.

Proof. Theorem 4.6.17 allows to identify M'_{3m+1} with a subvariety of $\mathfrak{U}(3)$. Corollary 4.6.20 says that a point $(\langle f\rangle, x) \in \mathfrak{U}(3) \cong M_{3m+1}$ corresponds to an isomorphism class of a singular sheaf if and only if x is a singular point of the curve $Z(f)$. Using description (4.46), we have

$$\langle f\rangle = (C_{00} : C_{10} : C_{01} : C_{20} : C_{11} : C_{02} : C_{30} : C_{21} : C_{12} : C_{03})\ ,$$
$$x = (x_0 : x_1 : x_2)$$

and $\mathfrak{U}(3) \subseteq \mathbb{P}_9 \times \mathbb{P}_2$ is given by all points satisfying

$$\begin{aligned} &C_{00}x_0^3 + C_{10}x_0^2x_1 + C_{01}x_0^2x_2 + C_{20}x_0x_1^2 + C_{11}x_0x_1x_2 \\ &\quad + C_{02}x_0x_2^2 + C_{30}x_1^3 + C_{21}x_1^2x_2 + C_{12}x_1x_2^2 + C_{03}x_2^3 = 0\ . \end{aligned} \tag{4.47}$$

For a singular point, we need that all partial derivatives of f vanish at x, i.e.

$$\partial_0 f(x) = 0$$
$$\Leftrightarrow\ 3C_{00}x_0^2 + 2C_{10}x_0x_1 + 2C_{01}x_0x_2 + C_{20}x_1^2 + C_{11}x_1x_2 + C_{02}x_2^2 = 0\ , \qquad (e_0)$$
$$\partial_1 f(x) = 0$$
$$\Leftrightarrow\ C_{10}x_0^2 + 2C_{20}x_0x_1 + C_{11}x_0x_2 + 3C_{30}x_1^2 + 2C_{21}x_1x_2 + C_{12}x_2^2 = 0\ , \qquad (e_1)$$
$$\partial_2 f(x) = 0$$
$$\Leftrightarrow\ C_{01}x_0^2 + C_{11}x_0x_1 + 2C_{02}x_0x_2 + C_{21}x_1^2 + 2C_{12}x_1x_2 + 3C_{03}x_2^2 = 0\ . \qquad (e_2)$$

In particular this shows that M'_{3m+1} is indeed closed. However it does not mean that the codimension of M'_{3m+1} is 3. Indeed Euler's relation

$$x_0 \cdot \partial_0 f(x) + x_1 \cdot \partial_1 f(x) + x_2 \cdot \partial_2 f(x) = 3 \cdot f(x)$$

implies that a point also belongs to $\mathfrak{U}(3)$ as soon as $(e_0), (e_1), (e_2)$ are simultaneously satisfied. Thus M'_{3m+1} is of codimension 3 in $\mathbb{P}_9 \times \mathbb{P}_2$. $\mathfrak{U}(3)$ being described by the single equation (4.47), we conclude that $\mathfrak{U}(3)$ is closed in $\mathbb{P}_9 \times \mathbb{P}_2$ of codimension 1 and hence M'_{3m+1} is of codimension 2 in $\mathfrak{U}(3)$. Finally

$$\operatorname{codim}_{M_{3m+1}}\left(M'_{3m+1}\right) = 2 \ .$$

This can also be seen in the following way. Let $U_i = \{x_i \neq 0\} \subset \mathbb{P}_2$. Euler's relation implies that M'_{3m+1} is locally described by only 2 of the equations (e_i). Namely in $\mathfrak{U}(3) \cap U_0$, the equations of M'_{3m+1} are e_1 and e_2. Similarly in $\mathfrak{U}(3) \cap U_1$, the equations are e_0, e_2 and in $\mathfrak{U}(3) \cap U_2$, we have e_0, e_1. Finally M'_{3m+1} is a smooth subvariety of M_{3m+1} because the Jacobain matrix $J(e_0, e_1, e_2)$

$$\begin{pmatrix} 3x_0^2 & 2x_0x_1 & 2x_0x_2 & x_1^2 & x_1x_2 & x_2^2 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & x_0^2 & 0 & 2x_0x_1 & x_0x_2 & 0 & 3x_1^2 & 2x_1x_2 & x_2^2 & 0 & * & * & * \\ 0 & 0 & x_0^2 & 0 & x_0x_1 & 2x_0x_2 & 0 & x_1^2 & 2x_1x_2 & 3x_2^2 & * & * & * \end{pmatrix}$$

is always of rank 3 (each row contains a monomial x_0^2, x_1^2, x_2^2, so at least one of them is non-zero). □

Proposition 4.6.22. *$X' \subset X$ and $M'_{3m+1} \subset M_{3m+1}$ are irreducible subvarieties.*

Proof. Note that it is equivalent to say that X' is irreducible in X or in $\mathbb{A}_{18}$ since the induced topologies from $X' \subset X \subset \mathbb{A}_{18}$ coincide. It suffices to show that $Z(F_1, F_2)$ is irreducible as a subspace of $\mathbb{A}_{18}$. Then $X' \subset Z(F_1, F_2)$ is open by (4.43), hence irreducible as well.

First consider $Z(F_1) \subset \mathbb{A}_{18}$, which is irreducible since $F_1 \in \mathbb{K}[a_0, a_1, \ldots, B_{22}]$ is irreducible and $\mathbb{K}$ is algebraically closed. Similarly $\bar{F}_2 \in \mathbb{K}[a_0, a_1, \ldots, B_{22}]/\langle F_1 \rangle$ is irreducible since there does not exist a decomposition

$$\bar{F}_2 = \bar{g} \cdot \bar{h} \,, \quad \text{i.e.} \quad F_2 = g \cdot h + l \cdot F_1$$

for some non-constant polynomials g, h, l (here one uses that F_1 is linear in $A_{00}, \ldots, A_{22}$, which do not appear in F_2, and that F_2 is linear in $B_{00}, \ldots, B_{22}$, which do not appear in F_1). Therefore $Z(F_1, F_2) = Z(F_2) \cap Z(F_1)$ is irreducible in $Z(F_1)$, hence in $\mathbb{A}_{18}$.

Then we apply the morphism $\nu : X \to M_{3m+1}$ to the inclusions $X' \subset X \subset \mathbb{A}_{18}$ and conclude that $M_{3m+1} = \nu(X)$ and $M'_{3m+1} = \nu(X')$ are irreducible as well since they are continuous images of irreducible subspaces. □

Remark 4.6.23. relation with the space Ω from (4.13).
Recall that the Simpson moduli space was constructed in Section 4.2.2 as a quotient (4.13) of some space Ω of points that are semistable under the action of some $\mathrm{SL}(V)$. Looking at the above constructions for M_{3m+1} one may think that X is a concrete description of Ω and that the action of $\mathbb{P}G$ corresponds to the action of $\mathrm{SL}(V)$. However this is not true. For technical reasons one prefers to divide out the constants in the parameter space rather than in the group action in order to obtain an action of a linear group on a projective space. The general theory behind this idea is briefly explained in Appendix D.4. For M_{3m+1} it looks as follows:
Instead of X we consider the quasi-projective variety $\mathbb{P}X \subset \mathbb{P}_{17}$, so in some sense $\Gamma \cong \mathbb{K}^*$ is already divided out. Note that this is well-defined since the closed conditions that define the complement of X in $\mathbb{A}_{18}$ (zero determinant and linear dependence, see Lemma 4.6.8) are homogeneous polynomials of degree 2 in the 18 variables. Then Ω corresponds to $\mathbb{P}X$. Now we want to divide out a suitable group action. The action of G on X also induces an action of G on $\mathbb{P}X$ via

$$(g,h)\,.\,\bar{A} := \langle\,(g,h)\,.\,A\,\rangle$$

for $(g,h) \in G$ where $A \in \mathbb{K}^{18}$ is a representative of $\bar{A} \in \mathbb{P}_{17}$ and $\langle\,\cdot\,\rangle$ denotes the homogenous coordinates. This is well-defined as the initial action is linear. Since however we now work with projective spaces, we no longer have to care about multiplication by constants. In particular it suffices to consider matrices of determinant 1 (if $\det g \neq 0$, then $\exists\, \lambda \in \mathbb{K}^*$ such that $\det(\lambda g) = 1$). So we can take SL in both components of G. This yields the group $SG = \mathrm{SL}_2(\mathbb{K}) \times \mathrm{SL}(H)$, where $\mathrm{SL}(H)$ is the subgroup of matrices in H from (4.44) with determinant 1. The new construction is still compatible with dimensions since $\mathrm{Stab}_{SG}(\bar{A}) = \{\pm\,\mathrm{id}\}$ is finite, hence 0-dimensional for all $\bar{A} \in \mathbb{P}_{17}$ and SL_n has codimension 1 in GL_n. Thus $\mathbb{P}X/SG$ is of dimension $17-(4-1)-(5-1) = 10$. Finally we obtain that the action of this group SG is the concrete description of $\mathrm{SL}(V)$.

Similarly we also get a description for the space Ω^{sing} which defines the singular sheaves in the parameter space. It is given by $\mathbb{P}X'$, which is again possible since F_1 and F_2 are homogeneous polynomials of degree 5 in the 18 variables. In particular we see that $\operatorname{codim}_{\mathbb{P}X} \mathbb{P}X' = 2$ is compatible with Le Potier's result in Proposition 4.4.12.

Summary

$$\begin{array}{ccccc} X' & \xhookrightarrow[\text{codim=2}]{\text{closed,smooth,irreduc}} & X & \xhookrightarrow{\text{open}} & \mathbb{A}_{18} \\ \big\downarrow \nu & & \big\downarrow \nu & & \\ M'_{3m+1} & \xhookrightarrow[\text{codim=2}]{\text{closed,smooth,irreduc}} & M_{3m+1} \cong \mathfrak{U}(3) & \xhookrightarrow{\text{closed}} & \mathbb{P}_9 \times \mathbb{P}_2 \end{array}$$

where smoothness, codimension and irreducibility of the subspaces are preserved under the map ν.

4.6.6 The case $P_{\mathcal{F}}(m) = 3m+2$

By the results of Maican in Proposition 4.3.12 and Theorem 4.3.17 it is not necessary to study the moduli space M_{3m+2} since $M_{3m+2} \cong M_{3m-1} \cong M_{3m+1}$. Hence

Corollary 4.6.24. *M_{3m+2} is a smooth and irreducible projective variety of dimension 10. Moreover it is isomorphic to the universal cubic curve and the closed subvariety of singular sheaves M'_{3m+2} is isomorphic to the universal singular locus of a cubic curve, thus irreducible and of codimension 2.*

Remark 4.6.25. [[15], p.2]
Nevertheless let us mention that sheaves in M_{3m+2} are given by a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \oplus \mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0$$

and note that this looks indeed like a dual sequence to (4.36).

Chapter 5

Our case of interest

For the rest of the thesis we want to study the Simpson moduli space of stable sheaves on $\mathbb{P}_2$ with some general Hilbert polynomial $am \pm 1$. More precisely, let $n \in \mathbb{N}$, $n \geq 3$ and set $d := n + 1$. Our goal is to find some information such as codimension and smoothness of the subvariety of singular sheaves

$$M'_{dm-1} \subset M_{dm-1} \; .$$

This will be the content of Theorem 5.5.18, in which we prove that M'_{dm-1} is singular and of codimension 2.

5.1 Description of an open subset of M_{dm-1}

Since $\gcd(d, 1) = 1$, the results of Simpson and Le Potier from Theorem 4.2.14 and Theorem 4.3.10 again imply that M_{dm-1} is a smooth irreducible projective variety of dimension $d^2 + 1 = n^2 + 2n + 2$ and that its closed points are in 1-to-1 correspondence with isomorphism classes of stable sheaves of pure dimension 1. Due to the results of Maican from Section 4.3.3 we can also work with the moduli spaces M_{dm+1} and M_{dm+n} as these are isomorphic to the one we are studying. Actually we will switch from one point of view to the other whenever calculations can be made easier.

The goal of this section is to describe the sheaves $\mathcal{F}$ in a dense open subset of M_{dm-1} given by the condition $h^0(\mathcal{F}) = 0$. Such sheaves can be described by exact sequences, and hence by a parameter space (as in the case of M_{3m+1}) on which there is an action of a non-reductive group. Dividing out this group we aim again to obtain a geometric quotient whose points are in 1-to-1 correspondence with isomorphism classes of the sheaves we are interested in.

5.1.1 Semicontinuity

We start with some general results.

Definition 5.1.1. [[35], III, 12.7.1, p.287-288]
Let X be a topological space. A function $f : X \to \mathbb{Z}$ is *upper semicontinuous* if for all $x \in X$ there exists an open neighborhood U such that $f(y) \leq f(x)$, $\forall\, y \in U$. Intuitively this says that in a neighborhood of any point the function can only decrease. Equivalently, f is upper semicontinuous if and only if the subsets

$$Z_n := \left\{ x \in X \;\middle|\; f(x) \geq n \right\}$$

are closed in X for all $n \in \mathbb{Z}$.

Theorem 5.1.2. [[35], III, 12.8, p.288]
Let $\mathcal{X} \to \mathcal{Y}$ be a projective morphism of Noetherian schemes and $\mathcal{E}$ a coherent sheaf on $\mathcal{X}$ that is flat over $\mathcal{Y}$. Then for each $i \geq 0$, the map

$$y \longmapsto h^i(y, \mathcal{E}) := \dim_{\kappa(y)} H^i(\mathcal{X}_y, \mathcal{E}|_y)$$

is an upper semicontinuous function on $\mathcal{Y}$.

Corollary 5.1.3. *Assume that M_{am+b} is a fine moduli space. Then the sets $\{ \mathcal{F} \in M_{am+b} \mid h^i(\mathcal{F}) = 0 \}$ are open for all $i \geq 0$.*

Proof. Consider the universal family $\mathcal{U}$ on $\mathbb{P}_2 \times M_{am+b}$. For each $\mathcal{F} \in M_{am+b}$ given by a closed point y we have

$$\begin{array}{ccc} \mathbb{P}_2 & \longrightarrow & \mathbb{P}_2 \times M_{am+b} \\ \downarrow & & \downarrow \\ \{y\} & \longrightarrow & M_{am+b} \end{array}$$

and $\mathcal{U}|_y \cong \mathcal{F}$ by Corollary 4.2.9. Thus

$$h^i(y,\mathcal{U}) = \dim_{\kappa(y)} H^i(\mathbb{P}_2,\mathcal{U}|_y) = \dim_{\mathbb{K}} H^i(\mathbb{P}_2,\mathcal{F}) = h^i(\mathcal{F})$$

by Lemma 4.1.26. In other words the function $\mathcal{F} \mapsto h^i(\mathcal{F})$ is upper semicontinuous on M_{am+b} and the sets $\{ \mathcal{F} \in M_{am+b} \mid h^i(\mathcal{F}) \geq n \}$ are closed for all $n \in \mathbb{Z}$. But now we have

$$h^i(\mathcal{F}) = 0 \;\Leftrightarrow\; h^i(\mathcal{F}) < 1$$

since dimensions of vector spaces are non-negative. Hence $h^i(\mathcal{F}) = 0$ is an open condition. □

Remark 5.1.4. So for each $i \geq 0$ we obtain a stratification of the fine moduli space M_{am+b} into a dense open stratum given by the condition $h^i(\mathcal{F}) = 0$ and a closed stratum described by $h^i(\mathcal{F}) \neq 0$. It will turn out that the most useful choices are $i = 1$ and $i = 0$.

More generally, we even have

Proposition 5.1.5. [[48], 2.13, p.9]
Let $\mathcal{E} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be a locally free sheaf on $\mathbb{P}_2$ and fix $i, n \geq 0$. Then the set of isomorphism classes of stable sheaves $\mathcal{F} \in M^s_{am+b}$ given by the condition $h^i(\mathcal{F} \otimes \mathcal{E}) \geq n$ is a closed algebraic subset.

5.1.2 Errata and Corrigenda

Freiermuth has stated several errors regarding cohomological bounds in [23], which are the reason for further false statements in his paper. As these will however be important for us in order to use certain exact sequences, we point out and correct some of them here below.

Proposition 5.1.6. cf. [[23], 3.1, p.23-25] and [[48], 2.3, p.6][1]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ with Hilbert polynomial $P_{\mathcal{F}}(m) = am + b$ where $0 \leq b < a$. If

[1] The original statement in [23] is wrong ; Maican corrected it by adding an assumption and stating it differently, but does not provide a proof. Here we are going to undertake this task ; it is a modification of the proof in [23].

we assume that $h^0(\mathcal{F}(-1)) = 0$*, then*
1) We have the following bounds for the cohomology of $\mathcal{F}$*:*

$$b \leq h^0(\mathcal{F}) \leq a-1 \qquad \text{and} \qquad 0 \leq h^1(\mathcal{F}) \leq a-b-1 \ . \tag{5.1}$$

2) $h^1(\mathcal{F}(i)) = 0$ *for* $i \geq a-b-1$*.*

Proof. 1) As in the proof of Proposition 4.4.5 we choose an $\mathcal{F}$-regular section s with $H = V(s)$ and such that all points in the support of $\mathcal{F}|_H$ are simple. Tensoring the exact sequence (4.20) by $\mathcal{O}_{\mathbb{P}_2}(n)$ we get

$$0 \longrightarrow \mathcal{F}(n-1) \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{F}|_H \longrightarrow 0$$

since $\mathcal{F}|_H$ has constant Hilbert polynomial a (see Lemma 4.1.11). Applying the cohomology functor we obtain

$$\begin{aligned} 0 \longrightarrow H^0\big(\mathbb{P}_2, \mathcal{F}(n-1)\big) \longrightarrow H^0\big(\mathbb{P}_2, \mathcal{F}(n)\big) &\xrightarrow{f_n} \mathbb{K}^a \\ \longrightarrow H^1\big(\mathbb{P}_2, \mathcal{F}(n-1)\big) &\xrightarrow{g_n} H^1\big(\mathbb{P}_2, \mathcal{F}(n)\big) \longrightarrow 0 \end{aligned}$$

for all $n \in \mathbb{Z}$ since $\operatorname{supp}\mathcal{F}|_H$ consists of a simple points, whose global sections are given by $\mathbb{K}$, and $H^1(\mathbb{P}_2, \mathcal{F}|_H) = \{0\}$ because $\mathcal{F}|_H$ is 0-dimensional. Now we analyze the dimensions of these vector spaces. As g_n is surjective we get $h^1(\mathcal{F}(n)) \leq h^1(\mathcal{F}(n-1))$, hence the map $\mathbb{Z} \to \mathbb{N} : n \mapsto h^1(\mathcal{F}(n))$ is decreasing (not necessarily strictly). By a dimension count we also get

$$\begin{aligned} h^1(\mathcal{F}(n)) &= h^1(\mathcal{F}(n-1)) - a + h^0(\mathcal{F}(n)) - h^0(\mathcal{F}(n-1)) \\ \Rightarrow \ h^0(\mathcal{F}(n)) &= h^0(\mathcal{F}(n-1)) + a + h^1(\mathcal{F}(n)) - h^1(\mathcal{F}(n-1)) \\ &\leq h^0(\mathcal{F}(n-1)) + a \ . \end{aligned}$$

In particular for $n = 0$ we obtain $h^0(\mathcal{F}) \leq h^0(\mathcal{F}(-1)) + a = 0 + a$. From $b = h^0(\mathcal{F}) - h^1(\mathcal{F})$ we also conclude that $h^0(\mathcal{F}) \geq b$.
Next one can show that if f_n is surjective, then f_{n+1} is surjective as well. f_n being surjective means that the following map in the exact sequence is the zero map and hence g_n is an isomorphism, i.e. $h^1(\mathcal{F}(n)) = h^1(\mathcal{F}(n-1))$. Thus if f_{n_0} is surjective for some $n_0 \in \mathbb{Z}$, then

$$h^1(\mathcal{F}(n-1)) = h^1(\mathcal{F}(n)) = h^1(\mathcal{F}(n+1)) = \dots \ , \quad \forall\, n \geq n_0 \ . \tag{5.2}$$

By Serre's Theorem B all of these numbers are zero since $\mathcal{F}(n)$ becomes acyclic for n big enough. If f_n is not surjective, then g_n is not injective and

$$h^1(\mathcal{F}(n)) < h^1(\mathcal{F}(n-1)) \ .$$

So we conclude that the function $n \mapsto h^1(\mathcal{F}(n))$ is strictly decreasing until it reaches 0 and if it does not decrease at some step, then it is already 0 at that step. Also note that $\mathcal{F}(-1)$ has Hilbert polynomial $a(m-1)+b$, hence we obtain $-a+b = h^0(\mathcal{F}(-1)) - h^1(\mathcal{F}(-1))$ and $h^1(\mathcal{F}(-1)) = a-b$.
Now we show that $h^0(\mathcal{F}) \leq a-1$. Assume that $h^0(\mathcal{F}) = a$. Since $h^0(\mathcal{F}(-1)) = 0$, the morphism f_0 is injective and thus an isomorphism since both vector spaces have the same dimension. Hence (5.2) implies that $h^1(\mathcal{F}(-1)) = 0$, but this contradicts that $b < a$. Since $b \leq h^0(\mathcal{F}) \leq a-1$ and $b = h^0(\mathcal{F}) - h^1(\mathcal{F})$, we also see that $0 \leq h^1(\mathcal{F}) \leq a-b-1$.
2) As $h^0(\mathcal{F}) < a$, the morphism f_0 cannot be surjective and hence

$$h^1(\mathcal{F}(-1)) = a - b \neq 0 \ .$$

But since $n \mapsto h^1(\mathcal{F}(n))$ is strictly decreasing, it always decreases by at least 1 at each step. So in the worst case it reaches 0 after $a-b-1$ steps, i.e. $h^1(\mathcal{F}(n)) = 0$ for all $n \geq a-b-1$. □

Corollary 5.1.7. *For the Hilbert polynomial $dm+n$ we have the inclusion*

$$\left\{ \mathcal{F} \in M_{dm+n} \;\middle|\; h^0(\mathcal{F}(-1)) = 0 \right\} \subseteq \left\{ \mathcal{F} \in M_{dm+n} \;\middle|\; h^1(\mathcal{F}) = 0 \right\} . \qquad (5.3)$$

Proof. follows from Proposition 5.1.6 and (5.1) since $d-n-1=0$. □

Remark 5.1.8. In [[23], 4.1, p.42-43] Freiermuth states that every $[\mathcal{F}] \in M_{am+b}$ with $a > b \geq \frac{a}{2} > 0$ and $h^1(\mathcal{F}) = 0$ has a resolution of the type

$$0 \longrightarrow (2b-a)\,\mathcal{O}_{\mathbb{P}_2}(-1) \oplus (a-b)\,\mathcal{O}_{\mathbb{P}_2}(-2) \longrightarrow b\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0 \ .$$

This is wrong as a counter-example is presented in [[15], p.3]. Indeed e.g. in M_{4m+3} and M_{4m+2} one can construct two types of sheaves which both have zero first cohomology but which are given by different resolutions.

Remark 5.1.9. In [[23], 3.1, p.23 & 4.1, p.37-38] it is stated that every sheaf $[\mathcal{F}] \in M_{am+b}$ admits the bounds from (5.1), without the assumption $h^0(\mathcal{F}(-1)) = 0$. In particular this would mean that $h^1(\mathcal{F}) = 0$ for all $\mathcal{F} \in M_{dm+n}$. But for $d = 4$, Drézet and Maican proved the existence of a closed set in M_{4m+1} given by the condition $h^1(\mathcal{F}) = 1$. If all sheaves in M_{4m+3} would satisfy $h^1(\mathcal{F}) = 0$, then this closed stratum would not show up in $M_{4m+1} \cong M_{4m+3}$ neither.

Remark 5.1.10. In [[15], 3.2.3, p.20-21] it is also shown that in the case of M_{4m+1} there are no non-zero sheaves satisfying the relation $h^1(\mathcal{F}) \geq 2$. Hence the open condition $h^1(\mathcal{F}) = 0$ and the closed condition $h^1(\mathcal{F}) = 1$ are all possibilities in that case.

5.1.3 Isomorphisms of exact sequences

From (4.31) we know that all (isomorphism classes of) sheaves in M_{dm+n} or M_{dm-1} have a resolution of the form

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \ ,$$

where $\mathcal{E}_0, \mathcal{E}_1 \in \texttt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ are locally free. Now let us assume that two sheaves $\mathcal{F}_A, \mathcal{F}_B \in \texttt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ are both given as cokernels of such resolutions, i.e.

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \longrightarrow \mathcal{F}_A \longrightarrow 0 \qquad , \qquad 0 \longrightarrow \mathcal{E}_1 \xrightarrow{B} \mathcal{E}_0 \longrightarrow \mathcal{F}_B \longrightarrow 0 \ .$$

Under which conditions can we say that $\mathcal{F}_A \cong \mathcal{F}_B$?

Lemma 5.1.11. *A sufficient condition for $\mathcal{F}_A \cong \mathcal{F}_B$ is that there exist automorphisms $\varphi : \mathcal{E}_1 \to \mathcal{E}_1$ and $\psi : \mathcal{E}_0 \to \mathcal{E}_0$ such that $A \circ \varphi = \psi \circ B$.*

Proof. Consider

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{A} & \mathcal{E}_0 & \xrightarrow{\pi_A} & \mathcal{F}_A & \longrightarrow & 0 \\
 & & \uparrow \varphi & \nearrow & \uparrow \psi & \nearrow & \uparrow \exists f & & \\
0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{B} & \mathcal{E}_0 & \xrightarrow{\pi_B} & \mathcal{F}_B & \longrightarrow & 0
\end{array}$$

As $(\pi_A \circ \psi) \circ B = (\pi_A \circ A) \circ \varphi = 0$, the morphism $\pi_A \circ \psi$ factors through the cokernel of B and gives a morphism f, which is an isomorphism by diagram chasing.

− surjective : if $a \in \mathcal{F}_A$, then $a = \pi_A(b)$ for $b \in \mathcal{E}_0$ and $b = \psi(c)$ for $c \in \mathcal{E}_0$, so

$$a = \pi_A(\psi(c)) = f(\pi_B(c)) \ .$$

− injective : let $x \in \mathcal{F}_B$ such that $f(x) = 0$. Then $x = \pi_B(y)$ for $y \in \mathcal{E}_0$ and

$$\pi_A(\psi(y)) = f(\pi_B(y)) = f(x) = 0 \ ,$$

so $\psi(y) = A(b)$ for some $b \in \mathcal{E}_1$ and $b = \varphi(e)$ for some $e \in \mathcal{E}_1$. With

$$\psi(y) = A(\varphi(e)) = \psi(B(e)) \ ,$$

we get $y = B(e)$ by injectivity of ψ and finally $x = \pi_B(y) = \pi_B(B(e)) = 0$. □

We want to know if the converse is true as well, i.e. if there is an isomorphism $\mathcal{F}_A \cong \mathcal{F}_B$, does there exists automorphisms φ and ψ such that $\psi \circ B = A \circ \varphi$? Let $f : \mathcal{F}_B \xrightarrow{\sim} \mathcal{F}_A$ be an isomorphism. First we want to find $\psi \in \mathrm{Aut}(\mathcal{E}_0)$ such that $\pi_A \circ \psi = f \circ \pi_B$. For this we apply the covariant left exact functor $\mathrm{Hom}(\mathcal{E}_0, \cdot\,)$ to the exact sequence defining $\mathcal{F}_A$. This gives the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_1) \xrightarrow{A\circ} \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_0) \xrightarrow{\pi_A\circ} \mathrm{Hom}(\mathcal{E}_0, \mathcal{F}_A) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_1) \longrightarrow \dots \ . \end{aligned} \tag{5.4}$$

So if $\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_1) = \{0\}$, the morphism $\pi_A\circ$ would be surjective and using that $f \circ \pi_B \in \mathrm{Hom}(\mathcal{E}_0, \mathcal{F}_A)$, one gets a morphism $\psi \in \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_0)$ as desired. If in addition such a ψ exists, then the universal property of kernels immediately gives $\varphi \in \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_1)$ since $\pi_A \circ (\psi \circ B) = f \circ (\pi_B \circ B) = 0$, so $\psi \circ B$ factors through $\mathcal{E}_1 \cong \ker \pi_A$ and $\psi \circ B = A \circ \varphi$.
However φ and ψ are in general not automorphisms (a diagram chasing does not allow to show that they are bijective). Indeed if we also apply $\mathrm{Hom}(\mathcal{E}_0, \cdot\,)$ to the exact sequence defining $\mathcal{F}_B$ with $f^{-1} \circ \pi_A \in \mathrm{Hom}(\mathcal{E}_0, \mathcal{F}_B)$, we get a morphism $\rho \in \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_0)$ such that $\pi_B \circ \rho = f^{-1} \circ \pi_A$ and we obtain the relations

$$f \circ \pi_B \circ \rho = \pi_A \ \Leftrightarrow \ \pi_A \circ (\psi \circ \rho) = \pi_A \ ,$$
$$\pi_A \circ \psi = f \circ \pi_B \ \Leftrightarrow \ f^{-1} \circ \pi_A \circ \psi = \pi_B \ \Leftrightarrow \ \pi_B \circ (\rho \circ \psi) = \pi_B \ .$$

In general this does not imply that ψ and ρ are inverse to each other. This is where we introduce an additional condition, namely that $\mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_1) = \{0\}$ as well. If this is the case, then $\pi_A \circ$ and $\pi_B \circ$ in (5.4) are isomorphisms, from which we get that $\psi \circ \rho = \mathrm{id}$ and $\rho \circ \psi = \mathrm{id}$. Now ψ being an isomorphism, diagram chasing allows to show that φ is bijective too. $A \circ \varphi = \psi \circ B$ already implies that it is injective. For surjectivity let $y \in \mathcal{E}_1$. Then $A(y) = \psi(e)$ for some $e \in \mathcal{E}_0$ and $f(\pi_B(e)) = \pi_A(\psi(e)) = \pi_A(A(y)) = 0$, so $\pi_B(e) = 0$ and $e = B(x)$ for some $x \in \mathcal{E}_1$. Finally $A(y) = \psi(B(x)) = A(\varphi(x))$ and hence $y = \varphi(x)$.

Summarizing, we have proven the following result.

Proposition 5.1.12. *Let*

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{A} \mathcal{E}_0 \longrightarrow \mathcal{F}_A \longrightarrow 0$$

be an exact sequence of coherent sheaves. The group of automorphisms

$$\mathrm{Aut}(\mathcal{E}_0) \times \mathrm{Aut}(\mathcal{E}_1)$$

acts on the vector space of morphisms $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_0)$ *via* $(\psi, \varphi) \,.\, A = \psi \circ A \circ \varphi^{-1}$. *Assume that* $\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_1) = \mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_1) = \{0\}$. *Then* $\mathcal{F}_A \cong \mathcal{F}_B$ *if and only if* A *and* B *belong to the same orbit.*

In particular this means that the isomorphism classes of sheaves given as cokernels of such resolutions are in 1-to-1 correspondence with the orbits of the action of the automorphism group.

5.1.4 Our setting

Now we describe certain sheaves in M_{dm-1} for which we are going to study if they are singular. It has been shown in [[48], 4.2, p.12] that every $\mathcal{F} \in M_{dm+n}$ with $h^1(\mathcal{F}) = 0$ and $h^0(\mathcal{F}(-1)) = 0$ is obtained as the cokernel of a resolution of $\mathcal{O}_{\mathbb{P}_2}$–modules

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \oplus (n-1)\, \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{A} n\, \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0\,. \tag{5.5}$$

We will see in Theorem 5.1.28 that this is an equivalence if the sheaf obtained by such a resolution is indeed stable. The stability of $\mathcal{F}$ depends on the form of

A. The morphism A is given by an $n \times n$–matrix of the type

$$A = \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ z_{11} & z_{12} & \dots & z_{1,n} \\ z_{21} & z_{22} & \dots & z_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \dots & z_{n-1,n} \end{pmatrix} , \tag{5.6}$$

where $q_i \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$ and $z_{ij} \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ for all i, j such that $\langle \det A \rangle \neq 0$ (since A is injective). In order to define a stable sheaf the linear forms must satisfy some additional properties (compare e.g. with the case of M_{3m+1} in Proposition 4.6.4 where the linear forms had to be linearly independent). The general criterion due to Drézet and Maican will be stated in Proposition 5.1.20 and Theorem 5.1.28.

Remark 5.1.13. In order to obtain a resolution for sheaves in M_{dm-1}, we apply the isomorphism

$$M_{dm+n} \xrightarrow{\sim} M_{dm-1} \ : \ \mathcal{F} \longmapsto \mathcal{F}(-1) \tag{5.7}$$

from Proposition 4.3.12 to the exact sequence (5.5) and get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{G} \longrightarrow 0 \ , \tag{5.8}$$

where $\mathcal{G} = \mathcal{F}(-1) \in M_{dm-1}$ and A is again of the form (5.6). Indeed A does not change under this twist because Lemma 4.3.2 gives

$$\begin{aligned} &\operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}_2}(n), \mathcal{O}_{\mathbb{P}_2}(m)\big) = \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}_2}(n-1)(1), \mathcal{O}_{\mathbb{P}_2}(m)\big) \\ &\quad \cong \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}_2}(n-1), \mathcal{O}_{\mathbb{P}_2}(m)(-1)\big) = \operatorname{Hom}\big(\mathcal{O}_{\mathbb{P}_2}(n-1), \mathcal{O}_{\mathbb{P}_2}(m-1)\big) \ . \end{aligned}$$

Remark 5.1.14. Also note that the twist (5.7) shows that the subset of sheaves in M_{dm+n} satisfying $h^0(\mathcal{F}(-1)) = 0$ is open since it corresponds to the condition $h^0(\mathcal{G}) = 0$ in M_{dm-1}, which we know to be open from Corollary 5.1.3. In particular (5.3) is an inclusion of open sets. More generally this is just a particular case of Proposition 5.1.5.

For the following statements consider $\mathcal{F} \in M_{dm+n}$ with the resolution (5.5) or $\mathcal{F} \in M_{dm-1}$ with the resolution (5.8). $\det A$ is a homogeneous polynomial of degree $d = n+1$ in the variables X_0, X_1, X_2. The condition $\langle \det A \rangle \neq 0$ ensures injectivity of A. Let $C = \mathcal{Z}_f(\mathcal{F})$ be the Fitting support of $\mathcal{F}$, which is again a curve in $\mathbb{P}_2$ since $P_{\mathcal{F}}$ is linear. As a set we know from Proposition 4.5.9 that it is given by all the points $x \in \mathbb{P}_2$ such that $(\det A)(x) = 0$. Thus $C = Z(\det A) \subset \mathbb{P}_2$ is a curve of degree d (compare with Proposition 4.4.5). In particular it defines a projective variety $(C, \mathcal{O}_C)$ where the structure sheaf $\mathcal{O}_C$ has Hilbert polynomial $dm + \frac{3d-d^2}{2}$ by Proposition 4.3.9. The curve C is in general neither irreducible, nor reduced (i.e. it can have multiple structures), depending on the polynomial $\det A$.

Corollary 5.1.15. *There is a 1-to-1 correspondence between isomorphism classes of sheaves given by the resolution* (5.8) *and the orbits of the action of the group of automorphisms on* (5.8)*.*

Proof. We want to apply Proposition 5.1.12. For this we recall the following results from [35]. For $\mathcal{X} = \mathbb{P}_r$, we have
[5.1, p.225] $H^i\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(n)\big) = \{0\}$, $\forall\, n \in \mathbb{Z}$ and $0 < i < r$.
[6.3, p.234] $\mathrm{Ext}^i(\mathcal{O}_{\mathcal{X}}, \mathcal{F}) \cong H^i(\mathcal{X}, \mathcal{F})$, $\forall\, \mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ and $i \geq 0$.
[6.7, p.235] If $\mathcal{L} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$ is locally free of finite rank, then

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}\, , \,\mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}\, , \,\mathcal{L}^* \otimes \mathcal{G})\ .$$

Since Ext and Hom are additive in both arguments it suffices to check the conditions of Proposition 5.1.12 for a simple pair. For all $a, b \in \mathbb{Z}$, we have

$$\mathrm{Ext}^i\left(\mathcal{O}_{\mathbb{P}_2}(a), \mathcal{O}_{\mathbb{P}_2}(b)\right) \cong \mathrm{Ext}^i\left(\mathcal{O}_{\mathbb{P}_2}, \mathcal{O}_{\mathbb{P}_2}(b-a)\right) \cong H^i\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(b-a)\big) = \{0\}\ ,$$

so the condition on Ext^1 is always satisfied. For Hom we know that

$$\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}_2}(a), \mathcal{O}_{\mathbb{P}_2}(b)\right) = \{0\}$$

if $a > b$. In the case of (5.8) this is satisfied since $-1 > \max(-3, -2)$. $\square$

Notation 5.1.16. cf. [[48], p.10] and [[23], p.43]
Let $\mathbb{W}$ denote the $\mathbb{K}$-vector space of morphisms

$$\mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\, \mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\, \mathcal{O}_{\mathbb{P}_2}(-1)\ , \tag{5.9}$$

so that A is as in (5.6). How do automorphisms in (5.8) look like? We have

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{A} & n\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & \mathcal{F}_A & \longrightarrow 0 \\ & g \big\uparrow & & h \big\uparrow & & \cong \big\uparrow & \\ 0 \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{B} & n\,\mathcal{O}_{\mathbb{P}_2}(-1) & \longrightarrow & \mathcal{F}_B & \longrightarrow 0 \end{array}$$

where g and h are automorphisms. In particular, their determinants are non-zero polynomials by injectivity (Proposition 4.5.8). Since endomorphisms of $\mathcal{O}_{\mathbb{P}_2}(a)$ are just constants for all $a \in \mathbb{Z}$, we get $h \in \mathrm{GL}_n(\mathbb{K})$. However g has entries which are morphisms $\mathcal{O}_{\mathbb{P}_2}(-3) \to \mathcal{O}_{\mathbb{P}_2}(-2)$, i.e. linear forms and is thus of the form

$$g = \left(\begin{array}{c|ccc} \lambda & l_1 & \dots & l_{n-1} \\ \hline 0 & & & \\ \vdots & & \mathrm{GL}_{n-1}(\mathbb{K}) & \\ 0 & & & \end{array} \right),$$

where $\lambda \in \mathbb{K}^*$ and $l_i \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ for all $i \in \{1, \dots, n-1\}$. There is an action of the algebraic group

$$G' := \mathrm{Aut}\left(\mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2)\right) \times \mathrm{Aut}\left(n\,\mathcal{O}_{\mathbb{P}_2}(-1)\right) \tag{5.10}$$

on $\mathbb{W}$ via $(g,h)\,.\,A = g \cdot A \cdot h^{-1}$. Note that this is well-defined since quadratic forms can only appear in the first row. It follows from Corollary 5.1.15 that $\mathcal{F}_A \cong \mathcal{F}_B$ if and only if A and B are in the same orbit of the G'-action. This holds in particular if $A \sim B$ in the sense of linear algebra.

Remark 5.1.17. However, as in the case of M_{3m+1} in Remark 4.6.14, one prefers to divide out the 1-dimensional subgroup

$$\Gamma' = \left\{ (\lambda\,\mathrm{id}_n, \lambda\,\mathrm{id}_n) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G'$$

in order to get an action of $\mathbb{P}G' = G'/\Gamma'$ on $\mathbb{W}$. The orbits of this action are still in 1-to-1 correspondence with isomorphism classes of sheaves given by the resolution (5.8) as $\Gamma' \subseteq \mathrm{Stab}_G(A)$ for all $A \in \mathbb{W}$. On the other hand this inclusion may be strict, so the action of $\mathbb{P}G'$ is not free in general, see e.g. Example 5.1.24.

Now we want to describe (an open subset of) M_{dm-1} as a quotient of (an open subset of) $\mathbb{W}$ by the group action of $\mathbb{P}G'$ as in Theorem 4.6.15. For this we however first need to know under which conditions the sheaf $\mathcal{F}$ defined by a matrix $A \in \mathbb{W}$ in (5.8) is stable, and hence defines an element in M_{dm-1}. Indeed note that every sheaf $\mathcal{F}$ with a resolution of the form (5.8) has Hilbert polynomial

$$n \cdot \frac{m(m+1)}{2} - \frac{(m-1)(m-2)}{2} - (n-1) \cdot \frac{m(m-1)}{2} = (n+1)\,m - 1$$

by Example 4.3.1, which is $dm-1$, so the only condition that is missing for $\mathcal{F}$ to belong to M_{dm-1} is stability.

5.1.5 Kronecker modules

In order to obtain the condition for stability, we have to introduce the following objects. The idea of a Kronecker module is to generalize the notion of a matrix with entries in homogeneous polynomials. Let us first refer to Appendix D.4 for some general facts about Geometric Invariant Theory, which we are going to apply in this section.

Definition 5.1.18. cf. [[19], p.86]
Let E, F, V be finite-dimensional vector spaces over $\mathbb{K}$ with $q = \dim V \geq 3$ and

$$W = \mathrm{Hom}_{\mathbb{K}}(E, F \otimes V)\ .$$

A *Kronecker module* is a $\mathbb{K}$-linear map $\varphi \in W$.[2] After a choice of bases for E and F with $\dim E = n$, $\dim F = m$, a Kronecker module can hence be written as a $n \times m$–matrix

$$\varphi = \begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix}$$

[2]Some authors as Drézet [13] and Ellingsrud-Strømme [20] define Kronecker modules as elements in $\mathrm{Hom}_{\mathbb{K}}(E \otimes V, F)$. This is due to technical reasons which may simplify the proofs. Indeed both definitions are equivalent since all vector space are of finite dimension, hence $\mathrm{Hom}(E \otimes V, F) \cong \mathrm{Hom}(E, \mathrm{Hom}(V, F)) \cong \mathrm{Hom}(E, V^* \otimes F)$ and $V^* \cong V$ via a dual basis. Of course it is not canonical.

with entries $v_{ij} \in V$. Here we again consider row vectors and multiply them by the matrix on the right, i.e.

$$E \longrightarrow F \otimes V \ : \ e = (e_1, \ldots, e_n) \longmapsto e \cdot \varphi = (f_1, \ldots, f_m) \ .$$

The space of Kronecker modules W can be identified with an affine space, more precisely $W \cong \mathbb{A}_k$ for $k = n \cdot m \cdot q$.

Definition 5.1.19. [[19], p.86] and [[13], p.12]
There is an action of the group $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$ on W given by the rule $(g, h) \,.\, \varphi = (h \otimes \mathrm{id}_V) \circ \varphi \circ g^{-1}$. The 1-dimensional subgroup

$$\Gamma = \left\{ \left(\lambda \, \mathrm{id}_E \, , \, \lambda \, \mathrm{id}_F \right) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G$$

acts trivially, so there is an induced action of $\mathbb{P}G = G/\Gamma \cong G/\mathbb{K}^*$ on W. This action is in general not free as we e.g. illustrate in Example 5.1.24. Going to the projective space we deduce an action of the reductive group $S = \mathrm{SL}(E) \times \mathrm{SL}(F)$ on $\mathbb{P}(W) \cong \mathbb{P}_{k-1}$. Now we say that a non-zero Kronecker module $\varphi \in W$ is *(semi)stable* if its image in $\mathbb{P}(W)$ is a (semi)stable point under the action of S (in the sense of GIT as defined in Appendix D.4). We denote the open subsets of stable and semistable Kronecker modules in W by W^s and W^{ss}.

Drézet has shown the following useful characterization of (semi)stable Kronecker modules.

Proposition 5.1.20. [[13], Prop.15, p.12-14] and [[20], 6.2, p.176]
A non-zero Kronecker module $\varphi \in W$ is semistable if and only if for all vector subspaces $E' \subset E$ and $F' \subset F$ with $E' \neq \{0\}$, $F' \neq F$ and $\varphi(E') \subseteq F' \otimes V$, we have the inequality

$$\frac{\dim F'}{\dim E'} \geq \frac{\dim F}{\dim E} = \frac{m}{n} \ .$$

For stable Kronecker modules the same results holds true with a strict inequality.

Since G is a reductive algebraic group acting linearly on W, Theorem D.4.13 allows to conclude

Corollary 5.1.21. [[13], p.14] and [[20], 6.4, p.176]
There is a good quotient $N = W^{ss}/\!/\mathbb{P}G$ and a geometric quotient $N_s = W^s/\mathbb{P}G$ which is open in N. Moreover the action of $\mathbb{P}G$ on N_s is free.

Remark 5.1.22. If one wants to indicate the dimensions of the vector spaces, one also denotes the quotients by $N(q,n,m)$ and $N_s(q,n,m)$.

The Kronecker modules we are going to work with are of the following form.

Notation 5.1.23. Consider the linear part of the matrix A from (5.6), i.e. a matrix of the form

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1,n} \\ z_{21} & z_{22} & \cdots & z_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n} \end{pmatrix} .$$

This is a Kronecker module as defined above by taking $E = \mathbb{K}^{n-1}$, $F = \mathbb{K}^n$ and $V = \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ to be the 3-dimensional vector space of linear forms. The space of all such matrices is denoted by $\mathbb{V} = \mathrm{Hom}_{\mathbb{K}}(E, F \otimes V)$. Thus $\mathbb{V}$ is the $\mathbb{K}$-vector space of all morphisms of the form

$$(n-1)\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2} .$$

We have an action of $G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$ on $\mathbb{V}$ via $(g,h)\,.\,\Phi = g \cdot \Phi \cdot h^{-1}$ and the 1-dimensional subgroup

$$\Gamma = \left\{ \left(\lambda\,\mathrm{id}_{n-1}\,,\,\lambda\,\mathrm{id}_n\right) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G$$

is contained in the stabilizer of each $\Phi \in \mathbb{V}$, hence we also get an action of $\mathbb{P}G = G/\Gamma$ on $\mathbb{V}$.

Example 5.1.24. The action of $\mathbb{P}G$ is not free: consider e.g. $n = 3$ and the Kronecker module

$$\Phi = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

for some $z_1, z_2, z_3 \in V$. Then for all $\lambda, \mu \in \mathbb{K}$ with $\lambda \neq 0$, we have

$$\begin{pmatrix} \lambda+\mu & -\mu \\ \mu & \lambda-\mu \end{pmatrix} \cdot \begin{pmatrix} z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix} = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_1 & z_2 & z_3 \end{pmatrix} ,$$

so that $\Gamma \subsetneq \mathrm{Stab}_G(\Phi)$, i.e. $\mathrm{Stab}_{\mathbb{P}G}(\Phi)$ is still non-trivial.

Remark 5.1.25. Consider the exact sequence (5.5)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-2) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0\ .$$

Every Kronecker module $\Phi \in \mathbb{V}$ can be seen as a submatrix of such an $A \in \mathbb{W}$, so we may write $A = \binom{Q}{\Phi}$ where Q is a row vector consisting of n quadratic forms. If we set $\mathbb{U}_2 = n\,\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$, then the space of morphisms $\mathbb{W}$ from (5.9) may be identified with $\mathbb{V} \times \mathbb{U}_2$ via the isomorphism

$$\mathbb{V} \times \mathbb{U}_2 \xrightarrow{\sim} \mathbb{W} \ :\ (\Phi, Q) \longmapsto \begin{pmatrix} Q \\ \Phi \end{pmatrix}\ . \tag{5.11}$$

Definition 5.1.26. Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be a sheaf on $\mathbb{P}_2$ which is given as the cokernel of a matrix $A = \binom{Q}{\Phi}$ with $Q \in \mathbb{U}_2$ and $\Phi \in \mathbb{V}$ as in (5.5). We call Φ the Kronecker module *associated* to $\mathcal{F}$.

Remark 5.1.27. The Kronecker module associated to a sheaf depends on the chosen resolution. Indeed let $\mathcal{F}$ be given by the cokernel of some $A = \binom{Q}{\Phi}$ and $\mathcal{F}'$ be the cokernel of some $A' = \binom{Q'}{\Phi'}$. If $\mathcal{F} \cong \mathcal{F}'$, then Φ and Φ' may be different. However Proposition 5.1.12 implies that in this case A and A' lie in the same orbit under G', and hence that Φ and Φ' are in the same G-orbit.

The relation between stable Kronecker modules and stable sheaves in the moduli space M_{dm+n} is now given as follows.

Theorem 5.1.28. [[48], 4.2, p.12-14]
Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ be a sheaf on $\mathbb{P}_2$ with linear Hilbert polynomial $P_{\mathcal{F}}(m) = dm+n$ and given as a cokernel of a matrix $A = \binom{Q}{\Phi}$ as in resolution (5.5)*. Assume that*

$h^0\big(\mathcal{F}(-1)\big) = 0$, *i.e.* $\mathcal{F}(-1)$ *has no non-zero global sections. Then the following are equivalent:*

1) $\mathcal{F} \in M_{dm+n}$, *i.e.* $\mathcal{F}$ *is semistable (hence stable since the moduli space is fine).*

2) $\Phi \in \mathbb{V}^s$, *i.e. the Kronecker module* Φ *associated to* $\mathcal{F}$ *is stable*[3].

3) Φ *is not equivalent to (i.e. does not lie in the same orbit under* $\mathbb{P}G$ *as) a matrix of the form*

$$\begin{pmatrix} \psi & 0 \\ * & * \end{pmatrix} ,$$

where $\psi : m\,\mathcal{O}_{\mathbb{P}_2}(-1) \to m\,\mathcal{O}_{\mathbb{P}_2}$ *is an* $m \times m$*–matrix of linear forms for some* $m \in \{1, \dots, n-1\}$.

Remark 5.1.29. In other words, $\Phi \in \mathbb{V}^s$ is not allowed to lie in the same orbit under $\mathbb{P}G$ as a Kronecker module of the form

$$\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \psi & & 0 & 0 \\ & & & 0 & 0 \\ \hline * & * & * & * & * \\ * & * & * & * & * \end{array}\right)$$

with a zero block of size $j \times (n-j)$ for some $j \in \{1, \dots, n-1\}$.

Example 5.1.30. For e.g. $n = 4$, a matrix $A = \binom{Q}{\Phi}$ defines a stable sheaf $\mathcal{F}$ in (5.5) if its Kronecker module Φ does not lie in the same orbit as

$$\begin{pmatrix} \times & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \quad , \quad \begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ * & * & * & * \end{pmatrix} \quad , \quad \begin{pmatrix} \times & \times & \times & 0 \\ \times & \times & \times & 0 \\ \times & \times & \times & 0 \end{pmatrix} . \qquad (5.12)$$

Remark 5.1.31. By Remark 5.1.27 we know that the Kronecker modules associated to isomorphic sheaves lie in the same G-orbit. Hence if Φ is associated to some $\mathcal{F}$ as in (5.5), then the isomorphism class $[\mathcal{F}]$ is stable (i.e. belongs to M_{dm+n}) if and only if all Kronecker modules in the orbit $O(\Phi)$ are stable. So even if an associated Kronecker module depends on the chosen resolution, its stability does not (which is reasonable since the stability of $[\mathcal{F}]$ does not).

[3]as defined in Definition 5.1.19.

Remark 5.1.32. Similarly as $\gcd(d,n) = 1$ implies that all semistable sheaves in M_{dm+n} are stable, it implies that all semistable Kronecker modules are stables. Hence we have $\mathbb{V}^s = \mathbb{V}^{ss}$ in this case. Moreover the group $\mathbb{P}G$ acts freely on the open subset $\mathbb{V}^s$ of (semi)stable points in $\mathbb{V}$. We will prove a slightly weaker version of this statement in Corollary 5.2.38.

5.1.6 Parameter space

From now on we always denote $M := M_{dm-1}$ and $M' := M'_{dm-1}$. Hence the situation is as follows:
Consider the subvariety $M_0 \subset M$ of isomorphism classes of sheaves $\mathcal{F} \in M$ which are given by the condition $h^0(\mathcal{F}) = 0$. Corollary 5.1.3 implies that M_0 is open and dense. Next we restrict ourselves to the study of singular sheaves in the open stratum M_0. In other words if we denote $M'_0 = M' \cap M_0$, we want to compute

$$\operatorname{codim}_{M_0} M'_0 \,.$$

By a result from Yuan in [70] it turns out that this is actually sufficient. We will explain this more detailed in the proof of Theorem 5.5.18. By applying the twist $\mathcal{F} \mapsto \mathcal{F}(-1)$ of Proposition 4.3.12 to Theorem 5.1.28, we now obtain

Corollary 5.1.33. *Sheaves $\mathcal{F} \in M_0$ are exactly those that are given by a resolution* (5.8)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0\,,$$

where $A \in \mathbb{W}$ can be written as $A = \binom{Q}{\Phi}$ with $Q \in \mathbb{U}_2$ and $\Phi \in \mathbb{V}^s$.

Remark 5.1.34. Thus if we denote

$$\mathbb{W}_0 := \left\{ A \in \mathbb{W} \;\middle|\; A \text{ is injective and } A = \tbinom{Q}{\Phi} \text{ with } \Phi \in \mathbb{V}^s \right\},$$

we finally get the criterion

$$\mathcal{F} \in M_0 \;\Leftrightarrow\; A \in \mathbb{W}_0\,, \tag{5.13}$$

i.e. the sheaves in M without global sections are exactly the cokernels of morphisms in $\mathbb{W}_0$.

Our next goal is to parametrize all isomorphism classes of stable sheaves in M_0 in order to describe M_0 as a quotient of a parameter space by a certain group action.

Notation 5.1.35. The space $\mathbb{W}$ can be identified with a quasi-affine variety as follows. Recall from (5.6) that

$$A = \begin{pmatrix} Q \\ \Phi \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ z_{11} & z_{12} & \cdots & z_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n} \end{pmatrix},$$

where $q_i \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2))$ and $z_{ij} \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ for all i, j such that A is injective and $\Phi \in \mathbb{V}^s$. A consisting of quadratic and linear forms we may write as in (4.37)

$$\begin{aligned} z_{ij} &= a_{ij}^0 X_0 + a_{ij}^1 X_1 + a_{ij}^2 X_2 \\ q_i &= A_i^1 X_0^2 + A_i^2 X_0 X_1 + A_i^3 X_0 X_2 + A_i^4 X_1^2 + A_i^5 X_1 X_2 + A_i^6 X_2^2 \end{aligned} \tag{5.14}$$

for some $a_{ij}^k, A_i^k \in \mathbb{K}$, $\forall\, i \in \{1, \ldots, n\}$, $\forall\, j \in \{1, \ldots, n-1\}$. This allows to identify $\mathbb{W}$ with the affine space $\mathbb{A}_w$ for $w = 6n + 3n(n-1) = 3n(n+1)$. In order to obtain a description of $\mathbb{W}_0$ we have to include the conditions on A being injective and Φ being stable, i.e. using Proposition 4.5.8 we get

$$\mathbb{W}_0 = \left\{ A \in \mathbb{W} \;\middle|\; \det A \neq 0,\ \Phi \in \mathbb{V}^s \right\}.$$

Hence $\mathbb{W}_0$ may be seen as a subset of $\mathbb{A}_w$.

Proposition 5.1.36. $\mathbb{W}_0 \subset \mathbb{W}$ *is open.*

Proof. Similarly as in Lemma 4.6.8, we shall show that $\mathbb{A}_w \backslash \mathbb{W}_0$ is given by closed conditions. We may write

$$\mathbb{W}_0 = \{ A \mid \det A \neq 0 \} \cap \{ A \mid \Phi \in \mathbb{V}^s \} = V_1 \cap V_2 .$$

Let us first show that $\{ A \mid \det A = 0 \}$ is closed in $\mathbb{A}_w$. The coefficients of $\det A$ are polynomial expressions in the variables a_{ij}^k, A_i^k. Hence saying that $\det A$ is the zero polynomial gives $\binom{d+2}{2}$ closed conditions (as $\det A$ is homogeneous of

degree d) by putting all coefficients to zero. So V_1 is open since its complement is closed as it is an intersection of vanishing sets of polynomials on $\mathbb{A}_w$.
For V_2, we know from general GIT (Lemma D.4.11) that $\mathbb{V}^s$ is open in $\mathbb{V}$. Here we can also prove this directly by showing that $\mathbb{V} \setminus \mathbb{V}^s$ is closed in $\mathbb{V}$, which can be identified with $\mathbb{A}_v$ for $v = 3n(n-1)$. Let

$$Y_j = \left\{ \Phi \in \mathbb{V} \;\middle|\; \Phi \text{ is a Kronecker module with a zero block of size } j \times (n-j) \right\}$$

for all $j \in \{1, \ldots, n-1\}$ as in (5.12). Hence each Y_j is closed in $\mathbb{A}_v$ since some coefficients are put to zero. Now consider the orbits $\mathbb{P}G.Y_j$ under the action of $\mathbb{P}G$ on $\mathbb{V}$. These are still closed. But then we have

$$\mathbb{V} \setminus \mathbb{V}^s = (\mathbb{P}G.Y_1) \cup \ldots \cup (\mathbb{P}G.Y_{n-1})$$

since Kronecker modules which are not stable are exactly those that lie in the orbits under $\mathbb{P}G$ of Kronecker modules with zero blocks. Therefore $\mathbb{V}^s$ is open.
Now consider the isomorphism $\mathbb{V} \times \mathbb{U}_2 \cong \mathbb{W}$ from (5.11). It gives the inclusion of open sets

$$V_2 = \mathbb{V}^s \times \mathbb{U}_2 \hookrightarrow \mathbb{V} \times \mathbb{U}_2 \;\cong\; \mathbb{W}\ ,$$

i.e. V_2 is open in $\mathbb{W} \cong \mathbb{A}_w$. Hence $\mathbb{W}_0 = V_1 \cap V_2$ is open as well. □

Having found the quasi-affine variety $\mathbb{W}_0$ which parametrizes all sheaves in M_0 we are now interested in the subspace $\mathbb{W}_0' \subset \mathbb{W}_0$ of coefficients which describe the singular sheaves of M_0'. First we have the following criterion.

Proposition 5.1.37. cf. [[39], 4.1, p.6]
A sheaf $\mathcal{F} \in M_0$ given as a cokernel of some $A \in \mathbb{W}_0$ as in (5.8) *is singular if and only if there exists a point $p \in \mathbb{P}_2$ on which all submaximal minors of A vanish.*

Proof. The proof is similar as the one of Proposition 4.6.10.
$\underline{\Rightarrow}$: By contraposition, assume that for all points $x \in \mathbb{P}_2$ there is always at least one minor of order $(n-1) \times (n-1)$ which does not vanish at x. Hence this minor is a unit in the local ring $\mathcal{O}_{\mathbb{P}_2,x}$. By performing row and column transformations

we may assume that the first submaximal minor does not vanish at x, i.e. A_x is of the form

$$A_x \sim \begin{pmatrix} & & & a_1 \\ & B & & \vdots \\ & & & a_{n-1} \\ b_1 & \dots & b_{n-1} & c \end{pmatrix}$$

with $\det B$ being invertible in $\mathcal{O}_{\mathbb{P}_2,x}$. Hence we can multiply

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} & & & a_1 \\ & B & & \vdots \\ & & & a_{n-1} \\ b_1 & \dots & b_{n-1} & c \end{pmatrix} = \begin{pmatrix} 1 & & & a'_1 \\ & 1 & & \vdots \\ & & 1 & a'_{n-1} \\ b_1 & \dots & b_{n-1} & c \end{pmatrix} \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & C \end{pmatrix}$$

with

$$C = c - b_1 a'_1 - \ldots - b_{n-1} a'_{n-1} = \frac{1}{\det B} \cdot \det A \ .$$

So up to a unit we have

$$A_x \sim \begin{pmatrix} \mathrm{id}_{n-1} & 0 \\ 0 & \det A \end{pmatrix}$$

and $\mathcal{F}_x \cong \operatorname{coker} A_x \cong \mathcal{O}_{C,x}$ since $C = Z(\det A)$. Therefore $\mathcal{F}$ is non-singular.

$\underline{\Leftarrow}$: Assume that there is a point $p \in \mathbb{P}_2$ on which all submaximal minors vanish. From Lemma 4.5.13 and the resolution (5.8) we obtain the exact sequence of $\mathbb{K}$-vector spaces

$$\mathbb{K}^n \xrightarrow{A(p)} \mathbb{K}^n \longrightarrow \mathcal{F}(p) \longrightarrow 0 \ .$$

If all submaximal minors of A vanish at p, then $A(p)$ is a matrix of rank $< n-1$ and $\dim_{\mathbb{K}} \mathcal{F}(p) \geq 2$. Hence $\mathcal{F}_p$ cannot be a free module over $\mathcal{O}_{C,p}$, otherwise it would be free of rank 1 because of Proposition 4.5.14 and the same proof as in (4.41) gives $\mathcal{F}(p) \cong \mathbb{K}$. Hence $\mathcal{F}$ is singular. □

Remark 5.1.38. Proposition 5.1.37 is actually a generalization of Proposition 4.6.10. Indeed the latter says that a sheaf in M_{3m+1} is singular if and only if $z_1(p) = z_2(p) = q_1(p) = q_2(p) = 0$. But for matrices of order 2, all submaximal minors are of order 1, i.e. saying that all submaximal minors vanish at p means that all entries vanish at p.

The only difference in the case of M_{3m+1} is that we have a precise description of p in Lemma 4.6.9.

Corollary 5.1.39. *The subspace $\mathbb{W}'_0 \subset \mathbb{W}_0$ describing singular sheaves is closed.*

Proof. Using Proposition 5.1.37 it comes down to show that the subspace

$$S = \left\{ A \in \mathbb{W} \;\middle|\; \exists\, p \in \mathbb{P}_2 \text{ such that all submaximal minors of } A \text{ vanish at } p \right\}$$

is closed in $\mathbb{A}_w$. It is non-empty and proper. As the coefficients of the submaximal minors of A are polynomial expressions in the variables a_{ij}^k, A_i^k, we already conclude that the set

$$S' = \left\{ (A,p) \in \mathbb{A}_w \times \mathbb{P}_2 \;\middle|\; \text{all submaximal minors of } A \text{ vanish at } p \right\}$$

is closed in the product variety $\mathbb{A}_w \times \mathbb{P}_2$. If $\pi : \mathbb{A}_w \times \mathbb{P}_2 \to \mathbb{P}_2$ denotes the projection, we have $S = \pi(S')$. Hence S is closed as well since $\mathbb{P}_2$ is complete and π is closed (Proposition D.1.16). $\square$

5.1.7 Group action and geometric quotient

Summarizing, we have that a sheaf $\mathcal{F} \in M_0$ is given by the resolution (5.8)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0\,,$$

with $A \in \mathbb{W}_0$ by (5.13). Moreover Corollary 5.1.15 says that two morphism in $\mathbb{W}_0$ define isomorphic sheaves via their cokernels if and only if they lie in the same orbit under the action of the non-reductive group $\mathbb{P}G'$ from (5.10). Hence we get $M_0 \cong \mathbb{W}_0/\mathbb{P}G'$, at least as a set bijection, since points in the open stratum are in 1-to-1 correspondence with the orbit space of the action.

Remark 5.1.40. This is also compatible with the dimensions of the spaces. Theorem 4.3.10 says that M and hence M_0 are of dimension $d^2+1 = n^2+2n+2$. The parameter space $\mathbb{W}_0 \subset \mathbb{A}_w$ has dimension $w = 3n(n+1)$ and the group $\mathbb{P}G'$ is of dimension $n^2 + (n-1)^2 + 1 + 3(n-1) - 1 = 2n^2 + n - 2$.
Similarly as the action of $\mathbb{P}G$ is free on $\mathbb{V}^s$ (Remark 5.1.32), the action of $\mathbb{P}G'$ is free on $\mathbb{W}_0$ (in general it is not free on $\mathbb{W}$). The dimension of the quotient $\mathbb{W}_0/\mathbb{P}G'$ is therefore $3n(n+1) - (2n^2+n-2) = n^2+2n+2$.

Remark 5.1.41. [[49], p.8] and [[23], p.69]
We would like to construct the quotient of $\mathbb{W}_0$ by the action of $\mathbb{P}G'$ as in Theorem 4.6.15. Unfortunately this does not immediately follows from GIT. Indeed the group G' is not reductive. The reason for this is because it's first factor contains terms of the form $\mathcal{O}_{\mathbb{P}_2}(a) \oplus \mathcal{O}_{\mathbb{P}_2}(b)$ with $a < b$. Indeed if we denote $V = \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(b-a)\big)$, the unipotent radical of

$$\left\{ \begin{pmatrix} \lambda_1 & q \\ 0 & \lambda_2 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{K}^*,\; q \in V \right\}$$

is

$$\left\{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \;\middle|\; q \in V \right\}$$

because the matrices $\mathrm{id} - \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right)$ are nilpotent (see Definition D.4.7), and hence non-trivial.

So GIT does not ensure that the orbit space $\mathbb{W}_0/\mathbb{P}G'$ is again a projective variety. However Maican managed to show by some ad hoc construction that

Proposition 5.1.42. [[48], 7.6, p.39-40]
$\mathbb{W}_0$ admits a geometric quotient modulo $\mathbb{P}G'$ which is isomorphic to the dense open subset of M_{dm-1} given by the condition $h^0(\mathcal{F}) = 0$.[4]

It is constructed by using that M_{dm-1} is a fine moduli space to get a morphism $\mathbb{W}_0 \to M_{dm-1}$. In other words, we indeed have a geometric quotient

$$M_0 \cong \mathbb{W}_0/\mathbb{P}G' \;.$$

Remark 5.1.43. Finally let us also see our parameter spaces in the sense of (4.13). The classifying space Ω corresponds to $\mathbb{P}(\mathbb{W}_0) \subset \mathbb{P}_{w-1}$ and the subvariety Ω^{sing} giving singular sheaves is given by $\mathbb{P}(\mathbb{W}_0')$. The action of $\mathrm{SL}(V)$ follows the same idea as in Remark 4.6.23 and is hence given by the one of SG', the closed subgroup of G' consisting of pairs of matrices with determinant 1.

[4] Maican actually proves the statement for M_{dm+n} and uses G to denote our non-reductive group $\mathbb{P}G'$. We also point out that [48] contains a misprint as it writes the closed condition $h^0(\mathcal{F}(-1)) \neq 0$.

5.2 Computations with Kronecker modules

In the following we want to develop some properties of Kronecker modules in terms of their maximal minors. As usual here below $\mathbb{V} \cong \mathbb{A}_v$ with $v = 3n(n-1)$ denotes the affine variety of Kronecker modules of type $(n-1) \times n$ with $n \geq 3$ on which we have the action of $G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$ via $(g, h) \,.\, \Phi := g \cdot \Phi \cdot h^{-1}$. This just means that we perform linear transformations of the rows (described by g) and the columns (described by h^{-1}) of Φ. Hence the elements in the orbit of a fixed Kronecker module $\Phi \in \mathbb{V}$ are exactly all the matrices which are similar to Φ, i.e. $\Phi \sim \Phi' \Leftrightarrow \Phi' \in O(\Phi)$. Two matrices belong to the same G-orbit if and only if they can be obtained one from the other by linear transformations of the rows and columns.

5.2.1 Linear independence of the maximal minors

First let us study certain properties of the Kronecker modules. In particular we are interested in its maximal minors and want to analyze how they behave under the action of G.

Definition 5.2.1. The *maximal minors* of a Kronecker module $\Phi \in \mathbb{V}$ are defined as $d_i := (-1)^{i+1} \cdot \det(\Phi_i)$, where Φ_i is the $(n-1) \times (n-1)$–submatrix of Φ obtained by erasing the i^{th} column.

Lemma 5.2.2. *Every matrix in* $\mathrm{GL}_n(\mathbb{K})$ *can be written (not uniquely) as a product of elementary matrices*

$$E_i(\lambda) := \begin{pmatrix} 1 & & \\ & \lambda & \\ & & 1 \end{pmatrix} \qquad \text{and} \qquad F_{ij}(\mu) := \begin{pmatrix} 1 & & \mu \\ & 1 & \\ & & 1 \end{pmatrix} \tag{5.15}$$

with $\lambda \in \mathbb{K}^*$ *at position* (i, i) *and* $\mu \in \mathbb{K}$ *at position* (i, j), $i \neq j$.

Proof. This follows by the Gaussian elimination process. Let T be any matrix with n rows. Computing $E_i(\lambda) \cdot T$ means that the i^{th} row of T is multiplied by the non-zero scalar λ. Computing $F_{ij}(\mu) \cdot T$ means that we add μ times the j^{th} row of T to the i^{th} one. Hence the $E_i(\lambda)$ and $F_{ij}(\mu)$ describe all possible linear transformations of the rows. Let $C \in \mathrm{GL}_n(\mathbb{C})$. We know that $C \sim \mathrm{id}_n$ by only performing linear transformations of the rows. Hence there exist elementary matrices B_i as in (5.15) such that $B_r \cdot \ldots \cdot B_1 \cdot C = \mathrm{id}_n$, where

$$B_r \cdot \ldots \cdot B_1 = C^{-1} \qquad \text{and} \qquad C = B_1^{-1} \cdot \ldots \cdot B_r^{-1} \,.$$

Note that $E_i(\lambda)^{-1} = E_i\big(\frac{1}{\lambda}\big)$ and $F_{ij}(\mu)^{-1} = F_{ij}(-\mu)$ since $i \neq j$, so each B_k^{-1} is again an elementary matrix. □

Example 5.2.3. Consider the matrix

$$S_{ij} := \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} ,$$

which interchanges the i^{th} and the j^{th} row ($i < j$). The zeros on the diagonal are at position (i, i) and (j, j) and the additional 1's at position (i, j) and (j, i). A possible decomposition (it is not unique) of S_{ij} according to the algorithm described in the proof of Lemma 5.2.2 would be e.g.

$$S_{ij} = F_{ij}(1)^{-1} \cdot F_{ji}(-1)^{-1} \cdot F_{ij}(1)^{-1} \cdot E_j(-1)^{-1} = F_{ij}(-1) \cdot F_{ji}(1) \cdot F_{ij}(-1) \cdot E_j(-1) \,.$$

Remark 5.2.4. Similar results hold true if we perform linear transformations of the columns. The only difference is that we have to switch indices and multiply by the elementary matrices on the right. More precisely, if T is a matrix with n columns, then $T \cdot E_i(\lambda)$ multiplies the i^{th} column of T by λ and $T \cdot F_{ji}(\mu)$ adds μ times the j^{th} column to the i^{th} one.

Proposition 5.2.5 (Leytem)**.** *Let $\Phi, \Phi' \in \mathbb{V}$ be Kronecker modules such that $\Phi' \sim \Phi$, given by $\Phi' = g \cdot \Phi \cdot h^{-1}$ for some $(g, h) \in G$. If $d_1, \ldots, d_n$ are the maximal minors of Φ and $d'_1, \ldots, d'_n$ those of Φ', then*

$$\begin{pmatrix} d'_1 \\ \vdots \\ d'_n \end{pmatrix} = \frac{\det(g)}{\det(h)} \cdot h \cdot \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \,. \tag{5.16}$$

Proof. Let us first study the cases where g and h are elementary matrices as in (5.15). We denote the rows of Φ by $r_1, \dots, r_{n-1}$ and its columns by $c_1, \dots, c_n$.
1) If $g = E_i(\lambda) \in \mathrm{GL}_{n-1}(\mathbb{K})$ for some $\lambda \neq 0$, then $g \cdot \Phi$ is equal to Φ with r_i replaced by $\lambda\, r_i$, hence all minors are multiplied by λ, i.e. $d'_k = \lambda\, d_k$, $\forall\, k$.
2) If $h = E_i(\lambda) \in \mathrm{GL}_n(\mathbb{K})$ for some $\lambda \neq 0$, then $\Phi \cdot h^{-1}$ is equal to Φ with c_i replaced by $\frac{1}{\lambda}\, c_i$, hence

$$d'_i = d_i \qquad , \qquad d'_k = \tfrac{1}{\lambda}\, d_k,\ \forall\, k \neq i\ .$$

3) If $g = F_{ij}(\mu) \in \mathrm{GL}_{n-1}(\mathbb{K})$ for some $\mu \in \mathbb{C}$, then $g \cdot \Phi$ is equal to Φ with r_i replaced by $r_i + \mu\, r_j$. So we shall compute the maximal minors of the matrix

$$\begin{pmatrix} r_1 \\ \vdots \\ r_i + \mu\, r_j \\ \vdots \\ r_{n-1} \end{pmatrix} .$$

Now we use that the determinant is a multilinear mapping in the rows. Hence $d'_k = d_k + e_k$, where e_k is the k^{th} maximal minor of the matrix Φ with r_i replaced by $\mu\, r_j$. We see that all maximal minors of this matrix are zero since $j \neq i$, so the matrix always contains the (shortened) rows r_j and $\mu\, r_j$, which are linearly dependent. Thus $e_k = 0$ and $d'_k = d_k$, $\forall\, k$.
4) If $h = F_{ji}(\mu) \in \mathrm{GL}_n(\mathbb{K})$ for some $\mu \in \mathbb{C}$, then $\Phi \cdot h^{-1}$ is equal to Φ with c_i replaced by $c_i - \mu\, c_j$:

$$\begin{pmatrix} c_1 & \dots & c_i - \mu\, c_j & \dots & c_n \end{pmatrix} .$$

Hence we already get $d'_i = d_i$. For $k \neq i$, we argue similarly as above by using multilinearity in the columns. This allows to write again $d'_k = d_k + f_k$, where f_k is the k^{th} maximal minor of the matrix Φ with c_i replaced by $-\mu\, c_j$. If $k \neq j$, then $f_k = 0$ since the columns c_j and $-\mu\, c_j$ are linearly dependent. Moreover $f_j = (-1)^{j+1} \cdot (-1)^{i-j-1} \cdot (-1)^{i+1} \cdot (-\mu) \cdot d_i = \mu\, d_i$, so that

$$d'_k = d_k,\ \forall\, k \neq j \qquad , \qquad d'_j = d_j + \mu\, d_i\ .$$

So in each of the 4 cases, the elementary matrices behave as described in (5.16) because

$$\det\big(E_i(\lambda)\big) = \lambda \qquad \text{and} \qquad \det\big(F_{ij}(\mu)\big) = 1 \ .$$

The 4 cases of the transformations of the d_i's correspond to

$$1)\ \det\big(E_i(\lambda)\big) \quad , \quad 2)\ \frac{E_i(\lambda)}{\lambda} \quad , \quad 3)\ \det\big(F_{ij}(\mu)\big) \quad , \quad 4)\ \frac{F_{ji}(\mu)}{1} \ .$$

Formula (5.16) now follows because the determinant is multiplicative and from Lemma 5.2.2, which says that every invertible matrix writes as a product of elementary matrices. □

Remark 5.2.6. A similar statement can also be found in [[61], Thm.3, p.7]. It states that the minors of any order of two equivalent matrices of type $p \times q$ generate the same ideals.

Corollary 5.2.7. *Let $\Phi, \Phi' \in \mathbb{V}$ such that they belong to the same G-orbit. Then the maximal minors of Φ are linearly independent if and only if the maximal minors of Φ' are linearly independent.*

Proof. By formula (5.16) we see that the G-action can only perform $\mathbb{K}$-linear combinations on the maximal minors. This will not affect their linear independence. Vice-versa we can invert the formula since h in (5.16) is invertible. □

Corollary 5.2.8. *Fix $\Phi \in \mathbb{V}$ and assume that its maximal minors $d_1, \ldots, d_n$ are linearly independent, so that the $\mathbb{K}$-vector space $V := \langle\, d_1, \ldots, d_n \,\rangle$ is of dimension n. Proposition 5.2.5 gives an assignment*

$$\psi \ : \ G \longrightarrow \mathrm{GL}(V) \ : \ (g,h) \longmapsto \frac{\det(g)}{\det(h)} \cdot h \ ,$$

where we identify $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{K})$ via the basis $\{d_1, \ldots, d_n\}$. This defines a group representation of G on V. Moreover ψ is surjective.

Proof. The fact that ψ is a group homomorphism follows from multiplicativity of the determinant. For surjectivity, we shall prove that the image of ψ contains every invertible $n \times n$–matrix. But this is clear since for $T \in \mathrm{GL}_n(\mathbb{K})$, one can choose $\psi(g,T) = T$ where g is a matrix in $\mathrm{GL}_{n-1}(\mathbb{K})$ such that $\det(g) = \det(T)$, e.g. $g = \lambda\, \mathrm{id}_{n-1}$ for some $\lambda \in \mathbb{K}$ such that $\lambda^n = \det(T)$. □

Remark 5.2.9. Linear independence of the maximal minors is only needed to ensure well-definedness of h as a linear map on V. If we would consider $d_1, \dots, d_n$ as formal expressions, the same proof shows that every linear transformation of the d_i's given by some $T \in \mathrm{GL}_n(\mathbb{K})$ can be written in such a way.

Remark 5.2.10. Surjectivity of ψ implies that every invertible linear transformation on the space of maximal minors, i.e. every basis change in V, is actually induced by the G-action of an element $(g, h) \in G$ and formula (5.16) gives a constructive way to find the matrices g and h.

Example 5.2.11. The maximal minors of

$$\Phi = \begin{pmatrix} z_1 & z_2 & z_3 \\ l_1 & l_2 & l_3 \end{pmatrix}$$

are

$$d_1 = z_2 l_3 - z_3 l_2 \quad , \quad d_2 = z_3 l_1 - z_1 l_3 \quad , \quad d_3 = z_1 l_2 - z_2 l_1 \ .$$

Now we would like to find a Kronecker module $\Phi' \in \mathbb{V}$ with maximal minors

$$\begin{cases} d_1' = d_1 + d_2 \\ d_2' = d_3 \\ d_3' = 3d_2 - d_1 \end{cases} \Leftrightarrow \begin{pmatrix} d_1' \\ d_2' \\ d_3' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} .$$

The determinant of the invertible matrix T is -4. Hence one can take for example

$$\Phi' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \Phi \cdot T^{-1} = \begin{pmatrix} \frac{3}{2}z_1 + \frac{1}{2}z_2 & 2z_3 & \frac{1}{2}z_2 - \frac{1}{2}z_1 \\ -\frac{3}{2}l_1 - \frac{1}{2}l_2 & -2l_3 & \frac{1}{2}l_1 - \frac{1}{2}l_2 \end{pmatrix} .$$

Corollary 5.2.12. *If $\Phi \in \mathbb{V}$, then its maximal minors are linearly dependent if and only if there exists $\Phi' \in O(\Phi)$ such that the first maximal minor of Φ' is zero (in other words: if and only if there exists $(g, h) \in G$ such that the first maximal minor of $g \cdot \Phi \cdot h^{-1}$ is zero).*

Proof. Sufficiency follows from Corollary 5.2.7: if $\exists\, \Phi' \in O(\Phi)$ such that the first maximal minor of Φ' is zero, then these are linearly dependent, hence so are the maximal minors of Φ.

Vice-versa, let $d_1, \ldots, d_n$ be the maximal minors of Φ. If one of them is already zero, we are done. Otherwise assume that $d_i = \sum_{k \neq i} \lambda_k d_k$ for some $\lambda_k \in \mathbb{K}$ and some $i \in \{1, \ldots, n\}$. Now we want to perform the transformation

$$\begin{cases} d'_1 = d_i - \sum_{k \neq i} \lambda_k d_k \\ d'_2 = d_2 \\ \ldots \\ d'_i = d_1 \\ \ldots \\ d'_n = d_n \end{cases} \Leftrightarrow \begin{pmatrix} d'_1 \\ d'_2 \\ \vdots \\ d'_i \\ \vdots \\ d'_n \end{pmatrix} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & \ldots & 1 & \ldots & -\lambda_n \\ 0 & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & 1 & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & 1 & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 1 \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{pmatrix},$$

so that $d'_1 = 0$. The determinant of this invertible matrix h is -1, hence the Kronecker module

$$\Phi' = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{id}_{n-2} \end{pmatrix} \cdot \Phi \cdot h^{-1}$$

satisfies the required condition. □

Corollary 5.2.13. *Let $\Phi \in \mathbb{V}$ and assume that its maximal minors $d_1, \ldots, d_n$ span a $\mathbb{K}$-vector space of dimension $k < n$. Then there exists $\Phi' \in O(\Phi)$ such that the maximal minors of Φ' are $d'_1, \ldots, d'_k, 0, \ldots, 0$, where $d'_1, \ldots, d'_k$ are linearly independent.*

Proof. The proof is similar as in Corollary 5.2.12. First we perform a permutation such that the first k maximal minors are linearly independent and then a transformation by substracting the linear combinations from the remaining ones to get 0 for $i \in \{k+1, \ldots, n\}$. □

Now we are able to generalize a result from [19] which states that Kronecker modules of the type 2×3 with linearly independent maximal minors are stable.

Proposition 5.2.14 (Leytem)**.** cf. [[19], Lemma 1 & Lemma 2, p.87-88][5]
If $\Phi \in \mathbb{V}$ is such that its maximal minors are linearly independent, then Φ is

[5]The statement in [19] is only for the case $n = 3$ and is proven by the means of 1-parameter subgroups (consider Definition D.4.16).

a stable Kronecker module, i.e. $\Phi \in \mathbb{V}^s$. *For* $n = 3$, *the conditions are even equivalent.*

Proof. If Φ is not semistable, then $\exists\, m \in \{1, \ldots, n-1\}$ and a matrix ψ of the type $m \times m$ whose entries are linear forms such that

$$\Phi \sim \begin{pmatrix} \psi & 0 \\ * & * \end{pmatrix} .$$

We show that the first maximal minor (see Definition 5.2.1) of the matrix on the RHS is zero. Corollary 5.2.12 then implies that the maximal minors of Φ are linearly dependent as well. This will prove the statement by contraposition.
When erasing the first column for $m = 1$, the resulting $(n-1)\times(n-1)$–submatrix has its first row equal to zero, hence the determinant will be zero. So let $m \geq 2$. We will prove by induction on $n \geq 3$ that the first maximal minor of such a Kronecker module is zero. For $n = 3$, this is clear as Φ is then of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \end{pmatrix} .$$

Now let $n \geq 4$ and assume that the result holds true for $n-1$ and for any $m' \in \{2, \ldots, n-2\}$ (the case $m = 1$ was discussed above). Consider n and fix $m \in \{2, \ldots, n-1\}$. We expand the determinant of

$$\begin{pmatrix} \psi_{11} & \ldots & \psi_{1m} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \ldots & \psi_{mm} & 0 & \ldots & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

along the first row, so the first maximal minor of this $(n-1)\times n$–matrix is equal to

$$d = \psi_{12} \cdot a_2 - \psi_{13} \cdot a_3 + \ldots \pm \psi_{1m} \cdot a_m + 0 + \ldots + 0 ,$$

where a_i is the determinant of the $(n-2)\times(n-2)$–submatrix associated to ψ_{1i}. However each a_i is also the first maximal minor of some Kronecker module of the type $(n-2)\times(n-1)$ whith a block ψ' of size $(m-1)\times(m-1)$. For example

$$a_2 = \det\begin{pmatrix} \psi_{23} & \dots & \psi_{2m} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{m3} & \dots & \psi_{mm} & 0 & \dots & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

is the first maximal minor of the Kronecker module

$$\begin{pmatrix} \psi_{22} & \psi_{23} & \dots & \psi_{2m} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{m2} & \psi_{m3} & \dots & \psi_{mm} & 0 & \dots & 0 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} = \begin{pmatrix} \psi' & 0 \\ * & * \end{pmatrix} .$$

$a_3, \dots, a_m$ are of a similar form. Hence all a_i are zero by induction hypothesis. It follows that d is zero as well.

To prove the converse for $n = 3$, let $\Phi \in \mathbb{V}$ such that its first maximal minor is zero. Then Φ is of the form

$$\Phi \sim \begin{pmatrix} * & z_1 & z_2 \\ * & z_3 & z_4 \end{pmatrix} ,$$

where the z_i are linear forms such that $z_1z_4 - z_2z_3 = 0$. Since $\mathbb{K}[X_0, X_1, X_2]$ is a UFD and all the z_i are irreducible, the equality $z_1z_4 = z_2z_3$ implies that either $z_1 = \pm\, z_2$ and $z_4 = \pm\, z_3$, in which case

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} \pm\, z_2 & z_2 \\ z_3 & \pm\, z_3 \end{pmatrix} \sim \begin{pmatrix} 0 & z_2 \\ 0 & \pm\, z_3 \end{pmatrix} ,$$

or $z_1 = \pm\, z_3$ and $z_4 = \pm\, z_2$, in which case

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} \pm\, z_3 & z_2 \\ z_3 & \pm\, z_2 \end{pmatrix} \sim \begin{pmatrix} \pm\, z_3 & z_2 \\ 0 & 0 \end{pmatrix} .$$

Hence Φ is of the form

$$\Phi \sim \begin{pmatrix} 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad \text{or} \qquad \Phi \sim \begin{pmatrix} * & * & * \\ * & 0 & 0 \end{pmatrix}$$

and thus not semistable. □

Remark 5.2.15. We no longer have an equivalence for $n > 3$. A counter-example for $n = 4$ is given in Example 5.6.1.

Corollary 5.2.16. *Let $\mathbb{V}_l \subset \mathbb{V}$ denote the set of Kronecker modules with linearly independent minors. Then $\mathbb{V}_l \subseteq \mathbb{V}^s$ and this is an inclusion of open sets, so $\mathbb{V}_l$ is an open subvariety of $\mathbb{V} \cong \mathbb{A}_v$.*

Proof. The inclusion follows from Proposition 5.2.14. To show that $\mathbb{V}_l$ is open, note that $\mathbb{V} \setminus \mathbb{V}_l$ is closed since the minors of Φ being linearly dependent means that the matrix consisting of their coefficients, which are polynomial expressions in the variables a_{ij}^k from (5.14), has all its maximal minors equal to zero. This is an intersection of vanishing sets in $\mathbb{A}_v$. Hence $\mathbb{V}_l$ is open. □

5.2.2 The Hilbert-Burch Theorem

The following theorem is an important tool as it allows to construct kernels of Kronecker modules.

Theorem 5.2.17 (Hilbert-Burch)**.** [[16], 20.15, p.502], [[61], Th.15, p.228-229] *Let R be a local ring and assume that $I \trianglelefteq R$ is an ideal with a minimal projection resolution*

$$0 \longrightarrow F_2 \xrightarrow{\Phi} F_1 \xrightarrow{\varphi} R \longrightarrow R/I \longrightarrow 0 \ . \tag{5.17}$$

1) Assume that $F_1 \cong R^n$ is free of rank n in (5.17), so that φ writes as

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}$$

for some $\varphi_1, \ldots, \varphi_n \in R$. Then $F_2 \cong R^{n-1}$, so that $\Phi : R^{n-1} \to R^n$ is an $(n-1) \times n$–matrix with entries in R and maximal minors $d_1, \ldots, d_n$, and there exists a NZD $a \in R$ such that $\varphi_i = a \cdot d_i$ for all $i \in \{1, \ldots, n\}$. Moreover

$$I = a \cdot I_{n-1}(\Phi) \ ,$$

where $I_{n-1}(\Phi) = \langle\, d_1, \ldots, d_n \,\rangle$ is the ideal generated by the minors of Φ, I has depth 2 as an R-module and the quotient R/I is a Cohen-Macaulay ring of Krull dimension 0.

2) Conversely, if we are given a NZD $a \in R$ and an $(n-1) \times n$–matrix Φ with maximal minors $d_1, \dots, d_n$ such that the ideal $I = I_{n-1}(\Phi) = \langle d_1, \dots, d_n \rangle \trianglelefteq R$ has depth 2, then the morphism φ defined by $\varphi_i = a \cdot d_i$ for all i turns (5.17) into a free resolution of R/I with $I = a \cdot I_{n-1}(\Phi)$.

We will use this statement in order to include Kronecker modules into some exact sequences. From Proposition 4.5.8 and Proposition 4.5.9 we already know

Corollary 5.2.18. *Let $n, m \in \mathbb{N}$ and consider a morphism of sheaves*

$$\psi \ : \ \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}_2}(k_i) \longrightarrow \bigoplus_{j=1}^{m} \mathcal{O}_{\mathbb{P}_2}(l_j)$$

for some $k_i, l_j \in \mathbb{Z}$, so that ψ is a $n \times m$–matrix of homogeneous polynomials.

- *a) If $n \leq m$ and at least one of the $n \times n$–minors of ψ is non-zero, then ψ is injective.*
- *b) If $n \geq m$, then $\operatorname{coker} \psi$ is supported on the common vanishing set of all $m \times m$–minors of ψ. In particular, ψ is surjective if this vanishing set is empty.*
- *c) For $n = m$, we thus get: ψ is injective[6] if $\langle \det \psi \rangle \neq 0$ and ψ is surjective if $Z(\det \psi) = \emptyset$.*

In order to apply the Hilbert-Burch Theorem to this situation, we have to make sure that the maximal minors of Φ define a subscheme of dimension 0 (since R/I must be of Krull dimension 0).

Lemma 5.2.19. *Let $d_1, \dots, d_n$ be the maximal minors of a Kronecker module $\Phi \in \mathbb{V}$. Assume that they are not all identically zero and consider their greatest common divisor $g = \gcd(d_1, \dots, d_n)$. Then the zero scheme*

$$Z = Z(d_1, \dots, d_n)$$

given by the vanishing set of the minors is 0-dimensional if and only if $g = 1$.

[6] Recall that the notation $\langle f \rangle$ means that we consider the vector of coefficients of a homogeneous polynomial f.

Proof. Since not all maximal minors are zero, we already get $\dim Z < 2$ by Corollary 5.2.18.

$\underline{\Rightarrow}$: If $\dim Z = 0$, then $g = 1$, otherwise g is non-constant and $Z(g) \subseteq Z$ implies that $\dim Z \geq 1$.

$\underline{\Leftarrow}$: By contraposition, assume that $\dim Z \geq 1$. Then Z contains an irreducible subscheme of dimension 1, hence there exists a non-constant homogeneous irreducible polynomial f such that $Z(f) \subseteq Z$, which implies that each d_i is divisible by f. □

Definition 5.2.20. Let $\Phi \in \mathbb{V}$ with maximal minors $d_1, \ldots, d_n$. We say that these are *coprime* if $\gcd(d_1, \ldots, d_n) = 1$.

Lemma 5.2.21. *If $\Phi \in \mathbb{V}$ and $\varphi = {}^t(d_1, \ldots, d_n)$ is a column vector (here ^{t} denotes the transpose), then $\varphi \circ \Phi = 0$ (or $\Phi \cdot \varphi = 0$ in terms of matrices).*

Proof. We expand the determinant of Φ that has been augmented by its i^{th} row in the first row, so this determinant is zero:

$$0 = \det \begin{pmatrix} z_{i1} & z_{i2} & \cdots & z_{in} \\ z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n} \end{pmatrix}$$
$$= z_{i1}d_1 + z_{i2}d_2 + \ldots + z_{in}d_n = i^{\text{th}} \text{ entry of } \Phi \cdot \varphi \,.$$
□

Remark 5.2.22. But in general a complex defined by Φ and φ is not exact at the step $\varphi \circ \Phi = 0$. A condition for exactness is given in Proposition 5.2.23.

Proposition 5.2.23. *Assume that the maximal minors $d_1, \ldots, d_n$ of a Kronecker module $\Phi \in \mathbb{V}$ are linearly independent and coprime and let $\varphi = {}^t(d_1, \ldots, d_n)$. Then the sequence*

$$0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0 \qquad (5.18)$$

is exact. Moreover Z is a 0-dimensional scheme of length $\binom{n}{2} = \frac{n^2-n}{2}$.

Proof. Let $I = \langle\, d_1, \ldots, d_n \,\rangle$ and $R = \mathbb{K}[X_0, \ldots, X_n]$. Since the minors are linearly independent, they are a minimal set of generators of I and as they are coprime, the localizations of R/I are Cohen-Macaulay rings of Krull dimension 0. Hence by Hilbert-Burch, the kernel of

$$n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2}$$

is given by an $(n-1) \times n$–matrix Ψ with entries in linear forms such that $I_{n-1}(\Psi) = I$. Now by the universal property of the kernel there is a morphism

$$(n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\rho} (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n)$$

such that

$$\begin{array}{ccccccc}
0 \longrightarrow & (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) & \xrightarrow{\Psi} & n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{P}_2} \\
& \nwarrow^{\rho} & & \uparrow{\scriptstyle\Phi} & & \nearrow_{0} \\
& & & (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) & &
\end{array}$$

with $\varphi \circ \Phi = \varphi \circ \Psi = 0$ and $\Psi \circ \rho = \Phi$. Since the maximal minors of Φ are linearly independent, they are in particular not all zero, hence Φ is injective (Corollary 5.2.18) and so is ρ. Now we consider ρ as a linear map

$$(n-1) \cdot \mathbb{K}(X_0, X_1, X_2) \xrightarrow{\rho} (n-1) \cdot \mathbb{K}(X_0, X_1, X_2)$$

of vector spaces over the field of rational functions $K = \mathbb{K}(X_0, X_1, X_2)$. Since ρ is an injective map of finite-dimensional K-vector spaces of the same dimension, it is thus an isomorphism and hence so is its restriction

$$(n-1) \cdot \mathbb{K}[X_0, X_1, X_2] \xrightarrow{\rho} (n-1) \cdot \mathbb{K}[X_0, X_1, X_2]\ .$$

The sheaf $\mathcal{O}_Z$ is then defined as the cokernel of φ, hence it is supported on the common vanishing set of $d_1, \ldots, d_n$. As these are coprime, we get $\dim Z = 0$. To compute its Hilbert polynomial, we use Example 4.3.1 and additivity in the exact sequence (5.18) to get

$$P_{\mathcal{O}_Z}(m) = \tfrac{(m+2)(m+1)}{2} - n \cdot \tfrac{(m-n+3)(m-n+2)}{2} + (n-1) \cdot \tfrac{(m-n+2)(m-n+1)}{2} = \tfrac{n^2-n}{2}\ .$$

□

Example 5.2.24. The sequence (5.18) from Proposition 5.2.23 is not exact if the maximal minors of Φ are not coprime. Consider e.g.

$$\Phi = \begin{pmatrix} X_1 & X_0 & 0 \\ X_2 & 0 & X_0 \end{pmatrix} \qquad \Rightarrow \qquad \varphi = \begin{pmatrix} X_0^2 \\ -X_0X_1 \\ -X_0X_2 \end{pmatrix} .$$

Then $\Phi \cdot \varphi = 0$, but $(0, X_2, -X_1) \in \ker\varphi \setminus \operatorname{im}\Phi$ since

$$(f, g) \cdot \Phi = (fX_1 + gX_2 \, , \, fX_0 \, , \, gX_0) \, .$$

In order to obtain $(0, X_2, -X_1)$, we need $f = \frac{X_2}{X_0}$ and $g = -\frac{X_1}{X_0}$, but this choice is not possible for a stalk in $\mathbb{P}_2 \setminus U_0$, where U_0 is the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}_2$ with $x_0 \neq 0$.

The following result gives important invariants of every Hilbert-Burch resolution.

Proposition 5.2.25. [[17], 3.7 & 3.9, p.47-49] and [[18], III-61, p.133]
Let

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}_2}(-b_i) \xrightarrow{\Psi} \bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}_2}(-a_j) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z$$

be a minimal graded free resolution of an ideal with depth 2 (generated by n elements) defining a 0-dimensional scheme Z. We denote the degrees of the entries on the principal diagonals of Ψ by e_i and f_i, i.e.

$$\Psi = \begin{pmatrix} e_1 & f_1 & & * & * \\ & e_2 & f_2 & & * \\ * & & \ddots & \ddots & \\ * & * & & e_{n-1} & f_{n-1} \end{pmatrix} .$$

If we assume that $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_{n-1}$, we have the following bounds and relations:
1) $e_i \geq 1$ and $f_i \geq 1$, $\forall\, i \in \{1, \ldots, n-1\}$.
2) $e_i = b_i - a_i$ and $f_i = b_i - a_{i+1}$, $\forall\, i \in \{1, \ldots, n-1\}$.
3) $a_i = e_1 + \ldots + e_{i-1} + f_i + \ldots + f_{n-1}$, $\forall\, i \in \{1, \ldots, n-1\}$.
4) $b_1 + \ldots + b_{n-1} = a_1 + \ldots + a_n$.

5) $f_i \geq e_i$ and $f_i \geq e_{i+1}$, $\forall\, i \in \{1, \dots, n-2\}$.
Moreover, if N is the number of points in Z (with multiplicities), then

$$N = \sum_{i \leq j} e_i f_j \ .$$

5.2.3 Properties of coprime maximal minors

The results of the next two sections have mainly been pointed out by O. Iena.

Lemma 5.2.26. *Let $\Phi, \Phi' \in \mathbb{V}$ be two Kronecker modules such that their maximal minors $d_1, \dots, d_n$, resp. $d_1', \dots, d_n'$ are linearly independent and coprime. If $\{d_i\}_i$ and $\{d_j'\}_j$ span the same vector space over $\mathbb{K}$, then Φ and Φ' lie in the same orbit of the G-action on $\mathbb{V}$.*

Proof. Since the maximal minors span the same vector space over $\mathbb{K}$, there is an invertible matrix $h \in \mathrm{GL}_n(\mathbb{K})$ such that

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = h \cdot \begin{pmatrix} d_1' \\ \vdots \\ d_n' \end{pmatrix} \ \Leftrightarrow \ \varphi = h \cdot \varphi' \ .$$

By Corollary 5.2.8 we thus can define a Kronecker module $\Phi'' := g' \cdot \Phi' \cdot h^{-1}$ for some $g' \in \mathrm{GL}_{n-1}(\mathbb{K})$ such that $\det(g') = \det(h)$ which has the same maximal minors as Φ, i.e. $\varphi'' = \varphi$. Now consider

$$\begin{array}{ccccccc} 0 \longrightarrow \ker\varphi = (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) & \xrightarrow{\Phi} & n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{P}_2} \\ \uparrow g & & \| & & \| \\ 0 \longrightarrow \ker\varphi'' = (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) & \xrightarrow{\Phi''} & n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) & \xrightarrow{\varphi''} & \mathcal{O}_{\mathbb{P}_2} \end{array}$$

which is exact by Proposition 5.2.23. The morphism $g \in \mathrm{GL}_{n-1}(\mathbb{K})$ exists by the universal property of $\ker\varphi$ (as $\Phi'' \cdot \varphi = 0$, it factorizes through the kernel) and is an isomorphism by uniqueness of kernels. So we get

$$\Phi'' = g \cdot \Phi \ \Leftrightarrow \ \Phi = (g^{-1} \cdot g') \cdot \Phi' \cdot h^{-1} \ ,$$

i.e. Φ and Φ' are in the same G-orbit. □

Example 5.2.27. Lemma 5.2.26 does not hold if the maximal minors are not coprime. Consider

$$\Phi = \begin{pmatrix} X_1 & 0 & -X_2 & 0 \\ 0 & X_1 & 0 & -X_2 \\ 0 & 0 & -X_1 & X_0 \end{pmatrix} \qquad \text{and} \qquad \Phi' = \begin{pmatrix} 0 & X_1 & 0 & -X_2 \\ -X_1 & X_0 & 0 & 0 \\ 0 & 0 & -X_1 & X_0 \end{pmatrix}$$

$$\Rightarrow \quad d_1 = d_1' = X_0X_1X_2 \qquad , \qquad d_2 = d_2' = X_1^2X_2 \ ,$$
$$d_3 = d_3' = X_0X_1^2 \qquad , \qquad d_4 = d_4' = X_1^3 \ .$$

Both Kronecker modules have the same maximal minors, which have X_1 as a common divisor and are hence not coprime, but they do not lie in the same G-orbit. Indeed if we are looking for square-matrices g and h with entries in $\mathbb{K}$ such that $g \cdot \Phi = \Phi' \cdot h$, then $g = 0$ and $h = 0$, i.e. $\Phi' \notin O(\Phi)$.

Remark 5.2.28. A geometric interpretation illsutrating the difference between Φ and Φ' is given in Example 5.6.2.

Proposition 5.2.29. *Let $\Phi \in \mathbb{V}^s$ be a stable Kronecker module with maximal minors $d_1, \ldots, d_n$. If these are coprime, then they are linearly independent.*

Proof. Let $\gcd(d_1, \ldots, d_n) = 1$, so that we can apply Hilbert-Burch and Proposition 5.2.25. Assume that $d_1, \ldots, d_n$ are not linearly independent and that they span a vector space of dimension $k < n$. Then by Corollary 5.2.13 we may assume that the maximal minors of Φ are $d_1, \ldots, d_k, 0, \ldots, 0$ such that $d_1, \ldots, d_k$ are linearly independent. Moreover these are still coprime, otherwise the linear combinations that have been removed would have the same non-trivial divisor, so that the initial minors were not coprime. Hence $d_1, \ldots, d_k$ define a 0-dimensional subscheme Z and if $\varphi = {}^t(d_1, \ldots, d_k)$, then the minimal Hilbert-Burch resolution of $\mathcal{O}_Z$ reads

$$0 \longrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}_2}(-n+1-b_i) \longrightarrow k\, \mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

for some $e_i \geq 1$ with $e_1 \geq \ldots \geq e_{k-1}$ by Proposition 5.2.25 since all d_i are of degree $n-1$. Moreover we have the relation

$$\sum_{i=1}^{k}(n-1+e_i) = \sum_{i=1}^{k}(n-1) \;\Leftrightarrow\; (k-1)\cdot(n-1) + \sum_{i=1}^{k-1} e_i = k\cdot(n-1)$$
$$\Leftrightarrow\; \sum_{i=1}^{k-1} e_i = n-1 \;.$$

Hence at most $k-2$ of the e_i may be equal to 1, which means that there are at most $\ell \leq k-2$ linear syzygies of φ, i.e. there are exactly ℓ linear relations between $d_1,\ldots,d_k$. However we have

$$\Phi\cdot\begin{pmatrix}\varphi\\0\end{pmatrix} = 0\;,$$

so the Kronecker module φ contains $n-1$ linear syzygies of φ. Since there are only $\ell \leq k-2 \leq n-3$, we can perform linear transformations of the rows of Φ and obtain the form

$$\Phi \sim \begin{pmatrix}\mathrm{sy} & *\\ 0 & *\end{pmatrix}\;,$$

where the block of syzygies is of size $\ell\times k$ and the block of zeroes has size $(n-1-\ell)\times k$. We want to show that this contradicts stability of Φ, for which we need a block of zeros of size $j\times(n-j)$ for some $j\geq 1$. We have

$$j := n-1-\ell \geq n-1-(k-2) = n-k+1 \geq n-(n-1)+1 = 2$$

and it remains to show that $k\geq n-j$. But $n-j=\ell+1\leq k-1$, so this is satisfied. Finally this contradiction shows that $d_1,\ldots,d_n$ cannot be linearly dependent. □

Now we study the dual situation to the one presented in (5.18). Let $\Phi\in\mathbb{V}$ with maximal minors $d_1,\ldots,d_n$ and consider its dual ${}^t\Phi$, which is a matrix of the type $n\times(n-1)$. If ${}^t\varphi = (d_1,\ldots,d_n)$ is the row vector consisting of the maximal minors, then ${}^t\varphi\cdot{}^t\Phi = 0$ from Lemma 5.2.21 implies that

$$\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{{}^t\varphi} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{{}^t\Phi} (n-1)\,\mathcal{O}_{\mathbb{P}_2}$$

is a complex. We want to compute its kernel.

Proposition 5.2.30. *Assume that not all maximal minors of* Φ *are identically zero and denote* $g = \gcd(d_1, \ldots, d_n)$*. Then the sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-n + \deg g) \xrightarrow{\psi} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{{}^t\Phi} (n-1)\,\mathcal{O}_{\mathbb{P}_2} \tag{5.19}$$

is exact, where $\psi = (d'_1, \ldots, d'_n)$ *with* $d'_i = \frac{d_i}{g}$ *for all* $i \in \{1, \ldots, n\}$*.*

Proof. ψ is injective by Corollary 5.2.18 since not all its maximal minors are zero. $\psi \cdot {}^t\Phi = 0$ implies that (5.19) is moreover a complex. To prove exactness, we consider ${}^t\Phi$ as a linear map of K-vector spaces

$$n \cdot \mathbb{K}(X_0, X_1, X_2) \xrightarrow{{}^t\Phi} (n-1) \cdot \mathbb{K}(X_0, X_1, X_2) ,$$

where $K = \mathbb{K}(X_0, X_1, X_2)$. Since not all maximal minors of ${}^t\Phi$ are zero, the matrix is of rank $n-1$ and hence the kernel of this linear map is a 1-dimensional vector space over K. But $(d'_1, \ldots, d'_n)$ is a non-zero element of that kernel, hence it is a generator and we get the exact sequence

$$0 \longrightarrow \mathbb{K}(X_0, X_1, X_2) \xrightarrow{\psi} n \cdot \mathbb{K}(X_0, X_1, X_2) \xrightarrow{{}^t\Phi} (n-1) \cdot \mathbb{K}(X_0, X_1, X_2) .$$

Taking the intersection

$$\mathbb{K}[X_0, X_1, X_2] \cap \langle\, d'_1, \ldots, d'_n \,\rangle_K = (d'_1, \ldots, d'_n) \cdot \mathbb{K}[X_0, X_1, X_2] ,$$

we find that the kernel of the restricted $\mathbb{K}[X_0, X_1, X_2]$–module homomorphism

$$n \cdot \mathbb{K}[X_0, X_1, X_2] \xrightarrow{{}^t\Phi} (n-1) \cdot \mathbb{K}[X_0, X_1, X_2]$$

is also generated by $(d'_1, \ldots, d'_n)$, over the polynomial ring. □

Remark 5.2.31. The sequence (5.19) shows in particular that the dual sequence of (5.18) can be made exact if the maximal minors are not coprime. However this is not possible in the other case. Consider e.g. again Example 5.2.24 where

$$\Phi = \begin{pmatrix} X_1 & X_0 & 0 \\ X_2 & 0 & X_0 \end{pmatrix} \quad \text{with} \quad \varphi = \begin{pmatrix} X_0^2 \\ -X_0X_1 \\ -X_0X_2 \end{pmatrix} \quad \Rightarrow \quad \psi = \begin{pmatrix} X_0 \\ -X_1 \\ -X_2 \end{pmatrix}$$

by diving out the gcd X_0. We get $\Phi \cdot \psi = 0$, but it is still not exact at this step because of the same counter-example $(0, X_2, -X_1) \in \ker\psi \setminus \operatorname{im}\Phi$. Actually the syzygy of ψ consists of the 3 relations

$$(X_1, X_0, 0) \quad , \quad (X_2, 0, X_0) \quad , \quad (0, X_2, -X_1) .$$

5.2.4 Applications

Definition 5.2.32. We define $\mathbb{Y} \subset \mathbb{V}$ to be the set of all Kronecker modules with coprime maximal minors. Moreover we set $\mathbb{V}_0 := \mathbb{Y} \cap \mathbb{V}^s$. By Proposition 5.2.29 and Corollary 5.2.16 we thus have the inclusions

$$\mathbb{V}_0 \subseteq \mathbb{V}_l \subseteq \mathbb{V}^s \subseteq \mathbb{V} \ .$$

The first goals of this section are to prove that a generic $\Phi \in \mathbb{V}^s$ has coprime maximal minors and that the action of $\mathbb{P}G$ is free on $\mathbb{V}_0$.

Lemma 5.2.33. $\mathbb{Y} \subset \mathbb{V}$ *is a generic set.*

Proof. We have to show that the set of coefficients a_{ij}^k, A_i^k from (5.14) which define coprime minors in $\mathbb{A}_v$ for $v = 3n(n-1)$ contains an open set. For this note that a smooth curve in $\mathbb{P}_2$ is irreducible since the intersection point of 2 components would be singular (by Bézout's Theorem such an intersection point always exists). For all $i \in \{1, \ldots, n\}$ let $S_i \subset \mathbb{A}_v$ be the subset of all coefficients such that the curve defined by the i^{th} maximal minor is smooth. Similarly as in the proof of Proposition 4.4.18 one finds that S_i is open since $\mathbb{P}_2$ is complete. Hence we get $S_1 \cup \ldots \cup S_n \subseteq \mathbb{Y}$ since the gcd is 1 as soon as at least one of the maximal minors defines a smooth curve (and is thus irreducible). So $\mathbb{Y}$ is generic because each S_i is open and non-empty. $\square$

Lemma 5.2.34. $\mathbb{Y} \subset \mathbb{V}$ *is even open itself. In particular* $\mathbb{V}_0 \subset \mathbb{V}^s$ *is open.*

Proof. We show that $\mathbb{V} \setminus \mathbb{Y}$, consisting of Kronecker modules whose maximal minors have a common divisor, is a closed set. Let us first describe the idea:
Let $f_1, \ldots, f_n$ be homogeneous polynomials of degree $e \geq 2$. We are interested in the case where these have a common factor, i.e. when there exists $r \in \{1, \ldots, e-1\}$, a homogeneous polynomial h of degree r and homogeneous polynomials $g_1, \ldots, g_n$ of degree $e - r$ such that $f_i = h \cdot g_i$ for all $i \in \{1, \ldots, n\}$.
Let

$$t_1 = \binom{r+2}{2} \quad , \quad t_2 = \binom{e-r+2}{2} \quad , \quad E = \binom{e+2}{2} \ .$$

First we look at the affine space $X_r = \mathbb{A}_{t_1} \times \prod_1^n \mathbb{A}_{t_2} \times \prod_1^n \mathbb{A}_E$ and the subset

$$V_r := \left\{ (h, g_1, \ldots, g_n, f_1, \ldots, f_n) \;\middle|\; f_i = h \cdot g_i \text{ for all } i \right\} \subset X_r \ .$$

Writing out the equality and obtaining polynomial equations in the coefficients of the polynomials one sees that V_r is closed, hence that its complement $U_r = X_r \backslash V_r$, consisting of such tuples of polynomials which have no common factor of degree r, is open. Next we consider the projection $p_r : X_r \to \prod_1^n \mathbb{A}_E$. Then the subset of all polynomials $f_1, \ldots, f_n$ which have no common factor is equal to $p_1(U_1) \cup \ldots \cup p_{e-1}(U_{e-1})$. Since all projections p_r are open (see Lemma 5.2.35), we obtain that the subset of $\prod_1^n \mathbb{A}_E$ of polynomials which have no common factor is open.
The same idea also applies to $\mathbb{V} \setminus \mathbb{Y}$. By putting the coefficients of the maximal minors of a Kronecker module into polynomial equations and using similar open projections, one obtains that $\mathbb{Y}$ is an open subset of $\mathbb{A}_v$ for $v = 3n(n-1)$. □

Lemma 5.2.35. [[11], 10.40.10] and [[53], 463845]
Let $\mathcal{X}, \mathcal{Y}$ be Noetherian schemes over $\mathbb{K}$. Then the projection $p : \mathcal{X} \times_{\mathbb{K}} \mathcal{Y} \to \mathcal{X}$ is open.

Remark 5.2.36. [[11], 10.40.8 & 28.24.9] and [[53], 676533]
Actually an even stronger statement holds true: let $\mathcal{X}, \mathcal{Y}$ be Noetherian schemes over $\mathbb{K}$. Then every flat morphism $f : \mathcal{X} \to \mathcal{Y}$ which is locally of finite presentation is open.

As already mentioned in Remark 4.6.14 and Remark 5.1.32, Proposition 5.2.30 now allows to prove

Corollary 5.2.37. *Consider the G-action on $\mathbb{V}$. If $\Phi \in \mathbb{V}_0$, then the stabilizer of Φ is equal to the 1-dimensional subgroup $\Gamma = \{ (\lambda \operatorname{id}_{n-1}, \lambda \operatorname{id}_n) \mid \lambda \in \mathbb{K}^* \} \subset G$.*

Proof. Assume that $(g, h) \in G$ is such that $g \Phi h^{-1} = \Phi$. Then by dualizing we get $a \cdot {}^t\Phi \cdot b = {}^t\Phi$ for $a = ({}^t h)^{-1}$ and $b = {}^t g$. From $(d_1, \ldots, d_n) \cdot {}^t\Phi = 0$, we also have $(d_1, \ldots, d_n) \cdot a \in \ker({}^t\Phi)$ since

$$(d_1, \ldots, d_n) \cdot a \cdot {}^t\Phi = (d_1, \ldots, d_n) \cdot a \cdot {}^t\Phi \cdot b \cdot b^{-1} = (d_1, \ldots, d_n) \cdot {}^t\Phi \cdot b^{-1} = 0 \,.$$

By exactness of (5.19) we thus get $(d_1, \ldots, d_n) \cdot a = \mu \, (d_1, \ldots, d_n)$ for some $\mu \in \mathbb{K}^*$ (μ cannot be zero, otherwise all $d_i = 0$ would by zero since a is invertible). Therefore

$$(d_1, \ldots, d_n) \cdot \big(a - \mu \operatorname{id}_n \big) = 0 \qquad \Rightarrow \; a = \mu \operatorname{id}_n$$

because the d_i are linearly independent by Proposition 5.2.29. Then we have

$$a \cdot {}^t\Phi \cdot b - {}^t\Phi = 0 \;\Leftrightarrow\; {}^t\Phi \cdot \big(\mu b - \mathrm{id}_{n-1}\big) = 0 \;\Leftrightarrow\; \big(\mu \cdot {}^t b - \mathrm{id}_{n-1}\big) \cdot \Phi = 0 \ .$$

By injectivity of Φ (not all maximal minors are zero), we hence obtain that $\mu b = \mathrm{id}_{n-1}$. Finally if we set $\lambda = \frac{1}{\mu}$, then

$$g = {}^t b = \lambda \,\mathrm{id}_{n-1} \qquad \text{and} \qquad h = ({}^t a)^{-1} = \lambda \,\mathrm{id}_n \ ,$$

i.e. $(g,h) \in \Gamma$. On the other hand the inclusion $\Gamma \subseteq \mathrm{Stab}_G(\Phi)$ is always true, so we get equality. □

Corollary 5.2.38. *The action of $\mathbb{P}G = G/\Gamma$ on $\mathbb{V}_0$ is free.*

Proof. By Corollary 5.2.37 we know that the stabilizer of any $\Phi \in \mathbb{V}_0$ is equal to Γ. Hence by dividing it out, all stabilizers become trivial. □

Another application of Hilbert-Burch is that one can obtain a locally free resolution for the structure sheaf of every 0-dimensional scheme whose points are in general position.

Definition 5.2.39. Fix some $d \geq 1$, let $k = \binom{d+2}{2}$ and consider k (simple) points $p_1, \ldots, p_k \in \mathbb{P}_2$. We say that these points are *in general position* if they do not all lie on a curve of degree d. E.g. if $d = 1$ we have 3 non-collinear points and for $d = 2$ we get 6 points that do not lie on a conic.

Lemma 5.2.40. *Let $e > d$ and denote $E = \binom{e+2}{2}$. Then the $\mathbb{K}$-vector space of homogeneous polynomials of degree e which vanish at k points in general position has dimension $E - k$. Therefore the corresponding subspace in $\mathcal{C}_e(\mathbb{P}_2) \cong \mathbb{P}_{E-1}$ of curves of degree e passing through $p_1, \ldots, p_k$ has dimension $E - k - 1$.*

Proof. If $h \in \mathbb{K}[X_0, X_1, X_2]$ is a homogeneous polynomial of degree e, then $h(p_i) = 0$ gives one linear condition on the coefficients of h. As we have k different points, we obtain k linearly independent conditions (since the points are in general position) and a vector subspace of codimension k. □

Remark 5.2.41. The same statement actually also holds true for points with multiplicities. Indeed saying that a point of multiplicity m belongs to a curve gives m linearly independent conditions.

Remark 5.2.42. Lemma 5.2.40 does not hold true if the points are not in general position. Consider for example the 3 collinear points $(1:0:0)$, $(0:1:0)$ and $(1:1:0)$. They already lie on a curve of degree 1. Now we add the points $(1:2:0)$, $(1:3:0)$ and $(1:4:0)$, so that $d=2$. These $6=\binom{4}{2}$ points still lie on the line given by the equation $X_2=0$ and thus e.g. on the conic defined by $X_2^2+X_0X_2+X_1X_2$. But for cubics ($e=3$), we now have the 10 monomials

$$X_0^3\ ,\ X_1^3\ ,\ X_2^3\ ,\ X_0^2X_1\ ,\ X_0^2X_2\ ,\ X_0X_1^2\ ,\ X_0X_2^2\ ,\ X_0X_1X_2\ ,\ X_1^2X_2\ ,\ X_1X_2^2\ ,$$

and saying that a homogeneous polynomial of degree 3 should vanish at these 6 points only requires that the coefficients in front of X_0^3, X_1^3, $X_0^2X_1$ and $X_0X_1^2$ must vanish. So we obtain a vector space of dimension $10-4=6$. However $E-k=10-6=4$.

Proposition 5.2.43. [[17], 3.8, p.48] and [[18], III-62, p.133]
Let $I \trianglelefteq \mathbb{K}[X_0,X_1,X_2]$ be a homogeneous ideal defining a 0-dimensional subscheme $Z\subset\mathbb{P}_2$. If all points of Z lie on a curve of degree e or if I contains an element of degree e, then I can be generated by $e+1$ elements.

Corollary 5.2.44. *Let $\Phi\in\mathbb{V}$ such that its maximal minors $d_1,\dots,d_n$ are coprime. Then the points of the 0-dimensional scheme $Z(d_1,\dots,d_n)\subset\mathbb{P}_2$ of length $\binom{n}{2}$ do not lie on a curve of degree $n-2$.*

Proof. The ideal generated by the maximal minors only contains non-trivial elements of degree $n-1$ (by definition), hence Proposition 5.2.43 implies that it can be generated by n elements (the minors themselves if they are linearly independent). Thus the points of Z cannot lie on a curve of degree $n-2$, otherwise the minimal number of generators would be $n-1$. □

Proposition 5.2.45. *Fix $d \geq 1$, let $k = \binom{d+2}{2}$ and consider a 0-dimensional scheme Z of k points (with multiplicities). Assume that the points are in general position, i.e. they do not lie on a curve of degree d. Then $\mathcal{O}_Z$ has a resolution*

$$0 \longrightarrow (d{+}1)\,\mathcal{O}_{\mathbb{P}_2}(-d{-}2) \xrightarrow{\Phi} (d{+}2)\,\mathcal{O}_{\mathbb{P}_2}(-d{-}1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ . \quad (5.20)$$

Proof. Let $\mathcal{I}_Z$ be the ideal sheaf defining Z. Since Z does not lie on a curve of degree d, but on some curve of degree $d+1$ (there are always enough coefficients to satisfy this condition), Proposition 5.2.43 implies that $\mathcal{I}_Z$ can be generated by $d+2$ elements. Thus we get the Hilbert-Burch resolution

$$0 \longrightarrow \bigoplus_{i=1}^{d+1} \mathcal{O}_{\mathbb{P}_2}(-b_i) \xrightarrow{\Phi} \bigoplus_{j=1}^{d+2} \mathcal{O}_{\mathbb{P}_2}(-a_j) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ ,$$

where $e_i = b_i - a_i$, $f_i = b_i - a_{i+1}$ for all $i \in \{1, \ldots, d+1\}$ and $k = \sum_{i \leq j} e_i f_j$ by Proposition 5.2.25. Note that the number of summands in this sum is

$$\sum_{a=1}^{d+1} a = \tfrac{(d+1)(d+2)}{2} = \tbinom{d+2}{2} = k$$

since at each step there is one summand less. As $e_i \geq 1$ and $f_i \geq 1$, we thus need that $e_i = f_j = 1$ for all i, j, i.e. the matrix Φ consists of linear forms. Now consider the $\mathbb{K}$-vector space V of homogeneous polynomials of degree $d+1$ vanishing at Z. By Lemma 5.2.40 its dimension is $\binom{d+3}{2} - k = d+2$, hence V is generated by $d+2$ linearly independent homogeneous polynomials of degree $d+1$; denote them by $f_1, \ldots, f_{d+2}$. Since $Z \subseteq Z(f_i)$ for all $f_i \in V$, we get $\langle\, f_i \,\rangle = \mathcal{I}_{Z(f_i)} \subseteq \mathcal{I}_Z$, i.e. each f_i also belongs to $\mathcal{I}_Z$. But as $\mathcal{I}_Z$ can be generated by $d+2$ elements, these generators can be chosen to be exactly the $f_i \in V$. By Theorem 5.2.17 we now know that the generators of $\mathcal{I}_Z$ coincide (up to a NZD) with the maximal minors of the matrix Φ. So finally we obtain the resolution (5.20) with

$$\varphi = \begin{pmatrix} f_1 \\ \vdots \\ f_{d+2} \end{pmatrix}$$

since all minors are of degree $d+1$ (i.e. all a_j are equal) and Φ consists of linear forms. □

Example 5.2.46. Consider e.g. $d = 1$ and $k = 3$. So we are looking at three non-collinear points. Let us take

$$Z = \left\{ z_0 = (1:0:0)\ ,\ z_1 = (0:1:0)\ ,\ z_2 = (0:0:1) \right\} .$$

The vector space of all curves of degree 2 is

$$\left\{ f = a_0 X_0^2 + a_1 X_1^2 + a_2 X_2^2 + b_0 X_0 X_1 + b_1 X_0 X_2 + b_2 X_1 X_2 \;\middle|\; a_i, b_i \in \mathbb{K} \right\} .$$

If we require that $Z \subseteq Z(f)$, we get the conditions $a_0 = a_1 = a_2 = 0$, so the vector space of all conics in $\mathbb{P}_2$ containing Z is

$$V = \left\{ f = b_0 X_0 X_1 + b_1 X_0 X_2 + b_2 X_1 X_2 \;\middle|\; b_i \in \mathbb{K} \right\} .$$

The 3 generators X_0X_1, X_0X_2, X_1X_2 are coprime and also generated the ideal sheaf $\mathcal{I}_Z$ since their common vanishing set is exactly $Z = Z(X_0X_1, X_0X_2, X_1X_2)$.

Remark 5.2.47. A priori the resolution (5.20) seems to be uniquely determined by Z. However since every quotient $\mathcal{O}_{\mathbb{P}_2} \twoheadrightarrow \mathcal{O}_Z$ defines the same ideal sheaf by $0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_Z \to 0$, one can always construct a resolution of this type which is compatible with a given surjective morphism $\mathcal{O}_{\mathbb{P}_2} \twoheadrightarrow \mathcal{O}_Z$ (since the proof of Proposition 5.2.45 only uses $\mathcal{I}_Z$ in the computations).

Summary

The main results of Section 5.2 are the following. The cokernel of a Kronecker module $\Phi \in \mathbb{V}_0$ defines a 0-dimensional scheme $Z \subset \mathbb{P}_2$ of length $l = \binom{n}{2}$. Moreover the points of Z do not lie on a curve of degree $n-2$. The maximal minors $d_1, \ldots, d_n$ of Φ are linearly independent and we have a resolution

$$0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ , \qquad (5.21)$$

where $\varphi = {}^t(d_1, \ldots, d_n)$.

5.2.5 Remark: relation with perfect ideals

Definition 5.2.48. [[55], p.131-132] and [[53], 576487]
Let $I \trianglelefteq R$ be an ideal in a Noetherian ring R. The *projective dimension* of I,

denoted $pd(I)$, is the smallest length n of a projective resolution of the R-module R/I, i.e.

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow R/I \longrightarrow 0 \ ,$$

where all P_i are projective R-modules. The *grade* of I, denoted $\mathrm{grade}(I)$, is defined as

$$\mathrm{grade}(I) = \min \left\{ i \geq 0 \;\middle|\; \mathrm{Ext}^i(R/I, R) \neq \{0\} \right\} .$$

I is called a *perfect* ideal if $pd(I) = \mathrm{grade}(I)$.

We briefly want to illustrate where perfect and non-perfect ideals show up in the setting of Hilbert-Burch and Kronecker modules. Consider e.g. $n = 3$.

Example 5.2.49. Let $R = \mathbb{K}[X_0, X_1, X_2]$ and assume that a stable Kronecker module Φ of size 2×3 is such that its maximal minors d_1, d_2, d_3 define an ideal of 3 different points. So in particular they are coprime. Then for $I = \langle\, d_1, d_2, d_3 \,\rangle$, Hilbert-Burch gives the resolution

$$0 \longrightarrow 2R \xrightarrow{\Phi} 3R \xrightarrow{\varphi} R \longrightarrow R/I \longrightarrow 0 \ , \tag{5.22}$$

where $\varphi = {}^t(d_1, d_2, d_3)$, hence $pd(I) = 2$. Recall that

$$\mathrm{Ext}^i(R/I, R) = H^i\big(\mathrm{Hom}_R(P^\bullet, R)\big) \ ,$$

where $P^\bullet$ is a projective resolution of R/I. We take (5.22) and get the complex

$$0 \longrightarrow \mathrm{Hom}_R(R/I, R) \longrightarrow \mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Hom}_R(3R, R) \longrightarrow \mathrm{Hom}_R(2R, R)$$
$$\Leftrightarrow \ 0 \longrightarrow \mathrm{Hom}_R(R/I, R) \longrightarrow R \xrightarrow{{}^t\varphi} 3R \xrightarrow{{}^t\Phi} 2R \ .$$

${}^t\varphi$ is injective since not all d_i are zero, hence we have

$$\mathrm{Ext}^0(R/I, R) = \mathrm{Hom}_R(R/I, R) = \{0\} \ .$$

d_1, d_2, d_3 being coprime, Proposition 5.2.30 also implies that the sequence is exact at the step ${}^t\Phi \circ {}^t\varphi = 0$, hence $\mathrm{Ext}^1(R/I, R) = \{0\}$ as well. However $\mathrm{Ext}^2(R/I, R) \neq \{0\}$ because ${}^t\Phi$ is not surjective (e.g. linear forms cannot be obtained). Thus $\mathrm{grade}(I) = 2$, meaning that I is a perfect ideal.

Example 5.2.50. Now consider a Kronecker module Φ where the maximal minors are not coprime, e.g.

$$\Phi = \begin{pmatrix} X_0 & 0 & X_1 \\ 0 & X_0 & X_2 \end{pmatrix} ,$$

so that $d_1 = -X_0X_1$, $d_2 = -X_0X_2$ and $d_3 = X_0^2$. If $I = \langle d_1, d_2, d_3 \rangle$, a similar sequence as in (5.22) is not exact since

$$\begin{pmatrix} X_2 & -X_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -X_0X_1 \\ -X_0X_2 \\ X_0^2 \end{pmatrix} = 0 ,$$

but $(X_2, -X_1, 0)$ is not in the image of Φ. Note however that

$$\varphi = \begin{pmatrix} -X_0X_1 \\ -X_0X_2 \\ X_0^2 \end{pmatrix} = \begin{pmatrix} -X_1 \\ -X_2 \\ X_0 \end{pmatrix} \cdot X_0 = \psi \cdot X_0 ,$$

where ψ can be resolved using syzygies as in Example 4.5.10. We find

$$0 \longrightarrow R \xrightarrow{\ell} 3R \xrightarrow{A} 3R \xrightarrow{\psi} R \xrightarrow{\cdot X_0} R$$

with

$$\ell = (X_0, -X_2, X_1) \quad , \quad A = \begin{pmatrix} X_2 & -X_1 & 0 \\ X_0 & 0 & X_1 \\ 0 & X_0 & X_2 \end{pmatrix} \quad , \quad \psi = \begin{pmatrix} -X_1 \\ -X_2 \\ X_0 \end{pmatrix}$$

and hence

$$0 \longrightarrow R \xrightarrow{\ell} 3R \xrightarrow{A} 3R \xrightarrow{\varphi} R \longrightarrow R/I \longrightarrow 0 ,$$

which implies that $pd(I) = 3$. On the other hand we get $\text{Ext}^0(R/I, R) = \{0\}$ since ${}^t\varphi$ is injective. But

$$0 \longrightarrow R \xrightarrow{{}^t\varphi} 3R \xrightarrow{{}^tA} 3R$$

is not exact at the step ${}^tA \circ {}^t\varphi = 0$ since syzygies imply that the kernel of ${}^tA$ is ${}^t\psi = (-X_1, -X_2, X_0)$. Thus $\text{Ext}^1(R/I, R) \neq \{0\}$ and $\text{grade}(I) = 1$, i.e. such an ideal I is not perfect.

5.3 Classifying quotient bundle

In this section we follow the same strategy as in the proof of [[48], 7.7, p.40-41] in order to construct a projective bundle $\mathbb{B}$ over the quotient $N = \mathbb{V}^s/\mathbb{P}G$. It will turn out that this bundle is a geometric quotient of a subspace of $\mathbb{V}^s \times \mathbb{U}_2$, contains M_0 as an open subset and its points are in 1-to-1 correspondence with the orbits of the G'-action from (5.10) on $\mathbb{W}_0$. The more interesting fact about $\mathbb{B}$ is however that the sheaves in a certain open subvariety $\mathbb{B}_0 \subseteq M_0 \subseteq \mathbb{B}$ have a precise description as twisted ideals sheaves of curves of degree d. This will be of major interest in order to compute the codimension of the subvariety of singular sheaves in M and is the aim of Proposition 5.3.31, which is a slight variation of a less general result from Drézet and Maican in [14] and [15].

5.3.1 Elimination of the non-reductive group action

We want to modify the action of the non-reductive group G' from (5.10) on $\mathbb{W}_0$ in order to construct geometric quotients by the action of reductive groups. Indeed we know by (5.13) that the sheaves in M_0 are parametrized by matrices in $\mathbb{W}_0 \subset \mathbb{V}^s \times \mathbb{U}_2 \subset \mathbb{W}$, so we are interested in a quotient space whose points are parametrized by the orbits of the G'-action on $\mathbb{V}^s \times \mathbb{U}_2$. Fix the notation

$$\mathbb{U}_2 = n\,\Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(2)\big) \qquad , \qquad \mathbb{U}_1 = (n-1)\,\Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)\ .$$

The idea is to "forget" about the action of the terms $\lambda \in \mathbb{K}^*$ and $L \in \mathbb{U}_1$ in $(g', h) \in G'$, where

$$g' = \begin{pmatrix} \lambda & L \\ 0 & g \end{pmatrix}$$

with $g \in \mathrm{GL}_{n-1}(\mathbb{K})$, so that all we have to care about is the reductive part of G', for which we can use the results from GIT.

Remark 5.3.1. The reductive group $G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$ can be seen as a subgroup of G' via the injection

$$i \ :\ G \hookrightarrow G' \ :\ (g,h) \longmapsto \left(\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} , h \right)\ .$$

So in particular every G'-action also induces an action of G.

Definition 5.3.2. [[48], p.41] and [[40], 2.1, p.2]
We consider the trivial vector bundles $\mathbb{V} \times \mathbb{U}_1$ and $\mathbb{V} \times \mathbb{U}_2$ over the space of Kronecker modules $\mathbb{V}$ and define a morphism of vector bundles over $\mathbb{V}$ by

$$F \;:\; \mathbb{V} \times \mathbb{U}_1 \longrightarrow \mathbb{V} \times \mathbb{U}_2 \;:\; (\Phi, L) \longmapsto (\Phi, L \cdot \Phi) \;. \tag{5.23}$$

Lemma 5.3.3. [[40], 2.1, p.2-3]
F is injective over $\mathbb{V}^s$.

Proof. Let $(\Phi, L), (\Phi', L') \in \mathbb{V} \times \mathbb{U}_1$ such that $F(\Phi, L) = F(\Phi', L')$. Thus $\Phi = \Phi'$ and we are left with

$$L \cdot \Phi = L' \cdot \Phi \;\Leftrightarrow\; (L - L') \cdot \Phi = 0 \;.$$

So it remains to show that the linear map $\mathbb{U}_1 \to \mathbb{U}_2 : L \mapsto L \cdot \Phi$ between the fibers in injective for stable Kronecker modules. For this, let $\Phi \in \mathbb{V}^s$ and assume that $L \cdot \Phi = 0$ for some non-zero $L = (l_1, \ldots, l_{n-1}) \in \mathbb{U}_1$.

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1,n} \\ z_{21} & z_{22} & \cdots & z_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n} \end{pmatrix}$$

Since $l_i \in \Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$ are linear forms, the dimension of the $\mathbb{K}$-vector space generated by the l_i is at most 3. So one there exists some $B \in \mathrm{GL}_{n-1}(\mathbb{K})$ such that $L \cdot B = (l'_1, l'_2, l'_3, 0, \ldots, 0)$ where the first non-zero entries are linearly independent over $\mathbb{K}$. Moreover Theorem 5.1.28 implies that Φ is semistable if and only if $B^{-1}\Phi$ is semistable (since both Kronecker modules lie in the same orbit). Thus we can write

$$0 = L \cdot \Phi = (LB) \cdot (B^{-1}\Phi)$$

and may assume without loss of generality that $L = (l_1, l_2, l_3, 0, \ldots, 0)$ such that the first non-zero entries of L are linearly independent. Now we distinguish 3 cases:

1) If $l_1 \neq 0$ and $l_2 = l_3 = 0$, then $L \cdot \Phi = 0$ implies that $l_1 \cdot z_{1j} = 0$ and thus $z_{1j} = 0$ for all $j \in \{1, \ldots, n\}$, i.e. the first row of Φ is zero. But this contradicts

stability of Φ.
2) If $l_1 \neq 0$, $l_2 \neq 0$ and $l_3 = 0$, then $L \cdot \Phi = 0$ gives $l_1 z_{1j} + l_2 z_{2j} = 0$ for all $j \in \{1, \ldots, n\}$. But the Koszul resolution from Example 4.5.10 implies that syzygy module of (l_1, l_2) is generated by $\binom{-l_2}{l_1}$. Hence $\forall\, j$, $\exists\, \alpha_j \in \mathbb{K}$ such that

$$\begin{pmatrix} z_{1j} \\ z_{2j} \end{pmatrix} = \alpha_j \cdot \begin{pmatrix} -l_2 \\ l_1 \end{pmatrix} ,$$

i.e. the columns of the first two rows of Φ are all scalar multiples of $\binom{-l_2}{l_1}$. Thus after performing elementary transformations of the columns of Φ, Φ is equivalent to a matrix with a zero block of size $2 \times (n-1)$.

$$\Phi \sim \begin{pmatrix} -l_2 & 0 & \ldots & 0 \\ l_1 & 0 & \ldots & 0 \\ * & * & * & * \end{pmatrix}$$

But this again contradicts stability of Φ.
3) Now assume that $l_i \neq 0$ for $i = 1, 2, 3$. Then we have $l_1 z_{1j} + l_2 z_{2j} + l_3 z_{3j} = 0$ for all j and the Koszul resolution says that the syzygy module of (l_1, l_2, l_3) is generated by the 3 linearly independent generators

$$v_1 = \begin{pmatrix} 0 \\ l_3 \\ -l_2 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} -l_3 \\ 0 \\ l_1 \end{pmatrix} \quad , \quad v_3 = \begin{pmatrix} l_2 \\ -l_1 \\ 0 \end{pmatrix} .$$

Hence the columns of the first 3 rows of ϕ are linear combinations of v_1, v_2, v_3, i.e. $\forall\, j \in \{1, \ldots, n\}$,

$$\begin{pmatrix} z_{1j} \\ z_{2j} \\ z_{3j} \end{pmatrix} = \sum_{i=1}^{3} \beta_{ij} \cdot v_i$$

for some $\beta_{ij} \in \mathbb{K}$. But then Φ is equivalent to a matrix with a zero block of size $3 \times (n-3)$, which contradicts again stability.

$$\Phi \sim \begin{pmatrix} v_1 & v_2 & v_3 & 0 & \ldots & 0 \\ * & * & * & * & * & * \end{pmatrix}$$

Finally we conlude that $L \cdot \Phi = 0$ with $\Phi \in \mathbb{V}^s$ is only possible for $L = 0$. $\square$

Remark 5.3.4. For fixed $L \in \mathbb{U}_1$, the morphism

$$\mathbb{V} \longrightarrow \mathbb{U}_2 \ : \ \Phi \longmapsto L \cdot \Phi$$

maps a Kronecker module to a $\Gamma(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1))$–linear combination of its rows. Indeed if we look for example at $n = 3$, we get

$$\begin{aligned}\begin{pmatrix} l_1 & l_2 \end{pmatrix} \cdot \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix} &= \begin{pmatrix} l_1 z_{11} + l_2 z_{21} & l_1 z_{12} + l_2 z_{22} & l_1 z_{13} + l_2 z_{23} \end{pmatrix} \\ &= l_1 \cdot \begin{pmatrix} z_{11} & z_{12} & z_{13} \end{pmatrix} + l_2 \cdot \begin{pmatrix} z_{21} & z_{22} & z_{23} \end{pmatrix} \ .\end{aligned}$$

In particular this implies that

$$\det \begin{pmatrix} L \cdot \Phi \\ \Phi \end{pmatrix} = 0 \ , \tag{5.24}$$

i.e. matrices which are in the image of F have zero determinant.

Corollary 5.3.5. *The cokernel of F, denoted $\mathbb{E}$, is a vector bundle of rank $3d$.*

Proof. Since F is injective over $\mathbb{V}^s$, we may see $\mathbb{V}^s \times \mathbb{U}_1$ as a vector subbundle of $\mathbb{V}^s \times \mathbb{U}_2$ and get an exact sequence

$$0 \longrightarrow \mathbb{V}^s \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V}^s \times \mathbb{U}_2 \xrightarrow{\pi} \mathbb{E} \longrightarrow 0 \ ,$$

where $\mathbb{E} = \operatorname{coker} F$. Recall that the rank of a vector bundle is given by the dimension of its fiber. As $\dim_{\mathbb{K}} \mathbb{U}_1 = 3(n-1)$ and $\dim_{\mathbb{K}} \mathbb{U}_2 = 6n$, additivity in exact sequences of the rank gives

$$\operatorname{rk} \mathbb{E} = \operatorname{rk}(\mathbb{V}^s \times \mathbb{U}_2) - \operatorname{rk}(\mathbb{V}^s \times \mathbb{U}_1) = 6n - 3(n-1) = 3n + 3 = 3d \ .$$

□

Remark 5.3.6. The fibers of $\mathbb{E}$ are given by

$$\mathbb{E}_\Phi \ \cong \ \operatorname{coker} F_\Phi = \big(\{\Phi\} \times \mathbb{U}_2\big) / \operatorname{im} F_\Phi \ \cong \ \{ \, Q \in \mathbb{U}_2 \, \} / \{ \, L\Phi \mid L \in \mathbb{U}_1 \, \} \ ,$$

so they parametrize quadratic forms which cannot be written as a product of Φ with linear forms.

The idea of defining $\mathbb{E}$ as the cokernel of F is that the action of the linear part of G' should become invisible. We will make this clear in the following steps.
We know that there is an action of G on $\mathbb{V}$ via $(g,h)\,.\,\Phi = g\cdot\Phi\cdot h^{-1}$. Moreover we have the action of G' on $\mathbb{W}$ via $(g',h)\,.\,A = g'\cdot A\cdot h^{-1}$.

Lemma 5.3.7. *There is an action of G on $\mathbb{V}^s$.*

Proof. The action of G on $\mathbb{V}$ preserves the stable Kronecker modules by definition (since Kronecker modules which are not stable lie in certain orbits under $\mathbb{P}G$, see Theorem 5.1.28), thus this action restricts to an action of G on $\mathbb{V}^s$. □

Lemma 5.3.8. *There is an action of G' on $\mathbb{V}^s\times\mathbb{U}_2$.*

Proof. The isomorphism (5.11) allows to consider $\mathbb{V}^s\times\mathbb{U}_2$ as an open subvariety in $\mathbb{W}$, on which we already have a G'-action. Let us write

$$g' = \begin{pmatrix} \lambda & \ell \\ 0 & g \end{pmatrix}$$

for $(g',h)\in G'$, where $\lambda\in\mathbb{K}^*$, $\ell\in\mathbb{U}_1$ and $g\in \mathrm{GL}_{n-1}(\mathbb{K})$. Hence if we take $(\Phi,Q)\in\mathbb{V}^s\times\mathbb{U}_2$ we have

$$\begin{aligned} (g',h)\,.\,(\Phi,Q) &= \begin{pmatrix} \lambda & \ell \\ 0 & g \end{pmatrix}\cdot\begin{pmatrix} Q \\ \Phi \end{pmatrix}\cdot h^{-1} \\ &= \begin{pmatrix} \lambda & \ell \\ 0 & g \end{pmatrix}\cdot\begin{pmatrix} Q\cdot h^{-1} \\ \Phi\cdot h^{-1} \end{pmatrix} = \begin{pmatrix} \lambda Q h^{-1} + \ell\Phi h^{-1} \\ g\cdot\Phi\cdot h^{-1} \end{pmatrix} \\ \Rightarrow\ (g',h)\,.\,(\Phi,Q) &= \big(\, g\Phi h^{-1}\,,\ \lambda Q h^{-1} + \ell\Phi h^{-1}\,\big)\ . \end{aligned}$$

In particular, $(g,h)\,.\,(\Phi,Q) = (g\Phi h^{-1}, Qh^{-1})$ for $(g,h)\in G$. □

Lemma 5.3.9. *There is an action of G' on $\mathbb{V}^s\times\mathbb{U}_1$ which is compatible with F and* (5.23)*.*

Proof. We want to define the action in a way such that $F:\mathbb{V}^s\times\mathbb{U}_1\to\mathbb{V}^s\times\mathbb{U}_2$ is a G'-morphism. In other words F should respect the actions of G' on $\mathbb{V}^s\times\mathbb{U}_1$ and $\mathbb{V}^s\times\mathbb{U}_2$, i.e.

$$F\big((g',h)\,.\,(\Phi,L)\big) = (g',h)\,.\,F(\Phi,L)$$

for all $(g', h) \in G'$, $\Phi \in \mathbb{V}^s$ and $L \in \mathbb{U}_1$. Writing out the RHS, we get

$$(g', h) \,.\, F(\Phi, L) = (g', h) \,.\, (\Phi, L \cdot \Phi) = \big(g\Phi h^{-1} \,,\, \lambda L \Phi h^{-1} + \ell \Phi h^{-1} \big)$$
$$= \big(g\Phi h^{-1} \,,\, (\lambda L + \ell)\Phi h^{-1} \big) \qquad (5.25)$$

since $L \cdot \Phi \in \mathbb{U}_2$. We set $(g', h) \,.\, (\Phi, L) = (\Phi', L')$ for some $\Phi' \in \mathbb{V}^s$ and $L' \in \mathbb{U}_1$ which have to be determined. Then

$$F\big((g', h) \,.\, (\Phi, L)\big) = F(\Phi', L') = (\Phi', L' \cdot \Phi') \ ,$$

which has to be equal to (5.25). Hence $\Phi' = g\Phi h^{-1}$ and

$$(\lambda L + \ell)\Phi h^{-1} = L'\Phi' = L' g \Phi h^{-1} \qquad \Rightarrow \ L' = (\lambda L + \ell) g^{-1} \ .$$

Finally the action of G' on $\mathbb{V}^s \times \mathbb{U}_1$ is given by

$$(g', h) \,.\, (\Phi, L) = (g\Phi h^{-1}, (\lambda L + \ell) g^{-1}) \ .$$

The action of G can then be obtained by setting $\lambda = 1$ and $\ell = 0$. □

Lemma 5.3.10. *There is an action of G' on $\mathbb{E}$.*

Proof. First we prove a more general statement. Assume that we have a short exact sequence of vector bundles

$$0 \longrightarrow A \xrightarrow{F} B \xrightarrow{\pi} C \longrightarrow 0$$

over a variety X and a group G acting on A and B such that F is a G-morphism. We define an action of G on C as follows. Let $c \in C$; thus there is some $x \in X$ such that c is in the fiber C_x. Since π is surjective, there is some $b \in B_x$ such that $c = \pi_x(b)$. Then we set

$$g.c := \pi_x(g.b) \ .$$

This is well-defined. Indeed if $b' \in B$ is such that $\pi(b') = \pi(b)$, then b and b' belong to the same fiber B_x and $b' - b \in \ker \pi_x = \operatorname{im} F_x$, hence there is some $a \in A_x$ such that $b' = b + F_x(a)$. Thus

$$\pi_x(g.b') = \pi_x\big(g.(b + F_x(a))\big) = \pi_x\big(g.b + g.F_x(a)\big) = \pi_x\big(g.b + F_x(g.a)\big)$$

since F is a G-morphism and G acts within the fibers (by definition of an action on a vector bundle). This yields

$$\pi_x(g.b') = \pi_x(g.b) + \pi_x\big(F_x(g.a)\big) = \pi_x(g.b)$$

as $\pi \circ F = 0$. Moreover the definition immediately makes π a G-morphism since $g.\pi(b) = \pi(g.b)$. Now we apply this procedure to the case of the short exact sequence

$$0 \longrightarrow \mathbb{V}^s \times \mathbb{U}_1 \xrightarrow{F} \mathbb{V}^s \times \mathbb{U}_2 \xrightarrow{\pi} \mathbb{E} \longrightarrow 0$$

of vector bundles over $\mathbb{V}^s$. An element $e \in \mathbb{E}$ is represented by $(\Phi, Q) \in \mathbb{V}^s \times \mathbb{U}_2$, so the action of $(g', h) \in G'$ on e is

$$(g', h) \,.\, e = \pi\big(\big(\, g\Phi h^{-1} \,,\, \lambda Q h^{-1} + \ell \Phi h^{-1}\big)\big) \,. \qquad \square$$

Lemma 5.3.11. *Let $\Phi, \Phi' \in \mathbb{V}^s$ and $Q, Q' \in \mathbb{U}_2$. Then*

$$\pi(\Phi, Q) = \pi(\Phi', Q') \;\Leftrightarrow\; \Phi' = \Phi \text{ and } Q' = Q + L \cdot \Phi \text{ for some } L \in \mathbb{U}_1 \,.$$

Proof. If $\pi(\Phi, Q) = \pi(\Phi', Q')$, then $\Phi' = \Phi$ since π is a morphism of vector bundles and respects the fibers. Moreover

$$\begin{aligned} \pi(\Phi, Q) = \pi(\Phi, Q') \;&\Leftrightarrow\; \pi_\Phi(\Phi, Q') - \pi_\Phi(\Phi, Q) = 0 \\ &\Leftrightarrow\; (\Phi, Q') - (\Phi, Q) = (\Phi, Q' - Q) \in \ker \pi_\Phi \\ &\Leftrightarrow\; \exists\, L \in \mathbb{U}_1 \text{ such that } (\Phi, Q' - Q) = F_\Phi(\Phi, L) = (\Phi, L \cdot \Phi) \\ &\Leftrightarrow\; \exists\, L \in \mathbb{U}_1 \text{ such that } Q' = Q + L \cdot \Phi \,. \end{aligned} \qquad \square$$

Lemma 5.3.12. *Let $(\Phi, Q) \in \mathbb{V}^s \times \mathbb{U}_2$ and $L \in \mathbb{U}_1$. Then (Φ, Q) and $(\Phi, Q + L\Phi)$ lie in the same orbit under G'. Moreover they only differ by an element in $\operatorname{im} F$, so that $\pi(\Phi, Q) = \pi(\Phi, Q + L\Phi)$ as elements in $\mathbb{E}$.*

Proof. Let $L = (l_1, \dots, l_{n-1})$ and consider the matrix

$$g' = \begin{pmatrix} 1 & L \\ 0 & \mathrm{id}_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & l_1 & \dots & l_{n-1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \,. \tag{5.26}$$

Then

$$(g', \mathrm{id}_n) \,.\, (\Phi, Q) = \begin{pmatrix} 1 & L \\ 0 & \mathrm{id}_{n-1} \end{pmatrix} \cdot \begin{pmatrix} Q \\ \Phi \end{pmatrix} = \begin{pmatrix} Q + L\Phi \\ \Phi \end{pmatrix}$$

with $(g', \mathrm{id}_n) \in G'$, hence (Φ, Q) and $(\Phi, Q + L\Phi)$ lie in the same G'-orbit. Moreover

$$(\Phi, Q + L\Phi) = (\Phi, Q) + (\Phi, L\Phi)$$

as elements in the vector space $\{\Phi\} \times \mathbb{U}_2$, where $(\Phi, L\Phi) \in \operatorname{im} F$, hence they are equal in $\mathbb{E}$. □

Corollary 5.3.13. *There is a 1-to-1 correspondence between the orbits of* $\mathbb{V}^s \times \mathbb{U}_2$ *and* $\mathbb{E}$ *under* G'*.*

Proof. If $O(\Phi, Q)$ is an orbit in $\mathbb{V}^s \times \mathbb{U}_2$, we may map it to $O(\pi(\Phi, Q))$ in $\mathbb{E}$. This is well-defined since if $(\Phi', Q') = (g', h).(\Phi, Q)$, then $\pi(\Phi', Q') = (g', h).\pi(\Phi, Q)$ as π respects the action of G', so elements in the same orbits in $\mathbb{V}^s \times \mathbb{U}_2$ will be mapped to elements in the same orbit in $\mathbb{E}$.
Vice-versa, if $O(e)$ is an orbit in $\mathbb{E}$ with $e = \pi(\Phi, Q)$, we may map it to $O(\Phi, Q)$. Lemma 5.3.11 and Lemma 5.3.12 imply that this is independent of the representative of e. Moreover if $e' = (g', h).e$, then $e' = (g', h).\pi(\Phi, Q) = \pi((g', h).(\Phi, Q))$ and hence the representatives (Φ, Q) and $(g', h).(\Phi, Q)$ of e and e' again lie in the same orbit in $\mathbb{V}^s \times \mathbb{U}_2$. □

Corollary 5.3.14. *The elements* $(g', \mathrm{id}_n) \in G'$ *from* (5.26) *lie in the stabilizer of every* $e \in \mathbb{E}$*.*

Proof. If $e \in \mathbb{E}$ is represented by some $(\Phi, Q) \in \mathbb{V}^s \times \mathbb{U}_2$, we have seen in Lemma 5.3.12 that

$$(g', \mathrm{id}_n) \,.\, (\Phi, Q) = (\Phi, Q + L\Phi) \qquad \text{and} \qquad e = \pi(\Phi, Q) = \pi(\Phi, Q + L\Phi)\,.$$

Thus $(g', \mathrm{id}_n) \,.\, e = e$. □

Remark 5.3.15. By Corollary 5.3.14 we see that the action of the linear terms in G' is trivial, hence we don't need to take care of them any more. Moreover Corollary 5.3.13 says that the orbits of G' on $\mathbb{V}^s \times \mathbb{U}_2$ and $\mathbb{E}$ are the same, so if we want to study the orbits on $\mathbb{V}^s \times \mathbb{U}_2$, it suffices to study those on $\mathbb{E}$, on which we can forget about the action of the linear terms of G'.

Now it remains to also get rid of the constant $\lambda \in \mathbb{K}^*$ in

$$\begin{pmatrix} \lambda & \mathbb{U}_1 \\ 0 & \mathrm{GL}_{n-1}(\mathbb{K}) \end{pmatrix}$$

to eliminate the non-reductive part of G'. This is the aim of the next step.

Lemma 5.3.16. *There is an action of $\mathbb{P}G'$ on $\mathbb{P}\mathbb{E}$.*

Proof. The projectivisation $\mathbb{P}\mathbb{E}$ of $\mathbb{E}$ is a projective bundle over $\mathbb{V}^s$ whose fibers are of dimension $3d-1 = 3n+2$ since they are given by the projective spaces of the fibers of $\mathbb{E}$, i.e. $(\mathbb{P}\mathbb{E})_\Phi \cong \mathbb{P}(\mathbb{E}_\Phi)$ for all $\Phi \in \mathbb{V}^s$. The diagonal group

$$\Gamma := \left\{ (\lambda \,\mathrm{id}_n, \lambda \,\mathrm{id}_n) \;\middle|\; \lambda \in \mathbb{K}^* \right\} \subset G'$$

acts trivially on $\mathbb{V}^s \times \mathbb{U}_2$ and hence on $\mathbb{E}$, so there is an action of $\mathbb{P}G' = G'/\Gamma$ on $\mathbb{P}\mathbb{E}$ via

$$\langle (g', h) \rangle \,.\, \langle e \rangle := \langle (g', h) \,.\, e \rangle \;,$$

where $\langle \cdot \rangle$ denotes the homogeneous coordinates. This is well-defined since $(\lambda \,\mathrm{id}_n, \lambda \,\mathrm{id}_n) \,.\, \mu e = \mu e$ for all $\mu \in \mathbb{K}^*$, so it does not depend on the representative of $\langle e \rangle$, neither of the one of $\langle (g', h) \rangle$. □

Lemma 5.3.17. *Every element $(g', \mathrm{id}_n) \in G'$ of the form*

$$g' = \begin{pmatrix} \lambda & L \\ 0 & \mathrm{id}_{n-1} \end{pmatrix}$$

with $\lambda \in \mathbb{K}^*$ and $L \in \mathbb{U}_1$ acts trivially on $\mathbb{P}\mathbb{E}$.

Proof. Let $\langle e \rangle \in \mathbb{P}\mathbb{E}$ be represented by some $e \in \mathbb{E}$, which is represented by some $(\Phi, Q) \in \mathbb{V}^s \times \mathbb{U}_2$. Then

$$(g', \mathrm{id}_n) \,.\, (\Phi, Q) = \begin{pmatrix} \lambda & L \\ 0 & \mathrm{id}_{n-1} \end{pmatrix} \cdot \begin{pmatrix} Q \\ \Phi \end{pmatrix} = \begin{pmatrix} \lambda Q + L\Phi \\ \Phi \end{pmatrix} \;,$$

i.e. $(g', \mathrm{id}_n) \,.\, (\Phi, Q) = \lambda \cdot (\Phi, Q) + F(\Phi, L)$. Therefore

$$\begin{aligned} \langle (g', h) \rangle \,.\, \langle e \rangle &= \big\langle (g', h) \,.\, \pi(\Phi, Q) \big\rangle = \big\langle \pi\big((g', h).(\Phi, Q)\big) \big\rangle \\ &= \big\langle \pi\big(\lambda\, (\Phi, Q) + F(\Phi, L)\big) \big\rangle = \langle \lambda \cdot \pi(\Phi, Q) + 0 \rangle = \langle \lambda e \rangle = \langle e \rangle \;. \end{aligned}$$

□

Corollary 5.3.18. *There is a 1-to-1 correspondence between the orbits of* $\mathbb{E}$ *under* G' *and those of* $\mathbb{PE}$ *under* $\mathbb{P}G'$*.*

Proof. If $O(e)$ is an orbit in $\mathbb{E}$, we may map it to the orbit $O(\langle e\rangle)$ in $\mathbb{PE}$. This is well-defined since if $e' = (g', h).e$, then $\langle e'\rangle = \langle (g', h).e\rangle = (g', h).\langle e\rangle$, so their images are also in the same orbits.
Vice-versa, if $O(\langle e\rangle)$ is an orbit in $\mathbb{PE}$, we may map it to $O(e)$ in $\mathbb{E}$. This is independent of the representative since $\lambda e = (\lambda \operatorname{id}_n, \operatorname{id}_n).e$, so any other representative will be in the same orbit. Finally if $\langle e'\rangle = \langle (g', h)\rangle.\langle e\rangle = \langle (g', h).e\rangle$, then $e' = \lambda \cdot (g', h).e = (\lambda g', h).e$, so $e' \in O(e)$ as well. □

Remark 5.3.19. By Corollary 5.3.18 and Corollary 5.3.13, it thus suffices to study the orbits of $\mathbb{P}G'$ on $\mathbb{PE}$ in order to understand the orbits of G' on $\mathbb{V}^s \times \mathbb{U}_2$. The advantage of this point of view is that because of Lemma 5.3.17 the non-reductive part of G' acts trivially on $\mathbb{PE}$ and we are left with

$$g' = \begin{pmatrix} \lambda & L \\ 0 & g \end{pmatrix} \qquad \Rightarrow \qquad \langle (g', h)\rangle \,.\, \langle e\rangle = \langle (g, h)\rangle \,.\, \langle e\rangle$$

for $\langle e\rangle \in \mathbb{PE}$, $(g, h) \in G$, $\lambda \in \mathbb{K}^*$ and $L \in \mathbb{U}_1$. So the only non-trivial action of $\mathbb{P}G'$ on $\mathbb{PE}$ is the one of the reductive subgroup $\mathbb{P}G$. In some sense we naturally extended the action of $\mathbb{P}G$ on the base space $\mathbb{V}^s$ to an action of $\mathbb{P}G$ on the fibers of the bundle $p : \mathbb{PE} \to \mathbb{V}^s : \langle e\rangle \mapsto \Phi$. Moreover p is compatible with the $\mathbb{P}G$-actions on $\mathbb{PE}$ and $\mathbb{V}^s$ since $p : \langle (g, h)\rangle.\langle e\rangle \mapsto g\Phi h^{-1}$ by definition.

5.3.2 Construction of the projective classification bundle

Before continuing we need a one more notion and result.

Definition 5.3.20. [[38], p.97]
Let G be a reductive algebraic group over $\mathbb{K}$ acting on two $\mathbb{K}$-schemes of finite type $\mathcal{X}$ and $\mathcal{Y}$. Assume that we have a morphism $\pi : \mathcal{X} \to \mathcal{Y}$ that is compatible with the G-action. We say that a coherent sheaf $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}})$ *descends* to $\mathcal{Y}$ if there exists a sheaf $\mathcal{E} \in \texttt{Coh}(\mathcal{O}_{\mathcal{Y}})$ such that $\mathcal{F} \cong \pi^*\mathcal{E}$.

Theorem 5.3.21. [[38], 4.2.15, p.98]
Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a good quotient and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ be locally free. Then $\mathcal{F}$ descends to $\mathcal{Y}$ if and only if for any point $x \in \mathcal{X}$ in a closed G-orbit the stabilizer $\mathrm{Stab}_G(x)$ *acts trivially on the fiber over x of the vector bundle defined by $\mathcal{F}$.*

Proposition 5.3.22. $\mathbb{PE}$ *descends to a projective $\mathbb{P}_{3n+2}$-bundle $\mathbb{B}$ on N such that $\mathbb{PE} \cong \gamma^*\mathbb{B}$:*

$$\begin{array}{ccc} \mathbb{PE} & \xrightarrow{\Gamma} & \mathbb{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \nu} \\ \mathbb{V}^s & \xrightarrow{\gamma} & N \end{array}$$

Proof. G being reductive we know from Corollary 5.1.21 that there exists a geometric quotient $N = \mathbb{V}^s/\mathbb{P}G$ with $\mathbb{V}^s = \mathbb{V}^{ss}$ as all semistable sheaves are stable. So we can apply Theorem 5.3.21 to the quotient morphism $\gamma : \mathbb{V}^s \to N$ with the (projective) bundle $p : \mathbb{PE} \to \mathbb{V}^s$. Since the action of $\mathbb{P}G$ on $\mathbb{V}^s$ is free (see Remark 5.1.32), the stabilizer of every $\Phi \in \mathbb{V}^s$ is trivial and hence acts trivially on the fibers of $\mathbb{PE}$. Therefore $\mathbb{PE}$ descends to N. □

Remark 5.3.23. The idea behind this construction is the following: by definition of $\mathbb{E}$ we have the quotient $\pi : \mathbb{V}^s \times \mathbb{U}_2 \to \mathbb{E}$. Now consider the composition

$$\eta \ : \ \mathbb{X} = (\mathbb{V}^s \times \mathbb{U}_2) \setminus \operatorname{im} F \xrightarrow{\pi} \mathbb{E} \setminus \{0\} \xrightarrow{\rho} \mathbb{PE} \xrightarrow{\Gamma} \mathbb{B} \ ,$$

which gives a fiber bundle[7] $\eta : \mathbb{X} \to \mathbb{B}$.

Then $\mathbb{B}$ classifies the orbits we are interested in. Indeed Maican has shown that

Proposition 5.3.24. [[48], 7.7, p.40-41][8]
1) $\Gamma : \mathbb{PE} \to \mathbb{B}$ is a geometric quotient for the action of the reductive group $\mathbb{P}G$.
2) $\eta : \mathbb{X} \to \mathbb{B}$ is a geometric quotient under the action of the non-reductive group $\mathbb{P}G'$. Moreover the base space N of $\nu : \mathbb{B} \to N$ is a smooth projective variety of dimension $n^2 - n$.

[7] Maican uses the notation $W^{ss}(G, \Lambda)$ in [48] for the space $\mathbb{X} = (\mathbb{V}^s \times \mathbb{U}_2) \setminus \operatorname{im} F$. His goal is to extend the definition of semistability to the case of an action of a non-reductive group.
[8] The statements are actually contained in the proof of [[48], 7.7, p.40-41].

Remark 5.3.25. Hence we may see the morphism ν as

$$\nu \;:\; \mathbb{B} \longrightarrow N \;:\; [A] \longmapsto [\Phi] \;,$$

where the class of $A \in \mathbb{X}$ is with respect to $\mathbb{P}G'$ and the class of $\Phi \in \mathbb{V}^s$ is with respect to $\mathbb{P}G$. In particular the points in $\mathbb{B}$ (i.e. the fibers $\mathbb{X}_b$ of $\mathbb{X} \to \mathbb{B}$ for $b \in \mathbb{B}$) are in 1-to-1 correspondence with the orbits of the G'-action on $\mathbb{X} \subseteq \mathbb{V}^s \times \mathbb{U}_2$. [[53], 463845] moreover states that ν is open.

Remark 5.3.26. The dimension of $N = \mathbb{V}^s/\mathbb{P}G$ can also be computed directly. As $\mathbb{V}^s \subset \mathbb{V}$ is open and the action of $\mathbb{P}G$ on $\mathbb{V}^s$ is free (Remark 5.1.32), we find

$$\dim N = \dim \mathbb{V}^s - \dim \mathbb{P}G = 3n(n-1) - \big((n-1)^2 + n^2 - 1\big) = n^2 - n \;.$$

Corollary 5.3.27. *M_0 may be seen as an open subvariety of $\mathbb{B}$.*

Proof. Consider the open subvariety $\mathbb{W}_0 \subset \mathbb{W}$ of injective morphisms with stable Kronecker module which parametrize sheaves in M_0. We have

$$\mathbb{W}_0 \subseteq \mathbb{X} = (\mathbb{V}^s \times \mathbb{U}_2) \setminus \operatorname{im} F$$

since matrices in the image of F have determinant zero by Remark 5.3.4. Then we use that M_0 is a geometric quotient of $\mathbb{W}_0$ under $\mathbb{P}G'$ (Proposition 5.1.42) and that $\mathbb{B}$ is a geometric quotient of $\mathbb{X}$ under $\mathbb{P}G'$ (Proposition 5.3.24). This gives $M_0 = \mathbb{W}_0/\mathbb{P}G' \subseteq \mathbb{X}/\mathbb{P}G' = \mathbb{B}$. □

5.3.3 Sheaves in $\mathbb{B}_0$

Definition 5.3.28. Let $N_0 \subseteq N$ be the subvariety in the quotient space corresponding to $\mathbb{V}_0 \subseteq \mathbb{V}^s$, i.e. stable Kronecker modules with coprime maximal minors. N_0 is open by combining Lemma 5.2.34 and Remark 5.2.36. Alternatively we illustrate that it is open in Remark 5.3.29 below. We also denote by $\mathbb{B}_0 = \mathbb{B}|_{N_0}$ the restriction of $\mathbb{B}$ to the open subscheme $N_0 \subseteq N$.

Remark 5.3.29. By Lemma 5.2.19 and Proposition 5.2.23 we obtain a set theoretical map from $\mathbb{V}_0$ to the Hilbert scheme of 0-dimensional subschemes of $\mathbb{P}_2$

of length $l = \binom{n}{2}$, which sends $\Phi \in \mathbb{V}_0$ to the subscheme defined by its maximal minors. We denote

$$H = \mathbb{P}_2^{[l]} = \mathrm{Hilb}^l(\mathbb{P}_2) \ .$$

If we go to the quotient N_0, then this indeed gives a morphism of varieties $N_0 \to H$. Since every 0-dimensional scheme of length l whose points do not lie on a curve of degree $n-2$ has a minimal resolution of the type (5.21), see Proposition 5.2.45 and Corollary 5.2.44, it induces an isomorphism of N_0 to the open subvariety $H_0 \subseteq H$ consisting of Z not lying on a curve of degree $n-2$. It is e.g. mentioned in [[14], 4.7, p.46] that the complement of H_0, consisting of 0-dimensional schemes which are contained in such a curve, is a closed irreducible hypersurface. Thus $H_0 \cong N_0$ is open.

Proposition 5.3.30. *The fibers of $\mathbb{B}$ over N_0 are contained in M_0, i.e. we have the inclusions of open subschemes $\mathbb{B}_0 \subseteq M_0 \subseteq \mathbb{B}$. Hence the fibers of $\mathbb{B}_0$ also parametrize isomorphism classes of stable sheaves given as cokernels in the resolution* (5.8)*.*

Proof. $\mathbb{B}_0 = \nu^{-1}(N_0)$ is open in $\mathbb{B}$ (and hence in M_0) by continuity of ν. Now let us first show that a matrix $A = \binom{Q}{\Phi} \in \mathbb{W}$ with $Q \in \mathbb{U}_2$ and $\Phi \in \mathbb{V}_0$ has zero determinant if and only if A lies in the image of F. Necessity follows from (5.24). So assume that $\det A = 0$; let us denote the coprime maximal minors of Φ by $d_1, \ldots, d_n$ and $Q = (q_1, \ldots, q_n)$. Exactness of (5.21) then implies that

$$\begin{aligned} \det A = 0 \ &\Leftrightarrow \ q_1 d_1 + \ldots + q_n d_n = 0 \ \Leftrightarrow \ (q_1, \ldots, q_n) \in \operatorname{im} \Phi \\ &\Leftrightarrow \ \exists\, L \in \mathbb{U}_1 \text{ such that } Q = L \cdot \Phi \ \Leftrightarrow \ A \in \operatorname{im} F \ . \end{aligned}$$

However by definition, an element of $\mathbb{B}$ is represented by a matrix $A \in \mathbb{X}$ (geometric quotient), so it cannot belong to $\operatorname{im} F$. Therefore if $A = \binom{Q}{\Phi} \in \mathbb{X}$ is such that $\Phi \in \mathbb{V}_0$, then $\det A \neq 0$. So we get $\mathbb{B}_0 \subseteq M_0$, i.e. the points of $\mathbb{B}_0$ also represent isomorphism classes of sheaves in M_0, see (5.8). □

Our main result of this section is the following description of sheaves in $\mathbb{B}_0$. It is motivated by the corresponding assertion of Drézet in [14] and generalizes the description of Maican in [15].

Proposition 5.3.31. cf. [[14], Prop.4.5, p.43-44] and [[15], 3.3.1, p.21-22]
The sheaves $\mathcal{F}$ in $\mathbb{B}_0$ are exactly the twisted ideal sheaves $\mathcal{I}_{Z\subseteq C}(d-3)$ given by a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(d-3) \longrightarrow \mathcal{O}_Z \longrightarrow 0\ , \tag{5.27}$$

where $Z \subseteq C$ is a 0-dimensional subscheme of length $l = \binom{n}{2}$ lying on a curve C of degree d such that Z is not contained in a curve of degree $d-3$.

Proof. $\underline{\Rightarrow}$: Let $\mathcal{F}$ be a sheaf in $\mathbb{B}_0$. By Remark 5.3.25 its isomorphism class is given by the $\mathbb{P}G'$-orbit of some $A = \binom{Q}{\Phi} \in \mathbb{X}$ with $\Phi \in \mathbb{V}_0$, so that $\mathcal{F} \in M_0$ and has the resolution (5.8)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0\ .$$

In particular we obtain the curve $C = \operatorname{supp}\mathcal{F} = Z(\det A)$ of degree $d = n+1$. Denote the maximal minors of Φ by $d_1, \ldots, d_n$. As they are coprime homogeneous polynomials of degree $n-1$, Proposition 5.2.23 gives $Z = Z(d_1, \ldots, d_n)$ and the resolution

$$0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-n) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2}(-n+1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ , \tag{5.28}$$

where $\varphi = {}^t(d_1, \ldots, d_n)$ and Z satisfies the required properties by Corollary 5.2.44. Moreover $Z \subseteq C$ as a point in $\mathbb{P}_2$ which vanishes at all maximal minors of Φ also vanishes at $\det A = q_1 d_1 + \ldots + q_n d_n$. We also have the exact sequence defining the structure sheaf $\mathcal{O}_C$ of the curve C by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-n-1) \xrightarrow{\det A} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_C \longrightarrow 0\ . \tag{5.29}$$

Now we twist the sequences (5.28) and (5.29) by $\mathcal{O}_{\mathbb{P}_2}(n-2)$; this will not change $\mathcal{O}_Z$ as it has 0-dimensional support (see Lemma 4.1.11).
Since $A \cdot \varphi = \binom{\det A}{0}$ we can now put everything together into the following commutative diagram, where f is induced by the universal property of cokernels since $(k_2 \circ \varphi) \circ A = (k_2 \circ \det A) \circ \binom{1}{0} = 0$, so $k_2 \circ \varphi$ factors through $\mathcal{F} \cong \operatorname{coker} A$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \overset{\text{id}}{\dashrightarrow} & (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) & & \ker f & & \\
 & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) & \xrightarrow{A=\binom{Q}{\Phi}} & n\,\mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{k_1} & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \exists f} & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) & \xrightarrow{\det A} & \mathcal{O}_{\mathbb{P}_2}(n-2) & \xrightarrow{k_2} & \mathcal{O}_C(n-2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathcal{O}_Z & & \operatorname{coker} f & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The Snake Lemma (Proposition D.1.6) then gives an exact sequence

$$\begin{aligned} 0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) &\xrightarrow{\text{id}} (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \\ &\longrightarrow \ker f \longrightarrow 0 \longrightarrow \mathcal{O}_Z \longrightarrow \operatorname{coker} f \longrightarrow 0 \ , \end{aligned}$$

i.e. $\ker f = 0$ and $\operatorname{coker} f \cong \mathcal{O}_Z$, so finally we obtain $\mathcal{F}$ as an ideal sheaf

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(n-2) \longrightarrow \mathcal{O}_Z \longrightarrow 0 \ .$$

$\underline{\Leftarrow}$: Now assume that $\mathcal{F}$ is given by an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{O}_C(d-3) \xrightarrow{g} \mathcal{O}_Z \longrightarrow 0$$

for some curve C of degree d and a 0-dimensional subscheme $Z \subseteq C$ of length $l = \binom{n}{2}$ such that Z does not lie on a curve of degree $d-3$. Then (4.17) implies that $\mathcal{F}$ has Hilbert polynomial $dm-1$ since

$$\begin{aligned} P_{\mathcal{F}}(m) &= P_{\mathcal{O}_C(d-3)}(m) - P_{\mathcal{O}_Z}(m) = P_{\mathcal{O}_C}(m+d-3) - l \\ &= d \cdot (m+d-3) + \tfrac{3d-d^2}{2} - \tfrac{(d-1)(d-2)}{2} = d \cdot m - 1 \ . \end{aligned}$$

Let $C = Z(f)$ for some homogeneous polynomial f of degree $d = n+1$. This gives the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{f} \mathcal{O}_{\mathbb{P}_2}(d-3) \xrightarrow{k} \mathcal{O}_C(d-3) \longrightarrow 0 \ .$$

Putting $k' = g \circ k$, we thus get a surjective morphism $\mathcal{O}_{\mathbb{P}_2}(d-3) \twoheadrightarrow \mathcal{O}_Z$. As pointed out in Remark 5.2.47, there now exists a resolution of $\mathcal{O}_Z$ as in (5.20) which is compatible with k':

$$0 \longrightarrow (d-2)\,\mathcal{O}_{\mathbb{P}_2}(-d+1) \longrightarrow (d-1)\,\mathcal{O}_{\mathbb{P}_2}(-d+2) \longrightarrow \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_Z \longrightarrow 0\ .$$

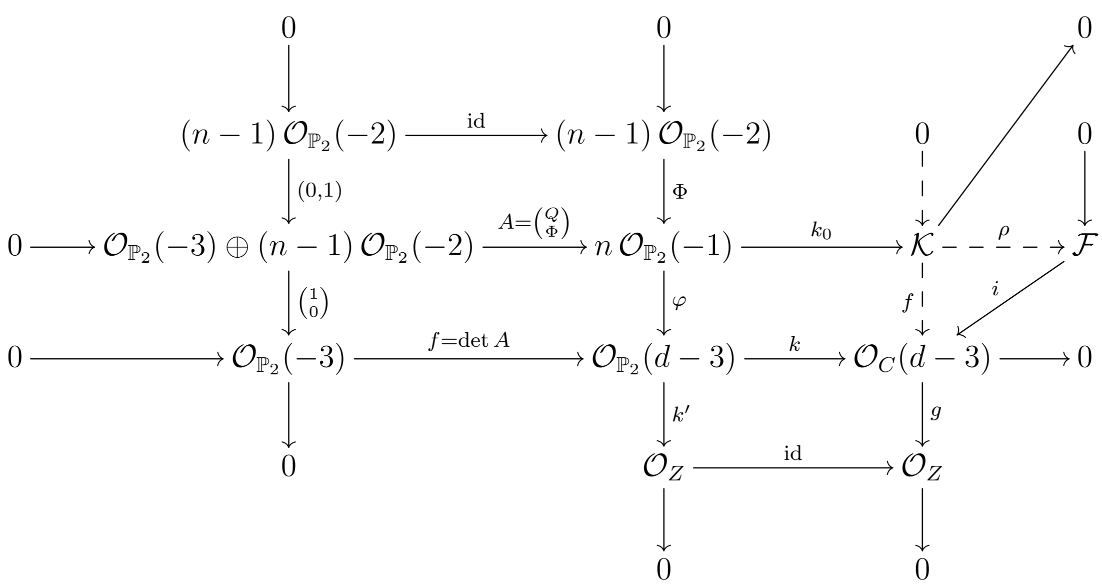
Replacing $d = n-1$ and twisting by $\mathcal{O}_{\mathbb{P}_2}(n-2)$, we then get a Kronecker module $\Phi \in \mathbb{V}_0$ such that

$$0 \longrightarrow (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{\Phi} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}_2}(n-2) \xrightarrow{k'} \mathcal{O}_Z \longrightarrow 0$$

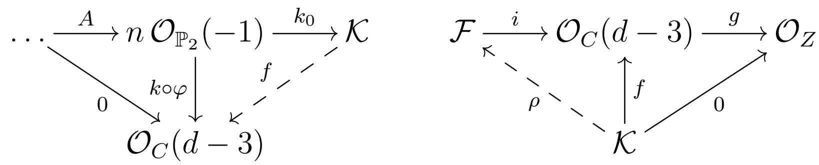
since $\mathcal{I}_Z$ can be generated by $(d-3)+2 = n$ elements. Denote $\varphi = {}^t(l_1, \ldots, l_n)$ for some coprime homogeneous polynomials l_i of degree $n-1$ such that we get $Z = Z(l_1, \ldots, l_n)$. $Z \subseteq C$ then implies that $f \in \mathcal{I}_Z = \langle\, l_1, \ldots, l_n \,\rangle$. Hence there exist homogeneous polynomials q_i of degree 2 such that $f = q_1 l_1 + \ldots + q_n l_n$. We set $Q := (q_1, \ldots, q_n)$ and $A := \binom{Q}{\Phi} \in \mathbb{U}_2 \times \mathbb{V}_0$, so that $\det A = f$. In particular $\det A$ is non-zero and we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{k_0} \mathcal{K} \longrightarrow 0\ ,$$

where $\mathcal{K} = \operatorname{coker} A$. Now it remains to show that $\mathcal{K} \cong \mathcal{F}$. Consider the commutative diagram



where the morphisms f and ρ are induced by the universal properties of cokernels and kernels



A Snake Lemma argument shows that $\ker f = 0$, hence f is injective. Moreover $g \circ f = 0$ since for computing $g(f(x))$, we have $x = k_0(a)$ by surjectivity of k_0, so $f(x) = f(k_0(a)) = k(\varphi(a))$ and

$$g\big(f(x)\big) = g\big(k(\varphi(a))\big) = k'\big(\varphi(a)\big) = 0 \ .$$

This induces the morphism $\rho : \mathcal{K} \to \mathcal{F}$, which is injective since f and i are. In order to prove that $\mathcal{F}$ has a resolution of the type (5.8) (and thus belongs to $\mathbb{B}_0$ since $\det A \neq 0$), it remains to show that ρ is surjective as well. Let $y \in \mathcal{F}$ be given. Then $i(y) = k(z)$ for some z and

$$0 = g\big(i(y)\big) = g\big(k(z)\big) = k'(z) \ \Rightarrow \ z = \varphi(a) \ ,$$
$$i\big(\rho(k_0(a))\big) = f\big(k_0(a)\big) = k\big(\varphi(a)\big) = k(z) \ .$$

By injectivity of i, we hence obtain that $y = \rho(k_0(a))$. Thus $\mathcal{F} \cong \mathcal{K}$. □

Proposition 5.3.32. [[40], p.4]
A fiber of $\nu : \mathbb{B}_0 \to N_0$ corresponds to the space of plane curves of degree d passing through the corresponding subscheme of $l = \binom{n}{2}$ points defined by (5.27). The identification is given by the map

$$\nu^{-1}([\Phi]) \longrightarrow \mathcal{C}_d(\mathbb{P}_2) \ : \ [A] \longmapsto \langle \det A \rangle \ . \tag{5.30}$$

Proof. Let $[\Phi] \in N_0$; an element in the preimage $\nu^{-1}([\Phi]) \subset \mathbb{B}_0$ corresponds to the G'-orbit of some $A = \binom{Q}{\Phi} \in \mathbb{X}$ with $\Phi \in \mathbb{V}_0$ (see Remark 5.3.25). Thus (5.30) is well-defined since if two matrices A, B over $[\Phi]$ are equivalent under the action of G', i.e. $B = g'Ah^{-1}$ for some $(g', h) \in G'$, then $\det A$ and $\det B$ only differ by a non-zero constant.

Vice-versa, assume that the determinants of two matrices $A = \binom{Q}{\Phi}$ and $B = \binom{Q'}{\Phi}$ over $[\Phi]$ only differ by a non-zero constant $\lambda \in \mathbb{K}^*$. If $Q = (q_1, \ldots, q_n)$ and $Q' = (q'_1, \ldots, q'_n)$, then we have

$$\begin{aligned} \det B = \lambda \cdot \det A \ &\Leftrightarrow\ q'_1 d_1 + \ldots + q'_n d_n = \lambda \cdot (q_1 d_1 + \ldots + q_n d_n) = 0 \\ &\Leftrightarrow\ (q'_1 - \lambda q_1) \cdot d_1 + \ldots + (q'_n - \lambda q_n) \cdot d_n = 0 \\ &\Leftrightarrow\ (Q' - \lambda Q) \cdot \varphi = 0 \ , \end{aligned}$$

where $\varphi = {}^t(d_1, \ldots, d_n)$ is the column vector consisting of the maximal minors of Φ. As these are coprime we have $Q' - \lambda Q \in \ker \varphi = \operatorname{im} \Phi$ by Proposition 5.2.23, i.e. $Q' = \lambda Q + L \cdot \Phi$ for some $L \in \mathbb{U}_1$. But then

$$\begin{pmatrix} Q' \\ \Phi \end{pmatrix} = \begin{pmatrix} \lambda Q + L \cdot \Phi \\ \Phi \end{pmatrix} = \begin{pmatrix} \lambda & L \\ 0 & \mathrm{id}_{n-1} \end{pmatrix} \cdot \begin{pmatrix} Q \\ \Phi \end{pmatrix} \ ,$$

i.e. $B = g' \cdot A$ with $(g', \mathrm{id}_n) \in G'$, similarly as in Lemma 5.3.12. Thus A and B lie in the same orbit under G' and we get $[A] = [B]$, showing that the map in (5.30) is injective.

To prove that it is surjective onto the space of curves of degree d passing through the subscheme Z defined by the maximal minors Φ, let C be such a curve. Writing $Z \subseteq C = Z(f)$ then implies that $f = q_1 d_1 + \ldots + q_n d_n$ for some quadratic forms q_i and hence we can take as preimage $[A]$ for $A = \binom{Q}{\Phi}$ with $Q = (q_1, \ldots, q_n)$. This will then be mapped to $\langle \det A \rangle = \langle f \rangle$. □

5.4 On the ideals of points on planar curves

Motivated by sequence (5.27) of twisted ideals sheaves, we are now going to characterize free ideals of points on planar curves in a local ring. Indeed we shall determine under which conditions the stalk $\mathcal{F}_p$ for $p \in C$ is a free module over $\mathcal{O}_{C,p}$. Here we restrict ourselves to so-called curvilinear points. In particular we are interested in simple and double points lying on a curve C. The main results of this section are Lemma 5.4.1 and Proposition 5.4.11, which characterize the free ideals at simple and double points. The utility of these criteria will be explained more precisely in Section 5.5.

5.4.1 Ideals of simple points on a curve

Let C be an arbitrary (abstract) curve and $p \in C$ a simple point. We denote the stalk at p by $R = \mathcal{O}_{C,p}$; it is a local $\mathbb{K}$-algebra of Krull dimension 1. Let $\mathfrak{M} = \mathfrak{M}_{C,p}$ be the unique maximal ideal of R and denote by $\Bbbk_p = R/\mathfrak{M}$ the field (thus a local ring) defining the structure sheaf of the closed one point subscheme $\{p\} \subset C$.
Let $I \trianglelefteq R$ be an ideal in R. As a submodule of the free module R, I is a torsion-free R-module. Moreover it is finitely generated since R is Noetherian. From Proposition D.1.12 and Proposition D.1.17 we know that R is a regular local ring if and only if R is a principal ideal domain if and only if p is a smooth point of C. Hence Theorem D.1.13 implies that I is free if R is regular. Therefore, non-regularity of R is a necessary condition for non-freeness of I. Now we observe the following elementary fact.

Lemma 5.4.1. *Consider the exact sequence of R-modules*

$$0 \longrightarrow \mathfrak{M} \longrightarrow R \longrightarrow \Bbbk_p \longrightarrow 0 \ .$$

Then $\mathfrak{M}$ is free (of rank 1) if and only if R is regular, i.e. if and only if p is a smooth point of C.

Proof. $\underline{\Leftarrow}$: If R is regular, then it is a principal ideal domain by Proposition D.1.12. As $\mathfrak{M}$ is a submodule of the free module R, it is torsion-free and thus free by Theorem D.1.13. Moreover it must be free of rank 1, otherwise $\mathfrak{M} \to R$ would not be injective.
$\underline{\Rightarrow}$: If $\mathfrak{M}$ is free, then we necessarily have $\mathfrak{M} \cong R$ (again because $\mathfrak{M} \to R$ is injective). This implies that $\mathfrak{M}$ is generated by one element as an R-module. Hence $\dim_{R/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2) = 1 = \dim R$ and Proposition D.1.12 implies that R is regular. □

Remark 5.4.2. Note that $\mathfrak{M}$ being free of rank 1 does not mean that $\mathfrak{M} \to R$ is an isomorphism. Indeed this cannot be true since $\mathfrak{M} \neq R$, so $\Bbbk_p \neq \{0\}$. In general it may however happen that a maximal ideal is free, even though it is different from the ring. Consider e.g.

$$0 \longrightarrow \mathbb{K}[X] \xrightarrow{\cdot X} \mathbb{K}[X] \longrightarrow \mathbb{K} \longrightarrow 0 \ ,$$

thus $\mathbb{K}[X] \cong \langle X \rangle$ by injectivity. In particular we see that $\langle X \rangle$ is free, even though multiplication by X is not an isomorphism.

5.4.2 Preliminaries

The next goal is to find a substitute for Lemma 5.4.1 in the case of double points. Intuitively a double point is a 1-point-space (on the topological level), but the scheme consists of two points which are infinitesimally close to each other. For this we first start with some easy observations.

Lemma 5.4.3. *Let*

$$0 \longrightarrow M_1 \xrightarrow{i} M \xrightarrow{\pi} M_0 \longrightarrow 0$$

be an exact sequence of R-modules. Assume that M_1 is generated by $m_1, \ldots, m_k$ and that M_0 is generated by $\bar{n}_1, \ldots, \bar{n}_\ell$. If we choose some $n_i \in M$ such that $\pi(n_i) = \bar{n}_i$, then M is generated by $i(m_1), \ldots, i(m_k), n_1, \ldots, n_\ell$.

Proof. Let $m \in M$. Since $\pi(m) \in M_0$, $\exists\, r_i \in R$ such that $\pi(m) = \sum_i r_i * \bar{n}_i$. Now set $u := \sum_i r_i * n_i$. Then

$$\pi(u) = \sum_i r_i * \pi(n_i) = \sum_i r_i * \bar{n}_i = \pi(m) \ ,$$

hence $\pi(m-u) = 0$ and $m-u \in \ker \pi = \operatorname{im} i$, i.e. $\exists\, z \in M_1$ such that $m-u = i(z)$. But then there exist $s_j \in R$ such that $z = \sum_j s_j * m_j$, so that

$$m = i(z) + u = \sum_j s_j * i(m_j) + \sum_i r_i * n_i \ .$$

□

Lemma 5.4.4. *Let $\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{X}$ be an inclusion of subschemes in $\mathbb{A}^n_{\mathbb{K}}$ such that $\mathcal{Z} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$ are closed. Then we have the exact sequence of the ideal sheaves*

$$0 \longrightarrow \mathcal{I}_{\mathcal{Y}} \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{I}_{\mathcal{Z}\subset\mathcal{Y}} \longrightarrow 0 \ . \tag{5.31}$$

Proof. We have $\mathcal{I}_{\mathcal{Y}} \subseteq \mathcal{I}_{\mathcal{Z}}$ and the exact sequences

$$0 \longrightarrow \mathcal{I}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow 0 \quad , \quad 0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}\subset\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \ .$$

Using that $(\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{Z}) \circ (\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y}) = (\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Z})$, which is just the corresponding property for projections, and the universal property of kernels we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{K} & & \mathcal{I}_\mathcal{Y} & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_\mathcal{Z} & \longrightarrow & \mathcal{O}_\mathcal{X} & \longrightarrow & \mathcal{O}_\mathcal{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_{\mathcal{Z}\subset\mathcal{Y}} & \longrightarrow & \mathcal{O}_\mathcal{Y} & \longrightarrow & \mathcal{O}_\mathcal{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{Q} & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where $\mathcal{K} = \ker f$ and $\mathcal{Q} = \operatorname{coker} f$. Using the Snake Lemma (Proposition D.1.6) we obtain $\mathcal{K} \cong \mathcal{I}_\mathcal{Y}$ and $\mathcal{Q} = 0$, which yields the exact sequence (5.31). □

Remark 5.4.5. Lemma 5.4.3 allows to find generators of $\mathcal{I}_\mathcal{Z}$ as an $\mathcal{O}_\mathcal{X}$–module if those of $\mathcal{I}_\mathcal{Y}$ and $\mathcal{I}_{\mathcal{Z}\subset\mathcal{Y}}$ are known. Note that the generators of $\mathcal{I}_{\mathcal{Z}\subset\mathcal{Y}}$ as an $\mathcal{O}_\mathcal{X}$–module and as an $\mathcal{O}_\mathcal{Y}$–module are the same since the $\mathcal{O}_\mathcal{X}$–module structure is precisely defined via the morphism $\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y}$.

Definition 5.4.6. Let $R[[X_1, \ldots, X_n]]$ denote the ring of formal power series

$$f(X_1, \ldots, X_n) = \sum_{|\alpha|=0}^{\infty} r_\alpha \cdot X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and coefficients $r_\alpha \in R$, $\forall\, \alpha$. Note that every formal power series f can be written as a sum of its homogeneous components

$$f(X_1, \ldots, X_n) = \sum_{i=0}^{\infty} f_i(X_1, \ldots, X_n) = \sum_{i=0}^{\infty} \left(\sum_{|\alpha|=i} r_\alpha \cdot X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n} \right).$$

The *order* of f, denoted by $\mathrm{ord}(f)$, is the smallest integer $i \geq 0$ such that $f_i \neq 0$. For $s = \mathrm{ord}(f)$ the homogeneous polynomial f_s is also called the *leading form* of f and denoted by f_*.

Corollary 5.4.7. *Consider the ring* $\mathbb{K}[X,Y]$ *and let* $\mathfrak{M} = \langle X, Y \rangle$.
The local ring $\mathbb{K}[X,Y]_{\mathfrak{M}}$ *can be embedded into* $\mathbb{K}[[X,Y]]$ *via* $\frac{f}{g} \mapsto f \cdot g^{-1}$.

Proof. $g \notin \mathfrak{M}$, so g has a non-zero constant term and is thus invertible as an element in $\mathbb{K}[[X,Y]]$ by Proposition D.1.19. Moreover this is well-defined since if $\frac{f}{g} = \frac{h}{\ell}$, then $f \cdot \ell = h \cdot g$ and thus $f \cdot g^{-1} = h \cdot \ell^{-1}$. □

5.4.3 Ideals of double points on a planar curve

Let $C = Z(f) = \{ (x_1, x_2) \in \mathbb{A}_2 \mid f(x_1, x_2) = 0 \}$ be a planar curve given by the vanishing set of some non-constant polynomial $f \in \mathbb{K}[X,Y]$. Its coordinate ring $\mathbb{K}[C] = \mathbb{K}[X,Y]/\langle f \rangle$ is a (not necessarily integral) Noetherian ring of Krull dimension 1. Assume that $p = (0,0) \in C$, i.e. f has no constant term. We denote the stalk of C at the origin by $R = \mathcal{O}_{C,p}$; it is given by the local $\mathbb{K}$-algebra

$$R = \Big(\mathbb{K}[X,Y]/\langle f \rangle\Big)_{\langle \bar{X}, \bar{Y} \rangle} \cong \Big(\mathbb{K}[X,Y]_{\langle X,Y \rangle}\Big)/\langle \tfrac{f}{1} \rangle \,, \tag{5.32}$$

which is obtained by localization of the coordinate ring of C.

Definition 5.4.8. The *tangent cone* of $C = Z(f)$ at the origin[9] is the zero set $Z(f_*)$ of the leading form f_* of f.[10]

Example 5.4.9. 1) The tangent cone of the nodal curve, defined by

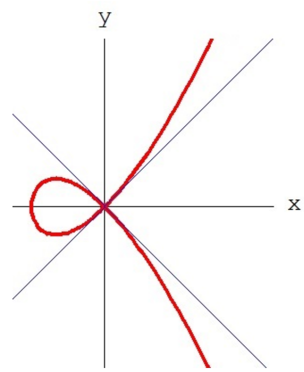
$$f(X,Y) = X^2 + X^3 - Y^2 \,,$$

at the origin is given by the reducible conic $f_*(X,Y) = X^2 - Y^2 = (X-Y)(X+Y)$ and thus consists of 2 lines.

[9] The tangent cone at an arbitrary point $p \in C$ is obtained by translating this point to the origin, and then back.

[10] There also exists a definition of the tangent cone to any affine algebraic variety by using the leading forms of all polynomials in the ideal that defines the variety.

Figure 5.1: The nodal curve (red) and its tangent cone at the origin (blue)



2) If $f(X,Y) = X^2 + XY + X^{15}$, then $f_*(X,Y) = X^2 + XY = X(X+Y)$, so we see that this tangent cone is reducible as well.
3) For $f(X,Y) = X^2 - Y^3$ we find $f_*(X,Y) = X^2$, hence the tangent cone consists of the double line given by $X^2 = 0$.

Remark 5.4.10. In $\mathbb{A}_2$ the tangent cone is always a union of finitely many lines (with multiplicities). Indeed every homogeneous polynomial $h \in \mathbb{K}[X,Y]$ can be written as a product of linear terms. Consider

$$h(X,Y) = \sum_{i=0}^{d} r_i \, X^i Y^{d-i}$$

$$\Rightarrow \quad \frac{h(X,Y)}{Y^d} = \sum_{i=0}^{d} r_i \, Z^i = \prod_{i=1}^{d} r_d \, (Z - a_i) \quad \text{where } Z = \tfrac{X}{Y}$$

$$\Rightarrow \quad h(X,Y) = Y^d \cdot \prod_{i=1}^{d} r_d \, (Z - a_i) = \prod_{i=1}^{d} r_d \, (X - a_i Y) \ ,$$

which works since $\mathbb{K}$ is algebraically closed. Multiplicities are obtained if some of the a_i are equal.

Next we assume that $p \in C$ is a fat double point given by the vanishing set $Z(X, Y^2)$. As an example of a curve containing this double point consider again the nodal curve, which satisfies the condition since $\langle \, X^2 + X^3 - Y^2 \, \rangle \subset \langle \, X, Y^2 \, \rangle$. This means that the subscheme $\{p\} \subset \mathbb{A}_2$ is given by

$$0 \longrightarrow J = \langle \, X, Y^2 \, \rangle \longrightarrow \mathbb{K}[X,Y] \longrightarrow \mathbb{K}[X,Y]/J \longrightarrow 0$$

together with

$$0 \longrightarrow \mathbb{K}[X,Y] \xrightarrow{\cdot f} \mathbb{K}[X,Y] \longrightarrow \mathbb{K}[C] = \mathbb{K}[X,Y]/\langle f \rangle \longrightarrow 0 \,,$$

which defines $C \subset \mathbb{A}_2$ and gives $\mathbb{K}[X,Y] \cong \langle f \rangle$ by injectivity. The subscheme $\{p\} \subset C$ also implies that $\langle f \rangle \subset J$. As in the proof of Lemma 5.4.4 the inclusion of subschemes $\{p\} \subset C \subset \mathbb{A}_2$ gives the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{I}_C & \xrightarrow{\sim} & \mathcal{O}_{\mathbb{A}_2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{\mathbb{A}_2} & \longrightarrow & \mathcal{O}_{dp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_{dp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\mathcal{I}_C \subset \mathcal{O}_{\mathbb{A}_2}$, $\mathcal{J} \subset \mathcal{O}_{\mathbb{A}_2}$, $\mathcal{I} \subset \mathcal{O}_C$ are the ideal sheaves and dp means “double point p”. Now we localize this diagram at the (topological) point p and get

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \langle f_p \rangle & \xrightarrow{\sim} & R' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{J}_p & \longrightarrow & R' & \longrightarrow & \mathcal{O}_{dp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{I}_p & \longrightarrow & R & \longrightarrow & \mathcal{O}_{dp} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $R' = \mathbb{K}[X,Y]_{\langle X,Y \rangle}$ and $f_p = \frac{f}{1} \in R'$. Here we use the RHS of the description in (5.32), so that R is the quotient of R' by $\langle f_p \rangle$. Denote $I = \mathcal{I}_p$; thus $I \subseteq R$ is the ideal defining the subscheme of the double point $\{p\} \subset C$ in the local ring. Let us also denote the classes of $X, Y \in \mathbb{K}[X,Y]$ in R' by $X_p = \frac{X}{1}$, $Y_p = \frac{Y}{1}$ and their classes in R by $x = \left[\frac{X}{1}\right]$ and $y = \left[\frac{Y}{1}\right]$. Hence we get

$$\mathcal{J}_p = J_{\langle X,Y \rangle} = \langle X_p, Y_p^2 \rangle \trianglelefteq R' \qquad \text{and} \qquad I = \langle x, y^2 \rangle \trianglelefteq R \,.$$

Proposition 5.4.11 (Leytem)**.** *Assume that R is a non-regular ring, i.e. that $p = (0,0)$ is a singular point of C. Then the following conditions are equivalent:*
1) I is a free R-module.
2) I is generated by one element (over R).
3) $\mathcal{J}_p$ is of the form $\langle \xi, f_p \rangle$ for some $\xi \in R'$.
4) f contains the monomial Y^2, i.e. the coefficient in front of Y^2 is non-zero.
5) The tangent cone of C at p consists of 2 lines (with multiplicities) not containing the line $X = 0$.
Moreover if I is free, then it is generated by x and there is an isomorphism

$$R \cong I \;:\; r \longmapsto r \cdot x \,.$$

Proof. Let us first collect what we know about f. For $d = \deg f$ we write

$$f(X,Y) = a_{00} + a_{10}\,X + a_{01}\,Y + \sum_{i=2}^{d} a_{i,d-i}\,X^i Y^{d-i} \,.$$

Since $p \in C$, we get $a_{00} = 0$. As p is a singular point we also need that $\frac{\partial f}{\partial X}$ and $\frac{\partial f}{\partial Y}$ vanish at $(0,0)$, i.e. $a_{10} = a_{01} = 0$. So in particular $\operatorname{ord}(f) \geq 2$. Moreover we have $f \in J$ because of the subscheme $\{p\} \subset C$, thus $\exists\, u, v \in \mathbb{K}[X,Y]$ such that $f = uX + vY^2$ with $u(0,0) = 0$ as $a_{10} = 0$. We may also assume that v only depends on Y since all terms containing X can be included in u.

1) $\Rightarrow$ 2) : If I is free, then it is necessarily generated by one element because of the exact sequence $0 \to I \to R \to R/I \to 0$. Note that the converse is not immediate even if I is torsion-free since the generator may be annihilated by a zero-divisor.
2) $\Rightarrow$ 3) : follows from Lemma 5.4.3 applied to the exact sequence of R'-modules

$$0 \longrightarrow \langle f_p \rangle \xrightarrow{\;i\;} \mathcal{J}_p \xrightarrow{\;\pi\;} I \longrightarrow 0 \,.$$

Note that $f_p \in \mathcal{J}_p$, so i is indeed the inclusion. $\langle f_p \rangle$ is generated by f_p and I is generated by one element (over R), so $\mathcal{J}_p$ is generated by two elements (over R'), one of them being f_p.

3) $\Rightarrow$ 2) : If $\mathcal{J}_p = \langle \xi, f_p \rangle$ for some $\xi \in R'$, then $I = \pi(\mathcal{J}_p)$ is generated by $\pi(\xi)$ and $\pi(f_p) = 0$, over R' and hence over R.

4) $\Rightarrow$ 3) : If f contains Y^2, then $v(0) \neq 0$ and $Y_p^2 = \frac{f}{v} - \frac{uX}{v}$ as elements in R'. Hence

$$\mathcal{J}_p = \langle X_p, Y_p^2 \rangle = \left\langle X_p \, , \, \tfrac{f}{v} - \tfrac{uX}{v} \right\rangle = \left\langle X_p \, , \, \tfrac{1}{v} \cdot \tfrac{f}{1} \right\rangle = \langle X_p, f_p \rangle \ .$$

3) $\Rightarrow$ 4) : Assume that $\mathcal{J}_p = \langle \xi, f_p \rangle$ for some $\xi \in R'$, thus $\exists\, a, b \in R'$ such that $\xi = a \cdot X_p + b \cdot Y_p^2$. By Corollary 5.4.7 we may see elements in R' as formal power series in X, Y. As $X_p \in \langle \xi, f_p \rangle$, there also exist $c, d \in R'$ such that $X = c \cdot \xi + d \cdot f = caX + cbY^2 + df$ (now seen as power series). But $\mathrm{ord}(cbY^2) \geq 2$ and $\mathrm{ord}(df) \geq 2$ with $\mathrm{ord}(X) = 1$, so we conclude that a and c are units in $\mathbb{K}[[X, Y]]$, i.e. $a(0,0) \neq 0$ and $c(0,0) \neq 0$. Then we set

$$\xi' := \tfrac{1}{a} \cdot \xi = X + \eta(X, Y)$$

for some $\eta \in \mathbb{K}[[X, Y]]$ with $\mathrm{ord}(\eta) \geq 2$. In order to show that f contains Y^2, we assume the contrary, i.e. that f does not contain Y^2.

As $Y_p^2 \in \langle \xi, f_p \rangle = \langle \xi', f_p \rangle$ but by our assumption f does not contain Y^2, we conclude that ξ' and hence η must contain Y^2. Substituting X by 0 in the equality $X = c \cdot \xi + d \cdot f$ we get

$$0 = c(0, Y) \cdot a(0, Y) \cdot \eta(0, Y) + d(0, Y) \cdot f(0, Y) \ .$$

c and a being units, they both have a non-zero constant term and hence the product $(c \cdot a \cdot \eta)(0, Y)$ contains Y^2. On the other hand, since f does not contain Y^2, the order of $f(0, Y)$ as a formal power series in Y is at least 3, so the order of $d(0, Y) \cdot f(0, Y)$ is ≥ 3 as well. This contradiction shows that our assumption was wrong.

4) $\Rightarrow$ 1) : If f contains Y^2 we conclude from the step 4) $\Rightarrow$ 3) that $\mathcal{J}_p = \langle X_p, f_p \rangle$ and hence from the step 3) $\Rightarrow$ 2) that I is generated by $\pi(X_p) = x$. It remains to show that there are no relations annihilating x. For this consider the surjective

morphism

$$R \longrightarrow I \ : \ r \longmapsto r \cdot x \ . \tag{5.33}$$

We want to show that it is also injective and hence that we have an isomorphism $R \cong I$. We write $r = \left[\frac{g_1}{g_2}\right]$ for some $g_1 \in \mathbb{K}[X,Y]$ and $g_2 \notin \langle X, Y \rangle$. Then

$$r \cdot x = 0 \ \Leftrightarrow \ \left[\tfrac{g_1}{g_2}\right] \cdot \left[\tfrac{X}{1}\right] = 0 \ \Leftrightarrow \ \left[\frac{g_1 X}{g_2}\right] = 0 \ \Leftrightarrow \ \frac{g_1 X}{g_2} = \frac{\alpha}{\beta} \cdot \frac{f}{1} \quad \text{for some } \tfrac{\alpha}{\beta} \in R' \ .$$

In order to get $\left[\frac{g_1}{g_2}\right] = 0$ we need that $\frac{g_1}{g_2} = \frac{h_1}{h_2} \cdot \frac{f}{1}$ for some $\frac{h_1}{h_2} \in R'$, i.e. that $g_1 h_2 = g_2 h_1 f$. We already have $g_1 X \beta = \alpha f g_2$. Note that one cannot choose $h_2 = X\beta$ since this vanishes at $(0,0)$. But $g_1 X \beta = \alpha f g_2$ is an equality in $\mathbb{K}[X,Y]$ where $\langle X \rangle$ is a prime ideal. Since f contains Y^2, it is not divisible by X, hence X must divide αg_2. But again $g_2(0,0) \neq 0$, so X cannot divide g_2. It follows that $\alpha = \alpha' X$ for some $\alpha' \in \mathbb{K}[X,Y]$. So we get

$$g_1 X \beta = \alpha f g_2 \ \Leftrightarrow \ g_1 X \beta = \alpha' X f g_2 \ \Leftrightarrow \ g_1 \beta = \alpha' f g_2 \ .$$

Now it suffices to choose $h_2 = \beta$, which is well-defined, and $h_1 = \alpha'$. Finally $I \cong \langle x \rangle$ is free.
4) $\Leftrightarrow$ 5) : Since $\mathrm{ord}(f) \geq 2$, we know that f is of the form

$$f(X,Y) = a_{20}\, X^2 + a_{11}\, XY + a_{02}\, Y^2 + g(X,Y) \quad \text{ with } \mathrm{ord}(g) \geq 3 \ .$$

If f contains Y^2, then $a_{02} \neq 0$ and its leading form is

$$f_*(X,Y) = a_{20} X^2 + a_{11} XY + a_{02} Y^2 \ .$$

This is not divisible by X, so the tangent cone does not contain the line $X = 0$. Vice-versa note that the tangent cone consisting of 2 lines (with multiplicities) means that f is of order 2, i.e. the leading form is of degree 2. By contraposition, if f does not contain Y^2, then

$$f_*(X,Y) = a_{20} X^2 + a_{11} XY = X \cdot (a_{20} X + a_{11} Y) \ ,$$

where a_{20} and a_{11} are not both zero, so the tangent cone contains the line given by $X = 0$. □

Remark 5.4.12. The assumption that the tangent cone consists of 2 lines with multiplicities is necessary. Consider for example $f \in \mathbb{C}[X,Y]$ given by

$$f(X,Y) = X^3 - (1+2i)\,X^2Y + (2i-1)\,XY^2 + Y^3 = (X-Y)\cdot(X-iY)^2\ .$$

f does not contain Y^2, but its tangent cone consists of 2 lines not containing the line $X = 0$. However one of them is a double line, so it actually consists of 3 lines (with multiplicities).

Remark 5.4.13. An alternative way to prove that $R \cong I$ via $r \mapsto r \cdot x$ is the following. Since f contains Y^2, it is not divisible by X. Now consider the morphism of $\mathbb{K}[X,Y]$–modules

$$\mathbb{K}[X,Y]/\langle\, f\,\rangle \longrightarrow \langle\, X, f\,\rangle/\langle\, f\,\rangle \cong \langle\, \bar{X}\,\rangle \;:\; \bar{g} \longmapsto \overline{gX}\ .$$

We want to show that it is injective. Indeed if $g \cdot X \in \langle\, f\,\rangle$, then $\exists\, h \in \mathbb{K}[X,Y]$ such that $h \cdot f = g \cdot X$. Again $\langle\, X\,\rangle$ is a prime ideal, but f is not divisible by X, thus X divides h and $h = h'X$ for some $h' \in \mathbb{K}[X,Y]$. But then $hf = h'Xf = gX$ which implies that $h'f = g$, i.e. $g \in \langle\, f\,\rangle$ and $\bar{g} = \bar{0}$. Now we localize this morphism at the maximal ideal $\langle\, \bar{X}, \bar{Y}\,\rangle$, hence we still have an injection

$$\Big(\mathbb{K}[X,Y]/\langle\, f\,\rangle\Big)_{\langle\, \bar{X},\bar{Y}\,\rangle} \longrightarrow \langle\, \bar{X}\,\rangle_{\langle\, \bar{X},\bar{Y}\,\rangle} \cong \langle\, x\,\rangle$$

by exactness of localization. Using the LHS of the description (5.32) we see that this is nothing but the map $R \to \langle\, x\,\rangle = I : r \mapsto r \cdot x$ from (5.33), which is hence injective.

5.4.4 Ideals of fat curvilinear points on a planar curve

After the case of a fat double point, let us also consider the more general situation of a point of multiplicity $n \geq 2$. Unfortunately we do not yet have a criterion which is valid for all fat points. So for the moment we restrict ourselves to so-called curvilinear points.

Definition 5.4.14. A fat *curvilinear* point of multiplicity $n \geq 2$ at the origin in $\mathbb{A}_2$ is a subscheme $\{p\} \subset \mathbb{A}_2$ where $p = (0,0)$ and its defining ideal is given by $\langle\, X - h(Y)\,,\, Y^n\,\rangle \trianglelefteq \mathbb{K}[X,Y]$ for some $h \in \mathbb{K}[Y]$ with $\deg h < n$ and $h(0) = 0$.

Intuitively this means that the fat point is obtained by intersecting the fat line of multiplicity n given by $Y^n = 0$ with the curve given by $X = h(Y)$, i.e. the fat point sits on a curve. For $h = 0$ we obtain a fat point sitting on the line given by $X = 0$.

Example 5.4.15. Examples of fat points which are not curvilinear are e.g. the triple point described by $\langle\, X^2, XY, Y^2 \,\rangle$ and the quadruple point given by $\langle\, X^2, Y^2 \,\rangle$; both do not sit on a smooth curve.

For curvilinear points, a similar result holds true as in the case of double points in Proposition 5.4.11. The proof is similar, so we are not going to develop all details again. However some of the facts from Proposition 5.4.11 will no longer be true and we will point out the main differences.
We will mostly keep the same notations as before. Let C be a planar curve defined as the vanishing set of some non-constant polynomial $f \in \mathbb{K}[X, Y]$ and assume that $p = (0, 0) \in C$. The subscheme $Z \subset C$ of a fat curvilinear point at p is given by the ideal $J = \langle\, X - h(Y)\,,\, Y^n \,\rangle$. As before we also consider the local rings $R' = \mathbb{K}[X, Y]_{\langle X,Y \rangle}$ and $R = \mathcal{O}_{C,p}$, in which we denote the classes of $X, Y \in \mathbb{K}[X, Y]$ by X_p, Y_p and x, y respectively. Thus we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \langle\, f_p \,\rangle & \xrightarrow{\sim} & R' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & J_p & \longrightarrow & R' & \longrightarrow & \mathcal{O}_{p^n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & \mathcal{O}_{p^n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $f_p = \frac{f}{1} \in R$, $J_p = \langle\, X_p - h(Y_p)\,,\, Y_p^n \,\rangle \trianglelefteq R'$, $I = \langle\, x - h(y)\,,\, y^n \,\rangle \trianglelefteq R$ and p^n means "point p of multiplicity n". $I \subseteq R$ is the ideal defining the subscheme of the fat curvilinear point $Z \subset C$ in the local ring. That subscheme moreover gives $\langle f \rangle \subset J$. For technical reasons we rewrite this as

$$f(X,Y) = \det \begin{pmatrix} X - h(Y) & Y^n \\ u(Y) & v(X,Y) \end{pmatrix} = v \cdot \big(X - h(Y)\big) - u \cdot Y^n \qquad (5.34)$$

for some $u \in \mathbb{K}[Y]$ and $v \in \mathbb{K}[X,Y]$.

Remark 5.4.16. Note that u may be chosen independent of X since all terms containing (multiples of) X can be eliminated by substracting a suitable multiple of the first row of the determinant.

Proposition 5.4.17 (Leytem)**.** *Assume that R is a non-regular ring, i.e. $(0,0)$ is a singular point of C. Then*

$$I \text{ is a free } R\text{-module} \;\Leftrightarrow\; u(0) \neq 0\ .$$

Moreover if I is free, then it is generated by $x - h(y)$ and there is an isomorphism $R \cong I$ given by

$$r \longmapsto r \cdot \big(x - h(y)\big)\ .$$

Proof. Along the same line as the proof in the case of a double point, we will proceed by showing the equivalence of the statements

1) I is free. , 2) I is generated by one element.

3) $J_p = \langle \xi, f_p \rangle$ for some $\xi \in R'$. , 4) $u(0) \neq 0$.

Proving 1) $\Rightarrow$ 2) , 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2) is done exactly as in the proof of Proposition 5.4.11.
4) $\Longrightarrow$ 3) : If $u(0) \neq 0$, then $Y_p^n = \frac{v(X-h(Y))}{u} - \frac{f}{u}$ as elements in R'. Hence

$$\begin{aligned} J_p = \langle X_p - h(Y_p)\,,\, Y_p^n \rangle &= \Big\langle X_p - h(Y_p)\,,\, \tfrac{v(X-h(Y))}{u} - \tfrac{f}{u} \Big\rangle \\ &= \Big\langle X_p - h(Y_p)\,,\, \tfrac{1}{u} \cdot \tfrac{f}{1} \Big\rangle = \langle X_p - h(Y_p)\,,\, f_p \rangle\ . \end{aligned}$$

3) $\Longrightarrow$ 4) : Assume that $J_p = \langle \xi, f_p \rangle$ for some $\xi \in R'$, thus $\exists\, a, b \in R'$ such that

$$\xi = a \cdot \big(X_p - h(Y_p)\big) + b \cdot Y_p^n\ .$$

Again we may see elements in R' as formal power series in X, Y. The fact that $X_p - h(Y_p) \in \langle\, \xi, f_p \,\rangle$ also implies that there exist $c, d \in R'$ such that we have an equality of power series

$$X - h(Y) = c \cdot \xi + d \cdot f = ca\big(X - h(Y)\big) + cbY^n + df \ .$$

But $\operatorname{ord}(cbY^n) \geq n \geq 2$ and $\operatorname{ord}(df) \geq 2$ since $\operatorname{ord}(f) \geq 2$. Together with $\operatorname{ord}(X - h(Y)) = 1$, we conclude that a and c are units in $\mathbb{K}[[X, Y]]$, so we can set

$$\xi' := \tfrac{1}{a} \cdot \xi = X - h(Y) + \eta(X, Y) \ ,$$

where $\eta \in \mathbb{K}[[X, Y]]$ is given by $\frac{b}{a} \cdot Y^n$, thus $\operatorname{ord}(\eta) \geq n$.
In order to show that $u(0) \neq 0$, we assume the contrary, i.e. $u(0) = 0$. As $Y_p^n \in \langle\, \xi, f_p \,\rangle = \langle\, \xi', f_p \,\rangle$, there are also $\gamma, \delta \in \mathbb{K}[[X, Y]]$ such that $Y^n = \gamma \cdot \xi' + \delta \cdot f$. Evaluating this equality at $X = h(Y)$ gives

$$\begin{aligned} Y^n &= \gamma\big(h(Y), Y\big) \cdot \xi'\big(h(Y), Y\big) + \delta\big(h(Y), Y\big) \cdot f\big(h(Y), Y\big) \\ &= \gamma\big(h(Y), Y\big) \cdot \underbrace{\eta\big(h(Y), Y\big)}_{\operatorname{ord} \geq n} + \delta\big(h(Y), Y\big) \cdot \underbrace{\big(-u(Y) \cdot Y^n\big)}_{\operatorname{ord} \geq n+1} \end{aligned}$$

since $u(0) = 0$ by our assumption. As $\operatorname{ord}(Y^n) = n$, γ must therefore have a constant term (i.e. γ is a unit) and $\operatorname{ord}\big(\eta(h(Y), Y)\big) = n$. Next we substitute $X = h(Y)$ in the equality $X - h(Y) = c \cdot \xi + d \cdot f$ to get

$$0 = c\big(h(Y), Y\big) \cdot a\big(h(Y), Y\big) \cdot \eta\big(h(Y), Y\big) + d\big(h(Y), Y\big) \cdot \big(-u(Y) \cdot Y^n\big) \ .$$

c and a being units, they have a non-zero constant term and hence the product $(c \cdot a \cdot \eta)(h(Y), Y)$ has order n. On the other hand, since $u(0) = 0$, the order of $d(h(Y), Y) \cdot u(Y)\, Y^n$ is at least $n + 1$. This contradiction shows that our assumption was wrong.
4) $\Rightarrow$ 1) : As in the proof of Proposition 5.4.11, we obtain that I is generated by $x - h(y)$. Note that the polynomial $X - h(Y)$ is irreducible in $\mathbb{K}[X, Y]$ since it is of degree 1 in X, so $\langle\, X - h(Y) \,\rangle$ is a prime ideal. As $u(0) \neq 0$, f is not divisible by $X - h(Y)$, otherwise

$$u \cdot Y^n = v \cdot \big(X - h(Y)\big) - f \ \in \ \langle\, X - h(Y) \,\rangle \ ,$$

which is impossible since $u \neq 0$ and u does not depend on X. Similarly as in Remark 5.4.13 the morphism of $\mathbb{K}[X,Y]$–modules

$$\mathbb{K}[X,Y]/\langle f \rangle \longrightarrow \langle X - h(Y) \, , \, f \rangle / \langle f \rangle \ : \ \bar{g} \longmapsto \overline{g \cdot (X - h(Y))}$$

is injective and after localization at the maximal ideal $\langle \bar{X}, \bar{Y} \rangle$ we obtain an isomorphism

$$R \overset{\sim}{\longrightarrow} I \ : \ r \longmapsto r \cdot \big(x - h(y)\big) \ . \qquad \square$$

Remark 5.4.18. A similar statement involving the tangent cone as in Proposition 5.4.11 is no longer true. Consider e.g. a triple point ($n = 3$) with $h(Y) = Y$, $v(X,Y) = X$ and $u(Y) = 1$, so that

$$\begin{aligned} f(X,Y) &= X \cdot (X - Y) - 1 \cdot Y^3 = X^2 - XY - Y^3 \\ \Rightarrow \ f_*(X,Y) &= X^2 - XY = X(X - Y) \ , \end{aligned}$$

i.e. the tangent cone would only consist of 2 lines, even though $u(0) \neq 0$.

Remark 5.4.19. The condition $u(0) \neq 0$ is moreover not equivalent to the one of f containing the monomial Y^n. Consider e.g. $n = 3$ with $h(Y) = -Y$, $v(X,Y) = Y^2$ and $u(Y) = 1$, which gives

$$f(X,Y) = Y^2 \cdot (X + Y) - 1 \cdot Y^3 = XY^2 \ .$$

If we want such a statement, (5.34) requires us to replace $f(X,Y)$ by $f(h(Y),Y)$:

$$u(0) \neq 0 \ \Leftrightarrow \ f\big(h(Y),Y\big) \text{ does not contain the monomial } Y^n \ .$$

5.5 Singular sheaves

Recall that we defined $M_0' = M' \cap M_0$ to be the closed subvariety of singular sheaves in M_0. Now we consider the restriction of $\nu : \mathbb{B}_0 \to N_0$ to M_0' and describe some of its fibers.

Lemma 5.5.1. *Let $\mathcal{F}$ be a sheaf in $\mathbb{B}_0$ with $C = \operatorname{supp} \mathcal{F}$ and let $Z \subseteq C$ be its corresponding 0-dimensional subscheme given as in (5.27). Then $\mathcal{F}$ is non-singular at all points $p \in C \setminus Z$.*

Proof. By Proposition 5.3.31 we have the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(d-3) \longrightarrow \mathcal{O}_Z \longrightarrow 0 \ .$$

If $p \in C \setminus Z$, then $\mathcal{O}_{Z,p} = \{0\}$ and $\mathcal{F}_p \cong \mathcal{O}_{C,p}$, which means that $\mathcal{F}_p$ is a free module over $\mathcal{O}_{C,p}$. □

Corollary 5.5.2. *If a sheaf $\mathcal{F} \in \mathbb{B}_0$ is singular at a point $p \in C$, then*

$$p \in \text{Sing}(C) \cap Z \ .$$

Proof. By Lemma 5.5.1 sheaves in $\mathbb{B}_0$ can only be singular at points of Z. Moreover those points have to be singular, otherwise if $p \in Z \subseteq C$ is a smooth point, then $\mathcal{O}_{C,p}$ is a regular local ring of Krull dimension 1, i.e. a PID and thus $\mathcal{F}_p$ is free. □

Hence in order to study singular sheaves in $\mathbb{B}_0$, it is important to understand under which conditions an ideal in the local ring of a singular point is not free. This is the reason why we established the characterizations of free ideals of points on planar curves in Section 5.4. Now we are going to apply these results in order to determine "how many" sheaves in $\mathbb{B}_0$ are singular, i.e. which subspace of $\mathbb{B}_0$ gives singular sheaves. For this we analyze the fibers of $\mathbb{B}_0$ over N_0 according to the multiplicity of the points in Z.

5.5.1 Generic fibers

Definition 5.5.3. Let N_c be the open subset of N_0 that corresponds to Kronecker modules whose maximal coprime minors defining a 0-dimensional subscheme of $l = \binom{n}{2}$ different points.[11] Intuitively one can see that this set is open since by slightly moving a configuration of l points, one still obtains a configuration of l points (i.e. slightly moving simple points does not create double points). Under the isomorphism $N_0 \cong H_0$ from Remark 5.3.29 it corresponds to the open subvariety $H_c \subseteq H_0$ of the configuration of l points on $\mathbb{P}_2$ which do not lie on a curve of degree $d-3$.

[11] The c in N_c stands for "configuration".

Let $[\Phi] \in N_c$; it defines a 0-dimensional subscheme $Z = \{p_1, \dots, p_l\}$. Since all points are different (i.e. we only have simple points), we obtain $\mathcal{O}_{Z,p_i} \cong \mathbb{K}$ for all $i \in \{1, \dots, l\}$. Let us denote $\mathbb{B}_c = \mathbb{B}|_{N_c}$. Then we get the following converse of Corollary 5.5.2.

Proposition 5.5.4. *Let $\mathcal{F} \in \mathbb{B}_c$ be a sheaf over $[\Phi] \in N_c$ with $C = \operatorname{supp} \mathcal{F}$. Then $\mathcal{F}$ is singular if and only if Z contains a singular point of C, i.e. if and only if $\operatorname{Sing}(C) \cap Z \neq \emptyset$.*

Proof. By Lemma 5.5.1 we know that it suffices to check the stalks of $\mathcal{F}$ at $p \in Z$. We localize the sequence (5.27) given by Proposition 5.3.31 at a point $p \in Z$, so

$$0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{O}_{C,p} \longrightarrow \mathcal{O}_{Z,p} \longrightarrow 0 \qquad \Leftrightarrow \qquad 0 \longrightarrow \mathfrak{M} \longrightarrow R \longrightarrow \Bbbk_p \longrightarrow 0 \; ,$$

where $\mathfrak{M} = \mathcal{F}_p$ is a maximal ideal of $R = \mathcal{O}_{C,p}$, otherwise the quotient is not a field. But this is exactly the situation as in Lemma 5.4.1. Hence we know that the R-module M is free if and only if p is a smooth point of the curve C. By negation, we conclude that $\mathcal{F}$ is singular if and only if there exists a singular point $p \in C \cap Z$. □

Remark 5.5.5. Proposition 5.5.4 does not hold true if the points are not simple. A counter-example involving a double point is given in Example 5.6.3.

Proposition 5.5.6. *The fibers of M_0' over N_c are unions of l different projective subspaces of $\mathbb{P}_{3d-1}$ of codimension 2.*

Proof. Let us study the fibers of $\nu : \mathbb{B}_c \to N_c$. We fix $[\Phi] \in N_c$ and denote $F := \nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$ (the fibers of $\mathbb{B}$ are of dimension $3n + 2 = 3d - 1$, see Proposition 5.3.22). By Proposition 5.3.32 F may be identified with the space of curves of degree d passing through Z via the assignment $\mathcal{F} \mapsto \operatorname{supp} \mathcal{F}$. Hence the words "curve" and "sheaf over $[\Phi]$" can be used equivalently. Without loss of generality we may also assume that $p_1 = (1 : 0 : 0)$.
Let $\mathcal{F} \in F$ and denote $C = \operatorname{supp} \mathcal{F} = Z(f)$ for some homogeneous polynomial f of degree d. Since $p_1 \in C$, we need that f does not contain the monomial X_0^d. Lemma 5.4.1 and Proposition 5.5.4 then say that $\mathcal{F}$ is singular at p_1 if and only if p_1 is a singular point of C. The latter holds if and only if the coefficients of f

in front of $X_0^{d-1}X_1$ and $X_0^{d-1}X_2$ vanish (compute the partial derivatives). So we obtain a condition which defines a closed subspace of codimension 2. Note that the condition about the vanishing of the coefficient in front of X_0^d does not count here since it is an immediate consequence of the setting and doesn't contribute to the criterion of singularity.

Next we need to show that the projective subspace $F_1 \subsetneq F$ of sheaves over $[\Phi]$ which are singular at p_1 is also of codimension 2 in the fiber F. Indeed we have the inclusions

$$F_1 \subset F \subset F_{p_1} \subset \mathcal{C}_d(\mathbb{P}_2) \ ,$$

where F_{p_1} is the space of curves of degree d passing through p_1. Note that

$$\operatorname{codim}_{\mathcal{C}_d(\mathbb{P}_2)} F_{p_1} = 1 \qquad \text{and} \qquad \operatorname{codim}_{\mathcal{C}_d(\mathbb{P}_2)} F = l$$

by Lemma 5.2.40 since Z does not lie on a curve of degree $d-3$. Above we have proved that the space $F'_{p_1} \subset F_{p_1}$ of curves passing through p_1 and being singular at p_1 is of codimension 2 in F_{p_1}. Now we shall prove that $F_1 = F'_{p_1} \cap F$ is still of codimension 2 in F (since the support of the sheaves in F contains all of Z).[12] For this we will show that there exists 2 linearly independent polynomials which vanish at Z and define non-singular sheaves in $F \subset \mathbb{B}_c$, so that we obtain 2 independent sheaves over $[\Phi]$ which are not in F_1. This will imply that the codimension of F_1 in F is still equal to 2. First note that

$$l = \tbinom{n}{2} = \tfrac{n(n-1)}{2} = \tfrac{(d-1)(d-2)}{2} = \tbinom{d-3+2}{2} \ ,$$

thus there are l linearly independent monomials of degree $d-3$ in the variables X_0, X_1, X_2. We denote these monomials e.g. in lexicographical order:

$$m_1 = X_0^{d-3} \quad , \quad m_2 = X_0^{d-4}X_1 \quad , \quad m_3 = X_0^{d-4}X_2 \quad , \quad \dots \ . \tag{5.35}$$

Hence an arbitrary homogeneous polynomial g of degree $d-3$ writes as a $\mathbb{K}$-linear combination $g = \sum_{i=1}^{l} c_i m_i$ for some $c_1, \dots, c_l \in \mathbb{K}$. If we want the corresponding

[12]Indeed the codimension of an intersection can decrease. Consider e.g. a complex line ℓ in $\mathbb{C}^3$; it is of codimension 2. But if we take a complex plane P containing ℓ, then the intersection $\ell \cap P = \ell$ is only of codimension 1 in $P \cong \mathbb{C}^2$.

curve $Z(g)$ to contain Z, then g must vanish at all points $p_1, \dots, p_l$, which gives the l conditions

$$\begin{cases} g(p_1) = 0 \\ \vdots \\ g(p_l) = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{l} c_i \cdot m_i(p_1) = 0 \\ \vdots \\ \sum_{i=1}^{l} c_i \cdot m_i(p_l) = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} m_1(p_1) & \dots & m_l(p_1) \\ \vdots & \ddots & \vdots \\ m_1(p_l) & \dots & m_l(p_l) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} .$$

However we know that Z does not lie on a curve of degree $d-3$, so this linear system should have no non-trivial solution $c_1, \dots, c_l$. This is the case if and only if the matrix $\big(m_j(p_i)\big)_{ij}$ is invertible. As we assumed that $p_1 = (1:0:0)$, we get $m_1(p_1) = 1$ and $m_j(p_1) = 0$ for all $j \geq 2$. Hence saying that the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ m_1(p_2) & m_2(p_2) & \dots & m_l(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(p_l) & m_2(p_l) & \dots & m_l(p_l) \end{pmatrix}$$

is invertible means that the matrix

$$m = \begin{pmatrix} m_1(p_2) & m_2(p_2) & \dots & m_l(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(p_l) & m_2(p_l) & \dots & m_l(p_l) \end{pmatrix}$$

has full rank (equal to $l-1$) since the first maximal minor is non-zero. Therefore the kernel of the linear map $m : \mathbb{K}^l \to \mathbb{K}^{l-1}$ is 1-dimensional and there exists a non-trivial solution of $m \cdot c = 0$, where $c = {}^t(c_1, \dots, c_l)$. This means that there exists a homogeneous polynomial q_1 of degree $d-3$ vanishing at the points $p_2, \dots, p_l$ and with a non-zero coefficient in front of the monomial $m_1 = X_0^{d-3}$ (otherwise q_1 also vanishes at p_1). Thus q_1 contains X_0^{d-3} and $Z \setminus \{p_1\} \subseteq Z(q_1)$. The forms $X_0^2 X_1 q_1$ and $X_0^2 X_2 q_1$ of degree d then vanish at all points of Z.

Now we use that the polynomial defining the support of a singular sheaf does not contain the monomials $X_0^{d-1} X_1$ and $X_0^{d-1} X_2$. However $X_0^2 X_1 q_1$ contains

$X_0^{d-1}X_1$ since q_1 contains m_1, but it does not have $X_0^{d-1}X_2$. Similarly, $X_0^2X_2q_1$ has $X_0^{d-1}X_2$, but not $X_0^{d-1}X_1$. Therefore the sheaves in F corresponding to the curves given by $X_0^2X_1q_1$ and $X_0^2X_2q_1$ are non-singular. As these polynomials are moreover linearly independent, we finally obtain that $\operatorname{codim}_F F_1 = 2$.

Repeating the same argument for each point $p_1, \ldots, p_l$, we conclude that the sheaves over $[\Phi]$ that are singular at p_i define a closed linear[13] projective subspace F_i of codimension 2 in the fiber F. In particular,

$$\mathcal{F} \in F \text{ is singular } \Leftrightarrow \mathcal{F} \in F_1 \cup \ldots \cup F_l$$

with $\operatorname{codim}_F(F_1 \cup \ldots \cup F_l) = 2$. However the 2 closed conditions defining F_i for $i \neq 1$ do not correspond to the vanishing of some coefficients in front of monomials ; this is only the case if the homogeneous coordinates of p_i are "nice". In general they are given by

$$(\partial_1 f)(p_i) = 0 \qquad \text{and} \qquad (\partial_2 f)(p_i) = 0\ . \tag{5.36}$$

Note that we do not need $(\partial_0 f)(p_i) = 0$ because of $f(p_i) = 0$ and Euler's relation for homogeneous polynomials, hence these 2 conditions are enough to get singular points. □

Remark 5.5.7. Now let us describe how the subspaces F_i in the fiber F corresponding to different points p_i intersect with each other. First note that Z contains a triple of non-collinear points because $d \geq 4$ and Z does not lie on a curve of degree $d-3$.[14] Hence in addition to the assumption $p_1 = (1:0:0)$, we may assume by Lemma D.1.20 that $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$. Similarly as in the proof of Proposition 5.5.6, the conditions for being singular at these 3 points correspond to the absence of the following monomials in the equation of the support C:

[13] "linear" means given by linear equations

[14] So the argument does not work for $d = 3$ and the moduli space M_{3m-1}, where Z just consists of 1 point.

$$X_0^{d-1}X_1 \ , \ X_0^{d-1}X_2 \quad \text{for the point } p_1 \ , \tag{5.37}$$

$$X_1^{d-1}X_0 \ , \ X_1^{d-1}X_2 \quad \text{for the point } p_2 \ , \tag{5.38}$$

$$X_2^{d-1}X_0 \ , \ X_2^{d-1}X_1 \quad \text{for the point } p_3 \ . \tag{5.39}$$

The conditions (5.37), (5.38) and (5.38) are clearly independent of each other. Moreover they are independent of the conditions imposed on the support by requiring that $Z \subseteq C$ since those equations involve the coefficients in front of X_0^d, X_1^d and X_2^d. We want to show that $\forall\, i \neq j$ in $\{1,2,3\}$,

$$\operatorname{codim}_F(F_i \cap F_j) = 4 \qquad \text{and} \qquad \operatorname{codim}_F(F_1 \cap F_2 \cap F_3) = 6 \ .$$

This is again achieved by constructing 4, resp. 6 linearly independent polynomials which define non-singular sheaves in $F \subset \mathbb{B}_c$. As in the proof of Proposition 5.5.6, we obtain homogeneous polynomials q_2 and q_3 of degree $d-3$ vanishing at $Z \setminus \{p_2\}$ and $Z \setminus \{p_3\}$ respectively such that

$$\begin{aligned}
&X_1^2X_0q_2 \ \text{ contains } X_1^{d-1}X_0 \text{ but not } X_1^{d-1}X_2 \ , \\
&X_1^2X_2q_2 \ \text{ contains } X_1^{d-1}X_2 \text{ but not } X_1^{d-1}X_0 \ , \\
&X_2^2X_0q_3 \ \text{ contains } X_2^{d-1}X_0 \text{ but not } X_2^{d-1}X_1 \ , \\
&X_2^2X_1q_3 \ \text{ contains } X_2^{d-1}X_1 \text{ but not } X_2^{d-1}X_0 \ ,
\end{aligned}$$

and all polynomials on the LHS vanish at Z. Together with $X_0^2X_1q_1$ and $X_0^2X_2q_1$ we thus have 6 linearly independent homogeneous polynomials of degree d which vanish at Z and define non-singular sheaves as they only contain exactly one of the monomials from (5.37), (5.38) and (5.38).

Corollary 5.5.8. *The fibers of M_0' over N_c are unions of l different linear subspaces of $F \cong \mathbb{P}_{3d-1}$ of codimension* 2 *such that each pair intersects in codimension* 4 *and each triple corresponding to 3 non-collinear points intersects in codimension* 6*. In particular the fibers are singular at the intersection points.*

Remark 5.5.9. Using Lemma D.1.20 for collinear points, one can also show (e.g. with `Singular`) that $\operatorname{codim}_F(F_1 \cap F_j \cap F_k) = 6$ for every triple of different indices i,j,k corresponding to 3 different points p_i, p_j, p_k from Z.

Remark 5.5.10. But in general the subspaces $F_i \subseteq F$ do not intersect transversally, i.e. in general

$$\operatorname{codim}_F(F_{i_1}, \dots, F_{i_r}) \neq 2r$$

for $r \in \{4, \dots, l\}$. For example, let $d = 6$ and $Z = \{p_1, \dots, p_{10}\}$ with

$$p_1 = (1:0:0) \quad , \quad p_2 = (0:1:0) \quad , \quad p_3 = (0:0:1)$$
$$p_4 = (0:1:1) \quad , \quad p_5 = (0:1:-1) \quad , \quad p_6 = (1:-2:0) \quad , \quad p_7 = (1:2:-1)$$
$$p_8 = (1:1:-2) \quad , \quad p_9 = (1:-1:1) \quad , \quad p_{10} = (1:1:-1)$$

Then one can compute the conditions (5.36) with `Singular` and obtain

$$\operatorname{codim}_F(F_1 \cap F_2 \cap F_3 \cap F_4) = 8 \quad \text{but} \quad \operatorname{codim}_F(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5) = 9 \ .$$

5.5.2 Fibers with a double point

Definition 5.5.11. Let N_1 be the open subset of $N_0 \setminus N_c$ that corresponds to $l-2$ different simple points and one double point. We denote $\mathbb{B}_1 = \mathbb{B}|_{N_1}$. Intuitively this can again be seen to be open since slightly moving this constellation inside of $N_0 \setminus N_c$ produces the same constellation (the simple points remain simple points and the double point cannot be "separated", otherwise we end up in N_c).

Remark 5.5.12. In order to show that the codimension of the singular sheaves in M_{dm-1} is 2, we also have to study this case. Indeed the codimension of the complement of N_c in N_0 is 1 and it may a priori happen that all sheaves over $N_0 \setminus N_c$ are singular, so the codimension would be equal to 1.

Let $[\Phi] \in N_1$ and denote its corresponding 0-dimensional subscheme by

$$Z = \{p_1\} \cup \{p_2, \dots, p_{l-1}\} \ ,$$

where p_1 is a double point. Without loss of generality we may assume that $p_1 = (1:0:0)$ and that it is given by the ideal $J = \langle X_1^2, X_2 \rangle$.
Note that if $\mathcal{F} \in \mathbb{B}_1$ is a sheaf over $[\Phi]$ with support $C = \operatorname{supp} \mathcal{F}$, then $p_1 \in Z \subseteq C$ does not mean that p_1 is an embedded double point of C. Indeed, if $\mathcal{F}$ is given by some $A \in \mathbb{X}$, then C is given by the homogeneous quotient $\mathbb{K}[X_0, X_1, X_2]/\langle \det A \rangle$ and this ring does not have embedded primes.

Proposition 5.5.13. *The sheaves over* $[\Phi]$ *that are singular at the double point* p_1 *constitute a closed linear projective subspace of codimension* 2 *in the fiber* $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

Proof. Let $m_1, \ldots, m_l$ again be all monomials of degree $d-3$ in lexicographical order as in (5.35). If $g = \sum_i c_i m_i$ is any polynomial of degree $d-3$, then saying that $Z \subseteq Z(g)$ means that

$$g(p_1) = g(p_2) = \ldots = g(p_{l-1}) = 0 \qquad \text{and} \qquad g \in I\ .$$

Since $m_1(p_1) = 1$ and $m_j(p_1) = 0$ for $j \geq 2$, we have $g(p_1) = 0 \Leftrightarrow c_1 = 0$. In addition we obtain $g \in I \Leftrightarrow c_1 = c_2 = 0$ because g writes as a combination

$$g = c_1 \cdot X_0^{d-3} + c_2 \cdot X_0^{d-4} X_1 + X_1^2 \cdot \big(\ldots\big) + X_2 \cdot \big(\ldots\big)\ .$$

Thus $g \in I$ if and only if the coefficients c_1 and c_2 in front of $m_1 = X_0^{d-3}$ and $m_2 = X_0^{d-4} X_1$ vanish. As Z does not lie on a curve of degree $d-3$, the linear system

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ m_1(p_2) & m_2(p_2) & \ldots & m_l(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(p_{l-1}) & m_2(p_{l-1}) & \ldots & m_l(p_{l-1}) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

does not have a non-trivial solution $c_1, \ldots, c_l$. In other words, the matrix

$$m = \begin{pmatrix} m_1(p_2) & m_2(p_2) & \ldots & m_l(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(p_{l-1}) & m_2(p_{l-1}) & \ldots & m_l(p_{l-1}) \end{pmatrix}$$

has full rank equal to $l-2$. Therefore the kernel of $m : \mathbb{K}^l \to \mathbb{K}^{l-2}$ has dimension 2, which means that there exist two linearly independent homogeneous polynomials q and q' of degree $d-3$ which vanish at the points $p_2, \ldots, p_{l-1}$. But since $Z(q)$ and $Z(q')$ are not allowed to contain all of Z, we can choose q to contain the monomial X_0^{d-3}, but not $X_0^{d-4}X_1$, and q' to contain $X_0^{d-4}X_1$, but not X_0^{d-3}. In particular the forms $X_0^2 X_2 q \in \langle\, X_2 \,\rangle$ and $X_0^2 X_1 q' \in \langle\, X_1^2 \,\rangle$ of degree d vanish at Z.

Now let $F = \nu^{-1}([\Phi])$ and $\mathcal{F} \in \mathbb{B}_1$ be a sheaf over $[\Phi] \in N_1$ with support $C = \operatorname{supp} \mathcal{F} = Z(f)$ for some homogeneous polynomial f of degree d. By Proposition 5.4.11 we conclude that $\mathcal{F}$ is singular at p_1 if and only if f does not contain the monomials $X_0^{d-2}X_1^2$ and $X_0^{d-1}X_2$. Indeed, Proposition 5.3.31 gives the exact sequence (5.27), which localized at the double point p_1 gives

$$0 \longrightarrow \mathcal{F}_{p_1} \longrightarrow \mathcal{O}_{C,p_1} \longrightarrow O_{Z,p_1} \longrightarrow 0 \quad \Leftrightarrow \quad 0 \longrightarrow I \longrightarrow R \longrightarrow \mathcal{O}_{dp_1} \longrightarrow 0 \, .$$

In the chart $U_0 \subset \mathbb{P}_2$ with $X_0 \neq 0$, the ideal I corresponds to $\langle X^2, Y \rangle$. Hence Proposition 5.4.11 says that I is a free R-module if and only if f contains X^2. However this condition is only correct in the case of singular points. So we have to add the condition on the vanishing of the coefficient in front of $X_0^{d-1}X_2$, which corresponds to Y in U_0 (otherwise p_1 would be a smooth point and $\mathcal{O}_{C,p_1}$ would be regular). The fact that the coefficient in front of $X_0^{d-1}X_1$ has to vanish as well is already contained in the assumption $f \in I$ and does not need to be added.
We obtained that a sheaf $\mathcal{F} \in F$ is singular at p_1 if and only if the coefficients in front of the monomials of f satisfy 2 closed conditions. Again one has to check that this 2-codimensional space of curves passing through p_1 properly intersects with F. This is the case since the polynomials $X_0^2X_2q$ and $X_0^2X_1q'$ above are linearly independent and define 2 different non-singular sheaves in $F \subset \mathbb{B}_1$. Indeed the first one contains $X_0^{d-1}X_2$ and the second one contains $X_0^{d-2}X_1^2$. □

It remains to study the case of singular sheaves at the simple points.

Proposition 5.5.14. *The projective subspace of sheaves over* $[\Phi]$ *which are singular at a simple point* p_i *is of codimension* 2 *in the fiber* $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

Proof. Let $i \in \{2, \ldots, l-1\}$. Here we are going to distinguish between the 2 following cases:
1) p_i is a simple point which lies on a line with p_1
2) p_i is a simple point which does not lie on a line with p_1.
Note that this makes sense since the double point p_1 also defines a "tangent direction", which already uniquely determines the line containing it.
If p_i lies on a line with p_1, then this line is necessarily given by the equation $X_2 = 0$ because if $h = aX_0 + bX_1 + cX_2$ has to satisfy $h(p_1) = 0$ and $h \in I$, then

$a = b = 0$. Hence we may assume without loss of generality that $p_i = (0 : 1 : 0)$. On the other hand since Z does not lie on a curve of degree $d-3$, it is in particular not contained in a line and hence there is always a point p_j which does not lie on the same line as p_1. In this case we may assume without loss of generality that $p_j = (0 : 0 : 1)$.

Let us first study the case where $p_i = (0 : 0 : 1)$. By renumbering the points we may assume that $i = 2$. Then we obtain the invertible matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ m_1(p_3) & m_2(p_3) & m_3(p_3) & \dots & m_l(p_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1(p_{l-1}) & m_2(p_{l-1}) & m_3(p_{l-1}) & \dots & m_l(p_{l-1}) \end{pmatrix}$$

since $m_l = X_2^{d-3}$. Hence there exists a homogeneous polynomial q'' of degree $d-3$ vanishing at $Z \setminus \{p_2\}$ with coefficient 1 in front of the monomial X_2^{d-3}. Then the polynomials $X_1X_2^2q''$ and $X_0X_2^2q''$ vanish at Z. Moreover the first one contains $X_1X_2^{d-1}$ but not $X_0X_2^{d-1}$, the latter one has $X_0X_2^{d-1}$ but not $X_1X_2^{d-1}$. As in Proposition 5.5.6 and Lemma 5.4.1 a sheaf $\mathcal{F}$ over $[\Phi] \in N_1$ is singular at p_2 if and only if the coefficients in front of $X_2^{d-1}X_0$ and $X_2^{d-1}X_1$ vanish, so we obtain a closed subspace of codimension 2. Since moreover $X_1X_2^2q''$ and $X_0X_2^2q''$ define 2 independent non-singular sheaves in $\mathbb{B}_1$, we obtain that the sheaves over $[\Phi]$ which are singular at p_2 constitute a linear projective subspace of codimension 2 in the fiber $\nu^{-1}([\Phi])$. Similarly the projective subspace of sheaves over $[\Phi]$ which are singular at some point p_i such that p_1 and p_i do not lie on a line is of codimension 2 in the fiber $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$.

Finally we analyze the case where $p_i = (0 : 1 : 0)$. In this case we obtain a homogeneous polynomial q''' of degree $d-3$ vanishing at $Z \setminus \{p_i\}$ with coefficient 1 in front of the monomial X_1^{d-3}. Then the polynomials $X_0X_1^2q'''$ and $X_1^2X_2q'''$ vanish at Z, the former one having $X_0X_1^{d-1}$ but not $X_1^{d-1}X_2$, the latter one containing $X_1^{d-1}X_2$ but not $X_0X_1^{d-1}$. Similarly as in the previous cases one concludes again that the sheaves over $[\Phi] \in N_1$ which are singular at some point

p_i such that p_1 and p_i lie on a line constitute a linear projective subspace of codimension 2 in the fiber $\nu^{-1}([\Phi]) \cong \mathbb{P}_{3d-1}$. □

Corollary 5.5.15. *The fibers of M_0' over N_1 are unions of $l-1$ different linear subspaces of $\mathbb{P}_{3d-1}$ of codimension 2. In particular the fibers are singular at the intersection points.*

5.5.3 Main result

In order to state our main theorem, we need the following result about generic smoothness of a morphism of varieties.

Theorem 5.5.16. [[68], 25.3.3, p.673]
Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of varieties over $\mathbb{K}$ such that $\mathcal{X}$ is smooth. Then there is a dense open subset $U \subseteq \mathcal{Y}$ such that $f_{|f^{-1}(U)} : f^{-1}(U) \to U$ is a smooth morphism, i.e. the fibers $f^{-1}(y) \subseteq \mathcal{X}$ are smooth varieties for all $y \in U$.

Remark 5.5.17. The set $f^{-1}(U) \subseteq \mathcal{X}$ and the fibers $f^{-1}(y)$ may be empty. Hence if we want the statement to be non-trivial, we have to add the assumption that f is a surjective morphism[15].

Theorem 5.5.18 (Iena-Leytem)**.** *Let $d \geq 4$ and $M = M_{dm-1}$ be the Simpson moduli space of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm-1$. If we denote by $M' \subset M$ the closed subvariety of singular sheaves in M, then M' is singular and of codimension 2.*

Proof. By Proposition 5.3.30 we have the inclusion of open subvarieties

$$\mathbb{B}_0 \subseteq M_0 \subseteq M \; .$$

It is shown in [[70], 4.7, p.11] that the codimension of the complement of $\mathbb{B}_0$ is M is ≥ 2. Hence in order to show that $\operatorname{codim}_M M' = 2$, it suffices to show that $\operatorname{codim}_{\mathbb{B}_0}(\mathbb{B}_0 \cap M') = 2$, where $\mathbb{B}_0 \cap M' \subseteq M_0'$. We also have

$$N_c \subseteq N_0 \text{ open} \qquad \text{and} \qquad N_1 \subseteq N_0 \setminus N_c \text{ open} \; .$$

[15] or, more generally a dominant morphism.

Since the codimension of the complement of N_1 in $N_0 \setminus N_c$ is 1, the complement of $N_c \cup N_1$ in N_0 is of codimension 2. Therefore it suffices to show that

$$\operatorname{codim}_{\mathbb{B}_c \cup \mathbb{B}_1} \big(M' \cap (\mathbb{B}_c \cup \mathbb{B}_1) \big) = 2 \; .$$

But this is exactly what we did in Proposition 5.5.6 and Corollary 5.5.15: the codimension of the fibers of M'_0 over $N_c \cup N_1$ is equal to 2.
To show that M' is a singular subvariety of M, it suffices to show that M'_0 is singular. Assume that M'_0 is smooth. Then we restrict the morphism $\nu : \mathbb{B} \to N$ to $M'_0 \subset M_0 \subseteq \mathbb{B}$, i.e. we consider

$$\nu \; : \; M'_0 \longrightarrow N \; ,$$

which is still surjective. By Theorem 5.5.16 we thus obtain a dense open subset $U \subseteq N$ such that the fibers $\nu^{-1}([\Phi])$ are smooth for all $[\Phi] \in U$. But in Corollary 5.5.8 and Corollary 5.5.15 we showed that a generic fiber of M'_0 over N is singular. Hence M'_0 cannot be smooth. □

Remark 5.5.19. The proof of Theorem 5.5.18 does not take care of all sheaves in M'. For example we will illustrate in Remark 5.6.5 that every sheaf in $\mathbb{B}_0$ is singular at a non-curvilinear triple point. These however only appear in subvarieties of N_0 of codimension ≥ 2 as they lie in the complement of $N_c \cup N_1$. Hence sheaves which are singular at such a triple point (and thus belong to M') may only arise in a subvariety of codimension ≥ 2 in the complement of $\mathbb{B}_c \cup \mathbb{B}_1$. This does not affect the fact that the codimension of M' is 2.

Finally we also want to find the smooth points of M', i.e. to determine the smooth locus of M'.

Definition 5.5.20. Let $l = \binom{n}{2}$ and consider the product variety $\prod_1^l \mathbb{P}_2$, on which we have an action of the group of permutations $\mathcal{S}_l$. The *symmetric product* $S^l\mathbb{P}_2$ is defined as the quotient

$$\textstyle\prod_1^l \mathbb{P}_2 \,/\, \mathcal{S}_l$$

and can be shown to be a projective variety. It consists of tuples of l points in $\mathbb{P}_2$ with no order.

Lemma 5.5.21. *Let $H = \mathbb{P}_2^{[l]}$ be the Hilbert schemes of l points in $\mathbb{P}_2$. Then there is an assignment $H \to S^l\mathbb{P}_2$ which induces a 1-to-1 correspondence between $H_c \cong N_c$ and an open subvariety of $S^l\mathbb{P}_2$.*

Proof. Let $Z \in H$ and denote $Z = \{p_1, \ldots, p_k\}$ for some $k \leq l$. If we denote the multiplicity of each p_i by n_i, then $n_1 + \ldots + n_k = l$. We define the map

$$H \longrightarrow S^l\mathbb{P}_2 \ : \ Z = \{p_1, \ldots, p_k\} \longmapsto n_1 \cdot p_1 + \ldots + n_k \cdot p_k \ .$$

Restricting this map to H_c means that all points in Z are simple and different, i.e. $k = l$ and $n_i = 1, \forall\, i \in \{1, \ldots, l\}$. This gives a 1-to-1 correspondence between H_c and the subvariety of $S^l\mathbb{P}_2$ consisting of tuples with all l entries being different, which is open since this is achieved by removing all (closed) diagonals. □

Proposition 5.5.22. *The smooth locus of M' over N_c consists of sheaves corresponding to a 0-dimensional subscheme $Z \subseteq C$ such that only one of the points in Z is a singular point of C.*

Proof. By Lemma 5.5.21 we may identify N_c with an open subscheme of $S^l\mathbb{P}_2$. Moreover we can choose a local section[16] $S^l\mathbb{P}_2 \to \prod_1^l \mathbb{P}_2$. We compose this one with the projection $\prod_1^l \mathbb{P}_2 \to \mathbb{P}_2$ to the j^{th} factor. Hence for a given $Z_0 \in N_c$ we obtain an open neighborhood $U \subseteq N_c$ and l different local choices $p_j : U \to \mathbb{P}_2$ for $j = 1, \ldots, l$ of a point in $Z \in U$.[17]
By shrinking U if necessary we may assume that $\mathbb{B} \to N_c$ is a trivial bundle over U, i.e. $\mathbb{B}|_U$ is isomorphic to the product variety $U \times \mathbb{P}_{3d-1}$. Consider the subvariety $S_j \subseteq \mathbb{B}|_U$ of those sheaves given by $Z \subseteq C$ that are singular at the point $p_j(Z) \in \mathbb{P}_2$. By Proposition 5.5.6 there thus exists an open subset $V_j \subseteq U$ such that $S_j|_{V_j}$ is isomorphic to a product of V_j with a linear subspace of $\mathbb{P}_{3d-1}$ of codimension 2, i.e. $S_j|_{V_j} \cong V_j \times \mathbb{P}_{3d-3}$. Therefore S_j is smooth (as products are). Taking $V = V_1 \cap \ldots \cap V_l$ we also get

$$M' \cap \mathbb{B}|_V \ \cong \ S_1|_V \cup \ldots \cup S_l|_V \ .$$

[16]In the analytic case this is clear. For the Zariski topology this has to be made precise by means of the étale topology since such a section may not be a regular map.

[17]Roughly speaking we just construct a map $\{p_1, \ldots, p_l\} \mapsto p_j$ which locally chooses a point from Z.

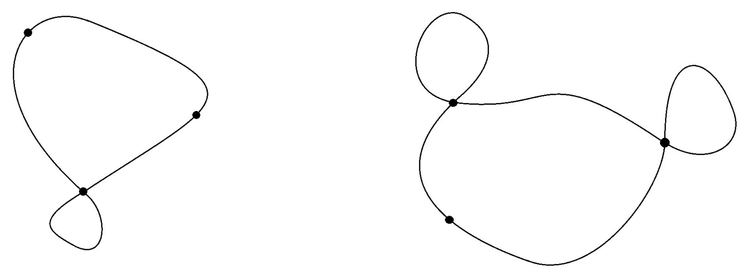
Hence the smooth points in M' over N_c are exactly those which are contained in

$$\left(\bigcup_j S_j\right) \setminus \left(\bigcup_{j\neq i}(S_j \cap S_i)\right) ,$$

i.e. sheaves that are singular at only one of the points in Z. $\square$

Example 5.5.23. Let $d = 4$ and $l = \binom{3}{2} = 3$. Consider two sheaves with singular support as given in Figure 5.2 below. The first one is a smooth point of M', the second one is not.

Figure 5.2: Supports of a smooth and of a singular point in M'



5.6 Examples, interpretations, open questions

Let us finally give a few examples and interpretations of the previous results.

5.6.1 Some computational examples

Example 5.6.1. In Proposition 5.2.14 we showed that the conditions for a Kronecker module $\Phi \in \mathbb{V}$ to be stable and to have linearly independent minors are equivalent for $n = 3$. Here we give an example to show that this is no longer true for $n > 3$. Consider e.g. $n = 4$ and

$$\Phi := \begin{pmatrix} X_0 & 0 & X_2 & -X_1 \\ X_1 & -X_2 & 0 & X_0 \\ X_2 & X_1 & -X_0 & 0 \end{pmatrix} .$$

Its first maximal minor is $X_2X_0X_1 - X_1X_2X_0 = 0$. But Φ is still semistable.

For this, we have to show that there are no matrices $g \in \mathrm{GL}_3(\mathbb{C})$ and $h \in \mathrm{GL}_4(\mathbb{C})$ such that

$$\underbrace{\begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix}}_{=g} \cdot \Phi \cdot \underbrace{\begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_5 & h_6 & h_7 & h_8 \\ h_9 & h_{10} & h_{11} & h_{12} \\ h_{13} & h_{14} & h_{15} & h_{16} \end{pmatrix}}_{=h}$$

is equal to

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{pmatrix} .$$

Computing the product and comparing the coefficients of the linear forms in X_0, X_1, X_2, we get the following conditions to obtain zero at position (i, j):

$$\begin{aligned}
(1,2) \ : \quad & g_1h_2 + g_2h_{14} - g_3h_{10} = 0 \quad , \quad & -g_1h_{14} + g_2h_2 + g_3h_6 = 0 \ , \\
& & g_1h_{10} - g_2h_6 + g_3h_2 = 0 \ . \\
(1,3) \ : \quad & g_1h_3 + g_2h_{15} - g_3h_{11} = 0 \quad , \quad & -g_1h_{15} + g_2h_3 + g_3h_7 = 0 \ , \\
& & g_1h_{11} - g_2h_7 + g_3h_3 = 0 \ . \\
(1,4) \ : \quad & g_1h_4 + g_2h_{16} - g_3h_{12} = 0 \quad , \quad & -g_1h_{16} + g_2h_4 + g_3h_8 = 0 \ , \\
& & g_1h_{12} - g_2h_8 + g_3h_4 = 0 \ . \\
(2,3) \ : \quad & g_4h_3 + g_5h_{15} - g_6h_{11} = 0 \quad , \quad & -g_4h_{15} + g_5h_3 + g_6h_7 = 0 \ , \\
& & g_4h_{11} - g_5h_7 + g_6h_3 = 0 \ . \\
(2,4) \ : \quad & g_4h_4 + g_5h_{16} - g_6h_{12} = 0 \quad , \quad & -g_4h_{16} + g_5h_4 + g_6h_8 = 0 \ , \\
& & g_4h_{12} - g_5h_8 + g_6h_4 = 0 \ . \\
(3,4) \ : \quad & g_7h_4 + g_8h_{16} - g_9h_{12} = 0 \quad , \quad & -g_7h_{16} + g_8h_4 + g_9h_8 = 0 \ , \\
& & g_7h_{12} - g_8h_8 + g_9h_4 = 0 \ .
\end{aligned}$$

1) For the first situation we consider $(1, 2)$, $(1, 3)$ and $(1, 4)$. Note that these equations only depend on g_1, g_2, g_3, which form the first row of g. So we can consider them as a linear system of equations in these variables. Rewriting the

equations in matrix form, we get the homogeneous linear system

$$\begin{pmatrix} h_2 & h_{14} & -h_{10} \\ h_3 & h_{15} & -h_{11} \\ h_4 & h_{16} & -h_{12} \\ -h_{14} & h_2 & h_6 \\ -h_{15} & h_3 & h_7 \\ -h_{16} & h_4 & h_8 \\ h_{10} & -h_6 & h_2 \\ h_{11} & -h_7 & h_3 \\ h_{12} & -h_8 & h_4 \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

As g is invertible we need a non-zero solution, so the 9×3–matrix must have rank at most 2. In particular, every 3×3–submatrix must have zero determinant. We see however that the three blocks of 3×3–matrices consist of maximal minors (up to sign) of the shortened matrix

$$h' := \begin{pmatrix} h_2 & h_3 & h_4 \\ h_6 & h_7 & h_8 \\ h_{10} & h_{11} & h_{12} \\ h_{14} & h_{15} & h_{16} \end{pmatrix} .$$

Hence the 2nd, 3rd and 4th maximal minor of h' are zero. Since h is invertible, the first maximal minor is not allowed to be zero as well, thus the 3 last rows of h' are linearly independent.

Next we show that at most one of h_2, h_3, h_4 can be zero (actually we only need that one of them is non-zero). Indeed assume e.g. that $h_2 = h_3 = 0$. Then we get the subsystem of equations

$$\begin{pmatrix} 0 & h_{14} & -h_{10} \\ 0 & h_{15} & -h_{11} \\ -h_{14} & 0 & h_6 \\ -h_{15} & 0 & h_7 \\ h_{10} & -h_6 & 0 \\ h_{11} & -h_7 & 0 \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

If any of

$$\det\begin{pmatrix} h_{14} & h_{10} \\ h_{15} & h_{11} \end{pmatrix} \quad , \quad \det\begin{pmatrix} h_{14} & h_6 \\ h_{15} & h_7 \end{pmatrix} \quad , \quad \begin{pmatrix} h_{10} & h_6 \\ h_{11} & h_7 \end{pmatrix} \tag{5.40}$$

is non-zero, then two of g_1, g_2, g_3 will be zero. To obtain a non-zero value for the third one, say g_i, we need that the i^{th} column is completely zero, which is impossible as it would imply that 2 of the last 3 rows in h' are linearly dependent and hence that its first maximal minor would be zero. So all determinants in (5.40) must be zero. But also this is impossible since then again the first maximal minor would be zero.

By permuting rows if necessary, we may thus assume that $h_2 \neq 0$. Since the fourth maximal minor of h' is zero, we know that its first 3 rows are linear dependent. Since however the first maximal minor is non-zero, the second and third row are linearly independent, so we get

$$(h_2, h_3, h_4) = \lambda \cdot (h_6, h_7, h_8) + \mu \cdot (h_{10}, h_{11}, h_{12})$$

for some $(\lambda, \mu) \neq (0, 0)$ since $h_2 \neq 0$. If $\lambda \neq 0$, then

$$0 \neq \det\begin{pmatrix} h_6 & h_7 & h_8 \\ h_{10} & h_{11} & h_{12} \\ h_{14} & h_{15} & h_{16} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{\lambda}h_2 - \frac{\mu}{\lambda}h_{10} & \frac{1}{\lambda}h_3 - \frac{\mu}{\lambda}h_{11} & \frac{1}{\lambda}h_4 - \frac{\mu}{\lambda}h_{12} \\ h_{10} & h_{11} & h_{12} \\ h_{14} & h_{15} & h_{16} \end{pmatrix} = 0 \; ,$$

and similarly if $\mu \neq 0$. This contradiction finally shows that 1) cannot happen.

2) Now consider the second case with $(1, 3)$, $(1, 4)$, $(2, 3)$ and $(2, 4)$. 6 of the equations only depend on g_1, g_2, g_3, while the other 6 only depend on g_4, g_5, g_6 and satisfy exactly the same conditions:

$$\begin{pmatrix} h_3 & h_{15} & -h_{11} \\ -h_{15} & h_3 & h_7 \\ h_{11} & -h_7 & h_3 \\ h_4 & h_{16} & -h_{12} \\ -h_{16} & h_4 & h_8 \\ h_{12} & -h_8 & h_4 \end{pmatrix} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} h_3 & h_{15} & -h_{11} \\ -h_{15} & h_3 & h_7 \\ h_{11} & -h_7 & h_3 \\ h_4 & h_{16} & -h_{12} \\ -h_{16} & h_4 & h_8 \\ h_{12} & -h_8 & h_4 \end{pmatrix} \cdot \begin{pmatrix} g_4 \\ g_5 \\ g_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; .$$

Since g is invertible, we thus need 2 linearly independent solutions of this homogeneous linear system. This is only possible if the 6×3–matrix has rank at most 1. Hence there exist $\alpha, \beta, \gamma \in \mathbb{K}$ such that each row is a scalar multiple of (α, β, γ). They are not all zero, otherwise $h_3 = h_7 = h_{11} = h_{15} = 0$ and h would not be invertible. In particular $\exists\, \lambda, \mu, \nu \in \mathbb{K}$ such that

$$(h_3, h_{15}, -h_{11}) = \lambda \cdot (\alpha, \beta, \gamma) \qquad , \qquad (-h_{15}, h_3, h_7) = \mu \cdot (\alpha, \beta, \gamma) \; ,$$
$$(h_{11}, -h_7, h_3) = \nu \cdot (\alpha, \beta, \gamma) \; .$$

From $h_3 = \lambda\alpha = \mu\beta$ and $h_{15} = \lambda\beta = -\mu\alpha$, we conclude that h_3 and h_{15} are either simultaneously zero or simultaneously non-zero. From $h_3 = \lambda\alpha = \nu\gamma$, $h_{11} = -\lambda\gamma = \nu\alpha$ and $h_3 = \mu\beta = \nu\gamma$, $h_7 = \mu\gamma = -\nu\beta$. we conclude that they are all non-zero since h is invertible. Comparing some more rows, we find

$$(h_4, h_{16}, -h_{12}) = c \cdot (h_3, h_{15}, -h_{11}) \quad \text{and} \quad (-h_{16}, h_4, h_8) = c' \cdot (-h_{15}, h_3, h_7)$$

for some $c, c' \in \mathbb{K}$. Since $h_3 \neq 0$, we get $c = c'$, hence

$$(h_4, h_8, h_{12}, h_{16}) = c \cdot (h_3, h_7, h_{11}, h_{16}) \; .$$

But this implies that $\det h = 0$, so 2) is impossible as well.

3) Finally we look at $(1, 4)$, $(2, 4)$ and $(3, 4)$. There we obtain the homogeneous linear system

$$\begin{pmatrix} h_4 & h_{16} & -h_{12} \\ -h_{16} & h_4 & h_8 \\ h_{12} & -h_8 & h_4 \end{pmatrix} \cdot \begin{pmatrix} g_{1+3i} \\ g_{2+3i} \\ g_{3+3i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for $i = 0, 1, 2$. In order for g to be invertible we need 3 linearly independent solutions, the matrix of type 3×3 must have rank 0, i.e. all entries must vanish, which is impossible as it consists of all the entries of the last column of h.

Finally we conclude that $\Phi \in \mathbb{V}$ is a semistable Kronecker module, but its maximal minors are linearly dependent (as one of them is zero).

Example 5.6.2. We have seen in Example 5.2.27 that the Kronecker modules

$$\Phi = \begin{pmatrix} X_1 & 0 & -X_2 & 0 \\ 0 & X_1 & 0 & -X_2 \\ 0 & 0 & -X_1 & X_0 \end{pmatrix} \qquad \text{and} \qquad \Phi' = \begin{pmatrix} 0 & X_1 & 0 & -X_2 \\ -X_1 & X_0 & 0 & 0 \\ 0 & 0 & -X_1 & X_0 \end{pmatrix}$$

have the same maximal minors

$$d_1 = X_0X_1X_2 \quad , \quad d_2 = X_1^2X_2 \quad , \quad d_3 = X_0X_1^2 \quad , \quad d_4 = X_1^3 \; ,$$

but do not lie in the same G-orbit. We are interested in a geometric description of Φ and Φ'. First note that permutations of rows and columns give

$$\Phi' \sim \begin{pmatrix} 0 & 0 & X_1 & -X_2 \\ -X_1 & 0 & X_0 & 0 \\ 0 & -X_1 & 0 & X_0 \end{pmatrix} \sim \begin{pmatrix} -X_1 & 0 & X_0 & 0 \\ 0 & -X_1 & 0 & X_0 \\ 0 & 0 & X_1 & -X_2 \end{pmatrix}$$
$$\sim \begin{pmatrix} X_1 & 0 & -X_0 & 0 \\ 0 & X_1 & 0 & -X_0 \\ 0 & 0 & -X_1 & X_2 \end{pmatrix} .$$

Hence Φ and Φ' only differ by a coordinate change (interchange X_0 and X_2) and it suffices to study only one of them. The situation of the other one is then obtained by a geometric reflection.

Let $g = \gcd(d_1, d_2, d_3, d_4) = X_1$. As the maximal minors of Φ are not coprime, a sequence as in (5.18) involving Φ cannot be made exact (see Remark 5.2.31), e.g. we also have

$$\begin{pmatrix} X_1 & -X_0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_0X_2 \\ X_1X_2 \\ X_0X_1 \\ X_1^2 \end{pmatrix} = 0 \, .$$

So let us consider the dual situation as in Proposition 5.2.30. With $n = 4$ and $\deg g = 1$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\psi} 4\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{{}^t\Phi} 3\,\mathcal{O}_{\mathbb{P}_2} \; ,$$

where $\psi = \big(X_0X_2 \, , \, X_1X_2 \, , \, X_0X_1 \, , \, X_1^2\big)$. Let us also consider the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\cdot X_1} \mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \xrightarrow{\psi'} 3\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{\Psi} 2\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \operatorname{coker}\Psi \longrightarrow 0 \; ,$$

where

$$\psi' = \big(X_0X_2\,,\, X_0X_1\,,\, X_1^2\big) \qquad \text{and} \qquad \Psi = \begin{pmatrix} X_1 & 0 \\ -X_2 & -X_1 \\ 0 & X_0 \end{pmatrix} .$$

By Example 4.3.1 coker Ψ has Hilbert polynomial

$$2 \cdot \frac{(m+2)(m+1)}{2} - 3 \cdot \frac{m(m+1)}{2} + \frac{(m-1)(m-2)}{2} = 3\,,$$

so we can write coker $\Psi = \mathcal{O}_Z$, where $Z \subset \mathbb{P}_2$ consists of finitely many points. Proposition 4.5.9 implies that Z is given by the common vanishing set of the maximal minors of Ψ, i.e.

$$Z = Z(X_0X_2, X_0X_1, X_1^2) = \big\{\, p_1 = (1:0:0)\,,\; p_2 = (0:0:1) \,\big\}\,.$$

But $P_{\mathcal{O}_Z}(m) = 3$, so one of these points must be a double point. Indeed,

$$X_0 \neq 0 \;\;\Rightarrow\;\; \begin{pmatrix} X_1 & 0 \\ -X_2 & -X_1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} X_1 & 0 \\ X_2 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

$$X_2 \neq 0 \;\;\Rightarrow\;\; \begin{pmatrix} X_1 & 0 \\ -1 & -X_1 \\ 0 & X_0 \end{pmatrix} \sim \begin{pmatrix} 0 & -X_1^2 \\ 1 & X_1 \\ 0 & X_0 \end{pmatrix} \sim \begin{pmatrix} 0 & X_1^2 \\ 1 & 0 \\ 0 & X_0 \end{pmatrix} \sim \begin{pmatrix} X_0 & 0 \\ X_1^2 & 0 \\ 0 & 1 \end{pmatrix} .$$

Hence p_2 is a double point and we have $Z = \{p_1, dp_2\}$ with $\mathcal{O}_Z = \mathcal{O}_{p_1} \oplus \mathcal{O}_{dp_2}$. The above exact sequences can now be put into the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{\cdot X_1} & \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{O}_{L_1} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) & \xrightarrow{\psi} & 4\,\mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{{}^t\Phi} & 3\,\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-3) & \xrightarrow{\psi'} & 3\,\mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{\Psi} & 2\,\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}$$

with

$$i_1 = \big(0,1,0,0\big) \quad , \quad i_2 = \big(0,1,0\big) \quad , \quad j_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad j_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Due to exactness of the rows and the first two columns the 9-Lemma gives the exact sequence

$$0 \longrightarrow \mathcal{O}_{L_1} \longrightarrow \mathcal{F} \longrightarrow O_Z \longrightarrow 0 \,,$$

i.e. $\mathcal{F} = \operatorname{coker}({}^t\Phi)$ is an extension of the structure sheaf of a line and the structure sheaf of a point and a double point. $\mathcal{F}$ is supported on the common vanishing set of d_1, d_2, d_3, d_4. Topologically this is just the line L_1, but we have again multiplicities at p_1 and p_2. Indeed

$$\begin{aligned} X_0 \neq 0 \quad &\Rightarrow \quad \langle\, d_1, d_2, d_3, d_4 \,\rangle = \langle\, X_1X_2, X_1^2X_2, X_1^2, X_1^3 \,\rangle = \langle\, X_1X_2, X_1^2 \,\rangle \,, \\ X_2 \neq 0 \quad &\Rightarrow \quad \langle\, d_1, d_2, d_3, d_4 \,\rangle = \langle\, X_0X_1, X_1^2, X_1^2X_0, X_1^3 \,\rangle = \langle\, X_0X_1, X_1^2 \,\rangle \,, \end{aligned}$$

so that the support of $\mathcal{F}$ consists of the line L_1 with double points at p_1 and p_2. Now we want to see how $\mathcal{F}$ looks like in a neighborhood of the p_i. For $p_1 = (1:0:0)$ and $X_0 \neq 0$ we have

$$\begin{aligned} {}^t\Phi \sim \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_1 & 0 \\ -X_2 & 0 & -X_1 \\ 0 & -X_2 & 1 \end{pmatrix} &\sim \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_1 & 0 \\ -X_2 & -X_1X_2 & 0 \\ 0 & -X_2 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_1 & 0 \\ -X_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

Let $R = \mathbb{K}[X_1, X_2]$; this means that $\mathcal{F}$ can locally around p_1 be described by the R-module

$$\mathbb{K}[X_1, X_2]/\langle\, X_1, X_2 \,\rangle \oplus \mathbb{K}[X_1, X_2]/\langle\, X_1 \,\rangle \;\cong\; \mathbb{K} \oplus \mathbb{K}[X_2]\,;$$

which corresponds to the direct sum of a simple point and a line. On the other hand, $p_2 = (0:0:1)$ with $X_2 \neq 0$ gives

$$
{}^t\Phi \sim \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_1 & 0 \\ -1 & 0 & -X_1 \\ 0 & -1 & X_0 \end{pmatrix} \sim \begin{pmatrix} X_1 & 0 & 0 \\ 0 & 0 & X_0X_1 \\ 1 & 0 & X_1 \\ 0 & 1 & -X_0 \end{pmatrix}
$$
$$
\sim \begin{pmatrix} X_1 & 0 & -X_1^2 \\ 0 & 0 & X_0X_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} X_0X_1 & 0 & 0 \\ X_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,
$$

so one obtains the module $\mathbb{K}[X_0, X_1]/\langle X_0X_1, X_1^2 \rangle$, which describes a line with an embedded double point. Hence even though p_1 and p_2 are both double points in the Fitting support of $\mathcal{F}$, we see that their natures are different.

Example 5.6.3. Proposition 5.5.4 does not hold true over $N_0 \setminus N_c$, i.e. if Z contains points of multiplicity > 1. Let for example $n = 3$ and $d = 4$ with

$$
A = \begin{pmatrix} X_2^2 & 0 & X_1^2 \\ X_0 & X_1 & 0 \\ 0 & X_0 & X_2 \end{pmatrix} \qquad \Rightarrow \qquad \Phi = \begin{pmatrix} X_0 & X_1 & 0 \\ 0 & X_0 & X_2 \end{pmatrix}
$$

The maximal minors of Φ are $d_1 = X_1X_2$, $d_2 = -X_0X_2$ and $d_3 = X_0^2$. These are coprime, hence $\Phi \in \mathbb{V}_0$. Moreover $\det A = X_0^2X_1^2 + X_1X_2^3$ is non-zero. By Proposition 5.3.31 the sheaf $[\mathcal{F}] \in \mathbb{B}_0$ corresponding to the matrix $A \in \mathbb{W}_0$ is thus given as an ideal sheaf

$$
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{O}_Z \longrightarrow 0 ,
$$

where $C = Z(\det A)$ is the support of $\mathcal{F}$ and $Z \subset \mathbb{P}_2$ consists of $\binom{3}{2} = 3$ points not lying on a line. By Proposition 4.5.9 the latter is given by

$$
Z = Z(d_1, d_2, d_3) = \left\{ p_1 = (0:1:0) \ , \ p_2 = (0:0:1) \right\} ,
$$

where p_1 is a double point since

$$
\begin{aligned}
X_1 \neq 0 \quad &\Rightarrow \quad \langle d_1, d_2, d_3 \rangle = \langle X_2, X_0X_2, X_0^2 \rangle = \langle X_0^2, X_2 \rangle , \qquad (5.41) \\
X_2 \neq 0 \quad &\Rightarrow \quad \langle d_1, d_2, d_3 \rangle = \langle X_1, X_0, X_0^2 \rangle = \langle X_0, X_1 \rangle .
\end{aligned}
$$

In particular the 2 points do not lie on a line. So we have $Z = \{dp_1, p_2\}$ and $\mathcal{O}_Z = \mathcal{O}_{dp_1} \oplus \mathcal{O}_{p_2}$. This is not a configuration, so $[\mathcal{F}] \in \mathbb{B}_0 \setminus \mathbb{B}_c$. On the other we see that $[\mathcal{F}] \in \mathbb{B}_1$ since there is just 1 double point. Note that C is a reducible curve because $\det A = X_1(X_0^2 X_1 + X_2^3)$. We also have

$$\frac{\partial(\det A)}{\partial X_0} = 2X_0X_1^2 \quad , \quad \frac{\partial(\det A)}{\partial X_1} = 2X_0^2X_1 + X_2^3 \, , \quad \frac{\partial(\det A)}{\partial X_2} = 3X_1X_2^2 \, ,$$

which allows to see that $p_1 \in Z$ is a singular point of C. However $\mathcal{F}$ is a non-singular sheaf. For this is suffices to check freeness of the stalks $\mathcal{F}_{p_1}$ and $\mathcal{F}_{p_2}$. Let $\{U_0, U_1, U_2\}$ be the standard open covering of $\mathbb{P}_2$. For $p_1 = (0:1:0) \in U_1$ we find

$$A|_{U_1} \sim \begin{pmatrix} X_2^2 & 0 & 1 \\ X_0 & 1 & 0 \\ 0 & X_0 & X_2 \end{pmatrix} \sim \begin{pmatrix} X_2^2 & 0 & 1 \\ X_0 & 1 & 0 \\ -X_0^2 - X_2^3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} X_0^2 + X_2^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

where $X_0^2 + X_2^3$ is equal to $\det A$ with $X_1 = 1$. Thus $\mathcal{F}_{p_1} \cong \mathcal{O}_{C,p_1}$.
For $p_2 = (0:0:1) \in U_2$ one gets

$$\begin{aligned} A|_{U_2} &\sim \begin{pmatrix} 1 & 0 & X_1^2 \\ X_0 & X_1 & 0 \\ 0 & X_0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ X_0 & X_1 & -X_0X_1^2 \\ 0 & X_0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & X_1 & -X_0X_1^2 \\ 0 & X_0 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & X_1 + X_0^2X_1^2 & -X_0X_1^2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} X_1 + X_0^2X_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

We obtain again $\mathcal{F}_{p_2} \cong \mathcal{O}_{C,p_2}$ since $X_1 + X_0^2X_1^2$ is equal to $\det A$ with $X_2 = 1$. Together with $\mathcal{F}|_{U_0} = 0$ (as $C \cap U_0 = \emptyset$) this finally gives $\mathcal{F} \cong \mathcal{O}_C$. We conclude that $\mathcal{F}$ is a non-singular sheaf even though $p_1 \in Z \cap \mathrm{Sing}(C)$.

Remark 5.6.4. On the other hand let us check that Proposition 5.4.11 is indeed satisfied. As p_1 is a double point we have to look at the affine chart U_1, in which the coordinates of p_1 become $(0,0)$. In this case we know from (5.41) that the double point is defined by $\langle X_0^2, X_2 \rangle$ and the curve $C|_{U_1}$ is given by the polynomial

$f = X_0^2 + X_2^3$ (so it defines a cusp at p_1). We also see that $\langle f \rangle \subset \langle X_0^2, X_2 \rangle$. Now the criterion of Proposition 5.4.11 is satisfied since p_1 is a singular point of C and f contains the monomial X_0^2, i.e. A defines a non-singular sheaf.

5.6.2 Interpretations of the Simpson moduli spaces

To close the thesis let us discuss several geometric and physical interpretations of $M = M_{dm-1}(\mathbb{P}_2)$.

Geometry of M

Consider the morphism

$$\sigma \;:\; M \longrightarrow \mathcal{C}_d(\mathbb{P}_2) \;:\; [\mathcal{F}] \longmapsto \mathcal{Z}_f(\mathcal{F})$$

that sends the isomorphism class of a sheaf to its Fitting support. We want to look at its fibers. Fix a curve $C \in \mathcal{C}_d(\mathbb{P}_2)$. First assume that C is smooth. Then the fiber $\sigma^{-1}(C)$ consists of (isomorphism classes of) stable line bundles on the non-singular curve C. Hence all such fibers are Jacobians over smooth curves[18]. Over singular curves the fibers can thus be seen as compactified Jacobians ; indeed the fibers are closed sets in the compact space M, hence compact as well. Sheaves that are line bundles on their support constitute an open subvariety $M_B \subset M$ (it may be bigger than the union of all fibers over smooth curves as sheaves on singular curves may also be non-singular). Then the closed subvariety of sheaves that are not locally free on their support is equal to the boundary $M' = M \setminus M_B$. In general it is non-empty, so one can consider M as a compactification of M_B. In other words, M' “measures the failure” of M_B to be a moduli space. The codimension of M' then gives information about the glueing of this compactification. We have shown that $\operatorname{codim}_M M' = 2$; the unsatisfactory aspect of this fact is that this is not the minimal codimension. So we are losing information as we glue together too many directions at infinity.[19]

[18] The Jacobian of a non-singular curve C is the moduli space of degree 0 line bundles on C. It can also be described as the connected component of the identity in the Picard group of C.

[19] Consider for example $\mathbb{C}^2$. A 1-point compactification (codimension 2) glues together all points at infinity to give a sphere. But if we glue along a line (codimension 1) we obtain the projective line, which already contains more information about “how to approach infinity”.

A possible way out of this problem is to consider the blow-up of M along M', which can be seen as a modification of the boundary M' by vector bundles. This process is described more precisely in [41] for $3m+1$ and more generally in [40] for $dm-1$.

Applications in physics

The geometry of M is also studied in other research fields, such as curve counting theory, strange duality and birational geometry.
1) The virtual curve counting theory focuses on computing certain BPS-invariants in terms of topological invariants of M, such as Betti numbers, Poincaré polynomials and the top Chern class of its cotangent bundle. The study of M and its cohomology ring provides explicit and computable examples of this theory.
2) In the case of strange duality, one is interested in comparing the space of theta divisors in two different moduli spaces of sheaves. By the Grothendieck-Riemann-Roch Theorem, the cohomology ring and the Chern classes of M provide some numerical data for this duality.
3) Birational geometry finally studies a connection of M with certain moduli spaces of objects in the derived category of coherent sheaves.
More precisely information about these 3 approaches can e.g. be found in [8].

5.6.3 Open questions

For the future we are still interested in answering the following questions:

1) Is M' irreducible and / or connected ?

2) Can we find a characterization of free ideals of fat non-curvilinear points ?

For the second one, the setting is as follows:
Let $C = Z(f)$ be a planar curve defined by a polynomial f, $p = (0,0)$ and $\mathcal{J} \subset \mathcal{O}_{\mathbb{A}_2}$ be a fat point given by an ideal $J \trianglelefteq \mathbb{K}[X,Y]$. Let also $\mathcal{I} \subset \mathcal{O}_C$ be the corresponding ideal sheaf of the fat point in C. Assume that the fat point at p is non-curvilinear, i.e. $\dim_{\mathbb{K}}(\mathbb{K}[X,Y]/J) = n \geq 3$ but J cannot be written of the form $\langle X - h(Y), Y^n \rangle$ for some $h \in \mathbb{K}[Y]$ with $\deg h < n$ and $h(0) = 0$.

Let $R' = \mathcal{O}_{\mathbb{A}_2,p} = \mathbb{K}[X,Y]_{\langle X,Y \rangle}$ and $R = \mathcal{O}_{C,p}$.
We want to know under which conditions it is possible that $I = \mathcal{I}_p$ can be a free R-module (in which case it is generated by 1 element). Equivalently, when do we have $\mathcal{J}_p = \langle \xi, f_p \rangle$ for some $\xi \in R'$?

Remark 5.6.5. If the minimal number of generators of J is ≥ 3, then it is clearly not possible. So e.g. for the triple point given by $J = \langle X^2, XY, Y^2 \rangle$, the ideal $\mathcal{I}_p$ can never be free. Therefore the question only makes sense if J can be generated by 2 elements.

Appendices

Appendix A

Basic facts on localization

In this appendix we want to recall some facts about localization of rings and modules. The results can be found in almost every textbook on Commutative Algebra. Our main references here are Atiyah-MacDonald [2], Gathmann [26] and [11], Section 10.9. At some places we also added more technical details to get some explicit formulas which are useful in certain computations.

A.1 Definition and first properties

Definition A.1.1. A subset $S \subseteq R$ is called *multiplicatively closed* if $1 \in S$ and $a \cdot b \in S$ for all $a, b \in S$. For such a multiplicatively closed subset S we define $S^{-1}R := (R \times S)/\sim$ as the set of equivalence classes $\frac{r}{s}$ where $r \in R$, $s \in S$ with $\frac{r}{s} = \frac{a}{b} \Leftrightarrow \exists\, t \in S$ such that $t \cdot (rb - as) = 0$. This is a ring with respect to the operations

$$\frac{r}{s} + \frac{a}{b} = \frac{rb + as}{sb} \qquad \text{and} \qquad \frac{r}{s} \cdot \frac{a}{b} = \frac{ra}{sb}\ ,$$

where $\frac{0}{1}$ and $\frac{1}{1}$ are the neutral elements for addition and multiplication.
$S^{-1}R$ is called the *localization of R with respect to S*. In the following we always assume that $0 \notin S$, otherwise $S^{-1}R$ is the zero ring as all fractions would be equal to $\frac{0}{1}$ by choosing $t = 0$. In particular S does not contain nilpotent elements.
We also define the ring homomorphism $i_S : R \to S^{-1}R : r \mapsto \frac{r}{1}$, which is in general neither injective, nor surjective if $S \neq \{1\}$. But it has the following properties.

Lemma A.1.2. [[2], p.37] and [[6], II.§2.n°1, p.77]
1) Elements in $i_S(S)$ are units in $S^{-1}R$ (but in general there may be more).
2) i_S is injective if and only if S does not contain zero-divisors of R.
3) i_S is bijective if and only if $S \subseteq R^\times$, i.e. every $s \in S$ is a unit in R.

Proof. 1) If $s \in S$, then $\frac{s}{1}$ is invertible in $S^{-1}R$ with inverse $\frac{1}{s}$: $i_S(s) \in (S^{-1}R)^\times$.
2)

$$\ker i_S = \left\{ r \in R \;\middle|\; \tfrac{r}{1} = \tfrac{0}{1} \right\} = \left\{ r \in R \;\middle|\; \exists\, s \in S \text{ such that } s \cdot r = 0 \right\} .$$

This is $\{0\}$ if S does not contain zero-divisors. Vice-versa, if $\exists\, s \in S$ which is a zero-divisor, then $\exists\, r \in R$, $r \neq 0$ such that $s \cdot r = 0$ and $i_S(r) = \frac{r}{1} = \frac{0}{1}$, so i_S is not injective.
3) $\underline{\Rightarrow}$: If i_S is bijective, then the inverse of $s \in S$ is given by $i_S^{-1}(\frac{1}{s})$ because

$$s \cdot i_S^{-1}\big(\tfrac{1}{s}\big) = i_S^{-1}\big(i_S(s)\big) \cdot i_S^{-1}\big(\tfrac{1}{s}\big) = i_S^{-1}\big(i_S(s) \cdot \tfrac{1}{s}\big) = i_S^{-1}\big(\tfrac{s}{1} \cdot \tfrac{1}{s}\big) = i_S^{-1}\big(\tfrac{1}{1}\big) = 1 .$$

$\underline{\Leftarrow}$: If all elements in S are invertible, then S does not contain zero-divisors, so i_S is already injective. Surjectivity follows from $\frac{r}{s} = \frac{rs^{-1}}{1} = i_S(rs^{-1})$ since $r \cdot 1 - rs^{-1} \cdot s = 0$. □

Definition A.1.3. Fundamental examples of multiplicatively closed subsets are $S = R \setminus P$ for some prime ideal $P \trianglelefteq R$ or $S = \{ r^n \,|\, n \in \mathbb{N}_0 \}$ for some $r \in R$ which is not nilpotent. We denote the corresponding localizations by R_P, called the *localization of R at P*, respectively R_r, which we call the *localized ring at r*. It turns out that R_P is a local ring with unique maximal ideal given by

$$P_P = \left\{ \tfrac{r}{s} \;\middle|\; r \in P,\, s \notin P \right\} = R_P \setminus (R_P)^\times .$$

If R is an integral domain, then $P = \{0\}$ is a prime ideal and $R_{\{0\}} = \mathrm{Quot}(R)$ is a field since all non-zero elements are invertible. It is called the *quotient field* (or *fraction field*) of R.

Proposition A.1.4. [[2], 3.1, p.37] , [[6], II.§2.n°1.Prop.1, p.75], [[11], 10.9.3]
Let $S \subset R$ be a multiplicatively closed subset and consider the covariant functor

$$\mathcal{F} \;:\; \texttt{Ring} \longrightarrow \texttt{Set} \;:\; T \longmapsto \left\{ \varphi : R \to T \text{ ring homomorphism} \;\middle|\; \varphi(S) \subseteq T^\times \right\} ,$$

where `Ring` *denotes the category of commutative unital rings. The functor* $\mathcal{F}$ *is representable by* $S^{-1}R$*, i.e.*

$$\mathcal{F}(T) \cong \mathrm{Hom}_{\mathtt{Ring}}(S^{-1}R, T) \ , \tag{A.1}$$

functorially with respect to T*. In other words,* $S^{-1}R$ *satisfies the following universal property: For any ring homomorphism* $\varphi : R \to T$ *such that* φ *maps* S *to units in* T*, there exists a unique ring homomorphism* $\phi : S^{-1}R \to T$ *such that* $\phi \circ i_S = \varphi$*.*

$$\begin{array}{ccc} S^{-1}R & \overset{\exists!\,\phi}{\dashrightarrow} & T \\ {\scriptstyle i_S}\uparrow & \nearrow_{\varphi} & \\ S \subseteq R & & \end{array}$$

Proof. $\mathcal{F}$ is covariant since a ring homomorphism $f : T \to T'$ induces a morphism by composition

$$f\circ \ : \ \mathcal{F}(T) \longrightarrow \mathcal{F}(T') \ : \ g \longmapsto f \circ g \ ,$$

where ring homomorphisms map units to units, so $f\big(g(S)\big) \subseteq f(T^\times) \subseteq (T')^\times$. In order to prove formula (A.1) first note that $i_S \in \mathcal{F}(S^{-1}R)$ by Lemma A.1.2. A ring homomorphism $\phi : S^{-1}R \to T$ defines an element in $\mathcal{F}(T)$ via

$$i_S \in \mathcal{F}(S^{-1}R) \xrightarrow{\phi\circ} \mathcal{F}(T) \ .$$

Now let $\varphi \in \mathcal{F}(T)$ be any ring homomorphism that maps elements from S to units in T. Since we want that $\phi \circ i_S = \varphi$, the only possible definition of ϕ is $\varphi(r) = \phi\big(i_S(r)\big) = \phi\big(\frac{r}{1}\big)$ for $r \in R$ with the restriction

$$1 = \phi(1) = \phi\big(\tfrac{s}{s}\big) = \phi\big(\tfrac{s}{1}\big) \cdot \phi\big(\tfrac{1}{s}\big) = \varphi(s) \cdot \phi\big(\tfrac{1}{s}\big)$$

for some $s \in S$, so $\varphi(s)$ is invertible and

$$\phi\big(\tfrac{r}{s}\big) = \phi\big(\tfrac{r}{1}\big) \cdot \phi\big(\tfrac{1}{s}\big) := \varphi(r) \cdot \varphi(s)^{-1} \ , \quad \forall\, \tfrac{r}{s} \in S^{-1}R \ . \tag{A.2}$$

This is also well-defined since for $\frac{r}{s} = \frac{a}{b}$ with $t \cdot (rb - as) = 0$ for some $t \in S$, we get

$$\varphi(t) \cdot \big(\varphi(r) \cdot \varphi(b) - \varphi(a) \cdot \varphi(s)\big) = 0 \ ,$$

where $\varphi(t)$ is a unit, hence multiplying by its inverse gives

$$\varphi(r) \cdot \varphi(s)^{-1} = \varphi(a) \cdot \varphi(b)^{-1} \ .$$

It remains to show that both constructions are inverse to each other. Given a morphism $\varphi : R \to T$ one constructs ϕ as in (A.2) and then

$$(\phi \circ i_S)(r) = \phi\big(\tfrac{r}{1}\big) = \varphi(r) \ ,$$

so we recover φ. Vice-versa, given $\phi : S^{-1}R \to T$ one sets $\varphi = \phi \circ i_S$ and definition (A.2) implies

$$\varphi(r) \cdot \varphi(s)^{-1} = \phi\big(\tfrac{r}{1}\big) \cdot \phi\big(\tfrac{s}{1}\big)^{-1} = \phi\big(\tfrac{r}{1}\big) \cdot \phi\big(\tfrac{1}{s}\big) = \phi\big(\tfrac{r}{s}\big) \ ,$$

which gives again the initial ϕ. Functoriality in T finally follows from the fact that the functors $\mathcal{F}$ and $\mathrm{Hom}(S^{-1}R, \,\cdot\,)$ are both functorially covariant via $f\circ$. □

A.2 Localization of modules

Definition A.2.1. Similarly as for rings one can also consider localizations of modules. If M is an R-module and $S \subset R$ a multiplicatively closed subset, we define $S^{-1}M := (M \times S)/\sim$ as the set of fractions $\frac{m}{s}$ where $m \in M$, $s \in S$ and $\frac{m}{s} = \frac{n}{u} \Leftrightarrow \exists\, t \in S$ such that $t * (u * m - s * n) = 0$, i.e. if and only if $ut * m = st * n$. This is a module over $S^{-1}R$ with respect to the operations

$$\frac{m}{s} + \frac{n}{u} = \frac{u * m + s * n}{s \cdot u} \qquad \text{and} \qquad \frac{r}{s} * \frac{m}{a} = \frac{r * m}{s \cdot a} \ .$$

Along the same definition as R_P and R_r, we also have the localizations M_P and M_r for $r \in R$ and a prime ideal $P \trianglelefteq R$. The goal of the next sections is to study some properties of the assignment $S^{-1} : M \mapsto S^{-1}M$.

A.2.1 Functoriality and exactness

Lemma A.2.2. [[2], 3.5, p.39] and [[26], 6.19, p.57]
For any multiplicatively closed subset $S \subset R$ and any R-module M, there is an isomorphism of $S^{-1}R$–modules

$$S^{-1}M \;\cong\; M \otimes_R S^{-1}R \ . \tag{A.3}$$

The isomorphism[1] *is given by $\varphi : \frac{m}{s} \mapsto m \otimes \frac{1}{s}$ with inverse $\psi : m \otimes \frac{r}{s} \mapsto \frac{r*m}{s}$.*

[1] In Bourbaki [6] (A.3) is actually taken as the definition of $S^{-1}M$.

Proof. φ is well-defined: if $\frac{m}{s} = \frac{n}{u}$, then $\exists\, t \in S$ such that $ut * m = st * n$ and

$$m \otimes \tfrac{1}{s} = m \otimes \tfrac{ut}{uts} = (ut * m) \otimes \tfrac{1}{uts} = (st * n) \otimes \tfrac{1}{uts} = n \otimes \tfrac{st}{uts} = n \otimes \tfrac{1}{u} \ .$$

The assignment $(m, \frac{r}{s}) \mapsto \frac{r*m}{s}$ is bilinear, so ψ is well-defined as well. It remains to check that

$$\psi\big(\varphi(\tfrac{m}{s})\big) = \psi\big(m \otimes \tfrac{1}{s}\big) = \tfrac{1*m}{s} = \tfrac{m}{s} \ ,$$
$$\varphi\big(\psi(m \otimes \tfrac{r}{s})\big) = \varphi\big(\tfrac{r*m}{s}\big) = (r * m) \otimes \tfrac{1}{s} = m \otimes \tfrac{r}{s} \ ,$$

hence $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identity maps. □

Lemma A.2.3. [[2], p.38] , [[26], 6.20, p.57] and [[61], p.38]
The assignment $S^{-1} : \mathrm{Mod}(R) \to \mathrm{Mod}(S^{-1}R) : M \mapsto S^{-1}M$ *is functorial.*

Proof. Let $f : M \to N$ be an R-module homomorphism and consider the map

$$f_S \ : \ S^{-1}M \longrightarrow S^{-1}N \ : \ \frac{m}{s} \longmapsto \frac{f(m)}{s} \ , \tag{A.4}$$

which is well-defined since if $\frac{m}{s} = \frac{n}{u}$ and $ut * m = st * n$ for some $t \in S$, then $ut * f(m) = st * f(n)$, so $\frac{f(m)}{s} = \frac{f(n)}{u}$. Moreover f_S is a morphism of modules over $S^{-1}R$ and the assignment $f \mapsto f_S$ behaves well with respect to composition, i.e. $(g \circ f)_S = g_S \circ f_S$ for all $g \in \mathrm{Hom}_R(N, L)$. □

Lemma A.2.4. cf. [[2], 3.4, p.39]
The functor $S^{-1} : \mathrm{Mod}(R) \to \mathrm{Mod}(S^{-1}R)$ *is additive, i.e. localization commutes with direct sums.*

Proof. Let $\{M_i\}_i$ be a family of R-modules. Using Lemma A.2.2 and the fact that tensor products commute with direct sums we find

$$S^{-1}\big(\textstyle\bigoplus_i M_i\big) \ \cong \ (\textstyle\bigoplus_i M_i) \otimes_R S^{-1}R \ \cong \ \textstyle\bigoplus_i \big(M_i \otimes_R S^{-1}R\big) \ \cong \ \textstyle\bigoplus_i (S^{-1}M_i) \ .$$

How does this isomorphism look like ? Recall that $(\bigoplus_i M_i) \otimes_R N \cong \bigoplus_i \big(M_i \otimes_R N\big)$ is given by

$$\{m_i\}_i \otimes n \longmapsto \{m_i \otimes n\}_i \quad \text{with inverse} \quad \{m_i \otimes n_i\}_i \longmapsto \textstyle\sum_i \big(\varepsilon_i(m_i) \otimes n_i\big) \ ,$$

where $\varepsilon_j : M_j \hookrightarrow \bigoplus_i M_i$ are the canonical injections. Hence we obtain the map

$$S^{-1}\big(\bigoplus_i M_i\big) \xrightarrow{\sim} \bigoplus_i (S^{-1}M_i) \;:\; \tfrac{\{m_i\}_i}{s} \mapsto \{m_i\}_i \otimes \tfrac{1}{s} \mapsto \big\{m_i \otimes \tfrac{1}{s}\big\}_i \mapsto \big\{\tfrac{m_i}{s}\big\}_i$$

with inverse

$$\big\{\tfrac{m_i}{s_i}\big\}_i \mapsto \big\{m_i \otimes \tfrac{1}{s_i}\big\}_i \mapsto \sum_i \big(\varepsilon_i(m_i) \otimes \tfrac{1}{s_i}\big) \mapsto \sum_i \Big(\tfrac{\varepsilon_i(m_i)}{s_i}\Big) .$$

For example, in the case of $S^{-1}(M \oplus N) \cong S^{-1}M \oplus S^{-1}N$, the isomorphism is given by

$$\frac{(m,n)}{s} \mapsto \Big(\frac{m}{s}\,,\,\frac{n}{s}\Big)$$

with inverse

$$\Big(\frac{m}{s}\,,\,\frac{n}{u}\Big) \mapsto \frac{(m,0)}{s} + \frac{(0,n)}{u} = \frac{(u*m, s*n)}{s \cdot u} .$$

□

Remark A.2.5. [[53], 860087]
In general localization does not commute with infinite direct products. Consider e.g. $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$ and $M_n = \mathbb{Z}/n\mathbb{Z}$ for $n \geq 2$. Then $S^{-1}M_n = \{0\}$, $\forall\, n \geq 2$ since $n * \bar{x} = \overline{n \cdot x} = \bar{0}$ with $n \in S$, hence $\frac{\bar{x}}{s} = \frac{\bar{0}}{1}$ for all $\bar{x} \in M_n$. But $S^{-1}\big(\prod_n M_n\big) \neq \{0\}$ because the element

$$\frac{(\bar{1}, \bar{1}, \bar{1}, \ldots)}{1}$$

is non-zero. Indeed if it was zero, $\exists\, s \in S$ such that $s * (\bar{1}, \bar{1}, \bar{1}, \ldots) = (\bar{0}, \bar{0}, \bar{0}, \ldots)$ and thus $(\bar{s}, \bar{s}, \bar{s}, \ldots) = (\bar{0}, \bar{0}, \bar{0}, \ldots)$, which means that $s \in \mathbb{Z}$ is divisible by every $n \geq 2$, i.e. $s = 0$. But $s \neq 0$ by definition of S.

Proposition A.2.6. [[2], 3.3, p.39] , [[11], 10.9.12] and [[26], 6.2.1, p.58]
The functor $S^{-1} : \mathrm{Mod}(R) \to \mathrm{Mod}(S^{-1}R)$ *is exact.*

Proof. Let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

be a short exact sequence of R-modules[2]. We shall show that the sequence of $S^{-1}R$–modules

$$0 \longrightarrow S^{-1}M \xrightarrow{f_S} S^{-1}N \xrightarrow{g_S} S^{-1}L \longrightarrow 0 \tag{A.5}$$

[2]Actually it suffices to check exact sequences of the form $M \to N \to L$, but for completion let us also check that injectivity and surjectivity are preserved.

is still exact. By definition (A.4) we see that $g_S \circ f_S = 0$, so the sequence is still a complex.
− f_S is injective: if $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s} = 0$, then $\exists\, t \in S$ such that $t * f(m) = 0$ and $f(t*m) = 0$. Hence by injectivity of f we get $t*m = 0$ and hence $\frac{m}{s} = 0$.
− g_S is surjective: let $\frac{l}{s} \in S^{-1}L$. As g is surjective, we know that $\exists\, n \in N$ such that $l = g(n)$ and we get

$$g_S\left(\tfrac{m}{s}\right) = \tfrac{g(n)}{s} = \tfrac{l}{s} \ .$$

− $\ker g_S = \operatorname{im} f_S$: if $\frac{n}{s} \in \ker g_S$, then $\frac{g(n)}{s} = 0$ and $\exists\, t \in S$ such that $t * g(n) = 0$ and $g(t*n) = 0$. By exactness of the initial sequence, this gives $t*n = f(m)$ for some $m \in M$, hence we can take

$$f_S\left(\tfrac{m}{s\cdot t}\right) = \tfrac{f(m)}{s\cdot t} = \tfrac{t*n}{s\cdot t} = \tfrac{n}{s} \ .$$

□

Corollary A.2.7. *If*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} L \longrightarrow 0$$

is a short exact sequence of R-modules, then the sequence of localizations

$$0 \longrightarrow M_P \xrightarrow{f_P} N_P \xrightarrow{g_P} L_P \longrightarrow 0$$

is an exact sequence of R_P-modules for all prime ideals $P \trianglelefteq R$.

Corollary A.2.8. [[2], 3.6, p.40] and [[11], 10.38.19]
$S^{-1}R$ is a flat R-module.

Proof. $S^{-1}R$ is an R-module via the ring homomorphism $i_S : R \to S^{-1}R$, see Lemma D.1.2. Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

be a short exact sequence of R-modules. Combining Lemma A.2.2 and Proposition A.2.6 we obtain that

$$0 \longrightarrow M \otimes_R S^{-1}R \longrightarrow N \otimes_R S^{-1}R \longrightarrow L \otimes_R S^{-1}R \longrightarrow 0$$

is isomorphic to the sequence (A.5), which is exact. Hence the functor $\otimes_R S^{-1}R$ is exact. □

Proposition A.2.9. [[6], II.§2.n°7.Prop.18, p.97-98] , [[53], 41750 & 271057]
Localization commutes with tensor products. More precisely, if M and N are R-modules, then

$$S^{-1}(M \otimes_R N) \cong (S^{-1}M) \otimes_{S^{-1}R} (S^{-1}N) \tag{A.6}$$

via the canonical isomorphism $\frac{m\otimes n}{s} \mapsto \frac{m}{1} \otimes \frac{n}{s}$ *with inverse* $\frac{m}{s} \otimes \frac{n}{t} \mapsto \frac{m\otimes n}{s\cdot t}$.

Proof. We will prove a slightly more general result[3]. Let $\varphi : A \to B$ be a ring homomorphism and M, N be A-modules. We denote $M_B = M \otimes_A B$, which hence becomes a module over B. Then

$$\begin{aligned} M_B \otimes_B N_B &\cong (M \otimes_A B) \otimes_B (N \otimes_A B) \cong M \otimes_A (N \otimes_A B) \\ &\cong (M \otimes_A N) \otimes_A B = (M \otimes_A N)_B \ . \end{aligned}$$

Hence formula (A.6) follows by taking $i_S : R \to S^{-1}R$ and applying (A.3). The explicit form of the isomorphism can then be found by following the steps above:

$$\begin{aligned} \tfrac{m\otimes n}{s} \mapsto (m \otimes n) \otimes \tfrac{1}{s} \mapsto m \otimes (n \otimes \tfrac{1}{s}) \mapsto (m \otimes \tfrac{1}{1}) \otimes (n \otimes \tfrac{1}{s}) \mapsto \tfrac{m}{1} \otimes \tfrac{n}{s} = \tfrac{m}{s} \otimes \tfrac{n}{1} \ , \\ \tfrac{m}{s} \otimes \tfrac{n}{t} \mapsto (m \otimes \tfrac{1}{s}) \otimes (n \otimes \tfrac{1}{t}) \mapsto m \otimes \left(\tfrac{1}{s} * (n \otimes \tfrac{1}{t})\right) = m \otimes (n \otimes \tfrac{1}{st}) \\ \mapsto (m \otimes n) \otimes \tfrac{1}{st} \mapsto \tfrac{m\otimes n}{st} \ . \end{aligned}$$

Note that is does not matter whether we take $\frac{m}{1} \otimes \frac{n}{s}$ or $\frac{m}{s} \otimes \frac{n}{1}$ since $\frac{1}{s}$ can freely change its position within the tensor product over $S^{-1}R$. □

Lemma A.2.10. [[2], 3.4, p.39] , [[26], 6.22, p.58] and [[11], 10.9.13]
1) For all $f \in \mathrm{Hom}_R(M, N)$, $\ker(f_S) = S^{-1}(\ker f)$ and $\mathrm{im}(f_S) = S^{-1}(\mathrm{im}\, f)$.
2) If $N \leq M$ is a submodule, then $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$.

Proof. 1) The inclusions $\supseteq$ are immediate: If $\frac{m}{s}$ is such that $m \in \ker f$, then $f_S(\frac{m}{s}) = \frac{f(m)}{s} = 0$. If $\frac{n}{s}$ is such that $n = f(x)$ for $x \in M$, then $\frac{n}{s} = \frac{f(x)}{s} = f_S(\frac{x}{s})$. Now assume that $f_S(\frac{m}{s}) = 0$, i.e. $\exists\, t \in S$ such that $t * f(m) = f(t * m) = 0$. Then $\frac{m}{s} = \frac{t*m}{s\cdot t}$ with $t * m \in \ker f$. Finally if $\frac{n}{s} = f_S(\frac{m}{a})$, then

$$\tfrac{n}{s} = \tfrac{f(m)}{a} \ \Leftrightarrow\ \exists\, t \in S \text{ such that } at * n = st * f(m) = f(st * m) \ ,$$

[3] An even more general statement can be found in Bourbaki [4], II.§5.n°1.Prop.3, p.83-84.

hence $\frac{n}{s} = \frac{at*n}{at\cdot s}$ with $at * n \in \operatorname{im} f$.
2) If we localize the exact sequence $0 \to N \to M \to M/N \to 0$, Proposition A.2.6 gives the exact sequence

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \to 0 ,$$

which allows to see $S^{-1}N$ as a submodule of $S^{-1}M$. Moreover we then obtain $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$ as $S^{-1}R$–modules by uniqueness of cokernels. □

A.2.2 Local properties

Proposition A.2.11. [[2], 3.8, p.40] and [[26], 6.27, p.60]
Let M be an R-module. Then M is the zero module if and only if the localization M_P is the zero module for all prime ideals $P \trianglelefteq R$.

Proof. Necessity is clear. Vice-versa, assume that $\exists\, m \in M$ such that $m \neq 0$ and set $N = \langle\, m \,\rangle \leq M$. Consider the R-module homomorphism $\varphi :\, R \to N$ defined by $\varphi(r) = r * m$ and the exact sequence

$$0 \longrightarrow I \longrightarrow R \overset{\varphi}{\longrightarrow} N \longrightarrow 0 ,$$

where $I = \ker \varphi$. Since $\varphi(1) = m \neq 0$, φ is not the zero map. Hence I is a proper ideal in R and thus contained in some maximal ideal $\mathfrak{M} \trianglelefteq R$. By exactness of localization the submodule $N \leq M$ gives an injection $N_{\mathfrak{M}} \hookrightarrow M_{\mathfrak{M}}$ with $M_{\mathfrak{M}} = \{0\}$ by assumption, so $N_{\mathfrak{M}} = \{0\}$ as well. But this module is generated by the element $\frac{m}{1} \in M_{\mathfrak{M}}$, so we need that $\frac{m}{1} = 0$, i.e. $\exists\, x \notin \mathfrak{M}$ such that $x * m = \varphi(x) = 0$. Thus $x \in \ker \varphi = I \subseteq \mathfrak{M}$, which contradicts that $x \notin \mathfrak{M}$. □

Corollary A.2.12. *Let $m \in M$. Then $m = 0$ if and only if $\frac{m}{1} = 0$ in M_P for all prime ideals $P \trianglelefteq R$. In other words, if $m \neq 0$, there is a prime ideal $P \trianglelefteq R$ such that $\frac{m}{1} \neq 0$ in M_P.*

Proof. Let $m \neq 0$ and consider the submodule $N = \langle\, m \,\rangle$. If $\frac{m}{1} = 0$ in M_P for all prime ideals P, then $N_P = \langle\, \frac{m}{1} \,\rangle = \{0\}$ for all P and Proposition A.2.11 implies that $N = \{0\}$, which contradicts $m \neq 0$. Thus there is at least some P such that $\frac{m}{1} \neq 0 \in M_P$. □

Proposition A.2.13. [[2], 3.9, p.40-41]
Let M, N be R-modules and $f \in \operatorname{Hom}_R(M,N)$. Then f is injective (resp. surjective, an isomorphism, or zero) if and only if $f_P : M_P \to N_P$ is injective (resp. surjective, an isomorphism, or zero) for all prime ideals $P \trianglelefteq R$.

Proof. 1) First we consider the case where f is injective. Necessity follows from Proposition A.2.6 (or alternatively from Lemma A.2.10). Vice-versa, denote $K = \ker f$, so that we have the exact sequences

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} N \qquad \text{and} \qquad 0 \longrightarrow K_P \longrightarrow M_P \xrightarrow{f_P} N_P$$

for all prime ideals $P \trianglelefteq R$, again because localization is exact. Now if f_P is injective for all P, then $\ker(f_P) = (\ker f)_P = K_P = \{0\}$ for all P by Lemma A.2.10 and Proposition A.2.11 implies that $K = \{0\}$, i.e. φ is injective.
2) The cases where f is surjective or an isomorphism are done similarly.
3) If $f = 0$, then f_P is zero too. Conversely if $f_P = 0$ for all P, then for any fixed $m \in M$ we find $\frac{f(m)}{1} = f_P(\frac{m}{1}) = 0$ for all P, i.e. $f(m) = 0$ by Corollary A.2.12 and thus $f = 0$. □

Corollary A.2.14. [[26], 6.27, p.60-61]
The converse of Corollary A.2.7 holds true as well.

Proof. Assume that

$$0 \longrightarrow M_P \xrightarrow{f_P} N_P \xrightarrow{g_P} L_P \longrightarrow 0 \tag{A.7}$$

is an exact sequence of R_P-modules for all prime ideals $P \trianglelefteq R$. Proposition A.2.13 already implies that f is injective and that g is surjective. Moreover $g_P \circ f_P = (g \circ f)_P = 0$, so $g \circ f = 0$ as well and we have $\operatorname{im} f \subseteq \ker g$. This allows to consider the quotient $\ker g / \operatorname{im} f$ and

$$(\ker g / \operatorname{im} f)_P \cong (\ker g)_P / (\operatorname{im} f)_P = \ker(g_P) / \operatorname{im}(f_P) = \{0\}$$

by Lemma A.2.10 and exactness of (A.7) for all P. Thus $\ker g / \operatorname{im} f = \{0\}$ by Proposition A.2.11 and we get $\ker g = \operatorname{im} f$. □

A.2.3 Localization and Hom

Next we are going to analyze the relation between the localization-functor and the Hom-bifunctor. More precisely we want to know if and/or under which conditions these functors commute.

Remark A.2.15. Let us first recall that for any R-module M and $n \in \mathbb{N}$, we have the isomorphism

$$\mathrm{Hom}_R(R^n, M) \cong M^n ,$$

given by $f \mapsto \big(f(e_1), \dots, f(e_n)\big)$, where $e_i = (0, \dots, 1, \dots, 0) \in R^n$ are the basis vectors. The inverse map is

$$(m_1, \dots, m_n) \longmapsto \big(u : R^n \to M : (r_1, \dots, r_n) \mapsto \textstyle\sum_i r_i * m_i \big) .$$

Proposition A.2.16. [[11], 10.10.2]
Let M, N be R-modules and $S \subset R$ a multiplicatively closed subset. If M is of finite presentation, then

$$S^{-1}\big(\mathrm{Hom}_R(M, N)\big) \cong \mathrm{Hom}_{S^{-1}R}\left(S^{-1}M, S^{-1}N\right) . \tag{A.8}$$

In particular, for every prime ideal $P \trianglelefteq R$, we have

$$\big(\mathrm{Hom}_R(M, N)\big)_P \cong \mathrm{Hom}_{R_P}(M_P, N_P) .$$

Proof. Choose a finite presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0 .$$

By left exactness of the contravariant functor $\mathrm{Hom}_R(\,\cdot\,, N)$ we get

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(R^n, N) \longrightarrow \mathrm{Hom}_R(R^m, N)$$
$$\Leftrightarrow \quad 0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow N^n \longrightarrow N^m .$$

Exactness and additivity of localization (Proposition A.2.6 and Lemma A.2.4) moreover give

$$0 \longrightarrow S^{-1}\big(\mathrm{Hom}_R(M, N)\big) \longrightarrow (S^{-1}N)^n \longrightarrow (S^{-1}N)^m .$$

On the other hand, we can also start by the finite presentation of M and first apply localization, which gives

$$(S^{-1}R)^m \longrightarrow (S^{-1}R)^n \longrightarrow S^{-1}M \longrightarrow 0 \ ,$$

and then apply the left exact functor $\mathrm{Hom}_{S^{-1}R}(\,\cdot\,, S^{-1}N)$, so that

$$0 \longrightarrow \mathrm{Hom}_{S^{-1}R}\big(S^{-1}M, S^{-1}N\big) \longrightarrow (S^{-1}N)^n \longrightarrow (S^{-1}N)^m \ .$$

Hence both modules are isomorphic as they are both kernels of the morphism $(S^{-1}N)^n \to (S^{-1}N)^m$. □

Remark A.2.17. One may ask whether the two constructions indeed give the same morphism. To check this we denote the basis vectors of R^n and R^m by $e_1, \ldots, e_n$, resp. $e'_1, \ldots, e'_m$. We start with

$$R^m \xrightarrow{\ \varphi\ } R^n \xrightarrow{\ f\ } M \longrightarrow 0 \ . \tag{A.9}$$

Denote $\varphi : R^m \to R^n$ with $\varphi(e'_i) = \sum_j r_{ij} e_j$, i.e. $\varphi(e'_i) = (r_{i1}, \ldots, r_{in})$ for all i. To get the morphism $\phi : N^n \to N^m$, consider

$$\begin{array}{ccc} \mathrm{Hom}_R(R^n, N) & \xrightarrow{\ \circ\varphi\ } & \mathrm{Hom}_R(R^m, N) \\ \big\downarrow \sim & & \big\downarrow \sim \\ N^n & \xrightarrow{\qquad \phi \qquad} & N^m \end{array}$$

Let $(a_1, \ldots, a_n) \in N^n$. This induces a morphism $u : R^n \to N$ which is defined by $u(r_1, \ldots, r_n) = \sum_i r_i * a_i$. Then let $v = u \circ \varphi : R^m \to N$. Evaluating at the basis vectors, this finally gives the element

$$v(e'_i) = u\big(\varphi(e'_i)\big) = u\big(\textstyle\sum_j r_{ij} e_j\big) = \sum_j r_{ij} * u(e_j) = \sum_j r_{ij} * (1 * a_j) = \sum_j r_{ij} * a_j$$

and

$$\phi \ : \ N^n \longrightarrow N^m \ : \ (a_1, \ldots, a_n) \longmapsto \big(\textstyle\sum_j r_{1j} * a_j \, , \ldots , \, \sum_j r_{mj} * a_j\big) \ .$$

Next we have to localize and commute localization with the direct sum, i.e.

$$\begin{array}{ccc} S^{-1}(N^n) & \xrightarrow{\ \phi_S\ } & S^{-1}(N^m) \\ \big\downarrow \sim & & \big\downarrow \sim \\ (S^{-1}N)^n & \xrightarrow{\ \Phi\ } & (S^{-1}N)^m \end{array}$$

Let $\left\{\frac{n_i}{s_i}\right\}_i \in (S^{-1}N)^n$. Using the identifications from Lemma A.2.4 and applying ϕ_S, we get

$$\begin{aligned}
\Phi \ : \ \left\{\tfrac{n_i}{s_i}\right\}_i &\longmapsto \sum_i \left(\tfrac{\varepsilon_i(n_i)}{s_i}\right) = \sum_i \frac{(0,\dots,n_i,\dots,0)}{s_i} \\
&\overset{\phi_S}{\longmapsto} \sum_i \frac{\phi(0,\dots,n_i,\dots,0)}{s_i} = \sum_i \frac{(r_{1i}*n_i,\dots,r_{mi}*n_i)}{s_i} \\
&\longmapsto \sum_i \left(\frac{r_{1i}*n_i}{s_i},\dots,\frac{r_{mi}*n_i}{s_i}\right) \\
&= \left(\sum_i r_{1i}*\tfrac{n_i}{s_i},\dots,\sum_i r_{mi}*\tfrac{n_i}{s_i}\right) = \left\{\sum_j r_{ij}*\tfrac{n_j}{s_j}\right\}_i .
\end{aligned}$$

Now let's go the other way round. We denote the basis vectors of $(S^{-1}R)^m$ by $E_1',\dots,E_m'$.

$$\begin{array}{ccc}
S^{-1}(R^m) & \xrightarrow{\varphi_S} & S^{-1}(R^n) \\
\downarrow\wr & & \downarrow\wr \\
(S^{-1}R)^m & \xrightarrow{\phi'} & (S^{-1}R)^n
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{S^{-1}R}\left((S^{-1}R)^n, S^{-1}N\right) & \xrightarrow{\circ\phi'} & \mathrm{Hom}_{S^{-1}R}\left((S^{-1}R)^m, S^{-1}N\right) \\
\downarrow\wr & & \downarrow\wr \\
(S^{-1}N)^n & \xrightarrow{\quad\Phi'\quad} & (S^{-1}N)^m
\end{array}$$

$$\begin{aligned}
\phi' \ : \ \left\{\tfrac{a_i}{s_i}\right\}_i &\longmapsto \sum_i \left(\tfrac{\varepsilon_i(a_i)}{s_i}\right) = \sum_i \frac{(0,\dots,a_i,\dots,0)}{s_i} \\
&\overset{\varphi_S}{\longmapsto} \sum_i \frac{\varphi(0,\dots,a_i,\dots,0)}{s_i} = \sum_i \frac{a_i\cdot(r_{i1},\dots,r_{in})}{s_i} \\
&\longmapsto \sum_i \left(\frac{r_{i1}\cdot a_i}{s_i},\dots,\frac{r_{in}\cdot a_i}{s_i}\right) \\
&= \left(\sum_i r_{i1}\cdot\tfrac{a_i}{s_i},\dots,\sum_i r_{in}\cdot\tfrac{a_i}{s_i}\right) = \left\{\sum_j r_{ji}\cdot\tfrac{a_j}{s_j}\right\}_i .
\end{aligned}$$

$$\begin{aligned}
\Phi' \ : \ \left\{\tfrac{n_i}{s_i}\right\}_i &\longmapsto \left(u' \ : \ \left(\tfrac{r_1}{t_1},\dots,\tfrac{r_n}{t_n}\right) \mapsto \sum_i \tfrac{r_i}{t_i}*\tfrac{n_i}{s_i}\right) \\
&\longmapsto u'\circ\phi' \longmapsto \left\{u'\big(\phi'(E_i')\big)\right\}_i = \left\{\sum_j r_{ij}*\tfrac{n_j}{s_j}\right\}_i
\end{aligned}$$

because $\phi'(E_i') = \left(\frac{r_{i1}}{1},\dots,\frac{r_{im}}{1}\right)$ and $u'\big(\phi'(E_i')\big) = \sum_j r_{ij}*\frac{n_j}{s_j}$. Thus $\Phi' = \Phi$, as expected.

Corollary A.2.18. *The isomorphism in* (A.8) *is given by*

$$\rho \;:\; S^{-1}\big(\operatorname{Hom}_R(M,N)\big) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}R}\left(S^{-1}M, S^{-1}N\right)$$
$$\tfrac{g}{s} \longmapsto \left(S^{-1}M \to S^{-1}N \;:\; \tfrac{m}{a} \mapsto \tfrac{g(m)}{a\cdot s} \right) .$$

Proof. Let us denote the LHS of (A.8) by S_1 and the RHS by S_2. In order to find the isomorphism ρ we have to use the universal property of kernels, i.e.

$$\begin{array}{ccccc} S_2 & \xrightarrow{F'} & (S^{-1}N)^n & \xrightarrow{\Phi} & (S^{-1}N)^m \\ & \nwarrow_{\exists!\,\rho} & \uparrow F & & \\ & & S_1 & & \end{array}$$

and ρ is the unique morphism such that $F' \circ \rho = F$, where F and F' are the transformations of the initial morphism $f : R^n \to M$ from (A.9), similarly as in Remark A.2.17. To get F, consider

$$\alpha \;:\; \operatorname{Hom}_R(M,N) \xrightarrow{\circ f} \operatorname{Hom}_R(R^n,N) \xrightarrow{\sim} N^n$$
$$g \longmapsto g\circ f \longmapsto \big\{g(f(e_i))\big\}_i \,,$$
$$F \;:\; S_1 \xrightarrow{\alpha_S} S^{-1}(N^n) \xrightarrow{\sim} (S^{-1}N)^n$$
$$\frac{g}{s} \longmapsto \frac{\alpha(g)}{s} = \frac{\{g(f(e_i))\}_i}{s} \longmapsto \left\{\frac{g(f(e_i))}{s}\right\}_i .$$

Let $E_1,\dots,E_n$ be a basis of $(S^{-1}R)^n$. Going the other way round we get F' by

$$\beta \;:\; (S^{-1}R)^n \xrightarrow{\sim} S^{-1}(R^n) \xrightarrow{f_S} S^{-1}M$$
$$\left\{\tfrac{r_i}{s_i}\right\}_i \longmapsto \sum_i \left(\tfrac{\varepsilon_i(r_i)}{s_i}\right) \longmapsto \sum_i \frac{f(\varepsilon_i(r_i))}{s_i} \,,$$
$$F' \;:\; S_2 \xrightarrow{\circ\beta} \operatorname{Hom}_{S^{-1}R}\big((S^{-1}R)^n, S^{-1}N\big) \xrightarrow{\sim} (S^{-1}N)^n$$
$$h \longmapsto h\circ\beta \longmapsto \big\{h(\beta(E_i))\big\}_i \,,$$

where $\beta(E_i) = \frac{f(\varepsilon_i(1))}{1} = \frac{f(e_i)}{1}$. Now let $\rho(\frac{g}{s}) = h$ where $h(\frac{m}{a}) = \frac{g(m)}{a\cdot s}$.
Then $F'\circ\rho = F$ because

$$\frac{g}{s} \overset{\rho}{\longmapsto} h \overset{F'}{\longmapsto} \left\{h\Big(\frac{f(e_i)}{1}\Big)\right\}_i = \left\{\frac{g(f(e_i))}{s}\right\}_i = F\big(\tfrac{g}{s}\big) .$$

This shows that ρ is the unique morphism making the diagram commute. Moreover it is an isomorphism since S_1 and S_2 are both kernels of Φ, hence they are canonically isomorphic via the unique isomorphism given by ρ. □

Remark A.2.19. In general formula (A.8) does not hold when M is just finitely generated instead of finitely presented. Indeed, if we have a presentation

$$R^{(I)} \longrightarrow R^n \longrightarrow M \longrightarrow 0 \; ,$$

the fact that localization does not commute with infinite direct products (see Remark A.2.5) will give an exact sequence

$$0 \longrightarrow S_1 \longrightarrow (S^{-1}N)^n \longrightarrow S^{-1}(N^I)$$

as $\mathrm{Hom}_R(R^{(I)}, N) \cong N^I$, whereas the other way contains $S^{-1}(R^{(I)}) \cong (S^{-1}R)^{(I)}$ and hence gives

$$0 \longrightarrow S_2 \longrightarrow (S^{-1}N)^n \longrightarrow (S^{-1}N)^I \; .$$

The last terms being different, one cannot conclude. Note however that every finitely generated module over a Noetherian ring is also finitely presented (see Proposition D.1.5), so in this case it will still work.

Remark A.2.20. If R is an integral domain, then formula (A.8) also holds true under an additional assumption if M is just finitely generated instead of finitely presented. The exact statement and proof are given in Proposition C.4.9.

Appendix B

Primary Ideal Decomposition and Associated Primes

B.1 Preliminaries

Let R be a ring and recall that the *annihilator* of an R-module M is given by all elements that act trivially on M:

$$\mathrm{Ann}_R(M) = \left\{ r \in R \;\middle|\; r * m = 0, \forall\, m \in M \right\} .$$

One checks that $\mathrm{Ann}_R(M)$ is an ideal in R. The annihilator of a specific element $m \in M$ is denoted by $\mathrm{Ann}_R(m)$. Thus

$$\mathrm{Ann}_R(M) = \bigcap_{m \in M} \mathrm{Ann}_R(m) .$$

Let $\mathrm{ZD}(R)$ denote the set of all zero-divisors in R; by convention $0 \in \mathrm{ZD}(R)$, although 0 is by definition not a zero-divisor. Hence R is an integral domain if and only if $\mathrm{ZD}(R) = \{0\}$. If we consider R as a module itself and take $x \in R$, then

$$x \in \mathrm{ZD}(R) \;\Leftrightarrow\; \mathrm{Ann}_R(x) \neq \{0\} .$$

Let us also recall that the *radical* of an ideal $I \trianglelefteq R$, which is again an ideal, is defined as

$$\mathrm{Rad}(I) := \left\{ r \in R \;\middle|\; \exists\, n \in \mathbb{N} \text{ such that } r^n \in I \right\} .$$

Proposition B.1.1. [[2], 1.15, p.9] and [[53], 371929]

$$\mathrm{ZD}(R) = \bigcup_{x\neq 0} \mathrm{Ann}_R(x) = \bigcup_{x\neq 0} \mathrm{Rad}\left(\mathrm{Ann}_R(x)\right) . \tag{B.1}$$

Proof. The first equality follows from the fact that $r \in R$ is a zero-divisor if and only if $\exists\, x \in R$, $x \neq 0$ such that $r \cdot x = 0$. For the second one, we obtain the inclusion $\subseteq$ because of $\mathrm{Ann}_R(x) \subseteq \mathrm{Rad}(\mathrm{Ann}_R(x))$. Vice-versa assume that r is not a zero-divisor with $r \in \mathrm{Rad}(\mathrm{Ann}_R(x))$ for some $x \neq 0$. Then $\exists\, n \in \mathbb{N}$ such that $r^n \in \mathrm{Ann}_R(x)$, i.e. $r^n \cdot x = 0$. Since r is not a zero-divisor, we need that $r^{n-1} \cdot x = 0$. Continuing the same way, we obtain that $r \cdot x = 0$ and finally that $x = 0$. This contradiction shows that $\bigcup_{x\neq 0} \mathrm{Rad}(\mathrm{Ann}_R(x)) \subseteq \mathrm{ZD}(R)$. □

Remark B.1.2. We do not necessarily have $\mathrm{Ann}_R(x) = \mathrm{Rad}(\mathrm{Ann}_R(x))$ for all $x \neq 0$ in (B.1).

We finish the preliminaries by the following important lemma which we will be using throughout the whole paper.

Lemma B.1.3 (Prime Avoidance)**.** [[2], 1.11, p.8]
1) Let $P_1, \ldots, P_k \trianglelefteq R$ be prime ideals and I an ideal such that $I \subseteq P_1 \cup \ldots \cup P_k$. Then there is an index $i \in \{1, \ldots, k\}$ such that $I \subseteq P_i$.
2) Let $I_1, \ldots, I_k$ be ideals and P a prime ideal such that $I_1 \cap \ldots \cap I_k \subseteq P$. Then $I_i \subseteq P$ for some $i \in \{1, \ldots, k\}$. And if $P = I_1 \cap \ldots \cap I_k$, then $P = I_i$ for some i.

Proof. 1) By contraposition we show that

$$I \nsubseteq P_i,\ \forall\, i \in \{1, \ldots, k\} \quad \Rightarrow \quad I \nsubseteq P_1 \cup \ldots \cup P_k .$$

The proof goes by induction on k. It is true for $k = 1$. Now assume that $k > 1$ and that the result holds true for $k - 1$. By induction hypothesis, for all fixed $j \in \{1, \ldots, k\}$ we have $I \nsubseteq \bigcup_{i\neq j} P_i$, i.e. $\exists\, x_i \in I$ such that $x_i \notin P_j$, $\forall\, j \neq i$. If there is some $\ell \in \{1, \ldots, k\}$ such that $x_\ell \notin P_\ell$, then

$$x_\ell \in I \setminus (P_1 \cup \ldots \cup P_k)$$

and we are done. If not, then $x_j \in P_j$ for all j. With this consider the element

$$y = \sum_{j=1}^{k} (x_1 \cdot \ldots \cdot x_{j-1} \cdot x_{j+1} \cdot \ldots \cdot x_k) .$$

We have $y \in I$ and each summand belongs to $\left(\bigcup_{i \neq j} P_i\right) \setminus P_j$ since P_j is a prime ideal. Hence $y \notin Pj$ for all $j \in \{1, \dots, k\}$ since all but one summands do. Finally $y \in I \setminus (P_1 \cup \dots \cup P_k)$.

2) By contraposition, assume that $I_i \nsubseteq P$ for all i, so that $\exists\, x_i \in I_i$ such that $x_i \notin P$, $\forall\, i \in \{1, \dots, k\}$. But $x_1 \cdot \ldots \cdot x_k \in (I_1 \cap \dots \cap I_k) \setminus P$ since P is prime, hence $I_1 \cap \dots \cap I_k \nsubseteq P$. And if $P = I_1 \cap \dots \cap I_k$, then $P \subseteq I_j$ for all j and by the previous result there is some i such that $I_i \subseteq P$, hence $I_i = P$. □

B.2 Primary Ideal Decomposition

Primary Ideal Decomposition is an important tool in Commutative Algebra. We will see our main application in Section 1.2. The general ideal is to write an ideal in a ring as a finite intersection of "easier" ideals, the so-called primary ideals. Several constructions are possible ; here we follow Atiyah-MacDonald [2].

B.2.1 Primary ideals

Definition B.2.1. cf. [[2], p.50]
Let R be a ring. An ideal $Q \trianglelefteq R$ is called a *primary* ideal if $Q \neq R$ and if for all $r, s \in R$,

$$r \cdot s \in Q \quad \Rightarrow \quad r \in Q \ \text{ or } \ s \in \operatorname{Rad}(Q)\ .$$

Since R is commutative, this definition is symmetric with respect to r and s and we have the equivalence

$$\begin{aligned} &\Big(\big(r \in Q\big) \text{ or } \big(s \in \operatorname{Rad}(Q)\big)\Big) \text{ and } \Big(\big(s \in Q\big) \text{ or } \big(r \in \operatorname{Rad}(Q)\big)\Big) \\ \Leftrightarrow\ &\big(r \in Q\big) \text{ or } \big(s \in Q\big) \text{ or } \Big(\big(r \in \operatorname{Rad}(Q)\big) \text{ and } \big(s \in \operatorname{Rad}(Q)\big)\Big)\ , \end{aligned} \tag{B.2}$$

hence $Q \neq R$ is a primary ideal if whenever $r \cdot s \in Q$, then either $r \in Q$ or $s \in Q$ or $r, s \in \operatorname{Rad}(Q)$.

Proposition B.2.2. [[2], 4.1 & 4.2, p.50-51] and [[53], 504551]
1) Prime ideals are primary.
2) $Q \trianglelefteq R$ is a primary ideal if and only if $R/Q \neq \{0\}$ and every zero-divisor in R/Q is nilpotent.

3) If Q is primary, then $\mathrm{Rad}(Q)$ *is a prime ideal. It is also the smallest prime ideal containing Q.*
4) An ideal is prime if and only if it is primary and radical.
5) If $Q \trianglelefteq R$ is such that $\mathrm{Rad}(Q)$ *is a maximal ideal, then Q is primary.*

Proof. 1) follows from (B.2).
2) If $\bar{r} \cdot \bar{s} = \bar{0}$ with $\bar{r} \neq \bar{0}$ and $\bar{s} \neq \bar{0}$, then $r \cdot s \in Q$ with $r, s \notin Q$, hence by (B.2) we have $r \in \mathrm{Rad}(Q)$, i.e. $\bar{r}$ is nilpotent in R/Q. Vice-versa if $r \cdot s \in Q$, then $\bar{r} \cdot \bar{s} = \bar{0}$ and $\bar{r}$ is a zero-divisor, hence nilpotent so that $r^n \in Q$, i.e. $r \in \mathrm{Rad}(Q)$.
3) We shall show that $\mathrm{Rad}(Q)$ is a prime ideal. Let $r \cdot s \in \mathrm{Rad}(Q)$, i.e. $\exists\, n \in \mathbb{N}$ such that $(rs)^n \in Q$. Therefore $r^n \in Q$ or $s^{nm} \in Q$ for some $m \in \mathbb{N}$ since Q is primary, which implies that $r \in \mathrm{Rad}(Q)$ or $s \in \mathrm{Rad}(Q)$. It is the smallest prime ideal containing Q since $\mathrm{Rad}(Q)$ is equal to the intersection of all prime ideals containing Q (see Lemma D.1.3).
4) Necessity follows from the definition and sufficiency follows from the fact that $\mathrm{Rad}(Q) = Q$ is prime.
5) Let $I \trianglelefteq R$ be such that $J = \mathrm{Rad}(I)$ is a maximal ideal in R. If $\pi : R \to R/I$, then

$$\mathrm{nil}(R/I) = \pi\big(\mathrm{Rad}(I)\big) = \pi(J)$$

and $\pi(J)$ is a maximal ideal in R/I since π is surjective. Indeed it is proper (as it is equal to the nilradical, which does not contain 1), hence contained in a maximal ideal M. But $\pi(J) \subseteq M$ implies that $J \subseteq \pi^{-1}(M)$ where J is maximal, so $J = \pi^{-1}(M)$ and $\pi(J) = \pi(\pi^{-1}(M)) = M$; here we use again surjectivity. Now since $\mathrm{nil}(R/I)$ is the intersection of all prime ideals in R/I (Lemma D.1.3), there cannot exist other maximal ideals, otherwise their intersection would be smaller. Hence R/I is a local ring with unique maximal ideal $\pi(J)$. In particular it only contains nilpotent elements and $R/I \setminus \pi(J)$ only consists of units (which are not zero-divisors). Thus all zero-divisors in R/I are nilpotent. □

Example B.2.3. [[2], p.51]
1) The primary ideals of $\mathbb{Z}$ are $\{0\}$ and $\langle\, p^n \,\rangle$ for each prime number p and each $n \in \mathbb{N}$. Indeed $\forall\, a \in \mathbb{Z}$ such that $a \notin \{0, 1, -1\}$,

$$\mathrm{Rad}\big(\langle\, a \,\rangle\big) = \mathrm{Rad}\big(\langle\, p_1^{n_1} \cdot \ldots \cdot p_k^{n_k} \,\rangle\big) = \langle\, p_1 \cdot \ldots \cdot p_k \,\rangle \qquad \text{(B.3)}$$

by factorizing a into a product of prime numbers, hence in order to obtain a prime ideal we need that $k = 1$. This necessary form is also sufficient since $r \cdot s \in \langle p^n \rangle \;\Rightarrow\; r, s \in \langle p \rangle = \text{Rad}\big(\langle p^n \rangle\big)$. Alternatively one can also use Proposition B.2.2 since non-zero prime ideals are maximal.

2) Not every power of a prime ideal is primary (although their radical is a prime ideal, see Lemma D.1.4). Consider for example $R = \mathbb{K}[X, Y, Z]/\langle XY - Z^2 \rangle$ with the ideals $P = \langle \bar{X}, \bar{Z} \rangle$, which is prime since $R/P \cong \mathbb{K}[\bar{Y}]$ is an integral domain, and $Q = P^2 = \langle \bar{X}^2, \bar{X}\bar{Z}, \bar{Z}^2 \rangle$. Q is not primary since

$$\bar{X} \cdot \bar{Y} = \bar{Z}^2 \in Q \qquad \text{but} \quad \bar{X} \notin Q \;,\; \bar{Y} \notin Q \;,\; \bar{Y} \notin \text{Rad}(Q) = P \;.$$

3) Combining 2) and Lemma D.1.4, we see that there are ideals which have a prime radical without being primary. Hence we need the assumption on $\text{Rad}(Q)$ being maximal in 5) of Proposition B.2.2.

4) There also exist primary ideals which are not powers of the prime ideal given by their radical. Let $R = \mathbb{K}[X, Y]$ and $Q = \langle X, Y^2 \rangle$. Q is primary since $R/Q \cong \mathbb{K}[Y]/\langle Y^2 \rangle$ is a ring in which all zero-divisors are multiples of $\bar{Y}$ and hence nilpotent. Alternatively, $P = \text{Rad}(Q) = \langle X, Y \rangle$ is a maximal ideal. Moreover $P^2 = \langle X^2, XY, Y^2 \rangle$, so $P^2 \subsetneq Q \subsetneq P$ and Q is not a power of $\text{Rad}(Q)$.

Definition B.2.4. [[2], p.51]
Let $P \trianglelefteq R$ be a prime ideal. An ideal $Q \trianglelefteq R$ is called *P-primary* if it is primary with $\text{Rad}(Q) = P$.
As shown in Example B.2.3 we see that not all powers of a prime ideal P are P-primary and not all P-primary ideals need to be powers of P.

Lemma B.2.5. [[2], 4.3, p.51]
If $Q_1, \ldots, Q_k \trianglelefteq R$ are P-primary, then $Q = Q_1 \cap \ldots \cap Q_k$ is P-primary as well.

Proof. From Lemma D.1.4 it follows that

$$\text{Rad}(Q) = \text{Rad}(Q_1 \cap \ldots \cap Q_k) = \text{Rad}(Q_1) \cap \ldots \cap \text{Rad}(Q_k) = P \cap \ldots \cap P = P \;.$$

To see that Q is still primary, let $r \cdot s \in Q$ with $r \notin Q$. Then $r \cdot s \in Q_i$ for some i with $r \notin Q_i$, so $s \in \text{Rad}(Q_i) = P = \text{Rad}(Q)$ since Q_i is primary. □

B.2.2 Primary decompositions

Definition B.2.6. [[2], p.51]
Let $I \trianglelefteq R$ be any ideal. A *primary decomposition* of I is an expression of I as a finite intersection of primary ideals

$$I = \bigcap_{i=1}^{\alpha} Q_i \ .$$

In general, such a primary decomposition may not exist and it does not need to be unique neither. We say that an ideal is *decomposable* if it admits such a decomposition. A primary decomposition is called *minimal* if α is minimal, i.e. if there does not exist a decomposition with less intersecting primary ideals.

Remark B.2.7. [[2], p.51-52]
A minimal primary decomposition satisfies the following properties:

- $\left(\bigcap_{i\neq j} Q_i\right) \nsubseteq Q_j$, $\forall\, j \in \{1, \ldots, \alpha\}$: no one of the Q_i is superfluous
- $\mathrm{Rad}(Q_i) \neq \mathrm{Rad}(Q_j)$, $\forall\, i, j \in \{1, \ldots, \alpha\}$: all radicals are distinct

The first one is obvious, since every superfluous Q_i can simply be omitted. To see why the second one holds true, assume e.g. that $\mathrm{Rad}(Q_i) = \mathrm{Rad}(Q_j) = P$ for some $i, j \in \{1, \ldots, \alpha\}$. This means that Q_i and Q_j are P-primary, hence $Q' := Q_i \cap Q_j$ is P-primary as well. But then one could replace Q_i and Q_j by Q', so α would not be minimal. Hence by removing and / or replacing the primary ideals causing problems, every primary decomposition can be modified in order to obtain a minimal one. So we may always assume that a given primary decomposition is minimal.

Example B.2.8. 1) If $Q \trianglelefteq R$ is already a primary ideal (e.g. prime or maximal), its minimal prime decomposition is trivial.
2) In $\mathbb{Z}$ we have $\langle\, 12 \,\rangle = \langle\, 3 \,\rangle \cap \langle\, 4 \,\rangle$ since an integer is a multiple of 12 if and only if it is a multiple of 3 and 4. Here $Q_1 = \langle\, 3 \,\rangle = \mathrm{Rad}(Q_1)$ is prime and $Q_2 = \langle\, 4 \,\rangle = \langle\, 2^2 \,\rangle$ is primary with $\mathrm{Rad}(Q_2) = \langle\, 2 \,\rangle$. Actually, we have the generalization

$$\langle\, a \,\rangle = \langle\, p_1^{n_1} \,\rangle \cap \ldots \cap \langle\, p_\alpha^{n_\alpha} \,\rangle$$

for $a \notin \{0, 1, -1\}$ with $p_1, \ldots, p_\alpha$ as in (B.3).

Example B.2.9. [[2], p.52-53]
1) Minimal primary decompositions still don't need to be unique. Consider e.g. $R = \mathbb{K}[X,Y]$ and $I = \langle X^2, XY \rangle$. I is not primary since $\bar{Y}$ is a zero-divisor in R/I which is not nilpotent. Then

$$I = \langle X \rangle \cap \langle X^2, Y \rangle = \langle X \rangle \cap \langle X^2, XY, Y^2 \rangle = \langle X \rangle \cap \langle X^2, X+Y \rangle \, . \quad \text{(B.4)}$$

All ideals on the right are primary as their radicals are $\langle X, Y \rangle$, which is maximal. Hence all decompositions in (B.4) are minimal with $\alpha = 2$. Moreover

$$\mathrm{Rad}(I) = \mathrm{Rad}(Q_1 \cap Q_2) = \mathrm{Rad}(Q_1) \cap \mathrm{Rad}(Q_2) = \langle X \rangle \cap \langle X, Y \rangle = \langle X \rangle \, .$$

2) Let $R = \mathbb{K}[X,Y]/\langle XY \rangle$ and $I = \{\bar{0}\}$, which is neither prime since $\bar{X} \cdot \bar{Y} = \bar{0}$, nor primary since $\bar{X}$ and $\bar{Y}$ are non-nilpotent zero-divisors in $R/I = R$. But we have $\{\bar{0}\} = \langle \bar{X} \rangle \cap \langle \bar{Y} \rangle$, where $\langle \bar{X} \rangle$ and $\langle \bar{Y} \rangle$ are both prime as $R/\langle \bar{X} \rangle \cong \mathbb{K}[\bar{Y}]$ and $R/\langle \bar{Y} \rangle \cong \mathbb{K}[\bar{X}]$ are integral domains. Note however that they are not maximal as $\langle \bar{X} \rangle, \langle \bar{Y} \rangle \subsetneq \langle \bar{X}, \bar{Y} \rangle$.
3) Consider the ideal $Q = \langle \bar{X}^2, \bar{X}\bar{Z}, \bar{Z}^2 \rangle$ in $R = \mathbb{K}[X,Y,Z]/\langle XY - Z^2 \rangle$. We already know from Example B.2.3 that Q is not primary. A possible (minimal) primary decomposition of Q can be given as $Q = \langle \bar{X} \rangle \cap \langle \bar{X}^2, \bar{X}\bar{Z}, \bar{Y} \rangle$. Note that $\langle \bar{X} \rangle$ is not a prime ideal since $\bar{Z}^2 = \bar{X}\bar{Y} \in \langle \bar{X} \rangle$, but $\bar{Z} \notin \langle \bar{X} \rangle$. However it is primary because $R/\langle \bar{X} \rangle \cong \mathbb{K}[Z]/\langle Z^2 \rangle$ only contains nilpotent zero-divisors. Moreover one can compute

$$\mathrm{Rad}\left(\langle \bar{X} \rangle\right) = \langle \bar{X}, \bar{Z} \rangle \qquad \text{and} \qquad \mathrm{Rad}\left(\langle \bar{X}^2, \bar{X}\bar{Z}, \bar{Y} \rangle\right) = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle \, ,$$

which shows that $\langle \bar{X}^2, \bar{X}\bar{Z}, \bar{Y} \rangle$ is primary since its radical is a maximal ideal.

Proposition B.2.10. [[2], p.53]
Let $I, J \trianglelefteq R$ be ideals such that $I \subseteq J$ and assume that J is decomposable with minimal primary decomposition $J = \bigcap_{i=1}^{\alpha} Q_i$. Then the ideal $\bar{J} \trianglelefteq R/I$ is also decomposable with primary decomposition $\bar{J} = \bigcap_{i=1}^{\alpha} \bar{Q}_i$, which can be made minimal. In particular for $J = I$, we obtain a minimal primary decomposition of the zero ideal $\{\bar{0}\} \trianglelefteq R/I$.

Remark B.2.11. If J does not contain I, one has to consider a primary decomposition of $J+I$ in order to get a decomposition of $\bar{J}$ in the quotient. For example if I and J are both finitely generated by $r_1, \ldots, r_n$, resp. $s_1, \ldots, s_m$, then $I+J$ is generated by $r_1, \ldots, r_n, s_1, \ldots, s_m$.

We have seen, e.g. in (B.4), that the primary ideals Q_i in a primary decomposition of I may not be unique. However this example also already illustrated the upcoming uniqueness result, for which we need the following auxiliary definition.

Definition B.2.12. Let $I \trianglelefteq R$ be an ideal and $x \in R$. We denote

$$(I : x) = \left\{ r \in R \;\middle|\; r \cdot x \in I \right\} ,$$

which is an ideal since we may consider R/I as an R-module and $\bar{x} \in R/I$, so

$$\begin{aligned} \mathrm{Ann}_R(\bar{x}) = \left\{ r \in R \;\middle|\; r * \bar{x} = \bar{0} \right\} &= \left\{ r \in R \;\middle|\; \bar{r} \cdot \bar{x} = \bar{0} \right\} \\ &= \left\{ r \in R \;\middle|\; r \cdot x \in I \right\} = (I : x) \; . \end{aligned} \tag{B.5}$$

Lemma B.2.13. [[2], 4.4, p.52]
Let $Q \trianglelefteq R$ be a P-primary ideal and $x \in R$. Then we have the 3 possibilities:
1) If $x \in Q$, then $(Q : x) = R$.
2) If $x \notin Q$, then $(Q : x)$ is P-primary.
3) If $x \notin P$, then $(Q : x) = Q$.
In particular, if $x \notin Q$, then $\mathrm{Rad}(Q : x) = P$.

Proof. 1) is clear.
3) follows from the fact that Q is a primary ideal. Indeed $Q \subseteq (Q : x)$ and

$$r \in (Q : x) \;\Rightarrow\; r \cdot x \in Q \text{ with } x \notin \mathrm{Rad}(Q) \;\Rightarrow\; r \in Q \; .$$

2) Let us first compute the radical of $(Q : x)$. If $y \in (Q : x)$, then $x \cdot y \in Q$ with $x \notin Q$, hence $y \in Q$ or $y \in \mathrm{Rad}(Q) = P$. In both cases we get the inclusions $Q \subseteq (Q : x) \subseteq P$. Taking radicals then gives $\mathrm{Rad}(Q : x) = P$. To see that $(Q : x)$ is primary, let $y \cdot z \in (Q : x)$ with $y \notin \mathrm{Rad}(Q : x) = P$. Then $xyz = y \cdot zx \in Q$ with $y \notin P$, so $zx \in Q$ and $z \in (Q : x)$. □

Theorem B.2.14 (First Uniqueness Theorem)**.** [[2], 4.5, p.52]
Let $I \trianglelefteq R$ be a decomposable ideal with minimal primary decomposition given by $I = \bigcap_{i=1}^{\alpha} Q_i$ and denote the radicals by $P_i := \mathrm{Rad}(Q_i)$. The set of primes ideals $\{P_1, \ldots, P_\alpha\}$ is independent of the chosen minimal primary decomposition of I. More precisely, the P_i are exactly the prime ideals that occur in the set of ideals

$$\left\{ \mathrm{Rad}(I : x) \;\middle|\; x \in R \right\} .$$

In particular, $\forall\, i \in \{1, \ldots, \alpha\}$, $\exists\, x_i \in R$ such that $P_i = \mathrm{Rad}\left(\mathrm{Ann}_R(\bar{x}_i)\right)$ for $\bar{x}_i \in R/I$ with $\bar{x}_i \neq \bar{0}$.

Proof. By definition we get $(I : x) = \left(\bigcap_i Q_i : x\right) = \bigcap_i (Q_i : x)$, $\forall\, x \in R$. Thus by Lemma B.2.13 and Lemma D.1.4,

$$\mathrm{Rad}(I : x) = \mathrm{Rad}\left(\bigcap_i (Q_i : x)\right) = \bigcap_i \mathrm{Rad}(Q_i : x) = \bigcap_{x \notin Q_j} P_j \;. \tag{B.6}$$

Now suppose that $\mathrm{Rad}(I : x)$ is a prime ideal. Then by (B.6) and Prime Avoidance (Lemma B.1.3), we have $\mathrm{Rad}(I : x) = P_j$ for some j. Conversely as the decomposition of I is minimal, Remark B.2.7 implies that there exists $x_i \notin Q_i$ with $x_i \in \bigcap_{j \neq i} Q_j$ for each $i \in \{1, \ldots, \alpha\}$. Then $\mathrm{Rad}(I : x_i) = P_i$.
$\subseteq$: if $r^n \cdot x_i \in I = \bigcap_j Q_j$ with $x_i \notin Q_i$, then $r^n \in Q_i$ or $r^n \in \mathrm{Rad}(Q_i)$, so in both cases we have $r \in P_i$.
$\supseteq$: if $r \in P_i = \mathrm{Rad}(Q_i)$, then $r^n \cdot x_i \in \bigcap_j Q_j$, i.e. $r^n \in (I : x_i)$. □

B.2.3 Properties

Proposition B.2.15. [[2], 4.7, p.53]
Let $I = \bigcap_{i=1}^{\alpha} Q_i$ be minimal primary decomposition with $P_i = \mathrm{Rad}(Q_i)$. Then

$$P_1 \cup \ldots \cup P_\alpha = \left\{ x \in R \;\middle|\; (I : x) \neq I \right\} = \left\{ x \in R \;\middle|\; \bar{x} \in \mathrm{ZD}(R/I) \right\} .$$

Proof. We always have $I \subseteq (I : x)$. But $(I : x) = \mathrm{Ann}_R(\bar{x})$ by (B.5), thus

$$\begin{aligned} I \subsetneq (I : x) \;\Leftrightarrow\; I \subsetneq \mathrm{Ann}_R(\bar{x}) \;&\Leftrightarrow\; \{\bar{0}\} \subsetneq \mathrm{Ann}_{R/I}(\bar{x}) = \left\{ \bar{r} \in R/I \;\middle|\; \bar{r} \cdot \bar{x} = \bar{0} \right\} \\ &\Leftrightarrow\; \bar{x} \in \mathrm{ZD}(R/I) \;. \end{aligned}$$

To see why the elements of the P_i's are zero-divisors modulo I and vice-versa, note that (B.1) gives the description

$$\mathrm{ZD}(R/I) = \bigcup_{\bar{x} \neq \bar{0}} \mathrm{Rad}\left(\mathrm{Ann}_{R/I}(\bar{x})\right) .$$

From Theorem B.2.14 we know that $P_i = \mathrm{Rad}(\mathrm{Ann}_R(\bar{x}_i))$ for some $x_i \in R \setminus I$. Moreover

$$r \in \mathrm{Rad}(\mathrm{Ann}_R(\bar{x})) \;\Leftrightarrow\; r^n * \bar{x} = \bar{0} \;\Leftrightarrow\; \bar{r}^{\,n} \cdot \bar{x} = \bar{0} \;\Leftrightarrow\; \bar{r} \in \mathrm{Rad}(\mathrm{Ann}_{R/I}(\bar{x}))$$

for all $x \in R \setminus I$. Hence we get $r \in P_1 \cup \ldots \cup P_\alpha \Leftrightarrow \bar{r} \in \mathrm{ZD}(R/I)$. Note that this includes the case $\bar{r} = \bar{0}$ since $I \subseteq \mathrm{Rad}(I) = P_1 \cap \ldots \cap P_\alpha \subseteq P_1 \cup \ldots \cup P_\alpha$. □

Corollary B.2.16. [[2], p.53]
Let R be a ring in which the zero ideal is decomposable with $\{0\} = \bigcap_{i=1}^{\alpha} Q_i$ and denote the corresponding prime ideals by $P_i = \mathrm{Rad}(Q_i)$. Then the set of zero-divisors and nilpotent elements can be described as

$$\mathrm{ZD}(R) = P_1 \cup \ldots \cup P_\alpha \qquad \textit{and} \qquad \mathrm{nil}(R) = P_1 \cap \ldots \cap P_\alpha .$$

Proof. The formula for zero-divisors follows from Proposition B.2.15. For the nilpotent elements, consider

$$\mathrm{nil}(R) = \mathrm{Rad}\left(\{0\}\right) = \mathrm{Rad}(Q_1 \cap \ldots \cap Q_\alpha\big) = P_1 \cap \ldots \cap P_\alpha$$

by using Lemma D.1.4. □

Definition B.2.17. [[2], p.52]
If $I \trianglelefteq R$ is a decomposable ideal with minimal primary decomposition

$$I = Q_1 \cap \ldots \cap Q_\alpha ,$$

the prime ideals $P_i = \mathrm{Rad}(Q_i)$ are called the *primes associated to* I (or *belonging to* I). We denote $\mathrm{Ass}(I) := \{P_1, \ldots, P_\alpha\}$. The minimal elements (with respect to inclusion) of $\mathrm{Ass}(I)$ are called the *minimal primes* (or *isolated primes*) of I. The other ones are called *embedded* prime ideals. The idea behind this terminology is explained in Section 1.2.

Lemma B.2.18. cf.[[2], Ex.1 & 2, p.55]
Let $I \trianglelefteq R$ be a decomposable ideal with $I = \bigcap_{i=1}^{\alpha} Q_i$ and $P_i = \mathrm{Rad}(Q_i)$. Then
1) $\mathrm{Spec}(R/I)$ has finitely many irreducible components (as a topological space).
2) In particular, $\mathrm{Spec}(R/Q)$ is irreducible for every primary ideal $Q \trianglelefteq R$.
3) If I is a radical ideal, then I has no embedded primes, i.e. all elements in $\mathrm{Ass}(I)$ are minimal.

Proof. 1) Recall that we have $V(I) \cup V(J) = V(I \cap J)$ and $V(I) = V\big(\mathrm{Rad}(I)\big)$ as topological spaces. Hence

$$\mathrm{Spec}(R/I) \cong V(I) = V\Big(\bigcap_{i=1}^{\alpha} Q_i\Big) = \bigcup_{i=1}^{\alpha} V(Q_i) = \bigcup_{i=1}^{\alpha} V(P_i) \;, \tag{B.7}$$

where each $V(P_i) \cong \mathrm{Spec}(R/P_i)$ is an irreducible scheme because R/P_i is an integral domain.
2) Take $\alpha = 1$ in (B.7): $V(Q) = V\big(\mathrm{Rad}(Q)\big)$, where $\mathrm{Rad}(Q)$ is a prime ideal.
3) Assume that $I = \bigcap_{i=1}^{\alpha} Q_i$ is a minimal primary decomposition. As I is radical, we obtain another decomposition

$$I = \mathrm{Rad}(I) = \mathrm{Rad}\Big(\bigcap_{i=1}^{\alpha} Q_i\Big) = \bigcap_{i=1}^{\alpha} \mathrm{Rad}(Q_i) = \bigcap_{i=1}^{\alpha} P_i \;.$$

If I has embedded primes, then $\exists\, j,k \in \{1,\dots,\alpha\}$ such that $P_j \subsetneq P_k$ and $I = \bigcap_{i \neq k} P_i$ would contradict minimality of α. Hence there are no embedded primes. □

Proposition B.2.19. [[2], 4.6 & 4.11, p.52-54]
Let $I \trianglelefteq R$ be a decomposable ideal with $I = \bigcap_{i=1}^{\alpha} Q_i$ and $P_i = \mathrm{Rad}(Q_i)$. Then
1) Every prime ideal containing I also contains a minimal prime belonging to I. Hence the minimal prime ideals of I are precisely the minimal ones among all prime ideals containing I.
2) If P_i is a minimal prime of I, then the corresponding Q_i in the decomposition is unique.

Proof. We only prove the first statement. If P is a prime ideal containing the ideal $I = Q_1 \cap \ldots \cap Q_\alpha$, then $Q_1 \cap \ldots \cap Q_\alpha \subseteq P$ implies that

$$\mathrm{Rad}\big(Q_1 \cap \ldots \cap Q_\alpha\big) = P_1 \cap \ldots \cap P_\alpha \subseteq \mathrm{Rad}(P) = P \;.$$

Prime Avoidance (Lemma B.1.3) then says that $P_i \subseteq P$ for some i. In particular, P contains a minimal prime. □

B.2.4 Behaviour under localization

Next we want to see how primary ideal decomposition behaves under localization.

Lemma B.2.20. [[2], 4.8, p.53]
Let $S \subset R$ be a multiplicatively closed subset and $Q \trianglelefteq R$ be P-primary.
1) If $S \cap P \neq \emptyset$, then $S^{-1}Q = S^{-1}R$.
2) If $S \cap P = \emptyset$, then $S^{-1}Q$ is $S^{-1}P$-primary.
3) The primary ideals in $S^{-1}R$ are in 1-to-1 correspondence with primary ideals in R whose radical does not meet S via the bijection $Q \mapsto S^{-1}Q$.

Proposition B.2.21. [[2], 4.9, p.54]
Let $I \trianglelefteq R$ be a decomposable ideal with minimal primary decomposition

$$I = \bigcap_{i=1}^{\alpha} Q_i \ .$$

Denote $P_i = \mathrm{Rad}(Q_i)$ for all i and assume that the Q_i are numbered in such a way that S has empty intersection with all $P_1, \ldots, P_\gamma$ for some $\gamma \in \{1, \ldots, \alpha\}$, but not with $P_{\gamma+1}, \ldots, P_\alpha$. Then

$$S^{-1}I = \bigcap_{i=1}^{\gamma} S^{-1}Q_i$$

is a minimal primary decomposition of the ideal $S^{-1}I \trianglelefteq S^{-1}R$.

Corollary B.2.22. *Let R be a ring in which the zero ideal is decomposable with associated primes $P_1, \ldots, P_\alpha$. So we know that $\mathrm{ZD}(R) = P_1 \cup \ldots \cup P_\alpha$. Let $r \in R$ and $P \trianglelefteq R$ be a prime ideal. Then the sets of zero-divisors in the localizations R_r and R_P are given by*

$$\mathrm{ZD}(R_r) = (P_1)_r \cup \ldots \cup (P_{\gamma_1})_r \qquad , \qquad \mathrm{ZD}(R_P) = (P_1)_P \cup \ldots \cup (P_{\gamma_2})_P$$

for some $\gamma_1, \gamma_2 \leq \alpha$ and the associated primes are numbered in such a way that $r \notin P_1, \ldots, P_{\gamma_1}$, resp. $P_1, \ldots, P_{\gamma_2} \subseteq P$.

B.2.5 The Noetherian case

An interesting question is to know under which conditions an ideal is decomposable. The answer is actually quite easy.

Theorem B.2.23 (Lasker–Noether)**.** [[2], 7.11–7.13, p.83]
If R is a Noetherian ring, then every proper ideal $I \trianglelefteq R$ is decomposable.

Lemma B.2.24. [[2], 7.14 & 7.15, p.83]
1) In a Noetherian ring R, every ideal $I \trianglelefteq R$ contains a power of its radical.
2) In particular, the nilradical of a Noetherian ring is nilpotent.

Proof. 1) Assume that $\mathrm{Rad}(I)$ is generated by $x_1, \ldots, x_k$ with $x_i^{n_i} \in I$ for all $i \in \{1, \ldots, k\}$ and set $m = \sum_{i=1}^{k}(n_i - 1) + 1$. Then $\mathrm{Rad}(I)^m$ is generated by monomials of the form $x_1^{r_1} \cdot \ldots \cdot x_k^{r_k}$ with $r_1 + \ldots + r_k = m$. So by definition of m, there is always at least one index i such that $r_i \geq n_i$. Hence all these monomials lie in I, i.e. $\mathrm{Rad}(I)^m \subseteq I$.
2) Take $I = \{0\}$, so $\exists\, m \in \mathbb{N}$ such that $\mathrm{nil}(R)^m = \mathrm{Rad}(\{0\})^m \subseteq \{0\}$. □

Proposition B.2.25. [[2], 7.17, p.83-84]
Let R be a Noetherian ring and $I \trianglelefteq R$ a proper ideal. Then the primes $P_1, \ldots, P_\alpha$ belonging to I are exactly the prime ideals that occur in the set of ideals

$$\big\{\, (I : x) \mid x \in R \,\big\} \;.$$

For $I = \{0\}$ we hence find element $y_1, \ldots, y_\alpha \in R$ with $y_i \neq 0$ such that

$$P_i = \mathrm{Ann}_R(y_i) = \big\{\, r \in R \;\big|\; r \cdot y_i = 0 \,\big\} \,, \quad \forall\, i \in \{1, \ldots, \alpha\} \;.$$

Proof. Let $I = \bigcap_i Q_i$ be a minimal primary decomposition with $\mathrm{Rad}(Q_i) = P_i$. If we fix i and denote $I_i := \bigcap_{j \neq i} Q_j$, then $I \subsetneq I_i$ as the decomposition is minimal. As in the proof of Theorem B.2.14 we have

$$\mathrm{Rad}(I : x) = \mathrm{Rad}\big(\mathrm{Ann}_R(\bar{x})\big) = P_i \,,\; \forall\, x \in I_i \setminus I \;, \tag{B.8}$$

hence $\mathrm{Ann}_R(\bar{x}) \subseteq P_i$. Since Q_i is P_i-primary, Lemma B.2.24 implies that P_i^m is contained in Q_i for some $m \in \mathbb{N}$ and hence

$$I_i \cdot P_i^m \subseteq I_i \cap P_i^m \subseteq I_i \cap Q_i = I \;.$$

Let $m \geq 1$ be the smallest integer such that $I_i \cdot P_i^m \subseteq I$ and let $x \in I_i \cdot P_i^{m-1}$ be such that $\bar{x} \neq \bar{0}$. So in particular $x \in I_i \setminus I$ and we have (B.8). But then $P_i \cdot x \subseteq I$ and $P_i \subseteq \mathrm{Ann}_R(\bar{x})$. So we showed that for each fixed i, the associated prime P_i can be written as some annihilator ideal.
Conversely, if $\mathrm{Ann}_R(\bar{x})$ for $x \in R$ is a prime ideal P, then $\mathrm{Rad}(\mathrm{Ann}_R(\bar{x})) = P$ and hence by Theorem B.2.14 this P is a prime ideal belonging to I. □

So in the case of a Noetherian ring we do no longer need to take radicals of annihilators for the associated primes. On the other hand, similarly as in Remark B.1.2, the elements $x_i, y_i \in R$ in the descriptions $P_i = \mathrm{Rad}(\mathrm{Ann}_R(\bar{x}_i))$ and $P_i = \mathrm{Ann}_R(\bar{y}_i)$ do not need to be the same. Also note that the proof of Proposition B.2.25 is independent of Theorem B.2.23 (once we know that I is decomposable).

We finish the section by an important corollary.

Corollary B.2.26. [[46], 13.23, p.431-432]
If R is a Noetherian ring and $I \trianglelefteq R$ entirely consists of zero-divisors, then $\mathrm{Ann}_R(I) \neq \{0\}$. *Hence if an ideal satisfies* $\mathrm{Ann}_R(I) = \{0\}$, *then it must contain an element which is not a zero-divisor.*

Proof. Let $P_1, \ldots, P_\alpha$ be the associated primes of $\{0\}$. Saying that I entirely consists of zero-divisors means that $I \subseteq P_1 \cup \ldots \cup P_\alpha$ by Proposition B.2.15. Thus $I \subseteq P_i$ for some $i \in \{1, \ldots, \alpha\}$ by Prime Avoidance and Proposition B.2.25 gives a non-zero element $y \in R$ such that $P_i = \mathrm{Ann}_R(y)$. But then $I \subseteq \mathrm{Ann}_R(y)$ and hence $y \in \mathrm{Ann}_R(I)$. It follows that $\{0\} \neq \langle\, y \,\rangle \subseteq \mathrm{Ann}_R(I)$. □

Remark B.2.27. One could think of proving this result by saying that I is finitely generated, i.e. $I = \langle\, r_1, \ldots, r_n \,\rangle$ since R is Noetherian and where each r_i is a zero-divisor, so $\exists\, s_1, \ldots, s_n \in R$ such that $r_i \cdot s_i = 0$, $\forall\, i \in \{1, \ldots, n\}$ and $s := s_1 \cdot \ldots \cdot s_n$ would satisfy $s \cdot I = \{0\}$. But this does not work since maybe $s = 0$. For example, consider the Noetherian ring

$$R = \mathbb{K}[X, Y, Z, T] / \langle\, XT, YT, ZT, XY \,\rangle \ .$$

The associated primes of $\{\bar{0}\}$ are $P_1 = \langle \bar{X}, \bar{T} \rangle$, $P_2 = \langle \bar{Y}, \bar{T} \rangle$ and $P_3 = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$, hence the ideal $I = \langle \bar{Y}, \bar{Z} \rangle \subseteq P_3$ only consists of zero-divisors. We also have $\bar{Y} \cdot \bar{X} = \bar{0}$ and $\bar{Z} \cdot \bar{T} = \bar{0}$, so $s_1 = \bar{X}$ and $s_2 = \bar{T}$, but $s_1 \cdot s_2 = \bar{X} \cdot \bar{T} = \bar{0}$.

B.3 Associated primes of a module

The idea of defining associated primes of an R-module M is to generalize the notion of the associated primes of an ideal $I \trianglelefteq R$ (as every ideal can also be seen as a module over R). The main references here are Igusa [42], Chapter 5.3, Matsumura [54], Chapter 3, Section 7, Matsumura [55], Chapter 2, Section 6 and Bourbaki [6], Chapter IV, Section §1.

B.3.1 Associated primes

Let R be a ring and M a module over R. The definition of an associated prime of M is inspired from the one of the prime ideals associated to the zero ideal in a Noetherian ring.

Definition B.3.1. [[42], 5.10, p.29] and [[54], 7.A, p.49]
We say that a prime ideal $P \trianglelefteq R$ is an *associated prime* of M if there exists an element $x \in M$ such that $P = \mathrm{Ann}_R(x)$. Hence the associated primes of M are the prime ideals in the set

$$\mathcal{M} = \left\{ \mathrm{Ann}_R(x) \;\middle|\; x \in M, \, x \neq 0 \right\} . \tag{B.9}$$

Note that we can remove $x = 0$ since $\mathrm{Ann}_R(0) = R$ is not prime. The set of associated primes of M is denoted by $\mathrm{Ass}_R(M)$.

Remark B.3.2. Thus if R is a Noetherian ring, we have $\mathrm{Ass}(I) = \mathrm{Ass}_R(R/I)$ for any ideal $I \trianglelefteq R$ since $P \in \mathrm{Ass}(I)$ is given as $P = \mathrm{Ann}_R(\bar{x})$ for some $\bar{x} \in R/I$. In particular $\mathrm{Ass}\big(\{0\}\big) = \mathrm{Ass}_R(R)$.

Lemma B.3.3. [[42], 5.10 & 5.11, p.29] and [[54], 7.A, p.49]
1) A prime ideal $P \trianglelefteq R$ belongs to $\mathrm{Ass}_R(M)$ if and only if there exists an injection of R-modules $R/P \hookrightarrow M$.

2) If $P \trianglelefteq R$ is a prime ideal and $M \leq R/P$ a non-zero submodule, then $\mathrm{Ass}_R(M) = \{P\}$.

Proof. 1) $\underline{\Rightarrow}$: If $P = \mathrm{Ann}_R(x)$ for some $x \in M$, we can consider the morphism $R/P \to M : \bar{r} \mapsto r * x$, which is well-defined and injective since

$$r * x = 0 \;\Leftrightarrow\; r \in \mathrm{Ann}_R(x) \;\Leftrightarrow\; \bar{r} = \bar{0} \;.$$

$\underline{\Leftarrow}$: Let $\varphi : R/P \hookrightarrow M$ and $x = \varphi(\bar{1})$. Then $P = \mathrm{Ann}_R(x)$ by injectivity of φ:

$$\begin{aligned} r \in \mathrm{Ann}_R(x) \;&\Leftrightarrow\; r * \varphi(\bar{1}) = 0 \;\Leftrightarrow\; \varphi(r * \bar{1}) = 0 \;\Leftrightarrow\; \varphi(\bar{r}) = 0 \\ &\Leftrightarrow\; \bar{r} \in \ker\varphi \;\Leftrightarrow\; \bar{r} = \bar{0} \;\Leftrightarrow\; r \in P \;. \end{aligned}$$

2) If $M = R/P$ this follows from Remark B.3.2 as P is already primary. In general we have to show that there are no other associated primes of M. For any $\bar{m} \in M$ with $\bar{m} \neq \bar{0}$, we get $\mathrm{Ann}_R(\bar{m}) = P$ because

$$r \in \mathrm{Ann}_R(\bar{m}) \;\Leftrightarrow\; r * \bar{m} = \bar{0} \;\Leftrightarrow\; r \cdot m \in P \;\Leftrightarrow\; r \in P \quad (\text{since } m \notin P) \;,$$

so $P \in \mathrm{Ass}_R(M)$ and this is the only associated prime since every $\bar{m}$ has the same annihilator. □

Proposition B.3.4. [[54], 7.B, p.49-50] and [[6], IV.§1.n°1.Prop.2, p.308]
1) If P is a maximal element in the set $\mathcal{M}$ from (B.9)*, then $P \in \mathrm{Ass}_R(M)$.*
2) Let R be a Noetherian ring. Then $\mathrm{Ass}_R(M) = \emptyset$ if and only if $M = \{0\}$.

Proof. 1) We show a maximal element $P \in \mathcal{M}$ is prime. Let $P = \mathrm{Ann}_R(x)$ for some $x \in M$, $x \neq 0$ and $r \cdot s \in P$. Assume that $s \notin P$, i.e. $s * x \neq 0$, but $0 = (r \cdot s) * x = r * (s * x)$, hence $r \in \mathrm{Ann}_R(s * x)$. On the other hand, we have $P = \mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(s * x)$. By maximality of P, we thus get

$$r \in \mathrm{Ann}_R(s * x) = P \;.$$

2) If $M = \{0\}$, then $\mathcal{M}$ is empty since $\mathrm{Ann}_R(0) = R$ is not prime, hence $\{0\}$ has no associated primes. Conversely, assume that $M \neq \{0\}$, so $\mathcal{M}$ is a non-empty set of ideals. R being Noetherian, it thus contains a maximal element, which is prime by 1) and hence belongs to $\mathrm{Ass}_R(M)$. □

Proposition B.3.5. [[42], 5.22, p.33] , [[54], 7.F, p.51-52] and [[55], 6.3, p.38]
Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

be a short exact sequence of R-modules. Then

$$\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(N) \cup \mathrm{Ass}_R(L) \ .$$

Proof. First we use Lemma B.3.3. The inclusion $\mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M)$ holds because R/P can also be embedded into M if it can be embedded into N.
Now let $P \in \mathrm{Ass}_R(M)$ with $P = \mathrm{Ann}_R(x)$ for some $x \in M$, so that we have an injection $R/P \hookrightarrow M$. Then $R/P \cong \langle\, x \,\rangle$ since the injection is given by $\bar{r} \mapsto r * x$ (Lemma B.3.3). Now there are 2 cases:
If $\langle\, x \,\rangle \cap N = \{0\}$, then $R/P \hookrightarrow M/N : \bar{r} \mapsto \overline{r * x}$ because

$$\overline{r * x} = \bar{0} \ \Leftrightarrow \ r * x \in N \ \Leftrightarrow \ r * x = 0 \ \Leftrightarrow \ r \in \mathrm{Ann}_R(x) = P \ \Leftrightarrow \ \bar{r} = \bar{0}$$

with $L \cong M/N$, thus $R/P \hookrightarrow L$ and $P \in \mathrm{Ass}_R(L)$. If $\langle\, x \,\rangle \cap N \neq \{0\}$, then $\exists\, y \in N$ such that $y = a * x \neq 0$ for some $a \in R$. But then $P = \mathrm{Ann}_R(y)$ since $a \notin \mathrm{Ann}_R(x) = P$ and

$$r \in \mathrm{Ann}_R(y) \ \Leftrightarrow \ r * y = r * (a * x) = 0 \ \Leftrightarrow \ r \cdot a \in \mathrm{Ann}_R(x) = P \ \Leftrightarrow \ r \in P \ .$$

Hence $P \in \mathrm{Ass}_R(N)$. We get $\mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(N) \cup \mathrm{Ass}_R(L)$ in both cases. □

Corollary B.3.6. [[42], 5.23, p.33] and [[6], IV.§1.n°1.Cor.1, p.309]
1) If M and N are two R-modules, then $\mathrm{Ass}_R(M \oplus N) = \mathrm{Ass}_R(M) \cup \mathrm{Ass}_R(N)$.
2) More generally, for a finite family of R-modules $\{M_i\}_{i=1,\dots,n}$ we have

$$\mathrm{Ass}_R\Big(\bigoplus_{i=1}^{n} M_i\Big) = \bigcup_{i=1}^{n} \mathrm{Ass}_R(M_i) \ . \tag{B.10}$$

Proof. 1) Consider the exact sequence

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0 \ .$$

Proposition B.3.5 gives the inclusions

$$\mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(M \oplus N) \qquad , \qquad \mathrm{Ass}_R(N) \subseteq \mathrm{Ass}_R(M \oplus N) \ ,$$
$$\mathrm{Ass}_R(M \oplus N) \subseteq \mathrm{Ass}_R(N) \cup \mathrm{Ass}_R(N) \ .$$

Hence both sets are equal. 2) then follows by induction. □

The behaviour of associated primes under localization is given as follows.

Proposition B.3.7. [[54], 7.C, p.50], [[55], p.38], [[6], IV.§1.n°2.Prop.5, p.310]
Let R be Noetherian and $S \subset R$ a multiplicatively closed subset. The assignment $P \mapsto S^{-1}P$ gives a bijection between prime ideals in $\mathrm{Ass}_R(M)$ not intersecting S and $\mathrm{Ass}_{S^{-1}R}(S^{-1}M)$. More precisely,

$$\mathrm{Ass}_R(S^{-1}M) = \left\{ P \in \mathrm{Ass}_R(M) \;\middle|\; P \cap S = \emptyset \right\} ,$$
$$\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = S^{-1}\big(\mathrm{Ass}_R(S^{-1}M)\big) .$$

Remark B.3.8. Proposition B.3.7 does not hold true if the ring is not Noetherian. In general the map $P \mapsto S^{-1}P$ may not be surjective ; an example is given in [[6], Ex.1, p.339].

Another reason why it is useful to consider Noetherian rings is the following.

Theorem B.3.9. [[54], 7.G, p.51], [[55], 6.5, p.39], [[6], IV.§1.n°4.Cor, p.313]
If M is a finitely generated module over a Noetherian ring R, then $\mathrm{Ass}_R(M)$ is finite.

B.3.2 Relations with the support

Definition B.3.10. The *support* of an R-module M is the set of all prime ideals with non-zero localization, i.e.

$$\mathrm{supp}\, M = \left\{ P \trianglelefteq R \text{ prime} \;\middle|\; M_P \neq \{0\} \right\} .$$

This is equal to the support of the quasi-coherent sheaf $\widetilde{M}$ on $\mathcal{X} = \mathrm{Spec}\, R$, hence the name.

Proposition B.3.11. [[61], Thm.13, p.42-43]
If M is a finitely generated R-module, then the prime ideals in $\mathrm{supp}\, M$ are precisely the prime ideals that contain $\mathrm{Ann}_R(M)$.

Proof. If P is a prime ideal such that $M_P \neq \{0\}$, then $\mathrm{Ann}_R(M) \subseteq P$, otherwise $\exists\, r \in \mathrm{Ann}_R(M) \setminus P$ with $r * m = 0$ and hence $\frac{m}{1} = 0$ for all $m \in M$ since $r \notin P$.

Vice-versa let $\mathrm{Ann}_R(M) \subseteq P$ and assume that $M_P = \{0\}$. If $m_1, \dots, m_n$ are generators of M, then $\frac{m_i}{1} = 0$ for all i, so there are elements $\exists\, r_1, \dots, r_n \in R \setminus P$ such that $r_i * m_i = 0$, $\forall\, i$. We set $r := r_1 \cdot \ldots \cdot r_n$. Then $r * m = 0$ for all $m \in M$, i.e. $r \in \mathrm{Ann}_R(M)$. Moreover $r \notin P$ since P is a prime ideal, so in particular $r \neq 0$. But this contradicts the assumption that $\mathrm{Ann}_R(M) \subseteq P$. Hence we need $M_P \neq \{0\}$. □

Proposition B.3.12. [[42], 5.18, p.31]
Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

be a short exact sequence of R-modules. Then $\operatorname{supp} M = \operatorname{supp} N \cup \operatorname{supp} L$.

Proof. By exactness of the localization functor (Proposition A.2.6), we have

$$\begin{aligned} P \in \operatorname{supp} M \;\Leftrightarrow\; M_P \neq \{0\} \;&\Leftrightarrow\; N_P \neq \{0\} \text{ or } L_P \neq \{0\} \\ &\Leftrightarrow\; P \in \operatorname{supp} N \text{ or } P \in \operatorname{supp} L\,. \end{aligned}$$

□

Proposition B.3.13. [[6], II.§4.n°4.Prop.18, p.134]
If M and N are two finitely generated R-modules, then

$$\operatorname{supp}\big(M \otimes_R N\big) = \operatorname{supp} M \cap \operatorname{supp} N\,. \tag{B.11}$$

Proof. By Proposition A.2.9 we have $(M \otimes_R N)_P \cong M_P \otimes_{R_P} N_P$ for all prime ideals $P \trianglelefteq R$. This already gives the inclusion $\subseteq$ since a tensor product being non-zero implies that both factors are non-zero as well. In order to prove formula (B.11), it now suffices to show the following:
Let R be a local ring and E, F two finitely generated R-modules such that $E \neq \{0\}$ and $F \neq \{0\}$. Then $E \otimes_R F \neq \{0\}$. This can e.g. be found in [[6], II.§4.n°4.Lemma.3, p.134]. □

(B.11) allows to prove the following interesting criterion to decide whether a tensor product is zero. The proof however uses some facts from Proposition 1.4.4.

Corollary B.3.14. [[53], 93228]
Let M and N be two finitely generated modules over R. Then

$$M \otimes_R N = \{0\} \;\Leftrightarrow\; \operatorname{Ann}_R(M) + \operatorname{Ann}_R(N) = R \,.$$

Proof. For sufficiency, assume that $1 \in R$ can be written as $1 = r + s$ for $r \in \operatorname{Ann}_R(M)$ and $s \in \operatorname{Ann}_R(N)$. Then for all $m \otimes n \in M \otimes_R N$,

$$m \otimes n = 1 \cdot (m \otimes n) = (r+s) \cdot (m \otimes n) = \big((r * m) \otimes n\big) + \big(m \otimes (s * n)\big) = 0 \,.$$

Necessity can be proven directly, but it is easier using formula (B.11). We have

$$\begin{aligned} \mathcal{Z} := V\big(\operatorname{Ann}_R(M \otimes_R N)\big) &= \operatorname{supp}\big(M \otimes_R N\big) = \operatorname{supp} M \cap \operatorname{supp} N \\ &= V\big(\operatorname{Ann}_R(M)\big) \cap V\big(\operatorname{Ann}_R(N)\big) = V\big(\operatorname{Ann}_R(M) + \operatorname{Ann}_R(N)\big) \,. \end{aligned}$$

Hence $M \otimes_R N = \{0\} \Leftrightarrow \operatorname{Ann}_R(M \otimes_R N) = R \Leftrightarrow \mathcal{Z} = \emptyset$, which is the case if and only if $\operatorname{Ann}_R(M) + \operatorname{Ann}_R(N) = R$. ☐

The relation between the associated primes and the support of a finitely generated R-module M is the content of the next two results.

Theorem B.3.15. [[42], 5.19, p.31-32] , [[54], 7.D, p.50-51] and [[55], 6.5, 39]

$$\operatorname{Ass}_R(M) \subseteq \operatorname{supp} M \,.$$

If R is moreover Noetherian, then the minimal elements of $\operatorname{supp} M$ *are associated primes of M.*

Proof. We only prove the inclusion. If $P \in \operatorname{Ass}_R(M)$, we have the short exact sequences

$$0 \longrightarrow R/P \xrightarrow{\varphi} M \longrightarrow \operatorname{coker}\varphi \longrightarrow 0 \,,$$
$$0 \longrightarrow (R/P)_P \longrightarrow M_P \longrightarrow (\operatorname{coker}\varphi)_P \longrightarrow 0 \,.$$

Now we have $(R/P)_P \neq \{0\}$ since

$$\tfrac{\bar r}{s} = \tfrac{\bar 0}{1} \;\Leftrightarrow\; \exists\, b \notin P \text{ such that } b * \bar r = \bar 0 \;\Leftrightarrow\; b \cdot r \in P \;\Leftrightarrow\; r \in P \;\Leftrightarrow\; \bar r = \bar 0 \,.$$

Hence $M_P \neq \{0\}$ as well, i.e. $P \in \operatorname{supp} M$. ☐

Corollary B.3.16. [[6], IV.§1.n°3.Cor.1, p.312] and [[55], 6.5, p.39]
If R is Noetherian, the minimal primes of $\operatorname{Ass}_R(M)$ *and* $\operatorname{supp} M$ *are the same.*

Proof. A minimal prime $P \in \operatorname{Ass}_R(M)$ is in $\operatorname{supp} M$ and it is minimal there as well, otherwise a minimal prime contained in P would also belong to $\operatorname{Ass}_R(M)$, so that P is no longer minimal. Conversely, if P is minimal in $\operatorname{supp} M$, then it belongs to $\operatorname{Ass}_R(M)$ by Theorem B.3.15 and is minimal there as well, otherwise an associated prime contained in P would also belong to $\operatorname{supp} M$ and P would no longer be minimal. □

Corollary B.3.17. [[42], 5.20, p.32-33]
If R is a Noetherian ring and M is finitely generated, then

$$\bigcap_{P \in \operatorname{Ass}_R(M)} P \;=\; \bigcap_{P \in \operatorname{supp} M} P \;= \operatorname{Rad}\big(\operatorname{Ann}_R(M)\big)\ .$$

Proof. The intersections are equal since $\operatorname{Ass}_R(M)$ and $\operatorname{supp} M$ have the same minimal primes by Corollary B.3.16. Also note that $\operatorname{Ass}_R(M)$ is finite by Theorem B.3.9, while $\operatorname{supp} M$ is usually not.
$\supseteq$: We have $\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(x)$ for all $x \in M$, so $\operatorname{Ann}_R(M)$ is contained in every associated prime of M, hence so is its radical.
$\subseteq$: Let $\{m_1, \ldots, m_n\}$ be a finite generating set of M and $r \notin \operatorname{Rad}(\operatorname{Ann}_R(M))$. Hence by Lemma D.1.3 there exists a prime ideal P containing $\operatorname{Ann}_R(M)$ such that $r \notin P$. Assume that $M_P = \{0\}$. Then $\forall\, i$, $\exists\, b_i \notin P$ such that $b_i * m_i = 0$ and the element $b := b_1 \cdot \ldots \cdot b_n \notin P$ satisfies $b * m = 0$, $\forall\, m \in M$, hence $b \in \operatorname{Ann}_R(M) \subseteq P$: contradiction. So we get $M_P \neq \{0\}$ and $P \in \operatorname{supp} M$. But since $r \notin P$, it cannot belong to the intersection neither. □

Remark B.3.18. [[6], IV.§1.n°3.Cor.2, p.312]
In particular, the nilradical of a Noetherian ring R is equal to the intersection of all prime ideals $P \in \operatorname{Ass}_R(R)$ because $\operatorname{Ann}_R(R) = \{0\}$. Alternatively this can be seen by the fact that $\operatorname{nil}(R)$ is the intersection of all prime ideals in R (Lemma D.1.3). But $\operatorname{Spec} R = \operatorname{supp} R$ since $R_P \neq \{0\}$ for all primes P, thus

$$\operatorname{nil}(R) = \bigcap_{P \text{ prime}} P = \bigcap_{P \in \operatorname{Spec} R} P = \bigcap_{P \in \operatorname{supp} R} P = \bigcap_{P \in \operatorname{Ass}_R(R)} P\ .$$

Appendix C

Complements on torsion and modules

In this appendix we want to analyze the notion of torsion. In the integral case, torsion elements of a module are just elements that are annihilated by some non-zero element from the ring. But this definition does no longer work when zero-divisors are involved. In the following we develop some basic properties about torsion elements in the general case. We also compare several notions of torsion and study their relation with reflexive and projective modules. Finally we establish some properties which hold true for integral domains and give counter-examples in the non-integral case. The notion of torsion is due to H. Bass.

Definition C.0.1. Let R be a ring and M a module over R. The set of all *torsion elements* of M is

$$\mathcal{T}_R(M) = \left\{ m \in M \;\middle|\; \exists\, r \in R,\ r \neq 0 \text{ such that } r \text{ is not a zero-divisor and } r * m = 0 \right\} .$$

In the following we write NZD for elements that are not zero-divisors (sometimes such elements are also called *regular*). $\mathcal{T}_R(M)$ is a submodule of M, called the *torsion submodule* of M. Indeed, 0 is a torsion element since $1 * 0 = 0$ and if $r, s \in \mathcal{T}_R(M)$ with $r * m = 0$ and $s * n = 0$, then

$$(r \cdot s) * (m + n) = 0 \quad \text{and} \quad r * (t * m) = 0,\, \forall\, t \in R\,,$$

where $r \cdot s$ is non-zero and still a NZD.

Remark C.0.2. The condition about including NZDs in the definition of $\mathcal{T}_R(M)$ is necessary, otherwise it may not be a submodule. Omitting "$r \neq 0$ is a NZD" is only possible if R is an integral domain. Consider the 2 following examples:
1) [[4], II.§7.n°10, p.115]
Take $R = \mathbb{Z}/6\mathbb{Z}$ and $M = R$ with $m = \bar{3}$ and $n = \bar{4}$. Then $\bar{2} * \bar{3} = \bar{0}$ and $\bar{3} * \bar{4} = \bar{0}$, where $\bar{2}$ and $\bar{3}$ are zero-divisors, but $\bar{3} + \bar{4} = \bar{1}$ and $\bar{1}$ cannot be annihilated by any non-zero element.
2) $R = \mathbb{K}[X,Y]/\langle\, XY \,\rangle$ and $M = R$ with $m = \bar{X}$ and $n = \bar{Y}$; they are annihilated by zero-divisors

$$\bar{Y} * m = \bar{Y} \cdot \bar{X} = \bar{0} \qquad , \qquad \bar{X} * n = \bar{X} \cdot \bar{Y} = \bar{0} \,,$$

but no element from R can annihilate $m + n$ since $\bar{X}\bar{Y}$ is zero (rigorous proof : we have the primary decomposition

$$\langle\, XY \,\rangle = \langle\, X \,\rangle \cap \langle\, Y \,\rangle \,,$$

so by Proposition B.2.10 the associated primes of $\{\bar{0}\}$ in R are $\langle\, \bar{X} \,\rangle$ and $\langle\, \bar{Y} \,\rangle$ and the zero-divisors of R are given by $\langle\, \bar{X} \,\rangle \cup \langle\, \bar{Y} \,\rangle$, see Proposition B.2.15, to which $m + n = \bar{X} + \bar{Y}$ does not belong).

C.1 Torsion-free modules

Definition C.1.1. An R-module M is called *torsion-free* if it contains no non-zero torsion elements, i.e. if $\mathcal{T}_R(M) = \{0\}$. We say that M *has torsion* if it is not torsion-free. If $\mathcal{T}_R(M) = M$, then M is called a *torsion module*. By convention, the zero module $\{0\}$ is considered to be a torsion module.
A submodule $N \leq M$ is a *torsion submodule* if it is itself a torsion module over R, i.e. $\mathcal{T}_R(N) = N$.

Lemma C.1.2. *1) $\{0\}$ and $\mathcal{T}_R(M)$ are always torsion submodules of M.*
2) $N \leq M$ is a torsion submodule $\Leftrightarrow$ $N \subseteq \mathcal{T}_R(M)$.
3) If $\operatorname{Ann}_R(M)$ contains a NZD, then M is a torsion module.
4) The converse of 3) holds true if M is finitely generated.
5) M is torsion-free $\Leftrightarrow$ $\{0\}$ is the only torsion submodule of M.

Proof. 1) clear for $\{0\}$; moreover

$$\mathcal{T}_R\big(\mathcal{T}_R(M)\big) = \big\{\, m \in \mathcal{T}_R(M) \;\big|\; \exists\, r \in R,\, r \neq 0 \text{ which is a NZD such that } r * m = 0 \,\big\} = \mathcal{T}_R(M)\ .$$

2)

$$\begin{aligned} \mathcal{T}_R(N) = N \;&\Leftrightarrow\; \forall\, n \in N,\, \exists \text{ a NZD } r \in R,\, r \neq 0 \text{ such that } r * n = 0 \\ &\Leftrightarrow\; N \subseteq \mathcal{T}_R(M)\ . \end{aligned}$$

3) If $\mathrm{Ann}_R(M)$ contains a NZD r, then $r * m = 0$, $\forall\, m \in M$, so every $m \in M$ is a torsion element.
4) Let $\{m_1, \ldots, m_n\}$ be a finite generating set of M. As M is a torsion module, there are NZDs $r_1, \ldots, r_n \in R$ such that $r_i * m_i = 0$, $\forall\, i$. Set $r := r_1 \cdot \ldots \cdot r_n$; then r is also a NZD (otherwise some r_i would be a zero-divisor) and $r * m = 0$ for all m since every $m \in M$ is an R-linear combination of the m_i. Thus $r \in \mathrm{Ann}_R(M)$.
5) $\underline{\Rightarrow}$: If $\mathcal{T}_R(M) = \{0\}$, then every torsion submodule satisfies $N \subseteq \{0\}$ by 2).
$\underline{\Leftarrow}$: Assume that $\exists\, m \in \mathcal{T}_R(M)$ with $m \neq 0$ and consider the submodule $N = \langle\, m \,\rangle$. Let $r \in R$ be a NZD such that $r * m = 0$. Hence $r \in \mathrm{Ann}_R(N)$ and 3) implies that N is a non-zero torsion submodule of M : contradiction. □

Remark C.1.3. A counter-example which shows that 4) may fail for modules that are not finitely generated is given in C.1.6.

Proposition C.1.4. [[4], II.§7.n°10.Prop.25, p.116]
Let $\{M_i\}_{i\in I}$ be a (not necessarily finite) family of R-modules. Then

$$\mathcal{T}_R\big(\textstyle\bigoplus_i M_i\big) = \textstyle\bigoplus_i \mathcal{T}_R(M_i)\ .$$

Proof. $\underline{\subseteq}$: Let $\{m_i\}_i$ be a torsion element with a NZD $r \in R$ which satisfies $r * \{m_i\}_i = 0 \Leftrightarrow r * m_i = 0$ for all i. Thus $m_i \in \mathcal{T}_R(M)$, $\forall\, i$.
$\underline{\supseteq}$: Let $\{m_i\}_i$ be such that $m_i \in \mathcal{T}_R(M)$, $\forall\, i$. Hence there are NZDs $r_i \in R$ such that $r_i * m_i = 0$, $\forall\, i$ (for $m_i = 0$, one may choose $r_i = 1$). Define $r := \prod_i r_i$; then r is a NZD such that $r * \{m_i\}_i = 0$. □

Example C.1.5. [[4], II.§7.n°10, p.115-116] and [[11], 15.16.12]
1) Free modules are torsion-free. In particular, vector spaces are torsion-free.
2) Direct sums and submodules of torsion(-free) modules are again torsion(-free).
3) In particular, projective modules are torsion-free.
4) $M/\mathcal{T}_R(M)$ and $M^* = \mathrm{Hom}_R(M,R)$ are always torsion-free.
5) More generally, if N is torsion-free, then $\mathrm{Hom}_R(M,N)$ is torsion-free as well.
6) Infinite products of torsion-free modules are again torsion-free. In particular, R^I is torsion-free.

Proof. 1) Let $\{e_i\}_{i\in I}$ be a basis of $M \cong R^{(I)}$. If $m \in \mathcal{T}_R(M)$ is a torsion element with NZD $r \in R$, then $\exists\, a_i \in R$ such that $m = \sum_i a_i * e_i$ and only finitely many terms are non-zero. Moreover

$$0 = r * m = \sum_{i\in I} (r \cdot a_i) * e_i \;\Rightarrow\; r \cdot a_i = 0,\ \forall\, i \in I\ ,$$

which implies that $a_i = 0$, $\forall\, i \in I$ since r is a NZD. Thus $m = 0$.
2) a) For directs sums, Proposition C.1.4 gives

$$\mathcal{T}_R\big(\textstyle\bigoplus_i M_i\big) = \bigoplus_i \mathcal{T}_R(M_i) = \bigoplus_i M_i \quad \text{or} \quad \mathcal{T}_R\big(\bigoplus_i M_i\big) = \bigoplus_i \mathcal{T}_R(M_i) = \{0\}\ .$$

b) Let $N \leq M$ and $n \in N$. If M is a torsion module, there is a NZD $r \in R$ such that $r * n = 0$, thus $n \in \mathcal{T}_R(N)$ and N is a torsion module itself. If M is torsion-free, there cannot be a NZD annihilating n for $n \neq 0$, otherwise this n would be a torsion element in M.
3) since projective modules are direct summands (hence submodules) of free modules.
4) a) Let $r \in R$, $r \neq 0$ which is a NZD such that $r * \bar{m} = \bar{0}$ in $M/\mathcal{T}_R(M)$. Then

$$\begin{aligned} r * \bar{m} = \bar{0} \;&\Leftrightarrow\; r * m \in \mathcal{T}_R(M) \\ &\Leftrightarrow\; \exists\, s \in R,\ s \neq 0,\ s \text{ is a NZD such that } s * (r * m) = 0\ , \end{aligned}$$

thus $(s \cdot r) * m = 0$, where $s \cdot r \neq 0$ and $s \cdot r$ is still a NZD. It follows that $m \in \mathcal{T}_R(M)$ and $\bar{m} = \bar{0}$.
b) Let $f \in \mathcal{T}_R(M^*)$, i.e. $f : M \to R$ and there is a NZD $r \in R$ such that $r * f = 0$, which means that $r \cdot f(m) = 0$ for all $m \in M$. As r is a NZD, this

implies that $f(m) = 0$, $\forall\, m \in M$, hence $f = 0$ and $\mathcal{T}_R(M^*) = \{0\}$.
5) Let $f : M \to N$ be a R-module homomorphism such that $r * f = 0$ for some NZD $r \in R$, which means that $r * f(m) = 0$, $\forall\, m \in M$. r being a NZD and N being torsion-free, we need that $f(m) = 0$, $\forall\, m \in M$, i.e. $f = 0$.
6) The same proof as in Proposition C.1.4 shows that $\mathcal{T}_R\big(\prod_i M_i\big) \subseteq \prod_i \mathcal{T}_R(M_i)$, hence if all M_i are torsion-free, then so is their product. □

Example C.1.6. cf. [[4], II.§7.n°10, p.115-116 & Ex.31, p.197]
1) Let R be an integral domain. Then the quotient field $K = \mathrm{Quot}(R)$ is a torsion-free R-module.
2) $\forall\, n \geq 2$, $\mathbb{Z}/n\mathbb{Z}$ is a torsion module over $\mathbb{Z}$.
3) More generally, all finite $\mathbb{Z}$-modules (finite abelian groups) are torsion modules.
4) If M and N are torsion-free R-modules, then $M \otimes_R N$ may not be torsion-free any more.
5) Infinite products of torsion modules may no longer be torsion modules.
6) There exists (infinitely generated) torsion modules with zero annihilator.

Proof. 1) The R-module structure on K is defined by $r * \frac{a}{b} = \frac{r \cdot a}{b}$. Let $r \neq 0$ (so it is a NZD) and

$$r * \tfrac{a}{b} = 0 \;\Leftrightarrow\; \tfrac{r \cdot a}{b} = \tfrac{0}{1} \;\Leftrightarrow\; \exists\, s \in R,\, s \neq 0 \text{ such that } s \cdot r \cdot a = 0 \;\Rightarrow\; a = 0$$

since R is an integral domain, hence $\frac{a}{b} = 0$ and $\mathcal{T}_R(K) = \{0\}$.
2) Note that $\bar{1}$ is a generator of $\mathbb{Z}/n\mathbb{Z}$ as a module over $\mathbb{Z}$, so it suffices to find a non-zero element (which is hence a NZD as $\mathbb{Z}$ is integral) that annihilates $\bar{1}$. But $n * \bar{1} = \bar{n} = \bar{0}$. More precisely, we have $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = n\mathbb{Z}$, which contains infinitely many NZDs.
3) Let G be any finite abelian group. By Fermat's Theorem, we know that $g^{|G|} = e$, $\forall\, g \in G$. Using the additive notation, this yields $|G| * g = 0$, $\forall\, g \in G$, hence $\mathcal{T}_{\mathbb{Z}}(G) = G$.
4) For example, let $\mathbb{K}$ be a field, $R = \mathbb{K}[X, Y]$, which is an integral domain, and $M = \langle\, X, Y \,\rangle$. Being a submodule of a free module (the ring R as a module over itself), we get that M is torsion-free. But $M \otimes_R M$ is not torsion-free any more: consider the element $X \otimes Y - Y \otimes X$. It is non-zero because $1 \notin M$. However

it is a torsion element as it is annihilated by $X \neq 0$:

$$X * (X \otimes Y - Y \otimes X) = X * (X \otimes Y) - X * (Y \otimes X)$$
$$= X \otimes XY - XY \otimes X = Y * (X \otimes X) - Y * (X \otimes X) = 0 \ .$$

5) The inclusion $\mathcal{T}_R\big(\prod_i M_i\big) \subseteq \prod_i \mathcal{T}_R(M_i)$ can be strict if all M_i are torsion modules. For example, let $I = \mathbb{N}$, $R = \mathbb{Z}$ and $M_n = \mathbb{Z}/n\mathbb{Z}$, $\forall\, n \in \mathbb{N}$, so that $\mathcal{T}_\mathbb{Z}(M_n) = M_n$, $\forall\, n$. But the element $(\bar{1}, \bar{1}, \bar{1}, \ldots)$ in the product

$$M = \prod_{n\in\mathbb{N}} M_n = \prod_{n\in\mathbb{N}} \big(\mathbb{Z}/n\mathbb{Z}\big)$$

is not torsion since each $\bar{1}$ is annihilated by a higher integer $n \geq 1$. Actually $\mathcal{T}_\mathbb{Z}\big(\prod_n M_n\big) = \bigoplus_n M_n$.

6) By Lemma C.1.2 we know that the annihilator of a finitely generated torsion module contains a NZD and is thus non-zero. So a torsion module with zero annihilator cannot be finitely generated. Consider $\bigoplus_n M_n$ with $M_n = \mathbb{Z}/n\mathbb{Z}$, which is a torsion module by Example C.1.5, so every element is annihilated by a NZD. But there is no NZD annihilating all elements as $\bigoplus_n M_n$ is generated by

$$e_1 = (\bar{1}, \bar{0}, \bar{0}, \ldots) \quad , \quad e_2 = (\bar{0}, \bar{1}, \bar{0}, \ldots) \quad , \quad e_3 = (\bar{0}, \bar{0}, \bar{1}, \ldots) \quad , \quad \ldots$$

with $\mathrm{Ann}_\mathbb{Z}(e_n) = n\mathbb{Z}$ for all $n \in \mathbb{N}$ and the intersection of all these ideals is zero. □

Remark C.1.7. The references given for the next 4 results actually just give statements in the case of integral domains. However they remain true in general as we are going to show here below.

Proposition C.1.8. cf. [[11], 15.16.9]
Flat modules are torsion-free.

Proof. Fix a NZD $r \in R$, $r \neq 0$, so that the morphism $R \to R : a \mapsto r \cdot a$ is injective. This gives an exact sequence $0 \to R \to R$. M being flat, the functor $\otimes_R M$ is exact, hence we obtain the exact sequence

$$0 \longrightarrow R \otimes_R M \longrightarrow R \otimes_R M \qquad \Leftrightarrow \qquad 0 \longrightarrow M \longrightarrow M \ ,$$

where $M \to M : m \mapsto r * m$ since

$$M \overset{\sim}{\longrightarrow} R \otimes_R M \longrightarrow R \otimes_R M \overset{\sim}{\longrightarrow} M \ : \ m \longmapsto 1 \otimes m \longmapsto r \otimes m \longmapsto r * m \ .$$

As the morphism $M \to M$ is injective, we have $r * m = 0 \Rightarrow m = 0$ where r is any NZD, thus M is torsion-free. □

Lemma C.1.9. cf. [[4], II.§7.n°10.Prop.24, p.115]
Let $\varphi : M \to N$ be an R-module homomorphism. Then $\varphi\big(\mathcal{T}_R(M)\big) \subseteq \mathcal{T}_R(N)$. If φ is moreover injective, we have in addition $\varphi\big(\mathcal{T}_R(M)\big) = \mathcal{T}_R(N) \cap \operatorname{im}\varphi$.

Proof. If $m \in \mathcal{T}_R(M)$ is such that $r * m = 0$ for a NZD $r \in R$, then $r * \varphi(m) = 0$ and $\varphi(r * m) = 0$, i.e. $\varphi(m) \in \mathcal{T}_R(N)$. Now assume that φ is injective. The inclusion $\varphi\big(\mathcal{T}_R(M)\big) \subseteq \mathcal{T}_R(N) \cap \operatorname{im}\varphi$ is clear. Vice-versa, let $n \in \mathcal{T}_R(N)$ such that $n = \varphi(x)$ for some $x \in M$ and $r * n = 0$ for a NZD $r \in R$. This implies $0 = r*n = r*\varphi(x) = \varphi(r*x)$, i.e. $r*x = 0$ by injectivity of φ and $x \in \mathcal{T}_R(M)$. □

Corollary C.1.10. cf. [[4], II.§7.Ex.29, p.197]
If M is any R-module, then

$$M^* \cong \big(M/\mathcal{T}_R(M)\big)^* \ .$$

Proof. Let $f : M \to R$ be an R-module homomorphism. As R is free (hence torsion-free) over itself, we know from Lemma C.1.9 that $f\big(\mathcal{T}_R(M)\big) \subseteq \{0\}$. Thus the morphism

$$\bar{f} \ : \ M/\mathcal{T}_R(M) \to R \ : \ \bar{m} \mapsto f(m)$$

is well-defined. Vice-versa, if $g : M/\mathcal{T}_R(M) \to R$ is given, it suffices to define $\hat{g} : M \to R : m \mapsto g(\bar{m})$, so that $\hat{g}\big(\mathcal{T}_R(M)\big) \subseteq \{0\}$. □

Corollary C.1.11. cf. [[11], 15.16.5]
If

$$0 \longrightarrow M \overset{\varphi}{\longrightarrow} N \overset{\psi}{\longrightarrow} L$$

is an exact sequence of R-modules where M and L are torsion-free, then N is torsion-free as well.

Proof. Let $n \in \mathcal{T}_R(N)$ with a NZD $r \in R$ such that $r * n = 0$. As n is a torsion element, we obtain $\psi(n) \in \mathcal{T}_R(L)$ by Lemma C.1.9, i.e. $\psi(n) = 0$ since L is torsion-free. Thus $n \in \ker \psi = \operatorname{im} \varphi$ and $\exists\, m \in M$ such that $n = \varphi(m)$. Then $0 = r * n = \varphi(r * m)$ implies that $r * m = 0$ by injectivity of φ. But as M is torsion-free, we have $m = 0$, hence $n = \varphi(0) = 0$ and N is torsion-free. □

Remark C.1.12. Proposition 1.3.1 gives an alternative way of proving Corollary C.1.11. Indeed if

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} L$$

is an exact sequence of R-modules where M and L are torsion-free, then by left exactness we obtain

$$0 \longrightarrow \mathcal{T}_R(M) \longrightarrow \mathcal{T}_R(N) \longrightarrow \mathcal{T}_R(L) \qquad \Leftrightarrow \qquad 0 \longrightarrow 0 \longrightarrow \mathcal{T}_R(N) \longrightarrow 0\ ,$$

which is exact and hence implies that $\mathcal{T}_R(N) = \{0\}$ as well.

Remark C.1.13. We finish the section about torsion-freeness by briefly mentioning some more results without proof.

1) We have seen in Example C.1.5 that projective modules are torsion-free. The converse is false, e.g. $\operatorname{Quot}(R)$ is not projective over R.

2) We also saw in Proposition C.1.8 that flat modules are torsion-free. The converse is false in general, but true over Dedekind rings.

3) The Structure Theorem of finitely generated modules over PIDs (see Theorem D.1.13) implies that a finitely generated module over a principal ideal domain is free if and only if it is torsion-free.

C.2 Torsionless modules

The notions of a torsion-free module and a torsionless module are often treated as equivalent, but in general they are not. In this section we want to point out the main differences.

Definition C.2.1. For an R-module M, we consider the canonical R-module homomorphism from M to its bidual

$$j \;:\; M \longrightarrow M^{**} \;:\; m \longmapsto \big(\, \mathrm{ev}_m \;:\; M^* \to R \;:\; f \mapsto f(m) \big)$$

and denote $K_R(M) := \ker j$, which is a submodule of M.
Note that $j(m) = \text{ev}_m = 0 \;\Leftrightarrow\; f(m) = 0$ for all $f \in M^*$, hence

$$K_R(M) = \bigcap_{f \in M^*} \ker f \; . \tag{C.1}$$

M is called *torsionless* if j is injective, i.e. if $K_R(M) = \{0\}$.

Lemma C.2.2. [[63], B.1.7, p.534]
We have $\mathcal{T}_R(M) \subseteq K_R(M)$. In particular, torsionless modules are torsion-free.

Proof. Let $m \in \mathcal{T}_R(M)$, i.e. $\exists\, r \in R$, $r \neq 0$ which is a NZD such that $r * m = 0$.

$$\forall\, f \in M^*, \quad 0 = f(0) = f(r * m) = r * f(m) = r \cdot f(m) \;\Rightarrow\; f(m) = 0 \; ,$$

so that $f(m) = 0$, $\forall\, f \in M^*$, hence $m \in K_R(M)$. □

Proposition C.2.3. [[44], 4.65, p.144] and [[53], 851485]
Torsionless modules can be characterized as follows. The conditions below are equivalent:

1) *M is torsionless.*
2) *$\forall\, m \in M$ with $m \neq 0$, $\exists\, f \in M^*$ such that $f(m) \neq 0$.*
3) *M can be embedded into some (maybe infinite) direct product R^I.*

Proof. 1) ⇔ 2) follows from (C.1).
3) ⇒ 2) : denote $g : M \hookrightarrow R^I$ and let $m \neq 0$. Hence $g(m) \neq 0$ by injectivity, but this means that

$$0 \neq g(m) = \big\{ g_i(m) \big\}_i$$
$$\Rightarrow\; \exists\, i \in I \text{ and } g_i : M \to R \; (g_i \in M^*) \text{ such that } g_i(m) \neq 0 \; .$$

2) ⇒ 3) : let $\{f_i\}_{i \in I}$ be a (maybe infinite) generating set of M^* as an R-module and define

$$g \;:\; M \longrightarrow R^I \;:\; m \longmapsto \big\{ f_i(m) \big\}_i \; .$$

g is injective : let $g(m) = 0 \;\Leftrightarrow\; f_i(m) = 0$, $\forall\, i \in I$. Since $\{f_i\}_i$ is a generating set of M^*, we may write every $f \in M^*$ as a finite sum $f = \sum_i r_i\, f_i$, hence $f(m) = 0$, $\forall\, f \in M^*$. As M is torsionless this implies that $m = 0$. □

Corollary C.2.4. [[53], 851485]
Let M be a finitely generated module over a Noetherian ring R. Then M is torsionless if and only if it can be embedded into a finite direct product R^m.

Proof. Since M is finitely generated there exists a surjective morphism $R^n \twoheadrightarrow M$. Applying the left exact functor $\mathrm{Hom}_R(\,\cdot\,, R)$ to the exact sequence $R^n \to M \to 0$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, R) \longrightarrow \mathrm{Hom}_R(R^n, R) \quad \Leftrightarrow \quad 0 \longrightarrow M^* \longrightarrow R^n \ ,$$

which means that M^* may be seen as an R-submodule of R^n. Being a submodule of the finitely generated module R^n over a Noetherian ring, M^* is also finitely generated (see Proposition D.1.5). So we can do the same proof as in Proposition C.2.3 with $I = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$. □

Remark C.2.5. [[10], 2.5, p.2-3]
1) The proof of Corollary C.2.4 shows that if a module over a Noetherian ring is finitely generated, then so is its dual. But the number of generators may not be the same. Consider e.g. the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$, which is generated by $\bar{1}$, so for a morphism $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, it suffices to know $f(\bar{1}) = a \in \mathbb{Z}$. Since $2 * \bar{1} = \bar{0}$, this gives

$$0 = f(\bar{0}) = f(2 * \bar{1}) = 2 * f(\bar{1}) = 2 \cdot a \ ,$$

hence $a = 0$ and $(\mathbb{Z}/2\mathbb{Z})^* = \{0\}$.
2) More generally, if G is any finite abelian group ($\mathbb{Z}$-module), then $G^* = \{0\}$. Again by Fermat, every $g \in G$ is annihilated by $|G|$, so if $f \in G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$,

$$0 = f(0) = f\big(|G| * g\big) = |G| * f(g) = |G| \cdot f(g) \, , \ \forall \, g \in G \ .$$

As $|G| \neq 0$, this gives $f(g) = 0$, $\forall \, g \in G$ and $G^* = \{0\}$.

Example C.2.6. cf. [[44], 4.65, p.144-145]
1) Free modules are torsionless. In particular, vector spaces are torsionless.
2) Direct sums and submodules of torsionless modules are torsionless.
3) In particular, projective modules are torsionless.

4) $M/K_R(M)$ and M^* are always torsionless.

5) $K_R(M) = M \Leftrightarrow M^* = \{0\}$. In particular,

a) If G is a finite abelian group, then $\mathcal{T}_{\mathbb{Z}}(G) = K_{\mathbb{Z}}(G) = G$.

b) If M is a torsion module, then $M^* = \{0\}$.

Proof. 1) 2) since they can all be embedded into a direct product. An alternatively proof for 1) is:

Let $\{e_i\}_{i\in I}$ be a basis of $M \cong R^{(I)}$ and consider the projections $\pi_i : M \to R$, $\pi_i(m) = r_i$ where $m = \sum_i (r_i * e_i)$ is the unique decomposition of m in the basis. If $m \in \ker j$, then $f(m) = 0$ for all $f \in M^*$, so in particular for $f = \pi_i$. But $\pi_i(m) = 0, \forall\, i$ means that $r_i = 0$ for all i, i.e. $m = 0$. Hence $\ker j = \{0\}$.

3) as they are direct summands (hence submodules) of free modules, which are torsionless.

4) a) Let $\bar{m} \neq \bar{0}$, i.e. $m \notin K_R(M) = \ker j$, hence $\exists\, f \in M^*$ such that $f(m) \neq 0$. We define

$$\bar{f} \;:\; M/K_R(M) \longrightarrow R \;:\; \bar{x} \mapsto f(x) \;,$$

which is well-defined since $f\big(K_R(M)\big) \subseteq \{0\}$. But then $\bar{f}(\bar{m}) \neq 0$, thus $M/K_R(M)$ is torsionless.

b) Let

$$R^{(J)} \longrightarrow R^{(I)} \longrightarrow M \longrightarrow 0$$

be a presentation of M (this always exists). Applying the left exact duality functor yields

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_R(M,R) \longrightarrow \mathrm{Hom}_R\left(R^{(I)},R\right) \longrightarrow \mathrm{Hom}_R\left(R^{(J)},R\right) \\ \Leftrightarrow \qquad 0 \longrightarrow M^* \longrightarrow R^I \longrightarrow R^J \;, \end{gathered}$$

so M^* can be embedded into a direct product R^I, i.e. M^* is torsionless.

5) If $K_R(M) = M$, then $M = \ker j$ and $\mathrm{ev}_m = 0, \forall\, m \in M$. Thus if we fix $f \in M^*$, we get $f(m) = 0$ for all $m \in M$, which means that $f = 0$. Conversely if $M^* = \{0\}$, then

$$\ker j = \bigcap_{f \in M^*} \ker f = \ker 0 = M \;.$$

a) From Example C.1.6 and Remark C.2.5 we know that $\mathcal{T}_{\mathbb{Z}}(G) = G$ and that $G^* = \{0\}$, thus $K_{\mathbb{Z}}(G) = G$.

b) If M is a torsion module, then $M = \mathcal{T}_R(M) \subseteq K_R(M) \subseteq M$ implies that $K_R(M) = M$ as well. $\square$

Example C.2.7. If $I \trianglelefteq R$ is an ideal that contains a NZD, then

$$\mathcal{T}_R(R/I) = K_R(R/I) = R/I \ .$$

In particular, if R is an integral domain, then $\mathrm{Hom}_R(R/I, R) = \{0\}$ for $I \neq \{0\}$.

Proof. Let $x \in I$, $x \neq 0$ be a NZD. Then $x * \bar{r} = \bar{0}$ for all $\bar{r}$, so R/I is a torsion module. To show directly that $K_R(R/I) = R/I$, we prove that its dual is zero. Let $f \in (R/I)^*$. Since R/I is generated by $\bar{1}$ as an R-module, it suffices to know the value of $y := f(\bar{1}) \in R$. We consider the projection $\pi : R \to R/I$ and define $\phi : R \to R$ by $\phi := f \circ \pi$, so that ϕ is an R-module homomorphism satisfying $\phi(I) \subseteq \{0\}$. In particular $\phi(x) = 0$. But then

$$0 = \phi(x) = \phi(x \cdot 1) = x \cdot \phi(1) = x \cdot f(\bar{1}) = x \cdot y \ .$$

If $y \neq 0$, this contradicts the fact that x is a NZD. Thus $y = 0$ and $f = 0$. $\square$

The relation $\mathcal{T}_R(M) \subseteq K_R(M)$ from Lemma C.2.2 shows that torsionless modules are torsion-free and that torsion modules also satisfy $K_R(M) = M$. However this inclusion can be strict, so the notions of a torsion-free module and a torsionless module are in general not equivalent.

Example C.2.8. [[10], 2.5, p.2-3] and [[63], B.1.7, p.534]
Consider $R = \mathbb{Z}$ and $M = \mathbb{Q}$ as a module over $\mathbb{Z}$. Then

$$\{0\} = \mathcal{T}_{\mathbb{Z}}(\mathbb{Q}) \subsetneq K_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q} \ .$$

By the general result about quotient fields of integral domains from Example C.1.6, we know that $\mathbb{Q} = \mathrm{Quot}(\mathbb{Z})$ is torsion-free over $\mathbb{Z}$. However, it is not torsionless. We will show that $\mathbb{Q}^* = \{0\}$ and thus $K_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$ by Example C.2.6. Let $f \in \mathbb{Q}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$. We have $f(0) = 0$ and $f(1) = k$ for some $k \in \mathbb{Z}$. Since f is a $\mathbb{Z}$-module homomorphism, we obtain $\forall\, n \in \mathbb{N}$,

$$k = f(1) = f\big(\tfrac{n}{n}\big) = f\big(n \cdot \tfrac{1}{n}\big) = n \cdot f\big(\tfrac{1}{n}\big) \ ,$$

which means that k is divisible by all n. Thus $k = 0$ and $f\left(\frac{1}{n}\right) = 0$ for all n as well. So if we write elements in $\mathbb{Q}$ as $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, then $f\left(\frac{a}{b}\right) = a \cdot f\left(\frac{1}{b}\right) = 0$, i.e. $f(q) = 0, \forall\, q \in \mathbb{Q}$. Note that it is important to consider the $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ because $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$.

Example C.2.9. cf. [[52], 11908]
Take again $R = \mathbb{Z}$ and consider the subring $\mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$, which consists of polynomial expressions in $\frac{1}{2} \in \mathbb{Q}$ with coefficients in $\mathbb{Z}$, i.e.

$$\mathbb{Z}[\tfrac{1}{2}] = \left\{ q\left(\tfrac{1}{2}\right) \;\middle|\; q \in \mathbb{Z}[X] \right\} \cong \mathbb{Z}[X]/\langle\, 2X - 1 \,\rangle \ .$$

This is a $\mathbb{Z}$-module via multiplication $\mathbb{Z} \times \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$. Moreover it is torsion-free since $\mathbb{Q}$ is an integral domain. But we will again show that $\left(\mathbb{Z}[\frac{1}{2}]\right)^* = \{0\}$, hence that $K_{\mathbb{Z}}\left(\mathbb{Z}[\frac{1}{2}]\right) = \mathbb{Z}[\frac{1}{2}]$.
A $\mathbb{Z}$-module homomorphism $f : \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}$ is uniquely determined by the values $f(1)$ and $f\left(\frac{1}{2^k}\right)$ for all $k \in \mathbb{N}$ since

$$f\Big(q\left(\tfrac{1}{2}\right)\Big) = f\Big(\sum_{k=0}^{n} a_k \cdot \left(\tfrac{1}{2}\right)^k\Big) = \sum_{k=0}^{n} a_k \cdot f\Big(\left(\tfrac{1}{2}\right)^k\Big) = a_0 \cdot f(1) + \sum_{k=1}^{n} a_k \cdot f\left(\tfrac{1}{2^k}\right) \ ,$$

where $a_k \in \mathbb{Z}$. But $2^k \cdot f\left(\frac{1}{2^k}\right) = f\left(\frac{2^k}{2^k}\right) = f(1)$, which means that $f(1)$ is divisible by 2^k for all k, i.e. $f(1) = 0$. It follows that $f\left(\frac{1}{2^k}\right) = 0$ for all k, hence $f = 0$.

Remark C.2.10. We will show in Section C.4 that being torsion-free and being torsionless is nevertheless equivalent in the case of finitely generated modules over integral domains. More precisely, we will get $\mathcal{T}_R(M) = K_R(M)$ in that case. An example of a finitely generated module (over a non-integral ring) that is torsion-free but not torsionless will be given in Example C.4.23.

Remark C.2.11. [[44], 4.65, p.145 & 19.38, p.519] and [[63], B.1.7, p.534]
We have seen in Example C.2.6 that M^* is always torsionless. This can also be proven using the canonical morphisms. Consider

$$\begin{aligned} &j \ : \ M \longrightarrow M^{**} \ : \ m \longmapsto \big(\mathrm{ev}_m \ : \ M^* \to R \ : \ f \mapsto f(m)\big) \ , \\ &j^* \ : \ M^{***} \longrightarrow M^* \ : \ (\varepsilon \ : \ M^{**} \to R) \longmapsto (\varepsilon \circ j \ : \ M \to R) \ , \\ &k \ : \ M^* \longrightarrow M^{***} \ : \ f \longmapsto \big(\mathrm{ev}_f \ : \ M^{**} \to R \ : \ g \mapsto g(f)\big) \ , \end{aligned} \tag{C.2}$$

where k is the canonical morphism associated to the R-module M^*. Then we have $j^* \circ k = \mathrm{id}_{M^*}$, which shows that k is injective. Indeed $\forall\, f \in M^*$,

$$j^*\big(k(f)\big) = j^*(\mathrm{ev}_f) = \mathrm{ev}_f \circ j = f$$

because

$$\mathrm{ev}_f\,\big(j(m)\big) = \mathrm{ev}_f(\mathrm{ev}_m) = \mathrm{ev}_m(f) = f(m),\ \forall\, m \in M\ .$$

It also follows that j^* is surjective. Actually it turns out that every torsionless module is a submodule of a dual module (no proof).

C.3 Reflexive and projective modules

Definition C.3.1. Let M be an R-module and consider again the canonical morphism $j : M \to M^{**}$. M is called *reflexive* if j is an isomorphism. Hence reflexive modules are in particular torsionless and thus torsion-free by Lemma C.2.2.

Remark C.3.2. There exist modules which are isomorphic to their bidual, however without being reflexive, see for example [[52], 76000]. Reflexivity always requires that the canonical morphism j is an isomorphism.

C.3.1 Standard facts

Remark C.3.3. In this section we shall always work with finitely generated modules.

1) Indeed even vector spaces of infinite dimension (i.e. free modules of infinite rank over fields!) are in general not reflexive. Consider e.g. $M \cong R^{(I)}$, so that

$$M^* = \mathrm{Hom}_R\left(R^{(I)}, R\right) \;\cong\; R^I\ .$$

As an example let $R = \mathbb{C}$ and $M = \mathbb{C}[X] \cong \mathbb{C}^{(\mathbb{N})}$, the complex polynomials in 1 variable, which is an infinite-dimensional $\mathbb{C}$-vector space. Then the dual is $M^* = \mathrm{Hom}_{\mathbb{C}}\big(\mathbb{C}[X], \mathbb{C}\big) \cong \mathbb{C}^{\mathbb{N}} \cong \mathbb{C}[[X]]$, the space of formal complex power series. So it is clear that $M^{**} \ncong M$. On the other hand we know by Example C.2.6 that all vector spaces are torsionless.

2) Actually it turns out that an infinite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$) is never isomorphic to its bidual. This is why in analysis one often redefines

the dual vector space as being the space of continuous linear maps $V \to \mathbb{K}$ (with respect to some norm), see e.g. [[10], 4.7 & 4.8, p.9].
3) But even under this continuity condition reflexivity is not always true. Consider for example the vector space $\mathbb{R}^{\mathbb{N}}$ of $\mathbb{R}$-valued sequences. We denote

$$c_0 = \left\{ (a_n)_n \in \mathbb{R}^{\mathbb{N}} \mid \lim_{n\to+\infty} a_n = 0 \right\} \quad , \quad \ell^1 = \left\{ (a_n)_n \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |a_n| < \infty \right\}$$

and let $\ell^\infty \subset \mathbb{R}^{\mathbb{N}}$ be the $\mathbb{R}$-vector subspace of bounded sequences. One can show that the dual space defined by continuous linear maps satisfies $(c_0)^* \cong \ell^1$ and $(\ell^1)^* \cong \ell^\infty$. Thus we get $c_0 \subset \ell^\infty$, which is also clear by definition since sequences converging to zero are bounded, but it is not an isomorphism as not every bounded sequence converges to zero.

Proposition C.3.4. [[10], 2.4, p.2 & 4.2, p.6-8]
Let M be a free module of finite rank. Then M^* is free of the same rank and M is reflexive.

Proof. If $M \cong R^n$, then $M^* \cong \mathrm{Hom}_R(R^n, R) \cong R^n$, so that $M^* \cong M$. Note however that this isomorphism is not canonical (as for vector spaces) since it needs the fixing of a basis first. If $\{e_1, \dots, e_n\}$ is a basis of M, then the dual basis is given by $\{\varepsilon^1, \dots, \varepsilon^n\}$ where the $\varepsilon^i : M \to R$ are defined by $\varepsilon^i(e_j) = \delta_{ij}$, $\forall\, i,j \in \{1, \dots, n\}$ since any R-module homomorphism $f : M \to R$ is uniquely determined by its values $f(e_1), \dots, f(e_n)$ and hence writes as a finite R-linear combination

$$f = \sum_{i=1}^{n} f(e_i) * \varepsilon^i \ .$$

To prove that free modules (of finite rank) are reflexive, there are 2 ways.
1) directly: let $M = \langle\, e_1, \dots, e_n \,\rangle$ and $M^* = \langle\, \varepsilon^1, \dots, \varepsilon^n \,\rangle$. By Example C.2.6 we already know that $j : M \to M^{**}$ is injective. To prove surjectivity, we take any R-module homomorphism $\phi : M^* \to R$ and denote $r_i := \phi(\varepsilon^i)$, $\forall\, i$. Then

$$j\big(\textstyle\sum_j r_j * e_j\big)(\varepsilon^i) = \varepsilon^i\big(\textstyle\sum_j r_j * e_j\big) = \textstyle\sum_j r_j \cdot \varepsilon^i(e_j) = \textstyle\sum_j r_j \cdot \delta_{ij} = r_i = \phi(\varepsilon^i) \ ,$$

i.e. ϕ and $j\big(\sum_j r_j * e_j\big)$ coincide on all basis elements and are hence equal. In particular, $\phi \in \mathrm{im}\, j$. Here we see that M must be finitely generated, otherwise the sum would be infinite.

2) Using that

$$M^* = \mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_R\big(R^n,R\big) \cong R^n \cong M \ ,$$

hence $M^{**} \cong M^* \cong M$. However here one needs to check that all these isomorphisms are compatible with the morphism j. Recall from Remark A.2.15 that we have the isomorphism

$$\mathrm{Hom}_R(R^n,R) \ \cong \ R^n \ : \ f \longmapsto \big(f(e_1),\dots,f(e_n)\big) \ ,$$

where $e_i = (0,\dots,1,\dots,0) \in R^n$. How do j and M^{**} look under this isomorphism ?
A functional $f \in M^*$ is identified with the n-tuple $\big(f(e_1),\dots,f(e_n)\big) \in R^n$. For $m = \sum_i r_i * e_i$, we get $\mathrm{ev}_m(f) = f\big(\sum_i r_i * e_i\big) = \sum_i r_i \cdot f(e_i)$. Since $M^* \cong R^n$, $\mathrm{ev}_m : M^* \to R$ can again be identified with such an n-tuple. The corresponding morphism $R^n \to R$ then looks like

$$g \ : \ R^n \longrightarrow R \ : \ (a_1,\dots,a_n) \longmapsto \textstyle\sum_i r_i \cdot a_i$$

and $g(e_1) = r_1, \dots, g(e_n) = r_n$, i.e. the map g is identified with $(r_1,\dots,r_n)$:

$$j \ : \ M \longrightarrow M^{**} \ : \ m \longmapsto \mathrm{ev}_m \simeq g$$
$$\Leftrightarrow \ j \ : \ R^n \longrightarrow R^n \ : \ (r_1,\dots,r_n) \longmapsto (r_1,\dots,r_n) \ ,$$

so under this identification j is nothing but the identity, which is obviously an isomorphism. □

Remark C.3.5. If M is free of infinite rank, then M^* is in general no longer free. An example with $R = \mathbb{Z}$, $M = \mathbb{Z}^{(\mathbb{N})}$, $M^* \cong \mathbb{Z}^{\mathbb{N}}$ is given in [[10], 3.5, p.5].

Proposition C.3.6. [[44], 19.38, p.519]
If M is a reflexive module, then M^* is reflexive as well.

Proof. Consider the same morphisms as in (C.2) where $j : M \to M^{**}$ is an isomorphism, hence so is $j^* : M^{***} \to M^*$ (as functors send isomorphisms to isomorphisms) and the relation $j^* \circ k = \mathrm{id}_{M^*}$ implies that

$$k = (j^*)^{-1} \ : \ M^* \longrightarrow M^{***}$$

is an isomorphism as well. Note again that it is not sufficient just to check that $M \cong M^{**}$ (obviously) implies $M^* \cong M^{***}$. □

Proposition C.3.7. *If M and N are reflexive, then $M \oplus N$ is reflexive as well.*

Proof. Denote the canonical morphisms by $j_1 : M \to M^{**}$ and $j_2 : N \to N^{**}$. By assumption they are isomorphisms. Taking the direct sum of the exact sequences

$$0 \longrightarrow M \xrightarrow{j_1} M^{**} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow N \xrightarrow{j_2} N^{**} \longrightarrow 0$$

gives

$$0 \longrightarrow M \oplus N \longrightarrow M^{**} \oplus N^{**} \longrightarrow 0 \; .$$

We shall show that the isomorphism $j_1 \oplus j_2$ is indeed the canonical map

$$j \;:\; M \oplus N \longrightarrow (M \oplus N)^{**} \; .$$

For this note that additivity of the Hom-functor gives

$$(M \oplus N)^* = \mathrm{Hom}_R(M \oplus N, R) \;\cong\; \mathrm{Hom}_R(M, R) \oplus \mathrm{Hom}_R(N, R) = M^* \oplus N^* \quad \text{(C.3)}$$

via $h \mapsto (h \circ \varepsilon_1, h \circ \varepsilon_2)$ where ε_i are the injections, with inverse

$$(f, g) \mapsto \big((m, n) \mapsto f(m) + g(n)\big) \; .$$

Hence $(M \oplus N)^{**} \cong M^{**} \oplus N^{**}$ as well. Now

$$\begin{aligned} j \;:\; M \oplus N &\longrightarrow (M \oplus N)^{**} \\ (m, n) &\longmapsto \Big(\mathrm{ev}_{(m,n)} \;:\; (M \oplus N)^* \to R \;:\; h \mapsto h(m, n) \Big) \; , \\ j_1 \oplus j_2 \;:\; M \oplus N &\longrightarrow M^{**} \oplus N^{**} \;:\; (m, n) \longmapsto (\mathrm{ev}_m, \mathrm{ev}_n) \; , \end{aligned}$$

and it remains to show that $\mathrm{ev}_{(m,n)}$ is equal to $(\mathrm{ev}_m, \mathrm{ev}_n)$ under the identification (C.3) which is used twice.

$$\begin{aligned} l \;:\; (M \oplus N)^* \xrightarrow{\sim} M^* \oplus N^* \;&:\; h \longmapsto (h \circ \varepsilon_1, h \circ \varepsilon_2) \; , \\ \circ\, l \;:\; (M^* \oplus N^*)^* \xrightarrow{\sim} (M \oplus N)^{**} \;&:\; \alpha \longmapsto \alpha \circ l \; . \end{aligned}$$

We have again $(M^* \oplus N^*)^* \cong M^{**} \oplus N^{**}$, under which $(\mathrm{ev}_m, \mathrm{ev}_n)$ is mapped to

$$(\mathrm{ev}_m, \mathrm{ev}_n) \longmapsto \Big(\psi \;:\; M^* \oplus N^* \to R \;:\; (f, g) \mapsto \mathrm{ev}_m(f) + \mathrm{ev}_n(g) = f(m) + g(n) \Big)$$

and

$$\begin{aligned} (\psi \circ l)(h) &= \psi(h \circ \varepsilon_1, h \circ \varepsilon_2) = h(\varepsilon_1(m)) + h(\varepsilon_2(n)) \\ &= h((m, 0) + (0, n)) = h(m, n) = \mathrm{ev}_{(m,n)}(h) \; , \end{aligned}$$

which finally is what we want. □

Definition C.3.8. An R-module P is called *projective* if the functor $\mathrm{Hom}_R(P, \cdot\,)$ is exact. So for example free modules are projective since $\mathrm{Hom}_R(R^{(I)}, \cdot\,) \simeq (\,\cdot\,)^I$ is an exact functor.

Proposition C.3.9. *1) An R-module P is projective if and only if it is a direct summand of a free module F, i.e. $\exists\, Q \in \mathrm{Mod}(R)$ such that $F \cong P \oplus Q$.*
2) Every exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

of R-modules where P is projective splits, i.e. $M \cong N \oplus P$.

Proposition C.3.10. [[9], 3.28, p.46], [[5], X.§1.n°4.Pro.5, p.10], [[53], 588311]
1) A direct summand of a finitely generated module is finitely generated as well.
2) If P is a finitely generated projective module, then it is a direct summand of a free module of finite rank.
3) Finitely generated projective modules are finitely presented (even if R is not Noetherian).
4) If P, Q are both finitely generated projective modules, then $\mathrm{Hom}_R(P, Q)$ is also finitely generated and projective.
5) In particular, if P is projective and finitely generated, then P^* is projective and finitely generated.

Proof. 1) Let $M \oplus N \cong L$ where L is finitely generated. Combining the surjections $R^n \twoheadrightarrow L$ and $L \twoheadrightarrow M$, we get $R^n \twoheadrightarrow M$, so M is finitely generated as well. Similarly for N.
2) P being finitely generated, there is a surjection $\varphi : R^n \twoheadrightarrow P$, which gives an exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow R^n \longrightarrow P \longrightarrow 0\,. \tag{C.4}$$

This exact sequence splits by Proposition C.3.9, hence $R^n \cong P \oplus \ker\varphi$ and P is a direct summand of the free module R^n.
3) As $\ker\varphi$ is a direct summand of R^n, which is finitely generated, it is finitely generated as well and we get a surjection $R^m \twoheadrightarrow \ker\varphi$. Combining with (C.4), we get an exact sequence

$$R^m \longrightarrow R^n \longrightarrow P \longrightarrow 0\,.$$

4) Let L_1, L_2 be two (finitely generated) modules that are the direct summands, i.e. $R^n \cong P \oplus L_1$ and $R^m \cong Q \oplus L_2$. Applying $\mathrm{Hom}_R(\,\cdot\,, Q)$, we get

$$\mathrm{Hom}_R(R^n, Q) \;\cong\; \mathrm{Hom}_R(P, Q) \oplus \mathrm{Hom}_R(L_1, Q) \;\Leftrightarrow\; Q^n \;\cong\; \mathrm{Hom}_R(P, Q) \oplus L_1' \;.$$

Thus $R^{mn} \cong Q^n \oplus L_2^n \;\Leftrightarrow\; R^{mn} \cong \mathrm{Hom}_R(P, Q) \oplus L_1' \oplus L_2^n$, which shows that $\mathrm{Hom}_R(P, Q)$ is a direct summand of the free module R^{mn}, hence projective. In particular, it is finitely generated as well.
5) Take $Q = R$, which is projective as it is free of rank 1. □

Proposition C.3.11. [[53], 620239]
Let P be a finitely generated projective module. Then P is reflexive.

Proof. Let Q be such that $R^n \cong P \oplus Q$. Let us denote this isomorphism by $\varphi : R^n \xrightarrow{\sim} P \oplus Q$ and consider the following commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\varphi} & P \oplus Q \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (j_P, j_Q)} \\ (R^n)^{**} & \xrightarrow{\varphi^{**}} & P^{**} \oplus Q^{**} \end{array}$$

Indeed, let $r \in R^n$ and denote $(p, q) := \varphi(r)$. Then

$$(j_P, j_Q)\big(\varphi(r)\big) = (j_P, j_Q)(p, q) = (\mathrm{ev}_p, \mathrm{ev}_q) \;:\; (f, g) \mapsto \big(f(p), g(q)\big) \;,$$

$$\begin{aligned} \varphi^{**}\big(j(r)\big) = \mathrm{ev}_r \circ \varphi^* \;:\; (f, g) &\mapsto \mathrm{ev}_r \left(\varphi^*(f, g)\right) \\ &= \mathrm{ev}_r \left((f, g) \circ \varphi\right) = (f, g)\big(\varphi(r)\big) = \big(f(p), g(q)\big) \;. \end{aligned}$$

Now we know that φ and hence φ^{**} are isomorphisms. Moreover j is an isomorphism by Proposition C.3.4 since R^n is free of finite rank. Thus we need that (j_P, j_Q) is an isomorphism as well, in particular $j_P : P \xrightarrow{\sim} P^{**}$. □

Proposition C.3.12. *Projective modules are flat.*

Proof. We will use the fact that if the direct sum of 2 sequences is exact, then the individual sequences are exact. Indeed if

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \qquad \text{and} \qquad X \xrightarrow{f} Y \xrightarrow{g} Z$$

are such that $A \oplus X \to B \oplus Y \to C \oplus Z$ is exact, then in particular $\psi \circ \varphi = 0$ and if $\psi(b) = 0$, then $(\psi, g)(b, 0) = (0, 0)$, so $(b, 0) = (\varphi, f)(a, x) = \big(\varphi(a), f(x)\big)$ and $b = \varphi(a)$. So let us consider an exact sequence of R-modules

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} L \, .$$

Let P be projective with $P \oplus Q \cong F$ where $F \cong R^{(I)}$. Since free modules are flat and tensoring commutes with direct sums, we get

$$\begin{aligned} & M \otimes_R (P \oplus Q) \longrightarrow N \otimes_R (P \oplus Q) \longrightarrow L \otimes_R (P \oplus Q) \\ \Leftrightarrow \;\; & (M \otimes P) \oplus (M \otimes Q) \longrightarrow (N \otimes P) \oplus (N \otimes Q) \longrightarrow (L \otimes P) \oplus (L \otimes Q) \, . \end{aligned}$$

In order to conclude that $M \otimes P \to N \otimes P \to L \otimes P$ is exact as well, it remains to show that the morphisms are indeed the ones that come from the direct sum of the individual tensored sequences. The compositions are

$$\begin{aligned} (M \otimes P) \oplus (M \otimes Q) &\xrightarrow{\sim} M \otimes_R (P \oplus Q) \longrightarrow N \otimes_R (P \oplus Q) \\ &\qquad\qquad\qquad\qquad\qquad \xrightarrow{\sim} (N \otimes P) \oplus (N \otimes Q) \\ (m \otimes p, n \otimes q) &\longmapsto m \otimes (p, 0) + n \otimes (0, q) \longmapsto \varphi(m) \otimes (p, 0) + \varphi(n) \otimes (0, q) \\ &\qquad\qquad\qquad\qquad\qquad \longmapsto \big(\varphi(m) \otimes p, 0\big) + \big(0, \varphi(n) \otimes q\big) \, , \end{aligned}$$

i.e. $(m \otimes p, n \otimes q) \mapsto \big(\varphi(m) \otimes p \, , \, \varphi(n) \otimes q\big) = (\varphi \otimes \mathrm{id}_P, \varphi \otimes \mathrm{id}_Q)(m \otimes p, n \otimes q)$. □

C.3.2 More advanced results

Here below we state some deeper theorems, mostly without proof. The goal is just to give an idea of what happens in some frequent situations.

Proposition C.3.13. [[9], 3.13 & 3.14, p.47]
A finitely generated projective module over a principal ideal domain is free.

Proof. This follows from Theorem D.1.13: finitely generated projective modules are direct summands of free modules, hence torsion-free and thus free. □

Remark C.3.14. Actually one can show that this is true for *any* projective module over a PID.

Theorem C.3.15. [[9], 3.16, p.48]
A finitely generated projective module over a local ring is free.

Remark C.3.16. [[9], 3.72, p.71]
I. Kaplansky actually showed that this is even true for *any* projective module over a local ring.

Remark C.3.17. [[9], p.47]
Theorem C.3.15 may fail for non-local rings. Let R be a non-trivial Noetherian ring, e.g. $R = \mathbb{K}[X]$, and define the product ring $T = R \times R$. Then consider the ideals $P = R \times \{0\}$ and $Q = \{0\} \times R$ as T-modules. These are finitely generated by 1 element (e.g. if $e = (1, 0) \in P$, then $\langle\, e \,\rangle = P$) and hence of finite presentation since R and T are Noetherian (see Lemma D.1.8). Moreover $P \oplus Q = T$, so P and Q are projective. However they cannot be free. Indeed, let $F \cong T^n$ be any free T-module of finite rank. Then $e * F \neq \{0\}$ since $(e, \ldots, e) \in F$. But $e * Q = \{0\}$ and similarly $(0, 1) * P = \{0\}$. So P and Q are examples of non-free projective modules.
Note that one cannot conclude via $e * Q = \{0\}$ that Q has torsion since e is a zero-divisor in T. Actually P and Q are torsion-free T-modules since they are submodules of T, which is free.

Theorem C.3.18. [[9], 7.26 & 7.27, p.151-152] and [[61], Thm.14, p.43-44]
Let M be an R-module. The following are equivalent:
1) M is a finitely generated projective module.
2) M is of finite presentation and M_P is a free R_P-module of finite rank for all prime ideals $P \trianglelefteq R$.
3) $\widetilde{M}$ is a locally finitely free sheaf on $\operatorname{Spec} R$ *(here on needs in addition that R is Noetherian).*

Proof. 2) $\Leftrightarrow$ 3) follows since $\widetilde{M}$ is coherent with stalks M_P.
For the rest we only prove 1) $\Rightarrow$ 2) :
M being finitely generated and projective, we know by Proposition C.3.10 that it is of finite presentation. Moreover $M \oplus Q \cong R^n$ for some (finitely generated) R-module Q. Now fix a prime ideal $P \trianglelefteq R$. Since localization commutes with

direct sums (Proposition A.2.4), we get $M_P \oplus Q_P \cong R_P^n$. Hence M_P is a finitely generated projective R_P-module. But R_P is a local ring, so M_P is a free R_P-module by Theorem C.3.15. □

Corollary C.3.19. [[9], 3.50, p.64]
Let R be a local ring and M an R-module of finite presentation. Then

$$M \text{ is free } \Leftrightarrow M \text{ is projective } \Leftrightarrow M \text{ is flat }.$$

Proof. The only part which still needs to be proven is flat ⇒ projective. Indeed the first equivalence follows from Theorem C.3.15 and the second implication is just Proposition C.3.12. □

Remark C.3.20. [[9], 7.29, p.152]
The second equivalence even holds true for any finitely generated module over a Noetherian ring.

Lemma C.3.21. [[10], 5.10, p.13-14]
If M, N are free modules of finite rank, then the assignment $f \mapsto f^*$ gives an isomorphism

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(N^*, M^*)\ .$$

Proposition C.3.22. [[10], 5.12 & 5.14, p.14-15]
1) Let $\varphi : M \to N$ be an R-module homomorphism. If φ is surjective, then φ^* is injective.
2) Let $\varphi : V \to W$ be a linear map of $\mathbb{K}$-vector spaces. If φ is injective, then φ^* is surjective.

Proof. We only prove 1). We have $\varphi^* : N^* \to M^*$ where φ is surjective. Let $f \in N^*$ such that $\varphi^*(f) = f \circ \varphi = 0$, i.e. $f\big(\varphi(m)\big) = 0$, $\forall\, m \in M$. As φ is surjective, $\forall\, n \in N$, $\exists\, m \in M$ such that $n = \varphi(m)$ and thus $f(n) = 0$, $\forall\, n \in N$, so $f = 0$. □

Remark C.3.23. [[10], 5.13, p.15]
The second statement of Proposition C.3.22 uses existence of the complement

of a vector subspace and is in general false for modules over a ring. A counter-example is e.g. the following:
Consider $R = \mathbb{Z}$, $M = 2\mathbb{Z}$ and $N = \mathbb{Z}$ with the inclusion map $\varphi : M \hookrightarrow N$. We will show that the dual map $\varphi^* : N^* \to M^*$ is not surjective. Consider for example $f \in M^*$ given by $M \to \mathbb{Z} : 2a \mapsto a$. If f is in the image of φ^*, then $\exists\, g \in N^*$ such that $f = g \circ \varphi$. In particular, $g\big(\varphi(2)\big) = g(2) = f(2) = 1$. On the other hand, we also have $g(2) = g(2 \cdot 1) = 2 \cdot g(1)$, so in the end $2 \cdot g(1) = 1$, which is impossible as $g(1) \in \mathbb{Z}$. Thus φ^* is not surjective. Actually,

$$\operatorname{im} \varphi^* = \left\{ f \in M^* \;\middle|\; \operatorname{im} f \subseteq 2\mathbb{Z} \right\} .$$

$\subseteq$: If $g \in N^*$, then $(g \circ \varphi)(2a) = g(2a) = 2 \cdot g(a) \in 2\mathbb{Z}$, thus $\operatorname{im}(g \circ \varphi) \subseteq 2\mathbb{Z}$.
$\supseteq$: Let $f \in M^*$ such that $\operatorname{im} f \subseteq 2\mathbb{Z}$ and set $g(b) := \frac{f(2b)}{2}$. Then $\varphi^*(g) = f$ since

$$(g \circ \varphi)(2a) = g(2a) = 2 \cdot g(a) = f(2a)\,, \ \forall\, 2a \in M .$$

C.4 The integral case

In this section we always assume that R is an integral domain. Hence the quotient field $K = \operatorname{Quot}(R)$ always exists and $\mathcal{T}_R(M)$ is given by all elements that are annihilated by some non-zero element from the ring as the condition about NZDs does not show up. Let $S = R \setminus \{0\}$, so $K = S^{-1}R$ and recall that $M_K \cong S^{-1}M$ by Lemma A.2.2. If M is an R-module we set $M_K := M \otimes_R K$, so we have the canonical morphism

$$\ell \;:\; M \longrightarrow M_K \;:\; m \longmapsto m \otimes \tfrac{1}{1} \,. \tag{C.5}$$

Proposition C.4.1. [[4], II.§7.n°10.Prop.26, p.116-117]
If R is an integral domain, then $\mathcal{T}_R(M) = \ker \ell$.

Proof. If we consider the map $i : M \to S^{-1}M : m \mapsto \frac{m}{1}$, then $\mathcal{T}_R(M) = \ker i$ since $\frac{m}{1} = 0$ if and only if $\exists\, s \in R$, $s \neq 0$ such that $s * m = 0$. Moreover we have the composition

$$\ell \;=\; \Big(\; M \xrightarrow{\;i\;} S^{-1}M \xrightarrow{\;\sim\;} M \otimes_R K \;:\; m \longmapsto \tfrac{m}{1} \longmapsto m \otimes \tfrac{1}{1} \;\Big)\,,$$

hence $\ker \ell = \ker i$ since the second map is an isomorphism. □

Remark C.4.2. There exists a similar description of $\mathcal{T}_R(M)$ in the non-integral case. That one can be found in Lemma 2.4.2.

Proposition C.4.3. [[4], II.§7.n°10.Cor.1, p.117]
Let R be an integral domain. Then
1) M is torsion-free $\Leftrightarrow$ $\ell : M \to M_K$ is injective.
2) M is a torsion module if and only if $M_K = \{0\}$.

Proof. 1) follows immediately from Proposition C.4.1 and $\mathcal{T}_R(M) = \ker \ell$.
2)

$$\begin{aligned} \mathcal{T}_R(M) = M \;&\Leftrightarrow\; \forall\, m \in M,\ \exists\, r \in R,\ r \neq 0 \text{ such that } r * m = 0 \\ &\Leftrightarrow\; \forall\, m \in M,\ \exists\, r \in S \text{ such that } r * m = 0 \\ &\Leftrightarrow\; \tfrac{m}{s} = 0,\ \forall\, \tfrac{m}{s} \in S^{-1}M \\ &\Leftrightarrow\; S^{-1}M = \{0\} \;\Leftrightarrow\; M_K = \{0\}\ . \end{aligned}$$

□

Lemma C.4.4. [[4], II.§7.n°10.Prop.27, p.118]
If $M \to N \to L$ is an exact sequence of R-modules, then $M_K \to N_K \to L_K$ is an exact sequence of K-vector spaces.

Proof. Exactness of the sequence follows from exactness of localization (Proposition A.2.6). The fact that M_K is a K-vector space can be seen in 2 ways:
1) $M_K \cong S^{-1}M$ is a module over $S^{-1}R = K$, thus a K-vector space.
2) $M_K = M \otimes_R K$ is given a vector space structure by $k * (m \otimes l) = m \otimes (k \cdot l)$. □

C.4.1 Equivalence of "torsion-free" and "torsionless"

Now we will see why it is useful to study M_K instead of M. As a vector space, we e.g. know that it always admits a basis. Let $\{m_i\}_{i \in I}$ be a (maybe infinite) generating set of M as an R-module. Then $\{m_i \otimes \frac{1}{1}\}_{i \in I}$ is a generating set of M_K as a K-vector space, so one can extract a basis $\{m_j \otimes \frac{1}{1}\}_{j \in J}$ for some $J \subseteq I$. Assume for example that we have a relation on the generators

$$a_1 * m_1 + \ldots + a_j * m_j + \ldots + a_k * r_k = 0 \;\Leftrightarrow\; a_j * m_j = -\sum_{i \neq j} (a_i * m_i)$$

with $a_j \neq 0$ (but not necessarily a unit, so we cannot extract m_j). However

$$m_j \otimes \tfrac{1}{1} = m_j \otimes (a_j \cdot \tfrac{1}{a_j}) = (a_j * m_j) \otimes \tfrac{1}{a_j} = \big(-\sum_{i \neq j} (a_i * m_i)\big) \otimes \tfrac{1}{a_j} = \sum_{i \neq j} -\frac{a_i}{a_j} * (m_i \otimes \tfrac{1}{1}) \ ,$$

so $m_j \otimes \frac{1}{1}$ can be omitted in the basis of M_K as it is a K-linear combination of the other $m_i \otimes \frac{1}{1}$ (whereas m_j cannot be omitted in the generating set of M). Thus we have $M_K \cong K^{(J)}$ given by $m_i \otimes \frac{1}{1} \mapsto e_i$. A priori this is only an isomorphism of K-vector spaces, but it is also compatible with the R-module structures since $R \subset K$:

$$m \otimes k = \sum_j k_j * (m_j \otimes \tfrac{1}{1}) \longmapsto \sum_j k_j \cdot e_j \ ,$$

$$r * (m \otimes k) = r * \sum_j k_j * (m_j \otimes \tfrac{1}{1}) = \sum_j (r \cdot k_j) * (m_j \otimes \tfrac{1}{1})$$

$$\longmapsto \sum_j (r \cdot k_j) \cdot e_j = r \cdot \sum_j k_j \cdot e_j \ .$$

Thus $M_K \cong K^{(J)}$ is also an isomorphism of R-modules. Moreover we showed

Corollary C.4.5. [[4], II.§7.n°10.Cor.4, p.117]
If M is a finitely generated module over an integral domain R, then M_K is a finite-dimensional K-vector space. More precisely, if M is generated by n elements, then $M_K \cong K^l$ for some $l \leq n$ (as R-modules and as K-vector spaces).

Remark C.4.6. [[4], II.§7.n°10, p.118]
The converse of this statement is false. Consider for example $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Thus $K = \text{Quot}(\mathbb{Z}) = \mathbb{Q}$ and $M_K = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a 1-dimensional vector space over $\mathbb{Q}$, generated by $1 \otimes 1$. However $\mathbb{Q}$ is not finitely generated over $\mathbb{Z}$.

In Section C.2 we have seen on several examples that the notions of torsion-freeness and torsionlessness are not equivalent in general. However

Proposition C.4.7. [[44], 4.65, p.145 & 2.31, p.44] and [[11], 15.16.7]
If R is an integral domain and M a finitely generated torsion-free module, then M is also torsionless.

Proof. Let M be torsion-free with generators $\{m_1, \dots, m_n\}$. We will show that M can be embedded into R^l for some $l \in \mathbb{N}$, hence that it is torsionless (see Proposition C.2.3).

As M is torsion-free and finitely generated, we get $\ell : M \hookrightarrow M_K$ and $M_K \cong K^l$ as R-modules for some $l \leq n$, hence we have an injective R-module homomorphism $M \hookrightarrow K^l$. Denote the images of the generators by

$$m_1 \longmapsto \left(\frac{r_1^1}{s_1^1}, \dots, \frac{r_l^1}{s_l^1} \right) \quad , \quad \dots \quad , \quad m_n \longmapsto \left(\frac{r_1^n}{s_1^n}, \dots, \frac{r_l^n}{s_l^n} \right)$$

and define $s := \prod_{ij} s_i^j$. Then $s * M \hookrightarrow R^l$ since s cancels all denominators. Moreover $M \hookrightarrow s * M$ since M is torsion-free. Combining everything, we get $M \hookrightarrow R^l$, hence M is torsionless. □

This shows that a finitely generated module over an integral domain is torsion-free if and only if it is torsionless. Actually an even stronger statement holds true.

Corollary C.4.8. *If M is a finitely generated module over an integral domain R, then*

$$\mathcal{T}_R(M) = K_R(M) \ .$$

Proof. By Lemma C.2.2 we already know that $\mathcal{T}_R(M) \subseteq K_R(M)$. To prove the other inclusion, let M be finitely generated, so that $M/\mathcal{T}_R(M)$ is torsion-free and still finitely generated, hence torsionless. If we assume that $\mathcal{T}_R(M) \subsetneq K_R(M)$, then $\exists\, m_0 \in K_R(M)$ such that $m_0 \notin \mathcal{T}_R(M)$, i.e. $\bar{m}_0 \neq \bar{0}$. $M/\mathcal{T}_R(M)$ being torsionless, there is an R-module homomorphism $\bar{f} : M/\mathcal{T}_R(M) \to R$ such that $\bar{f}(\bar{m}_0) \neq 0$. From Corollary C.1.10, we then get $f \in M^*$ such that $f(m) = \bar{f}(\bar{m})$ for all $m \in M$. In particular $f(m_0) \neq 0$, which contradicts $m_0 \in K_R(M)$. Thus $\mathcal{T}_R(M) = K_R(M)$. □

C.4.2 Some formulas holding true in the integral case

If R is an integral domain, we can prove some more formulas which do not hold true in general. We start with the following generalization of (A.8).

Proposition C.4.9. cf. [[4], II.§7.Ex.29, p.197]
Let M, N be modules over an integral domain R and $S \subset R$ a multiplicatively closed subset. If N is torsion-free and M is finitely generated, then

$$S^{-1}\big(\operatorname{Hom}_R(M,N)\big) \cong \operatorname{Hom}_{S^{-1}R}\left(S^{-1}M, S^{-1}N\right) . \tag{C.6}$$

Proof. Consider the morphism ρ as given in Corollary A.2.18:

$$\begin{aligned}\rho \;:\; S^{-1}\big(\operatorname{Hom}_R(M,N)\big) &\longrightarrow \operatorname{Hom}_{S^{-1}R}\left(S^{-1}M, S^{-1}N\right) \\ \tfrac{g}{s} &\longmapsto \left(\rho\big(\tfrac{g}{s}\big) \;:\; S^{-1}M \to S^{-1}N \;:\; \tfrac{m}{t} \mapsto \tfrac{g(m)}{s\cdot t}\right) .\end{aligned} \tag{C.7}$$

We will prove directly that it is an isomorphism under the given assumptions.
a) For injectivity, assume that $\rho\big(\frac{g}{s}\big) = 0$, i.e. $\frac{g(m)}{s\cdot t} = 0$ for all $\frac{m}{t} \in S^{-1}M$. In particular this means that

$$\forall\, m \in M,\ \exists\, s_m \in S \text{ such that } s_m * g(m) = 0 \text{ in } N .$$

As N is torsion-free and $s_m \neq 0$ is a NZD (here we use that R is an integral domain), we need $g(m) = 0$, $\forall\, m \in M$ and thus $\frac{g}{s} = 0$.
b) For surjectivity, let $\psi : S^{-1}M \to S^{-1}N$ be any morphism of $S^{-1}R$-modules. If $\{m_1, \ldots, m_n\}$ is a generating set of M, then $\big\{\frac{m_1}{1}, \ldots, \frac{m_n}{1}\big\}$ is a generating set of $S^{-1}M$ over $S^{-1}R$. We set $\frac{a_i}{s_i} := \psi\big(\frac{m_i}{1}\big)$ for some $a_i \in N$, $s_i \in S$, $\forall\, i \in \{1, \ldots, n\}$ and let $s := s_1 \cdot \ldots \cdot s_n$. Then we define

$$g \;:\; M \longrightarrow N \;:\; m \longmapsto s * \psi\big(\tfrac{m}{1}\big) ,$$

which is well-defined since s cancels all denominators, so the result indeed lies in N. Now we get

$$\rho\big(\tfrac{g}{s}\big) \;:\; \tfrac{m}{t} \longmapsto \frac{g(m)}{s\cdot t} = \frac{s * \psi\big(\frac{m}{1}\big)}{s \cdot t} = \frac{\psi\big(\frac{m}{1}\big)}{t} = \tfrac{1}{t} * \psi\big(\tfrac{m}{1}\big) = \psi\big(\tfrac{m}{t}\big) \tag{C.8}$$

by using $S^{-1}R$-linearity of ψ and hence $\rho\big(\frac{g}{s}\big) = \psi$. □

Corollary C.4.10. cf. [[52], 37497]
Let R be an integral domain and M a finitely generated R-module. Then

$$M^* \otimes_R K = \operatorname{Hom}_R(M,R) \otimes_R K \;\cong\; \operatorname{Hom}_K(M_K, K) . \tag{C.9}$$

Proof. If $S = R \setminus \{0\}$, we have $S^{-1}R = K$ and $M_K \cong S^{-1}M$, so taking $N = R$ in (C.6) gives

$$\begin{aligned}\operatorname{Hom}_R(M,R) \otimes_R S^{-1}R &\cong S^{-1}\big(\operatorname{Hom}_R(M,R)\big) \\ &\cong \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}R) \;\cong\; \operatorname{Hom}_K(M_K, K)\end{aligned}$$

by Lemma A.2.2. How does this isomorphism look like? We have the diagram

$$\begin{array}{ccc} S^{-1}\big(\operatorname{Hom}_R(M,R)\big) & \xrightarrow{\ \rho\ } & \operatorname{Hom}_{S^{-1}R}\left(S^{-1}M, S^{-1}R\right) \\ \big\uparrow\wr & & \big\downarrow\wr \\ \operatorname{Hom}_R(M,R)\otimes_R S^{-1}R & \longrightarrow & \operatorname{Hom}_{S^{-1}R}\left(M\otimes_R S^{-1}R\,,\ S^{-1}R\right) \end{array}$$

Using the isomorphism from (A.3) we get the composition

$$f\otimes\tfrac{r}{s}\longmapsto\tfrac{r*f}{s}\longmapsto\left(\psi\ :\ \tfrac{m}{t}\mapsto\tfrac{r\cdot f(m)}{s\cdot t}\right)\longmapsto\psi'\ ,$$

where

$$\psi'\ :\ M\otimes_R S^{-1}R\longrightarrow S^{-1}R\ :\ m\otimes\tfrac{a}{t}\longmapsto\psi\big(\tfrac{a*m}{t}\big)=\tfrac{r\cdot f(a*m)}{s\cdot t}=f(m)\cdot\tfrac{r}{s}\cdot\tfrac{a}{t}\ .$$

Hence the isomorphism (C.9) is given by $f\otimes\frac{r}{s}\longmapsto\psi'$. □

Remark C.4.11. In other words, Corollary C.4.10 says that taking the dual space of a finitely generated module commutes with localization at the prime ideal $\{0\}$, i.e.

$$S^{-1}(M^*)\ \cong\ (S^{-1}M)^* \qquad \text{resp.} \qquad (M^*)_K\ \cong\ (M_K)^*\ ,$$

where the dual on each RHS is taken with respect to $K=S^{-1}R=\operatorname{Quot}(R)$.

Proposition C.4.12. [[53], 858331]
If M is a module over an integral domain R, then $\mathcal{T}_R(M)\cong\operatorname{Tor}_1(M,K/R)$. In particular, we get $\mathcal{T}_R(M)=\{0\}$ if M is flat.

Proof. Consider the short exact sequence of R-modules

$$0\longrightarrow R\longrightarrow K\longrightarrow K/R\longrightarrow 0\ ,$$

from which we get the long exact sequence

$$\begin{aligned}\ldots\longrightarrow\operatorname{Tor}_1(M,R)\longrightarrow\operatorname{Tor}_1(M,K)\longrightarrow\operatorname{Tor}_1(M,K/R)&\\ \longrightarrow M\otimes_R R\longrightarrow M\otimes_R K\longrightarrow M\otimes_R K/R\longrightarrow 0\ .&\end{aligned}$$

As K is a flat R-module (Corollary A.2.8), we have $\mathrm{Tor}_1(M,K)=\{0\}$ since Tor_j is symmetric and vanishes for any flat module. Thus the sequence simplifies to

$$0 \longrightarrow \mathrm{Tor}_1(M,K/R) \longrightarrow M \stackrel{\ell}{\longrightarrow} M_K \longrightarrow M\otimes_R K/R \longrightarrow 0\ ,$$

where ℓ is the morphism (C.5) since $M\cong M\otimes_R R\to M_K$ via $m\mapsto m\otimes 1\mapsto m\otimes\frac{1}{1}$. It follows that $\mathrm{Tor}_1(M,K/R)\cong\ker\ell=\mathcal{T}_R(M)$ by Proposition C.4.1. □

Remark C.4.13. This is a generalization (in the integral case) of the fact that flat modules are torsion-free (see Proposition C.1.8). In Lemma 2.4.2 we even give a formula that holds true for any commutative ring.

Proposition C.4.14. cf. [[11], 15.16.3]
Let R be an integral domain, M an R-module and $S\subset R$ a multiplicatively closed subset. Then

$$S^{-1}\big(\mathcal{T}_R(M)\big)=\mathcal{T}_{S^{-1}R}\big(S^{-1}M\big)\ . \tag{C.10}$$

In particular, if M is torsion-free over R, then $S^{-1}M$ is torsion-free over $S^{-1}R$.

Proof. $\subseteq$: Let $m\in\mathcal{T}_R(M)$ with $r\in R$, $r\neq 0$ such that $r*m=0$ and $s\in S$. Then $\frac{r}{1}*\frac{m}{s}=0$ where $\frac{r}{1}\neq 0$ since R is an integral domain, i.e. $r\cdot t\neq 0$, $\forall\, t\in S$. So $\frac{m}{s}\in\mathcal{T}_{S^{-1}R}(S^{-1}M)$. Note that this includes the case where $\frac{m}{s}=0$ since 0 is a torsion element anyway.
$\supseteq$: Let $\frac{m}{s}$ be a torsion element with $\frac{r}{t}*\frac{m}{s}=0$ and $\frac{r}{t}\neq 0$. Thus $\frac{r*m}{s\cdot t}=0$, which means that $(a\cdot r)*m=0$ for some $a\in S$, $a\neq 0$. Moreover $a\cdot r\neq 0$ since $\frac{r}{t}\neq 0$. Hence $m\in\mathcal{T}_R(M)$ and $\frac{m}{s}\in S^{-1}\big(\mathcal{T}_R(M)\big)$. □

Remark C.4.15. The equality (C.10) does not hold true in the non-integral case since zero-divisors are involved in the computations (e.g. $a\cdot r$ may be a zero-divisor even if $\frac{r}{t}$ is none). An example is given in Section 2.3 where we consider a non-integral ring R with $S=R\setminus P$ for some prime ideal P and a torsion-free R-module M such that $\mathcal{T}_{R_P}(M_P)\neq\{0\}$.

C.4.3 Characterization of reflexive modules

Lemma C.4.16. [[11], 15.17.5]
Let R be an integral domain, $S=R\setminus\{0\}$ and $\varphi: M\to N$ an R-module

homomorphism and assume that M is torsion-free. If the induced K-linear map $\varphi_S : S^{-1}M \to S^{-1}N$ is injective, then φ is injective.

Proof. Let $m \in M$ such that $\varphi(m) = 0$. Then $0 = \frac{\varphi(m)}{1} = \varphi_S\big(\frac{m}{1}\big)$. φ_S being injective, we get $\frac{m}{1} = 0$, i.e. $\exists\, s \in S$ such that $s * m = 0$. Since $s \neq 0$ and M is torsion-free, we thus have $m = 0$. □

Lemma C.4.17. *Let R be an integral domain, $S = R\backslash\{0\}$ and M an R-module such that M and M^* are finitely generated. Then $S^{-1}M$ is reflexive.*

Proof. Consider the canonical morphism $j : M \to M^{**}$. Localizing this gives the K-linear map

$$j_S \ : \ S^{-1}M \to S^{-1}(M^{**}) \ .$$

Now we use Remark C.4.11 twice, which holds since M and M^* are finitely generated:

$$S^{-1}(M^{**}) \ \cong \ \big(S^{-1}(M^*)\big)^* \ \cong \ (S^{-1}M)^{**} \ .$$

Thus $j_S : S^{-1}M \to (S^{-1}M)^{**}$ and this is indeed the canonical morphism since similarly as the expression in (C.7) we have the isomorphisms

$$S^{-1}(M^*) \xrightarrow{\ \rho\ } (S^{-1}M)^* \qquad , \qquad (S^{-1}M)^{**} \xrightarrow{\ \circ\rho\ } (S^{-1}(M^*))^* \ ,$$
$$S^{-1}(M^{**}) \xrightarrow{\ \rho'\ } (S^{-1}(M^*))^* \ ,$$

so that $j_S\big(\frac{m}{s}\big) = \frac{\mathrm{ev}_m}{s}$ for $\frac{m}{s} \in S^{-1}M$ and

$$\begin{aligned} S^{-1}(M^{**}) &\longrightarrow (S^{-1}(M^*))^* \longrightarrow (S^{-1}M)^{**} \\ \tfrac{\mathrm{ev}_m}{s} &\longmapsto \rho'\big(\tfrac{\mathrm{ev}_m}{s}\big) \longmapsto \rho'\big(\tfrac{\mathrm{ev}_m}{s}\big) \circ \rho^{-1} \ . \end{aligned}$$

This is equal to $\mathrm{ev}_{m/s}$ since if $f \in (S^{-1}M)^*$ and we denote $\frac{g}{t} = \rho^{-1}(f)$, then $\rho\big(\frac{g}{t}\big) = f$ and

$$\rho'\big(\tfrac{\mathrm{ev}_m}{s}\big)\big(\rho^{-1}(f)\big) = \rho'\big(\tfrac{\mathrm{ev}_m}{s}\big)\big(\tfrac{g}{t}\big) = \tfrac{\mathrm{ev}_m(g)}{s\cdot t} = \tfrac{g(m)}{s\cdot t} = \rho\big(\tfrac{g}{t}\big)\big(\tfrac{m}{s}\big) = f\big(\tfrac{m}{s}\big) \ .$$

On the other hand, $S^{-1}M \cong M_K$ is a finite-dimensional vector space over K, hence reflexive by Proposition C.3.4 and it follows that j_S is an isomorphism (of K-vector spaces). □

Proposition C.4.18. [[11], 15.17.5] and [[36], 1.1, p.124]
Let R be a Noetherian integral domain and M a finitely generated R-module. Then M is reflexive if and only if there exists a short exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0 \ ,$$

where F, N are finitely generated, F is free (of finite rank) and N is torsion-free.

Proof. $\underline{\Rightarrow}$: Let M be reflexive. M being finitely generated, we know by Remark C.2.5 that the dual M^* is also finitely generated since R is Noetherian, thus also finitely presented (Proposition D.1.5). Choose a finite presentation

$$R^m \stackrel{\varphi}{\longrightarrow} R^n \stackrel{\psi}{\longrightarrow} M^* \longrightarrow 0 \ .$$

Dualizing, we obtain

$$0 \longrightarrow M^{**} \stackrel{\psi^*}{\longrightarrow} R^n \stackrel{\varphi^*}{\longrightarrow} R^m \ ,$$

where $M^{**} \cong M$, so

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow \operatorname{im}\varphi^* \longrightarrow 0$$

is an exact sequence and $\operatorname{im}\varphi^* \leq R^m$ is torsion-free as it is a submodule of R^m, which is free.
$\underline{\Leftarrow}$: Let an exact sequence

$$0 \longrightarrow M \stackrel{\varphi}{\longrightarrow} F \stackrel{\psi}{\longrightarrow} N \longrightarrow 0$$

as above be given. Note that this already implies that M is torsion-free since it is a submodule of a free module. Dualizing the sequence twice, we obtain the commutative diagram

$$\begin{array}{ccccccccc} & & M^{**} & \stackrel{a}{\longrightarrow} & F^{**} & \stackrel{b}{\longrightarrow} & N^{**} & & \\ & & j\big\uparrow & & \sim\big\uparrow l & & \big\uparrow k & & \\ 0 & \longrightarrow & M & \stackrel{\varphi}{\longrightarrow} & F & \stackrel{\psi}{\longrightarrow} & N & \longrightarrow & 0 \end{array}$$

where $a = \varphi^{**}$, $b = \psi^{**}$ and j, l, k are the canonical morphisms. Indeed $\forall\, f \in F^*$,

$$a\big(j(m)\big)(f) = (\mathrm{ev}_m \circ \varphi^*)(f) = \mathrm{ev}_m(f \circ \varphi) = f\big(\varphi(m)\big) \ ,$$
$$l\big(\varphi(m)\big)(f) = \mathrm{ev}_{\varphi(m)}(f) = f\big(\varphi(m)\big) \ ,$$

and similarly for $b \circ l = k \circ \psi$. We know that the top row is a complex (only the bottom row is exact) and that l is an isomorphism since F is free of finite rank. Now let us localize the diagram. By Remark C.4.11 we obtain the K-linear maps $j_S : S^{-1}M \to (S^{-1}M)^{**}$ and $k_S : S^{-1}N \to (S^{-1}N)^{**}$ since all modules and their duals are finitely generated (R being Noetherian). As in the proof of Lemma C.4.17, $S^{-1}M$ and $S^{-1}N$ are finite-dimensional K-vector spaces, thus reflexive, so that j_S and k_S are isomorphisms. Lemma C.4.16 now implies that j and k are injective since M and N are torsion-free. j being injective, we conclude that M is also torsionless. The same argument and torsion-freeness of M^{**} (duals are always torsion-free by Example C.1.5) show that a is injective as $a_S = l_S \circ \varphi_S \circ j_S^{-1}$ is injective.
Showing that j is surjective is done by some diagram chasing. Let $f \in M^{**}$; we want to find $m \in M$ such that $f = j(m)$. As $a(f) \in F^{**}$, surjectivity of l gives $r \in F$ such that $a(f) = l(r)$. Next we have

$$0 = b\big(a(f)\big) = b\big(l(r)\big) = k\big(\psi(r)\big)$$

by commutativity of the diagram. Injectivity of k implies that $\psi(r) = 0$, thus $\exists\, m \in M$ such that $r = \varphi(m)$ as the bottom row is exact. Then

$$a(f) = l(r) = l\big(\varphi(m)\big) = a\big(j(m)\big)$$

and hence $f = j(m)$ by injectivity of a. Finally j is an isomorphism, i.e. M is reflexive. □

Corollary C.4.19. *If R is a Noetherian integral domain, then kernels of morphisms between free modules of finite rank are reflexive: if $\varphi : R^n \to R^m$ is an R-module homomorphism, then $\ker\varphi$ is a reflexive R-module.*

Proof. Let $M = \ker\varphi$. Since R is Noetherian, M is finitely generated. Moreover we have an exact sequence

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{\varphi} R^m ,$$

from which we get

$$0 \longrightarrow M \longrightarrow R^n \longrightarrow \operatorname{im}\varphi \longrightarrow 0 ,$$

where $\operatorname{im}\varphi \leq R^m$ is torsion-free as submodule of a free module. Hence M is reflexive since there exists an exact sequence as in Proposition C.4.18. □

Remark C.4.20. Corollary C.4.19 is false in the non-integral case, see Example C.4.24.

Corollary C.4.21. [[36], 1.2, p.124-125]
Let R be a Noetherian integral domain. If M is a finitely generated R-module, then M^* is reflexive.

Proof. Since R is Noetherian, we can choose a finite presentation

$$R^m \xrightarrow{\varphi} R^n \xrightarrow{\psi} M \longrightarrow 0$$

of M and obtain by dualization

$$0 \longrightarrow M^* \longrightarrow R^n \longrightarrow R^m \qquad \text{and} \qquad 0 \longrightarrow M^* \longrightarrow R^n \longrightarrow N \longrightarrow 0 \; ,$$

where the last one is exact as in Proposition C.4.18 since $N = \operatorname{im} \varphi^* \leq R^m$ is torsion-free, hence M^* is reflexive. □

Corollary C.4.22. [[11], 15.17.6]
Let R be a Noetherian integral domain and M, N finitely generated R-modules. If N is reflexive, then $\operatorname{Hom}_R(M, N)$ is reflexive too.

Proof. If

$$R^m \xrightarrow{\varphi} R^n \xrightarrow{\psi} M \longrightarrow 0$$

is a finite presentation of M, we apply the left exact functor $\operatorname{Hom}_R(\,\cdot\,, N)$ and get

$$0 \longrightarrow \operatorname{Hom}_R(M, N) \xrightarrow{\psi^*} N^n \xrightarrow{\varphi^*} L_1 \longrightarrow 0 \; ,$$

where $L_1 = \operatorname{im} \varphi^*$. This already shows that $\operatorname{Hom}_R(M, N)$ is finitely generated as it is a submodule of N^n and R is Noetherian. Since N is reflexive, it is in particular torsion-free by Lemma C.2.2. Thus N^m and $L_1 \leq N^m$ are torsion-free as well. By Proposition C.4.18 there exists an exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow L_2 \longrightarrow 0 \; , \tag{C.11}$$

where F is free of finite rank and L_2 is torsion-free. Taking n copies of (C.11) gives

$$0 \longrightarrow N^n \xrightarrow{i} F^n \longrightarrow L_2^n \longrightarrow 0$$

and we get the injective morphism $\delta = i \circ \psi^* : \mathrm{Hom}_R(M,N) \to F^n$. Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M,N) & \xrightarrow{\psi^*} & N^n & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & F^n & \xrightarrow{\mathrm{id}} & F^n & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

The Snake Lemma (see Proposition D.1.6) gives an exact sequence

$$0 \longrightarrow \ker \delta \longrightarrow \ker i \longrightarrow \ker 0 \longrightarrow \operatorname{coker} \delta \longrightarrow \operatorname{coker} i \longrightarrow \operatorname{coker} 0 \longrightarrow 0$$
$$\Leftrightarrow \; 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow L_1 \longrightarrow \operatorname{coker} \delta \longrightarrow L_2^n \longrightarrow 0 \longrightarrow 0 \; .$$

Since L_1 and L_2^n are torsion-free, Corollary C.1.11 implies that $\operatorname{coker} \delta$ is torsion-free as well. Finally we get

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \xrightarrow{\delta} F^n \longrightarrow \operatorname{coker} \delta \longrightarrow 0 \; ,$$

where F^n is free of finite rank, hence $\mathrm{Hom}_R(M,N)$ is reflexive by Proposition C.4.18. □

C.4.4 Counter-examples in the non-integral case

Here below we give two counter-examples to show that Corollary C.4.8 and Corollary C.4.19 do in general not hold true for finitely generated modules over non-integral rings.

Example C.4.23. Consider the ring $R = \mathbb{K}[X,Y,Z]/\langle\, XY, X^2, XZ \,\rangle$ and the module $M = R/\langle\, \bar{Y}\bar{Z} \,\rangle$, which is generated by $[\bar{1}]$. Then M is an example of a finitely generated module over a non-integral ring which is torsion-free but not torsionless.

1) M is torsion-free[1]: The set of zero-divisors in R is given by $\mathrm{ZD}(R) = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle$. This can either be seen immediately by the definition of R or by computing its associated primes. Let $\bar{f} \in R$ be a NZD and $[\bar{g}] \in M$ such that

$$\bar{f} * [\bar{g}] = [\bar{0}] \;\Leftrightarrow\; \bar{f} \cdot \bar{g} \in \langle\, \bar{Y}\bar{Z} \,\rangle \; .$$

[1]In Section 2.3 we will give an alternative proof of torsion-freeness of M by using its associated primes and the criterion from Proposition 1.3.3.

If we denote $J := \langle\, XY, X^2, XZ, YZ \,\rangle \trianglelefteq \mathbb{K}[X,Y,Z]$, this gives the primary decompositions $J = \langle\, X,Z \,\rangle \cap \langle\, X,Y \,\rangle \cap \langle\, X^2, Y, Z \,\rangle$ and

$$\langle\, \bar{Y}\bar{Z} \,\rangle = \langle\, \bar{X}, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{Y}, \bar{Z} \,\rangle$$

because of Proposition B.2.10. So in particular $\bar{f}\cdot\bar{g} \in \langle\, \bar{X}, \bar{Z} \,\rangle$ and $\bar{f}\cdot\bar{g} \in \langle\, \bar{X}, \bar{Y} \,\rangle$. These ideals being prime (dividing them out gives an integral domain), we need that $\bar{g} \in \langle\, \bar{X}, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Y} \,\rangle$, otherwise $\bar{f}$ is a zero-divisor. Hence

$$\bar{g} \in \langle\, \bar{X}, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Y} \,\rangle = \langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle$$

and thus $[\bar{g}] \in \langle\, [\bar{X}] \,\rangle$, i.e. torsion elements are necessary multiples of $[\bar{X}]$. But $[\bar{X}]$ is not a torsion element as $\mathrm{Ann}_R(\bar{X}) = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle$ is a maximal ideal and $\mathrm{Ann}_R(\bar{X}) \subseteq \mathrm{Ann}_R([\bar{X}]) \neq R$, so $[\bar{X}]$ can only be annihilated by zero-divisors. Finally we obtain that $\mathcal{T}_R(M) = \{0\}$.

2) M is not torsionless: we first analyze how elements in M^* look like. Take any $f : M \to R$; since M is generated by $[\bar{1}]$, it suffices to know the value of $r = f([\bar{1}]) \in R$. The relation $[\bar{Y}\bar{Z}] = [\bar{0}]$ implies that

$$\bar{0} = f([\bar{0}]) = f([\bar{Y}\bar{Z}]) = f(\bar{Y}\bar{Z} * [\bar{1}]) = \bar{Y}\bar{Z} \cdot r \;,$$

thus $\bar{Y}\bar{Z} \cdot r \in \langle\, \bar{X} \,\rangle$, which is a prime ideal. As $\bar{Y} \notin \langle\, \bar{X} \,\rangle$ and $\bar{Z} \notin \langle\, \bar{X} \,\rangle$, we thus get $r \in \langle\, \bar{X} \,\rangle$ and $f([\bar{1}]) = r_0\bar{X}$ for some $r_0 \in R$ depending on f. Any $f \in M^*$ is necessarily of this form. Now consider $j : M \to M^{**} : m \mapsto \mathrm{ev}_m$ and take $m = [\bar{X}]$. Then

$$\mathrm{ev}_{[\bar{X}]}(f) = f([\bar{X}]) = f(\bar{X} * [\bar{1}]) = \bar{X} \cdot f([\bar{1}]) = r_0 \cdot \bar{X}^2 = \bar{0}$$

for all $f \in M^*$, i.e. $[\bar{X}] \in K_R(M)$. So we have $\mathcal{T}_R(M) = \{0\}$ and $K_R(M) \neq \{0\}$.

Example C.4.24. [[53], 221280]

Consider the Noetherian ring $R = \mathbb{K}[X,Y]/\langle\, X^2, XY, Y^2 \,\rangle$ and $M = \mathbb{K}$, where the R-module structure of M is given by $\bar{f} * \lambda = f(0) \cdot \lambda$. Actually $M = R/\langle\, \bar{X}, \bar{Y} \,\rangle$, so we see that M is generated by $1 \in \mathbb{K}$ and get an exact sequence

$$R^m \longrightarrow R \longrightarrow M \longrightarrow 0$$

for some $m \in \mathbb{N}$. Dualizing, this gives

$$0 \longrightarrow M^* \longrightarrow R \longrightarrow R^m \ ,$$

i.e. M^* is the kernel of a morphism $R \to R^m$. However we will show that $M^* \cong M \oplus M$, so M^* cannot be reflexive.
We have $R \cong \mathbb{K} \oplus \mathbb{K}\bar{X} \oplus \mathbb{K}\bar{Y}$ as $\mathbb{K}$-vector spaces, so any element $r \in R$ may be written as a $\mathbb{K}$-linear combination of $\bar{1}$, $\bar{X}$ and $\bar{Y}$. Let $g \in M^* = \mathrm{Hom}_R(M, R)$. As M is generated by 1, it suffices to know the value of $g(1) \in R$. Denote $g(1) = \alpha + \beta\bar{X} + \gamma\bar{Y}$ for some $\alpha, \beta, \gamma \in \mathbb{K}$. The relations $\bar{X} * 1 = 0$ and $\bar{Y} * 1 = 0$ imply that

$$\begin{aligned}\bar{0} = g(0) = g(\bar{X} * 1) = \bar{X} \cdot g(1) &= \bar{X} \cdot (\alpha + \beta\bar{X} + \gamma\bar{Y}) \\ &= \alpha\bar{X} + \beta\bar{X}^2 + \gamma\bar{X}\bar{Y} = \alpha\bar{X} \ ,\end{aligned}$$

and similarly $\alpha\bar{Y} = \bar{0}$. Thus $\alpha = 0$ and the values of β, γ determine the morphism g completely. In one formula,

$$g(\lambda) = g(\lambda * 1) = \lambda \cdot g(1) = \beta\lambda\bar{X} + \gamma\lambda\bar{Y} \ , \quad \forall\, \lambda \in M \ ,$$

where $\lambda \in R$ is seen as a constant function. So we have $M^* \cong \mathbb{K} \oplus \mathbb{K} = M \oplus M$ via the isomorphism $g \mapsto (\beta, \gamma)$. Note that this is indeed a morphism of R-modules since $\forall\, \bar{f} \in R$,

$$\begin{aligned}\bar{f} * g &\longmapsto (\bar{f} * g)(1) = \bar{f} \cdot g(1) = \bar{f} \cdot (\beta\bar{X} + \gamma\bar{Y}) = f(0) \cdot (\beta\bar{X} + \gamma\bar{Y}) \\ &\longmapsto \big(f(0) \cdot \beta \, , \, f(0) \cdot \gamma\big) = \bar{f} * (\beta, \gamma) \ ,\end{aligned}$$

where multiplication by $\bar{f}$ is the same as multiplication by $f(0)$ since all non-constant terms in $\bar{f}$ will vanish by either $\bar{X}$ or $\bar{Y}$. More generally we get that the n^{th} dual of M is equal to a direct sum of 2^n copies of M.

Remark C.4.25. The exact finite presentation of M can be found as follows. Consider the exact sequences of R-modules

$$0 \longrightarrow \langle\, \bar{X}, \bar{Y} \,\rangle \xrightarrow{\ i\ } R \xrightarrow{\ \pi\ } R/\langle\, \bar{X}, \bar{Y} \,\rangle \longrightarrow 0 \quad \text{and} \quad R^2 \xrightarrow{\ \phi\ } \langle\, \bar{X}, \bar{Y} \,\rangle \longrightarrow 0 \ ,$$

where $\phi(\bar{f}, \bar{h}) = \bar{f}\bar{X} + \bar{h}\bar{Y}$. Hence the finite presentation of M is given by

$$R^2 \longrightarrow R \longrightarrow M \longrightarrow 0 \ .$$

C.4.5 Alternative proofs of $\mathcal{T}_R(M) = K_R(M)$

In the previous sections we developed some tools which now allow some alternative proofs of the fact that torsion-freeness and torsionlessness are equivalent for finitely generated modules over integral domains. Unfortunately these need the additional assumption that M and M^* are both finitely generated. On the other hand, this is e.g. satisfied over Noetherian rings if we only assume that M is finitely generated.

alternative proof 1. Let M be a torsion-free module over an integral domain R such that M and M^* are finitely generated. Let $S = R \setminus \{0\}$ and consider the canonical morphism $j : M \to M^{**}$. As in the proof of Lemma C.4.17, we get that $j_S : S^{-1}M \to (S^{-1}M)^{**}$ is an isomorphism. M being torsion-free, Lemma C.4.16 then implies that j is injective, i.e. M is torsionless.

alternative proof 2. cf. [[52], 37497]
We shall show the inclusion $K_R(M) \subseteq \mathcal{T}_R(M)$. Tensoring $j : M \to M^{**}$ by K and using Remark C.4.11, we get

$$\begin{aligned} & j^{\otimes} \ : \ M \otimes_R K \longrightarrow M^{**} \otimes_R K \ : \ m \otimes k \longmapsto \mathrm{ev}_m \otimes k \\ \Leftrightarrow \ & j^{\otimes} \ : \ M_K \longrightarrow M^{**} \otimes_R K \cong (M^* \otimes_R K)^* \cong (M \otimes_R K)^{**} = (M_K)^{**} \end{aligned}$$

since M and M^* are finitely generated. Thus $j^{\otimes} : M_K \xrightarrow{\sim} (M_K)^{**}$ is an isomorphism (of K-vector spaces) since M_K is free over K, hence reflexive (we will check later that it is indeed the canonical morphism). Now let $m \in K_R(M)$, i.e. $\mathrm{ev}_m = 0$. Then

$$j^{\otimes}\big(m \otimes \tfrac{1}{1}\big) = \mathrm{ev}_m \otimes \tfrac{1}{1} = 0 \ ,$$

so $m \otimes \frac{1}{1} = 0$ by injectivity of $j^{\otimes}$. But $m \otimes \frac{1}{1} = 0$ means that $m \in \ker \ell = \mathcal{T}_R(M)$ by Proposition C.4.1. Now it only remains to check commutativity of the diagram

$$\begin{array}{ccccc} M \otimes K & \xrightarrow{j^{\otimes}} & M^{**} \otimes K & \xrightarrow{i_1} & (M^* \otimes K)^* \\ & \searrow^{j'} & & & \downarrow{\scriptstyle i_2} \\ & & & & (M \otimes K)^{**} \end{array}$$

where

$$j^{\otimes} \;:\; m\otimes k \longmapsto \mathrm{ev}_m \otimes k \qquad , \qquad j' \;:\; m\otimes k \longmapsto \mathrm{ev}_{m\otimes k}$$
$$i_1 \;:\; \varphi\otimes k \longmapsto \big(\,\psi_{\varphi k} \,:\, M^*\otimes K \to K \,:\, f\otimes l \mapsto \varphi(f)\cdot k\cdot l\,\big)$$
$$h \,:\, M^*\otimes K \xrightarrow{\sim} (M\otimes K)^* \,:\, f\otimes k \longmapsto \big(m\otimes l \mapsto f(m)\cdot k\cdot l\big)$$

and $i_2 = \circ\, h^{-1}$. All of them are given similarly as in (C.9) and in the proof of Corollary C.4.10. To conclude we want to get $(i_2\circ i_1\circ j^{\otimes})(m\otimes k) = \mathrm{ev}_{m\otimes k}$ for all $m\otimes k$. Let $g\in(M\otimes K)^*$ be arbitrary.

$$i_1\big(j^{\otimes}(m\otimes k)\big) = i_1(\mathrm{ev}_m\otimes k) = \big(\,\psi \,:\, f\otimes l \mapsto \mathrm{ev}_m(f)\cdot k\cdot l = f(m)\cdot k\cdot l\,\big)\ ,$$
$$i_2(\psi) = \psi\circ h^{-1}\ .$$

If $\{m_1,\ldots,m_n\}$ is a generating set of M, denote $\frac{a_i}{s_i} := g\big(m_i\otimes\frac{1}{1}\big)$ for all i and set $s := s_1\cdot\ldots\cdot s_n$. Now we define $f(m) = s * g\big(m\otimes\frac{1}{1}\big)$, so that $f(m)\in R$ and $f\in M^*$. Then $h\big(f\otimes\frac{1}{s}\big) = g$ since

$$h\big(f\otimes\tfrac{1}{s}\big)(m\otimes l) = f(m)\cdot\tfrac{1}{s}\cdot l = g\big(m\otimes\tfrac{1}{1}\big)\cdot l = g(m\otimes l)$$

by K-linearity of g. This construction is similar as the one in (C.8). Finally

$$\psi\big(h^{-1}(g)\big) = \psi\big(f\otimes\tfrac{1}{s}\big) = f(m)\cdot k\cdot\tfrac{1}{s} = k\cdot g\big(m\otimes\tfrac{1}{1}\big) = g(m\otimes k) = \mathrm{ev}_{m\otimes k}(g)$$

and both morphisms agree.

Appendix D

Collection of other results

In this appendix we state some well-known results which we are going to use at some places. Most of them are given without proof and can be found in almost every textbook on Commutative Algebra. The idea is just to recall the exact statements, so that the reader can immediately look them up.

D.1 General results

Lemma D.1.1. *The* Hom*-bifunctor commutes with finite products and direct sums in both arguments. But in the infinite case we only have*

$$\mathrm{Hom}_R\left(\bigoplus_i M_i, N\right) \cong \prod_i \mathrm{Hom}_R(M_i, N) \ ,$$
$$\mathrm{Hom}_R\left(M, \prod_i N_i\right) \cong \prod_i \mathrm{Hom}_R(M, N_i) \ .$$

In particular, $\mathrm{Hom}_R\left(R^{(I)}, M\right) \ncong M^{(I)}$ *and* $\mathrm{Hom}_R\left(R^I, M\right) \ncong M^I$*. However,*

$$\mathrm{Hom}_R\left(R^{(I)}, M\right) \cong M^I \ ,$$
$$\left(\bigoplus_i M_i\right)^* = \mathrm{Hom}_R\left(\bigoplus_i M_i, R\right) \cong \prod_i \mathrm{Hom}_R(M_i, R) = \prod_i M_i^* \ .$$

Lemma D.1.2. *1) Let* $\varphi : A \to B$ *be a ring homomorphism. If* M *is a module over* B*, then* M *is also a module over* A *via* $a * m := \varphi(a) * m$*.*
2) Similarly for sheaves: if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *and* $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ *are schemes such that there*

is a morphism of schemes $\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ *and* $\mathcal{F} \in \mathtt{Mod}(\mathcal{O}_{\mathcal{X}})$*, then* $\mathcal{F}$ *is also an* $\mathcal{O}_{\mathcal{Y}}$*–module.*

Lemma D.1.3. *Let* $I \trianglelefteq R$ *be an ideal. Its radical* $\mathrm{Rad}(I)$ *is given by the intersection of all prime ideals containing* I*, i.e.*

$$\mathrm{Rad}(I) = \bigcap_{I \subseteq P \, prime} P \; .$$

In particular, the nilradical of R *is equal to the intersection of all prime ideals.*

Lemma D.1.4. [[6], II.§2.n°6.Cor.2, p.95-96] and [[2], 1.13, p.9]
1) Let $I, J \trianglelefteq R$ *be two ideals. Then*

$$\mathrm{Rad}(I \cap J) = \mathrm{Rad}(I) \cap \mathrm{Rad}(J) = \mathrm{Rad}(I \cdot J) \; .$$

2) If I *is a radical ideal, then* $\mathrm{Rad}(I^n) = I$*,* $\forall \, n \in \mathbb{N}$*. In particular this holds true for prime ideals.*

Proof. 1) $I \cdot J \subseteq I \cap J$ already implies that $\mathrm{Rad}(I \cdot J) \subseteq \mathrm{Rad}(I \cap J)$. If $r^n \in I \cap J$ for some $n \in \mathbb{N}$, then $r^n \in I$ and $r^n \in J$. And if $r^n \in I$ and $r^m \in J$ for $n, m \in \mathbb{N}$, then $r^{nm} \in I \cdot J \subseteq I \cap J$.
2) $I^n \subseteq I$ implies that $\mathrm{Rad}(I^n) \subseteq \mathrm{Rad}(I) = I$ since I is radical. Moreover if $r \in I$, then $r^n \in I^n$, hence $r \in \mathrm{Rad}(I^n)$. □

Proposition D.1.5. *Let* R *be a Noetherian ring and* M *a finitely generated* R*-module. Then* M *is of finite presentation and every submodule of* M *is finitely generated as well.*

Proposition D.1.6 (Snake Lemma)**.** *Consider the commutative diagram of* R*-modules*

$$\begin{array}{ccccccc} & & M & \xrightarrow{f} & N & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & M' & \xrightarrow{f'} & N' & \xrightarrow{g'} & L' & & \end{array}$$

If the rows are exact, then there exists an exact sequence

$$\ker a \xrightarrow{F} \ker b \longrightarrow \ker c \longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \xrightarrow{G} \operatorname{coker} c \; .$$

If f *is injective, then* F *is injective and if* g' *is surjective, then* G *is surjective.*

Lemma D.1.7 (9-Lemma)**.** *Let M_i, N_i, L_i for $i = 1, 2, 3$ be R-modules and assume that the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_2 & \longrightarrow & N_2 & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_3 & \longrightarrow & N_3 & \longrightarrow & L_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

commutes such that all columns are exact. Then
1) If the two bottom rows are exact, then the top row is exact as well.
2) If the two top rows are exact, then the bottom row is also exact.
By symmetry, both statements also remain true when interchanging the words "columns" and "rows".

Lemma D.1.8. [[53], 868510]
If R and T are Noetherian rings, then $R \times T$ is Noetherian as well.

Proof. Let $I_1 \subseteq I_2 \subseteq \ldots$ be an increasing chain of ideals in $R \times T$. We identify R with the ideal $R \times \{0\}$ in the product and consider the projection morphism $\pi : R \times T \to T$. Then

$$(I_1 \cap R) \subseteq (I_2 \cap R) \subseteq \ldots \qquad \text{and} \qquad \pi(I_1) \subseteq \pi(I_2) \subseteq \ldots$$

are increasing chains of ideals in R and T respectively (here we need surjectivity of π to get ideals in T). As these are Noetherian, there exists $N \in \mathbb{N}$ such that $I_i \cap R = I_N \cap R$ and $\pi(I_i) = \pi(I_N)$, $\forall\, i \geq N$. Now fix $j \geq N$; we want to show that $I_j = I_N$ as well. Since $I_N \subseteq I_j$, we only have to prove the remaining inclusion. Let $(a, b) \in I_j$. Then $b \in \pi(I_j) = \pi(I_N)$, so $\exists\, r \in R$ such that $(r, b) \in I_N$. Moreover we have $(a - r, 0) \in R \times \{0\}$ and $(a - r, 0) = (a, b) - (r, b) \in I_j$, so that

$$(a - r, 0) \in I_j \cap R = I_N \cap R \ ,$$

hence $(a, b) = (a - r, 0) + (r, b) \in I_N$, i.e. $I_j \subseteq I_N$. Finally $I_i = I_N$, $\forall\, i \geq N$, so the chain becomes stationary. $\square$

Remark D.1.9. Not every ideal in $R \times T$ is of the form $I \times J$ for some ideals $I \trianglelefteq R$ and $J \trianglelefteq T$. Consider e.g. the principal ideal $\langle\, (2, 3)\, \rangle$ in $\mathbb{Z} \times \mathbb{Z}$ and recall that $\mathbb{Z}$ is a principal ideal domain. This ideal cannot be of the form $\langle\, a\, \rangle \times \langle\, b\, \rangle$ for some $a, b \in \mathbb{Z}$ since the latter e.g. contains infinitely many elements of the form $(\lambda a, \cdot\,)$ for every fixed $\lambda \in \mathbb{Z}$.

Proposition D.1.10. [[2], 7.3 & 7.4, p.80]
Let $S \subset R$ be a multiplicatively closed subset. If R is a Noetherian ring, then $S^{-1}R$ is Noetherian too. In particular, R_P is Noetherian for all prime ideals P.

Proof. Similarly as in Proposition 1.1.1 one shows that every ideal in $S^{-1}R$ is of the form $S^{-1}I$ for some ideal $I \trianglelefteq R$. This I is finitely generated since R is Noetherian, say by $r_1, \ldots, r_n \in R$. Then $S^{-1}I$ is generated by $\frac{r_1}{1}, \ldots, \frac{r_n}{1}$ over $S^{-1}R$. $\square$

Theorem D.1.11 (Nakayama's Lemma)**.** *Let R be a local ring with maximal ideal $\mathfrak{M}$ and M a finitely generated R-module. Assume that $\exists\, x_1, \ldots, x_n \in M$ such that the classes $\bar{x}_1, \ldots, \bar{x}_n$ are generators of the quotient $M/\mathfrak{M}M$ as a module over $R/\mathfrak{M}$. Then $x_1, \ldots, x_n$ generate M as an R-module.*

Proposition D.1.12. *Let R be a local Noetherian ring with unique maximal ideal $\mathfrak{M}$. Then*
1) $\dim_{R/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2)$ is the minimal number of generators of the ideal $\mathfrak{M}$.
2) If R is of Krull dimension 1, then R is regular if and only if R is a principal ideal domain.

Theorem D.1.13 (Structure Theorem of finitely generated modules over PIDs)**.** *Let R be a principal ideal domain and M a finitely generated R-module. Then there exists an integer $m \in \mathbb{N}$ such that*

$$M \cong \mathcal{T}_R(M) \oplus R^m\ ,$$

where $\mathcal{T}_R(M) \leq M$ is the torsion submodule. In particular, M is free if and only if it is torsion-free.

Theorem D.1.14 (Bézout's Theorem on $\mathbb{P}_2$)**.** *Let $f, g \in \mathbb{K}[X_0, X_1, X_2]$ be 2 homogeneous polynomials of degree $n, m \geq 1$ respectively and denote their vanishing sets in $\mathbb{P}_2$ by $C = Z(f)$ and $D = Z(g)$. Then $C \cap D \neq \emptyset$, i.e. C and D always intersect in at least 1 point. If moreover $\gcd(f, g) = 1$, then C and D intersect in exactly nm points with multiplicities.*

Corollary D.1.15. *Let $f \in \mathbb{K}[X_0, X_1, X_2]$ be an irreducible homogeneous polynomial of degree 2. Then there exists a projective change of variables such that f can be written as $X_0^2 - X_1X_2$. Hence all irreducible conics in $\mathbb{P}_2$ are smooth.*

Proposition D.1.16. *$\mathbb{P}_n$ is a complete variety, i.e. for every $m \geq 1$ and every quasi-projective variety $X \subseteq \mathbb{P}_m$, the projection map $\pi : X \times \mathbb{P}_n \to \mathbb{P}_n$ is closed.*

Proposition D.1.17. *Let $f \in \mathbb{K}[X, Y]$ be a non-constant irreducible polynomial and denote*

$$V = \left\{ (x, y) \in \mathbb{A}_2 \;\middle|\; f(x, y) = 0 \right\}$$

the resulting irreducible curve in the affine plane $\mathbb{A}_2$. Then
1) The coordinate ring $\mathbb{K}[V] = \mathbb{K}[X, Y]/\langle\, f \,\rangle$ is a Noetherian integral domain of Krull dimension 1.
2) Let $(x, y) \in V$ and $\mathfrak{M} = \langle\, \bar{X} - x \,,\, \bar{Y} - y \,\rangle \trianglelefteq \mathbb{K}[V]$ be the corresponding maximal ideal. The following statements are equivalent:

- *(a) (x, y) is a smooth point, i.e. the derivatives $\frac{\partial f}{\partial X}$ and $\frac{\partial f}{\partial Y}$ do not vanish simultaneously at (x, y).*
- *(b) $\mathbb{K}[V]_{\mathfrak{M}}$ is a regular local ring of Krull dimension 1.*
- *(c) $\mathbb{K}[V]_{\mathfrak{M}}$ is a principal ideal domain.*

Lemma D.1.18. *Let $f, g \in \mathbb{K}[X, Y]$ be non-constants polynomials with f irreducible and consider a point $p \in \mathbb{A}_2$ such that $f(p) = 0$, but $g(p) \neq 0$. If we denote $C = Z(f)$ and $D = Z(fg)$, then*

$$\mathbb{K}[D]_{\mathfrak{M}_p} \cong \mathbb{K}[C]_{\mathfrak{M}'_p} \;,$$

where $\mathfrak{M}_p \trianglelefteq \mathbb{K}[D]$ *and* $\mathfrak{M}'_p \trianglelefteq \mathbb{K}[C]$ *are the maximal ideals corresponding to* p*. In other words: The localization of the coordinate ring of a curve at a point which only belongs to one irreducible component of the curve is equal to the localization of the coordinate ring of this irreducible component at that point.*

Proof. Since $\langle\, fg \,\rangle \subset \langle\, f \,\rangle$, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{K}[X,Y] & & \\ \downarrow\!\!\!\downarrow & \searrow\!\!\!\searrow & \\ \mathbb{K}[D] & \xrightarrow{\ \varphi\ }\!\!\!\!\rightarrow & \mathbb{K}[C] \end{array}$$

where $\mathbb{K}[C] = \mathbb{K}[X,Y]/\langle\, f \,\rangle$ and $\mathbb{K}[D] = \mathbb{K}[X,Y]/\langle\, fg \,\rangle$, which is not an integral domain since $\langle\, fg \,\rangle$ is not a prime ideal. The map $\varphi : \mathbb{K}[D] \twoheadrightarrow \mathbb{K}[C] : \bar{a} \mapsto [a]$ is not injective in general. We localize the coordinate rings at the maximal ideals

$$\mathfrak{M}_p = \left\{ \bar{a} \in \mathbb{K}[D] \;\middle|\; \bar{a}(p) = 0 \right\} \trianglelefteq \mathbb{K}[D] \;,$$
$$\mathfrak{M}'_p = \left\{ [a] \in \mathbb{K}[C] \;\middle|\; [a](p) = 0 \right\} \trianglelefteq \mathbb{K}[C] \;,$$

where evaluations at p are well-defined since $f(p) = 0$. The morphism φ induces a ring homomorphism

$$\phi \;:\; \mathbb{K}[D]_{\mathfrak{M}_p} \longrightarrow \mathbb{K}[C]_{\mathfrak{M}'_p} \;:\; \frac{\bar{a}}{\bar{b}} \longmapsto \frac{[a]}{[b]} \;.$$

This is well-defined : if $\frac{\bar{a}}{\bar{b}} = \frac{\bar{c}}{\bar{d}}$, then $\exists\, \bar{t} \in \mathbb{K}[D] \setminus \mathfrak{M}_p$ such that

$$\bar{t} \cdot (\bar{a} \cdot \bar{d} - \bar{b} \cdot \bar{c}) = 0 \;\Leftrightarrow\; \bar{t} \cdot \overline{ad - bc} = 0 \;\Rightarrow\; [t] \cdot [ad - bc] = 0 \;\Rightarrow\; [ad] = [bc]$$

since $[t] \neq 0$, otherwise $t \in \langle\, f \,\rangle$ and $\bar{t}(p) = 0$, which contradicts $\bar{t} \notin \mathfrak{M}_p$. Moreover ϕ is surjective because $\bar{b} \notin \mathfrak{M}_p$ whenever $[b] \notin \mathfrak{M}'_p$ (since $\bar{b}(p) = 0$ implies that $[b](p) = 0$ as well). To show that ϕ is injective, let $\frac{[a]}{[b]} = 0$, which implies that $a \in \langle\, f \,\rangle$ since $\mathbb{K}[C]$ is an integral domain, hence $\overline{ag} = 0$. Note that $\bar{g} \notin \mathfrak{M}_p$ since $g(p) \neq 0$. But then $\frac{\bar{a}}{\bar{b}} = 0$ because $\bar{g} \cdot (\bar{a} \cdot 1 - \bar{b} \cdot 0) = \bar{g} \cdot \bar{a} = 0$. It follows that ϕ is a ring isomorphism between local rings, hence an isomorphism of local rings. □

Proposition D.1.19. *A formal power series* f *is a unit in* $R[[X_1, \ldots, X_n]]$ *if and only if its constant term is a unit in* R*. In particular a power series* $f \in \mathbb{K}[[X,Y]]$ *is invertible if and only if it has non-zero constant term.*

Lemma D.1.20. *Let $a, b, c \in \mathbb{P}_2$ be 3 different (simple) points. If they are non-collinear, then there exists a projective transformation which maps a, b, c to $(1:0:0)$, $(0:1:0)$, $(0:0:1)$.*
If they are collinear, then they can be mapped to $(1:0:0)$, $(0:0:1)$, $(1:0:1)$.

D.2 Some facts about intersections of ideals

Definition D.2.1. We say that 2 ideals $I, J \trianglelefteq R$ are *coprime* if $I + J = R$.

Lemma D.2.2. [[2], p.6-7]
Let $I, J \trianglelefteq R$ be ideals in a ring R. Then $(I+J)\cdot(I\cap J) \subseteq I\cdot J \subseteq I\cap J$. This follows from the distributive law

$$(I+J)\cdot(I\cap J) = I\cdot(I\cap J) + J\cdot(I\cap J) \subseteq I\cdot J\ . \tag{D.1}$$

In particular we obtain $I\cap J = I\cdot J$ for coprime ideals.

Remark D.2.3. The inclusion (D.1) may be strict and the converse in the case of coprime ideals is false as well. Take for example $I = \langle\, X\,\rangle$ and $J = \langle\, Y\,\rangle$ in $R = \mathbb{K}[X, Y]$. Then $I + J = \langle\, X, Y\,\rangle$ and

$$I\cap J = \langle\, XY\,\rangle \quad , \quad I\cdot J = \langle\, XY\,\rangle \quad , \quad (I+J)\cdot(I\cap J) = \langle\, X^2Y, XY^2\,\rangle\ .$$

We don't have equality and the product equals the intersection althought I and J are not coprime. In order to obtain strict inclusions everywhere, consider e.g. $I = \langle\, X^2\,\rangle$ and $J = \langle\, XY\,\rangle$. Then

$$I+J = \langle\, X^2, XY\,\rangle \quad , \quad I\cap J = \langle\, X^2Y\,\rangle \quad , \quad I\cdot J = \langle\, X^3Y\,\rangle\ ,$$
$$(I+J)\cdot(I\cap J) = \langle\, X^4Y, X^3Y^2\,\rangle\ .$$

Remark D.2.4. However (D.1) is an equality in Dedekind rings ; a proof is given in [[53], 263027].
We only prove the statement for $R = \mathbb{Z}$. Let $I = \langle\, a\,\rangle$, $J = \langle\, b\,\rangle$ for some $a, b \in \mathbb{Z}$. If we denote $d = \gcd(a, b)$ and $l = \operatorname{lcm}(a, b)$, then we have

$$I+J = \langle\, d\,\rangle \quad , \quad I\cap J = \langle\, l\,\rangle \quad , \quad I\cdot J = \langle\, a\cdot b\,\rangle$$

and thus

$$(I+J)\cdot(I\cap J)=\langle\, d\cdot l\,\rangle \qquad \text{and} \qquad I\cdot J=\langle\, a\cdot b\,\rangle\ .$$

These are equal since $\gcd(a,b)\cdot\mathrm{lcm}(a,b)=a\cdot b$ (consider the prime factorization of a and b).

Lemma D.2.5. [[53], 742473]
Let M be an R-module and $I\trianglelefteq R$ an ideal. Then $M\otimes_R R/I\cong M/(I*M)$.

Proof. Consider the morphisms

$$\varphi\ :\ M\otimes R/I\longrightarrow M/IM\ :\ m\otimes\bar{r}\longmapsto[r*m]\ ,$$
$$\psi\ :\ M/IM\longrightarrow M\otimes R/I\ :\ [m]\longmapsto m\otimes\bar{1}\ .$$

They are well-defined since $\varphi: m\otimes\bar{i}\mapsto[i*m]=0$ and

$$\psi\ :\ [i*m]\mapsto(i*m)\otimes\bar{1}=m\otimes(i*\bar{1})=m\otimes\bar{i}=0$$

for all $i\in I$. Moreover they are inverse to each other:

$$m\otimes\bar{r}\longmapsto[r*m]\longmapsto(r*m)\otimes\bar{1}=m\otimes\bar{r}\ ,$$
$$[m]\longmapsto m\otimes\bar{1}\longmapsto[1*m]=[m]\ .$$

□

Corollary D.2.6. *Let $I,J\trianglelefteq R$ be two ideals. Then*

$$I\otimes_R R/J\ \cong\ I/(I\cdot J) \qquad \textit{and} \qquad R/I\otimes_R R/J\ \cong\ R/(I+J)\ .$$

Proof. a) Taking $M=I$ in Lemma D.2.5 gives $I\otimes_R R/J\cong I/(J*I)=I/(I\cdot J)$.
b) If we take $M=R/I$ (and interchange the roles of I and J), then

$$R/I\otimes_R R/J\ \cong\ (R/I)/(J*R/I)\ .$$

We will show that $J*R/I\cong(I+J)/I$, so that

$$R/I\otimes_R R/J\ \cong\ \big(R/I\big)/\big((I+J)/I\big)\ \cong\ R/(I+J)\ .$$

Consider the morphism $J*R/I\to(I+J)/I: j*\bar{r}\mapsto\overline{j\cdot r}$, which is well-defined. It is injective since if $j\cdot r\in I$, then $j*\bar{r}=\overline{j\cdot r}=\bar{0}$. If $\bar{x}\in(I+J)/I$ is given, then $x=i+j$ for some $i\in I$, $j\in J$ and $j*\bar{1}=\bar{j}=\bar{x}$ since $\bar{i}=\bar{0}$, so we also get surjectivity. (Note that we need $(I+J)/I$ since J/I may not exist if $I\nsubseteq J$.) □

Proposition D.2.7. [[52], 49259]
Let $I, J \trianglelefteq R$ be two ideals. Then

$$I \cap J = I \cdot J \;\Leftrightarrow\; \mathrm{Tor}_1(R/I, R/J) = \{0\} \,.$$

Proof. Consider the sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \,,$$

which is exact. Tensoring by R/J we get

$$\begin{aligned} \ldots \longrightarrow \mathrm{Tor}_1(I, R/J) \longrightarrow \mathrm{Tor}_1(R, R/J) \longrightarrow \mathrm{Tor}_1(R/I, R/J) \\ \longrightarrow I \otimes_R R/J \longrightarrow R \otimes_R R/J \longrightarrow R/I \otimes_R R/J \longrightarrow 0 \,. \end{aligned}$$

Since R is flat (free modules are flat), we have $\mathrm{Tor}_1(R, R/J) = \{0\}$ and the sequence simplifies to

$$0 \longrightarrow \mathrm{Tor}_1(R/I, R/J) \longrightarrow I/(I \cdot J) \stackrel{\pi}{\longrightarrow} R/J \longrightarrow R/(I+J) \longrightarrow 0 \,. \qquad \text{(D.2)}$$

Let us compute the kernel of π. Under all identifications, we get

$$\begin{aligned} \pi \;:\; I/(I \cdot J) \stackrel{\sim}{\longrightarrow} I \otimes R/J \longrightarrow R \otimes R/J \stackrel{\sim}{\longrightarrow} R/J \\ [i] \longmapsto i \otimes \bar{1} \longmapsto i \otimes \bar{1} \longmapsto i * \bar{1} = \bar{i} \,, \end{aligned}$$

so that $\bar{i} = \bar{0} \Leftrightarrow i \in J$, thus $i \in I \cap J$ and $[i] \in (I \cap J)/(I \cdot J) = \ker \pi$. But this is also equal to $\mathrm{Tor}_1(R/I, R/J)$ by exactness of (D.2). Thus

$$I \cap J = I \cdot J \;\Leftrightarrow\; (I \cap J)/(I \cdot J) = \{0\} \;\Leftrightarrow\; \mathrm{Tor}_1(R/I, R/J) = \{0\} \,. \qquad \square$$

D.3 A useful application of essential ideals

Definition D.3.1. [[44], 3.26, p.74]
Let M be an R-module. We say that $N \leq M$ is an *essential submodule* if $N \cap M' \neq \{0\}$ for any non-zero submodule $M' \leq M$. In the case of $M = R$, we get the notion of an *essential ideal* $I \trianglelefteq R$. In other words, I is an essential ideal if $\forall\, J \trianglelefteq R$, the intersection is zero if and only if J is zero:

$$I \cap J = \{0\} \;\Leftrightarrow\; J = \{0\} \,.$$

Lemma D.3.2. [[44], 3.27, p.74]
$N \leq M$ is an essential submodule if and only if $\forall\, m \in M$, $m \neq 0$, $\exists\, r \in R$ such that $0 \neq r * m \in N$.
In particular, $I \trianglelefteq R$ is an essential ideal if and only if $\forall\, a \in R$, $a \neq 0$, $\exists\, r \in R$ such that $0 \neq a \cdot r \in I$.

Proof. $\underline{\Rightarrow}$: Let $m \neq 0$ and consider the submodule $N \cap \langle\, m \,\rangle$. This is non-zero since N is essential. Thus $\exists\, n \in N$, $n \neq 0$ such that $n = r * m$ for some $r \in R$.
$\underline{\Leftarrow}$: Let $M' \leq M$ be any non-zero submodule and $m \in M'$, $m \neq 0$. By assumption $\exists\, r \in R$ such that $0 \neq r * m \in N$. Thus $r * m \in N \cap M'$, which is hence non-zero. □

Lemma D.3.3. cf. [[28], 3.3, p.272]
1) Let $\varphi : M \to L$ be a morphism of R-modules. If $N \leq L$ is an essential submodule, then $\varphi^{-1}(N) \leq M$ is essential.
2) Let $\varphi : R \to T$ be a surjective ring homomorphism. If $J \trianglelefteq T$ is an essential ideal, then $\varphi^{-1}(J) \trianglelefteq R$ is essential.

Proof. We use the criterion from Lemma D.3.2.
1) Let $m \in M$, $m \neq 0$; we shall show that $\exists\, r \in R$ such that $0 \neq r*m \in \varphi^{-1}(N)$. If $\varphi(m) = 0$, one can choose $r = 1$ since $0 \neq m \in \varphi^{-1}(N)$. If $\varphi(m) \neq 0$, then $\exists\, r \in R$ such that $0 \neq r*\varphi(m) \in N$ since N is essential. $r*\varphi(m) = \varphi(r*m) \in N$, so $r * m \in \varphi^{-1}(N)$ and it is non-zero since $\varphi(r * m) \neq 0$.
2) Let $a \in R$, $a \neq 0$. If $\varphi(a) = 0$, one can choose $r = 1$ as above. If $\varphi(a) \neq 0$, then $\exists\, t \in T$ such that $0 \neq \varphi(a) \cdot t \in J$ since J is essential. By surjectivity we get $r \in R$ such that $t = \varphi(r)$ and hence $0 \neq \varphi(a) \cdot \varphi(r) = \varphi(a \cdot r) \in J$, so $a \cdot r \in \varphi^{-1}(J)$ and $a \cdot r \neq 0$. □

Proposition D.3.4. *Let $I \trianglelefteq R$ be an ideal. If I contains a NZD, then I is essential. The converse is true if R is reduced and Noetherian.*

Proof. Assume that I contains a NZD r and let $a \in R$, $a \neq 0$. Then $0 \neq a \cdot r \in I$, so I is essential.
Now let R be a reduced Noetherian ring and I an essential ideal. Assume that I

entirely consists of zero-divisors. By Corollary B.2.26 we have $\mathrm{Ann}_R(I) \neq \{0\}$. Now consider $I \cap \mathrm{Ann}_R(I)$. If this intersection is non-empty, there exists $i \in I$ which annihilates all elements in I, in particular it annihilates itself, i.e. $i^2 = 0$. But then $i = 0$ as R is reduced. Hence $I \cap \mathrm{Ann}_R(I) = \{0\}$, which contradicts that I is essential. □

Remark D.3.5. To see that the converse of Proposition D.3.4 may fail in the non-reduced case, we consider R as in Example E.4. The ideal $I = \langle \bar{X}, \bar{Y}, \bar{Z} \rangle$ only consists of zero-divisors. However it is essential. Let $\bar{f} \in R$, $\bar{f} \neq \bar{0}$. We have to find $\bar{g} \in R$ such that $\bar{0} \neq \bar{f} \cdot \bar{g} \in I$. If $\bar{f} \in I$, it suffices to choose $\bar{g} = \bar{1}$. If $\bar{f} \notin I$, then it has a non-zero constant term $f(0) \neq 0$. Multiplying by $\bar{g} = \bar{X}$ will cancel all non-constant terms by definition of R and we get $\bar{f} \cdot \bar{g} = f(0) \cdot \bar{X} \in I$.

Proposition D.3.6. cf. [[28], 3.4, p.272]
Let R be a reduced Noetherian ring, M an R-module and $I \subseteq \mathrm{Ann}_R(M)$. If M is torsion-free over R, then M is also torsion-free over R/I.

Proof. Let $m \in M$ be arbitrary and denote $\varphi : R \twoheadrightarrow R/I$. $\bar{r} * m = r * m = 0$ implies that

$$\varphi^{-1}\big(\mathrm{Ann}_{R/I}(m)\big) = \mathrm{Ann}_R(m) \ .$$

Now let $m \in M$ be a torsion element over R/I, i.e. $\mathrm{Ann}_{R/I}(m)$ contains a NZD and is thus essential. Taking the preimage under the surjective ring homomorphism φ, we find by Lemma D.3.3 that $\mathrm{Ann}_R(m)$ is an essential ideal in R. As R is reduced, we conclude that $\mathrm{Ann}_R(m)$ also contains a NZD, thus $m = 0$ since M is torsion-free over R. So M is also torsion-free over R/I. □

D.4 Basic facts on Geometric Invariant Theory

The main idea of GIT is to rigorously define the notion of a quotient space of a group action. We start by recalling some basic facts about group actions on algebraic varieties.

Definition D.4.1. [[60], p.42]
An *algebraic group* is a group G which also admits a structure of an algebraic variety such that the maps

$$G \times G \longrightarrow G \ : \ (g,h) \longmapsto g \cdot h \qquad , \qquad G \longrightarrow G \ : \ g \longmapsto g^{-1}$$

of multiplication and inversion are morphisms of algebraic varieties. A *morphism of algebraic groups* is a map that is simultaneously a group homomorphism and a morphism of algebraic varieties.

Definition D.4.2. [[60], p.43]
An *action* of an algebraic group G on a variety X is a morphism of varieties $G \times X \to X$ such that $e.x = x$ and $g.(h.x) = (g \cdot h) \, . \, x$, $\forall \, g, h \in G$, $x \in X$, where $e \in G$ is the identity element.

Example D.4.3. [[60], p.44]
Standard examples of algebraic groups are GL_n, which is an open subvariety of $\mathbb{A}_{n^2}$, and SL_n, being a closed subgroup of GL_n. They act on the affine space $\mathbb{A}_n$ by left multiplication.

Definition D.4.4. [[60], p.43]
Let G be an algebraic group acting on a variety X. The *orbit* of an element $x \in X$ is the subset

$$O(x) = G.X = \left\{ \, g.x \; \middle| \; g \in G \, \right\} \subseteq X$$

and its *orbit map* is given by $\sigma_x : G \to X : g \mapsto g.x$. In particular it is continuous and $\operatorname{im} \sigma_x = O(x)$. The *stabilizer* of x is the closed subgroup

$$\mathrm{Stab}_G(x) = \left\{ \, g \in G \; \middle| \; g.x = x \, \right\} = \sigma_x^{-1}\big(\{x\}\big) \; .$$

The *orbit space* of X is given by $X/G = \{ \, O(x) \, | \, x \in X \, \}$ and identifies points in X that belong to the same orbit. In general it is no longer a variety. The action is called *free* if $\mathrm{Stab}_G(x) = \{e\}$ for all $x \in X$, *transitive* if $O(x) = G$ for all $x \in X$ and *closed* if all orbits are closed.

Definition D.4.5. [[60], p.43-44]
A subset $W \subseteq X$ is called *G-invariant* if $G.W \subseteq W$ (hence $G.W = W$). In particular for $W = \{x\}$ one obtains $g.x = x$ and says that the point x is *invariant* under G. If G acts on two varieties X and Y, a morphism $\phi : X \to Y$ is called a *G-morphism* if $\phi(g.x) = g.\phi(x)$, $\forall\, x \in X$, $g \in G$. A *G-invariant* morphism satisfies $\phi(g.x) = \phi(x)$ for all $x \in X$ and $g \in G$. In other words, G-invariant morphisms are constant on orbits.

Definition D.4.6. [[60], p.44]
An algebraic group is called *linear* if it is isomorphic to a closed subgroup of GL_n for some $n \in \mathbb{N}$. In particular there is an injection $\rho : G \hookrightarrow \mathrm{GL}_n$.

In order to state the main results of GIT, we briefly mention the notion of a reductive group without explaining details.

Definition D.4.7. [[60], p.50]
The *radical* of a linear algebraic group G, denoted by $R(G)$, is the unique maximal closed connected normal and solvable subgroup of G. Equivalently it is the identity component of a maximal normal solvable subgroup. The group G is called *reductive* if $R(G)$ is isomorphic to a torus, i.e. a direct product of copies of $\mathbb{K}^*$ (with respect to multiplication).[1] One can show that e.g. GL_n and SL_n are reductive.
An alternative definition is the following : an element $g \in G$ is called *unipotent* if the endomorphism $\mathrm{id} - \rho(g)$ is nilpotent in GL_n. A group is unipotent if all its elements are unipotent. The *unipotent radical* of G, denoted by $R_u(G)$, is a maximal closed connected normal and unipotent subgroup of G. Equivalently it is the subgroup of all unipotent elements in $R(G)$. Then G is reductive if and only if $R_u(G) = \{e\}$.

Now we focus on actions of linear algebraic groups on projective spaces.

Definition D.4.8. [[7], 1.4, p.103]
Let G be a linear algebraic group acting on the vector space $\mathbb{K}^{n+1}$ with injection

[1] In the literature the multiplicative group $\mathbb{K}^*$ is often denoted by $\mathbb{G}_m = \mathrm{GL}(1)$.

$\rho: G \to \mathrm{GL}_{n+1}$, i.e. $g.v = \rho(g) \cdot v$ for $v \in \mathbb{K}^{n+1}$. This induces an action of G on the space of homogeneous polynomials $f \in \mathbb{K}[X_0, \ldots, X_n]$ by

$$(g.f)(v) := f\big(g^{-1}.v\big) .$$

So in particular we can consider *G-invariant polynomials*.

Definition D.4.9. [[60], p.73]
Let $X \subseteq \mathbb{P}_n$ be a projective variety and G a linear algebraic group acting on $\mathbb{K}^{n+1}$ via $\rho: G \to \mathrm{GL}_{n+1}$.
1) For any $z \in \mathbb{P}_n$, a *point lying over* z is a representative $\hat{z} \in \mathbb{K}^{n+1} \setminus \{0\}$.
2) G also acts on X via ρ: the action for $x \in X$ is given by $g.x = \langle\, \rho(g).\hat{x}\, \rangle \in X$. One also says that G acts *linearly* on X.
3) For a non-constant G-invariant homogeneous polynomial $f \in \mathbb{K}[X_0, \ldots, X_n]$, we define the set

$$X_f = \big\{\, x \in X \;\big|\; f(x) \neq 0 \,\big\} .$$

This is a G-invariant affine open subset of X.

Definition D.4.10. [[60], p.73-74]
Let $X \subseteq \mathbb{P}_n$ be a projective variety and G a linear algebraic group that linearly acts on X.
1) We say that a point $x \in X$ is *semistable* if there exists a G-invariant homogeneous polynomial f with $\deg f \geq 1$ such that $f(x) \neq 0$.
2) A point $x \in X$ is called *stable* if $\mathrm{Stab}_G(x)$ is finite and x is semistable such that the action of G on X_f (to which x belongs) is closed.[2] The condition of x having finite stabilizer is equivalent to the one that $\dim O(x) = \dim G$.

Lemma D.4.11. [[60], 3.13, p.74]
The sets X^{ss} and X^s of semistable, resp. stable points are G-invariant open subsets of X.

[2] In Mumford-Fogarty [58] this is actually the definition of a "properly stable" point. In order to be coherent with our results we keep the one used in [60].

Definition D.4.12. [[60], p.70] , [[7], 1.1, p.102] and [[15], 2.3.1, p.11-12]
Let G be a linear algebraic group that acts linearly on two projective varieties X and Y. A morphism $\phi : X \to Y$ of projective varieties is called a *good quotient* of X by G if ϕ is surjective, constant on orbits and satisfies some more technical conditions.[3] This is denoted by $Y = X /\!/ G$.
We say that it is a *geometric quotient* if it is a good quotient which is also an orbit space, denoted by $Y = X/G$. So in particular in case of a geometric quotient the orbit space of X is again a projective variety.

The essence of reductive groups will now become clear by the following deep results. Here below $\overline{Z}$ denotes the closure of a set Z.

Theorem D.4.13. [[60], 3.14, p.74-77] , [[7], 1.7, p.104] and [[67], 14.4, p.94]
Let G be a reductive algebraic group that acts linearly on a projective variety $X \subseteq \mathbb{P}_n$. Then
1) There exists a good quotient $\phi : X^{ss} \to Y$ where $Y = X^{ss} /\!/ G$ is a projective variety.
2) There is an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s$ and $Y^s = X^s/G$ is a geometric quotient.
3) If $x_1, x_2 \in X^{ss}$, then $\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{O(x_1)} \cap \overline{O(x_2)} \cap X^{ss} \neq \emptyset$.
4) If $x \in X^{ss}$, then x is stable if and only if it has finite stabilizer and $O(x)$ is closed in X^{ss}.

Remark D.4.14. The message of this theorem is that if the group G acting on X is reductive then we are able to construct "nice" quotient spaces of the sets of stable and semistable points.

Moreover we have the following useful criteria for determining (semi)stable points.

Theorem D.4.15. [[58], Prop. 2.2, p.50-51] and [[60], 4.7, p.101-102]
Let G be a reductive algebraic group that acts linearly on a projective variety $X \subseteq \mathbb{P}_n$. For any $x \in X$, let $\hat{x} \in \mathbb{K}^{n+1}$ be a point lying over x. Then, independent of the representative $\hat{x}$:

[3] They are not of our interest here ; for more information, we refer to the given references.

1) x is semistable $\Leftrightarrow 0 \notin \overline{O(\hat{x})}$.
2) x is stable $\Leftrightarrow$ *the stabilizer of $\hat{x}$ is finite and $O(\hat{x})$ is closed in $\mathbb{K}^{n+1}$.*
Condition 2) is moreover equivalent to the one that the morphism defined by $G \to \mathbb{K}^{n+1} : g \mapsto g.\hat{x}$ is proper.

Definition D.4.16. [[60], p.103-104]
A *1-parameter subgroup* (or for short a *1-PS*) of an algebraic group G is a non-trivial homomorphism of algebraic groups $\lambda : \mathbb{K}^* \to G$. If G acts linearly on a projective variety $X \subseteq \mathbb{P}_n$, one can associated to any 1-PS a *weight* $\mu_\lambda : X \to \mathbb{Z}$ defined by the condition

$$\mu_\lambda(x) := \text{the unique integer } \mu \text{ such that}$$
$$\lim_{t\to 0} \left(t^\mu \cdot \lambda(t).\hat{x}\right) \text{ exists in } \mathbb{K}^{n+1} \text{ and is non-zero}$$

One can show that this definition is independent of the choice of $\hat{x}$.

Using this we finally get a last criterion to determine (semi)stable points of X.

Theorem D.4.17 (Hilbert-Mumford)**.** [[58], Th. 2.1, p.49] and [[60], 4.9, p.105]
Let G be a reductive algebraic group that acts linearly on a projective variety $X \subseteq \mathbb{P}_n$. Then

$$x \in X \textit{ is semistable } \Leftrightarrow \mu_\lambda(x) \geq 0 \textit{ for every 1-PS } \lambda \textit{ of } G\,.$$

Moreover the same equivalence holds true with a strict inequality when x is stable.

The reason why it is interesting to consider such 1-PS is that they have a simple form e.g. in the case of SL_n.

Proposition D.4.18. [[60], 4.10, p.108] and [[7], p.117]
Every 1-PS of SL_n is conjugate to one of the form

$$\lambda(t) = \begin{pmatrix} t^{r_1} & 0 & \dots & 0 \\ 0 & t^{r_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{r_n} \end{pmatrix} ,$$

where $r_1 \geq \ldots \geq r_n$ such that $r_1 + \ldots + r_n = 0$ and not all r_i are zero.

Appendix E

Summary of the main examples

In this appendix we summarize all properties from the main examples in Part I. For a better visualization we also include figures to show the irreducible components of all schemes.

For each ring R, we give a module M and denote $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{F} = \widetilde{M}$, $\mathcal{Z} = \mathcal{Z}_a(\mathcal{F}) = \mathcal{Z}_f(\mathcal{F})$. Since all modules here are generated by 1 element, we alway get $\operatorname{Ann}_R(M) = \operatorname{Fitt}_0(M)$ and don't need to distinguish between annihilator support and Fitting support. Moreover all modules are of the type $M = R/I$, so

$$\operatorname{Ass}\big(\operatorname{Ann}_R(M)\big) = \operatorname{Ass}(I) = \operatorname{Ass}_R(R/I) = \operatorname{Ass}_R(M) \ .$$

In particular all sheaves are structure sheaves of some subschemes and thus torsion-free on their support, i.e. every $\mathcal{F}$ here below is torsion-free on $\mathcal{Z}$.

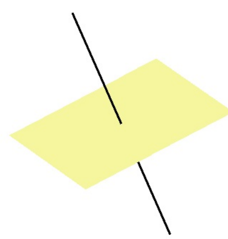
Example E.1. (line with a double point)

$$R = \mathbb{K}[X,Y]/\langle\, Y^2, XY \,\rangle \quad , \quad M = \mathbb{K} = R/\langle\, \bar{X}, \bar{Y} \,\rangle$$
$$\{\bar{0}\} = \langle\, \bar{X} \,\rangle \cap \langle\, \bar{Y} \,\rangle = Q_1 \cap Q_2 \quad , \quad \mathrm{Ann}_R(M) = \langle\, \bar{X}, \bar{Y} \,\rangle = Q'_1$$
$$\mathrm{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \ , \ P_2 = \langle\, \bar{Y} \,\rangle \,\big\} \quad , \quad \mathrm{Ass}_R(M) = \big\{\, P'_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \,\big\}$$

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = V(Q_1) \cup V(P_2) \quad , \quad \dim \mathcal{X} = 1$$
$$\mathcal{Z} = \mathcal{Z}_1 = V(P'_1) \quad , \quad \dim \mathcal{F} = 0$$

M is torsion-free over R and $\mathcal{F}$ is torsion-free on $\mathcal{X}$: $P'_1 = P_1$.
$\mathcal{F}$ is pure of dimension 0.

Example E.2. (plane and a line normal to it)

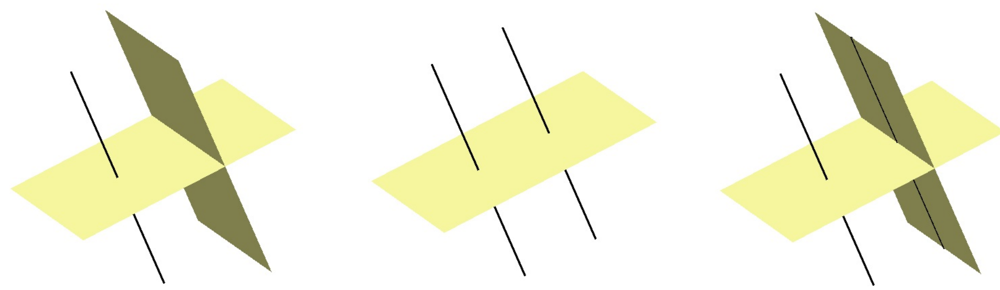


$$R = \mathbb{K}[X,Y,Z]/\langle\, ZX, ZY \,\rangle \quad , \quad M = R/\langle\, \bar{X}, \bar{Y} \,\rangle$$
$$\{\bar{0}\} = \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{Z} \,\rangle = Q_1 \cap Q_2 \quad , \quad \mathrm{Ann}_R(M) = \langle\, \bar{X}, \bar{Y} \,\rangle = Q'_1$$
$$\mathrm{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \ , \ P_2 = \langle\, \bar{Z} \,\rangle \,\big\} \quad , \quad \mathrm{Ass}_R(M) = \big\{\, P'_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \,\big\}$$
$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = V(P_1) \cup V(P_2) \quad , \quad \dim \mathcal{X} = 2$$
$$\mathcal{Z} = \mathcal{Z}_1 = V(P'_1) \quad , \quad \dim \mathcal{F} = 1$$

M is torsion-free over R and $\mathcal{F}$ is torsion-free on $\mathcal{X}$: $P'_1 = P_1$.
$\mathcal{F}$ is pure of dimension 1.

Example E.3. (2 perpendicular planes and a line)

Figure: $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{Z} = \operatorname{supp} \mathcal{F}$ and their superposition



$$R = \mathbb{K}[X,Y,Z]/\langle\, YZ(X-1), XZ(X-1)\,\rangle \qquad , \qquad M = R/\langle \bar{Y}\bar{Z}\,\rangle$$

$$\{\bar{0}\} = \langle\, \bar{Z}\,\rangle \cap \langle\, \bar{X}-1\,\rangle \cap \langle\, \bar{X},\bar{Y}\,\rangle = Q_1 \cap Q_2 \cap Q_3$$

$$\operatorname{Ass}_R(R) = \left\{\, P_1 = \langle\, \bar{Z}\,\rangle \ ,\ P_2 = \langle\, \bar{X}-1\,\rangle \ ,\ P_3 = \langle\, \bar{X},\bar{Y}\,\rangle \,\right\}$$

$$\operatorname{Ann}_R(M) = \langle\, \bar{Y}\bar{Z}\,\rangle = \langle\, \bar{Z}\,\rangle \cap \langle\, \bar{X}-1,\bar{Y}\,\rangle \cap \langle\, \bar{X},\bar{Y}\,\rangle = Q_1' \cap Q_2' \cap Q_3'$$

$$\operatorname{Ass}_R(M) = \left\{\, P_1' = \langle\, \bar{Z}\,\rangle \ ,\ P_2' = \langle\, \bar{X}-1,\bar{Y}\,\rangle \ ,\ P_3' = \langle\, \bar{X},\bar{Y}\,\rangle \,\right\}$$

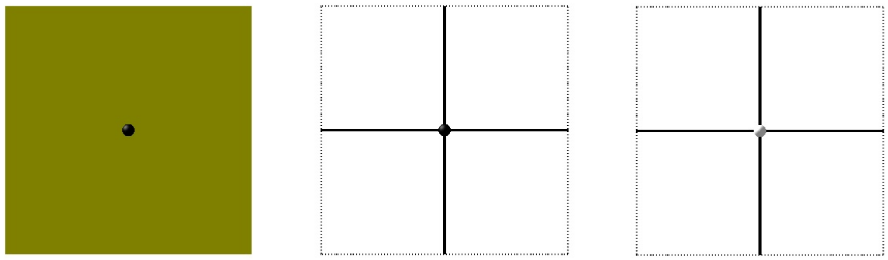
$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 = V(P_1) \cup V(P_2) \cup V(P_3) \quad , \quad \dim \mathcal{X} = 2$$

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 = V(P_1') \cup V(P_2') \cup V(P_3') \quad , \quad \dim \mathcal{F} = 2$$

M is not torsion-free over R: $P_2' \nsubseteq P_i$ for all i.
The torsion submodule is $\mathcal{T}_R(M) = \langle\, [\bar{X}\bar{Z}]\,\rangle$.
$\mathcal{F}$ is not torsion-free on $\mathcal{X}$: $P_2' \notin \operatorname{Ass}_R(R)$. $\mathcal{T}(\mathcal{F})$ is supported on $\mathcal{Z}_2$.
$\mathcal{F}$ is not pure of dimension 2: $\dim \mathcal{Z}_2 = \dim \mathcal{Z}_3 < \dim \mathcal{Z}_1$.

Example E.4. (plane with an embedded double point)

Figure: $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{Z} = \operatorname{supp} \mathcal{F}$ and $\operatorname{supp} \mathcal{T}(\mathcal{F})$



$$R = \mathbb{K}[X,Y,Z]/\langle\, XY, X^2, XZ \,\rangle \qquad , \qquad M = R/\langle\, \bar{Y}\bar{Z} \,\rangle$$

$$\{\bar{0}\} = \langle\, \bar{X} \,\rangle \cap \langle\, \bar{Y}, \bar{Z} \,\rangle = Q_1 \cap Q_2$$

$$\operatorname{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{X} \,\rangle \ , \ P_2 = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}$$

$$\operatorname{Ann}_R(M) = \langle\, \bar{Y}\bar{Z} \,\rangle = \langle\, \bar{X}, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{Y}, \bar{Z} \,\rangle = Q'_1 \cap Q'_2 \cap Q'_3$$

$$\operatorname{Ass}_R(M) = \big\{\, P'_1 = \langle\, \bar{X}, \bar{Z} \,\rangle \ , \ P'_2 = \langle\, \bar{X}, \bar{Y} \,\rangle \ , \ P'_3 = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}$$

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = V(P_1) \cup V(Q_2) \quad , \quad \dim \mathcal{X} = 2$$

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 = V(P'_1) \cup V(P'_2) \cup V(Q'_3) \quad , \quad \dim \mathcal{F} = 1$$

M is torsion-free over R: $P'_j \subseteq P_2$ for all j.

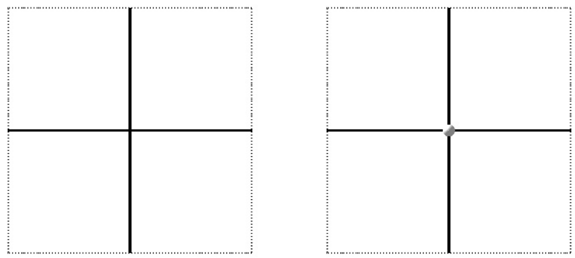
$\mathcal{F}$ is not torsion-free on $\mathcal{X}$: $P'_1, P'_2 \notin \operatorname{Ass}_R(R)$.

$\mathcal{T}(\mathcal{F})$ is not coherent and supported on $\mathcal{Z} \setminus \{P_2\}$.

$\mathcal{F}$ is not pure of dimension 1: $\dim \mathcal{Z}_3 < \dim \mathcal{Z}_1 = \dim \mathcal{Z}_2$.

Example E.5. (simple cross on a plane with embedded double point)

Figure: $\mathcal{Z} = \operatorname{supp} \mathcal{F}$ and $\operatorname{supp} \mathcal{T}(\mathcal{F})$



$$R = \mathbb{K}[X,Y,Z]/\langle\, XY, X^2, XZ \,\rangle \quad , \quad M = R/\langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle$$

$$\{\bar{0}\} = \langle\, \bar{X} \,\rangle \cap \langle\, \bar{Y}, \bar{Z} \,\rangle = Q_1 \cap Q_2$$

$$\operatorname{Ann}_R(M) = \langle\, \bar{X}, \bar{Y}\bar{Z} \,\rangle = \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{X}, \bar{Z} \,\rangle = Q'_1 \cap Q'_2$$

$$\operatorname{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{X} \,\rangle \;,\; P_2 = \langle\, \bar{X}, \bar{Y}, \bar{Z} \,\rangle \,\big\}$$

$$\operatorname{Ass}_R(M) = \big\{\, P'_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \;,\; P'_2 = \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\}$$

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = V(P_1) \cup V(Q_2) \quad , \quad \dim \mathcal{X} = 2$$

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 = V(P'_1) \cup V(P'_2) \quad , \quad \dim \mathcal{F} = 1$$

M is torsion-free over R: $P'_j \subseteq P_2$ for all j.

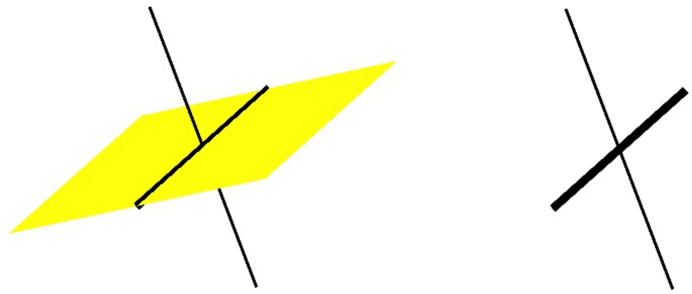
$\mathcal{F}$ is not torsion-free on $\mathcal{X}$: $P'_1, P'_2 \notin \operatorname{Ass}_R(R)$.

$\mathcal{T}(\mathcal{F})$ is not coherent and supported on $\mathcal{Z} \setminus \{P_2\}$.

$\mathcal{F}$ is pure of dimension 1: $\dim \mathcal{Z}_1 = \dim \mathcal{Z}_2$.

Example E.6. (plane with embedded double line and perpendicular line, I)

Figure: $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{Z} = \operatorname{supp} \mathcal{F}$



$$R = \mathbb{K}[X,Y,Z]/\langle\, XZ, YZ^2 \,\rangle \qquad , \qquad M = R/\langle\, \bar{X} \,\rangle$$

$$\{\bar{0}\} = \langle\, \bar{Z} \,\rangle \cap \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{X}, \bar{Z}^2 \,\rangle = Q_1 \cap Q_2 \cap Q_3$$

$$\operatorname{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{Z} \,\rangle \ , \ P_2 = \langle\, \bar{X}, \bar{Y} \,\rangle \ , \ P_3 = \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\}$$

$$\operatorname{Ann}_R(M) = \langle\, \bar{X} \,\rangle = \langle\, \bar{X}, \bar{Y} \,\rangle \cap \langle\, \bar{X}, \bar{Z}^2 \,\rangle = Q'_1 \cap Q'_2$$

$$\operatorname{Ass}_R(M) = \big\{\, P'_1 = \langle\, \bar{X}, \bar{Y} \,\rangle \ , \ P'_2 = \langle\, \bar{X}, \bar{Z} \,\rangle \,\big\}$$

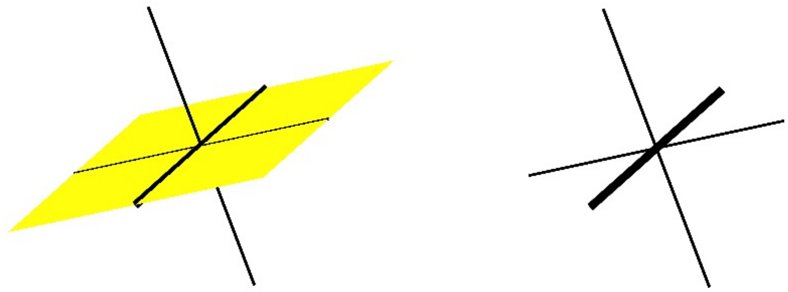
$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 = V(P_1) \cup V(P_2) \cup V(Q_3) \quad , \quad \dim \mathcal{X} = 2$$

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 = V(P'_1) \cup V(Q'_2) \quad , \quad \dim \mathcal{F} = 1$$

M is torsion-free over R and $\mathcal{F}$ is torsion-free on $\mathcal{X}$: $P'_1 = P_2$ and $P'_2 = P_3$.
$\mathcal{F}$ is pure of dimension 1 : $\dim \mathcal{Z}_1 = \dim \mathcal{Z}_2$.

Example E.7. (plane with embedded double line and perpendicular line, II)

Figure: $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{Z} = \operatorname{supp} \mathcal{F}$



$$R = \mathbb{K}[X,Y,Z]/\langle\, XZ, YZ^2\,\rangle \qquad , \qquad M = R/\langle\, \bar{X}\bar{Y}\,\rangle$$

$$\{\bar{0}\} = \langle\, \bar{Z}\,\rangle \cap \langle\, \bar{X},\bar{Y}\,\rangle \cap \langle\, \bar{X},\bar{Z}^2\,\rangle = Q_1 \cap Q_2 \cap Q_3$$

$$\operatorname{Ass}_R(R) = \big\{\, P_1 = \langle\, \bar{Z}\,\rangle \ , \ P_2 = \langle\, \bar{X},\bar{Y}\,\rangle \ , \ P_3 = \langle\, \bar{X},\bar{Z}\,\rangle\,\big\}$$

$$\operatorname{Ann}_R(M) = \langle\, \bar{X}\bar{Y}\,\rangle = \langle\, \bar{X},\bar{Y}\,\rangle \cap \langle\, \bar{Y},\bar{Z}\,\rangle \cap \langle\, \bar{X},\bar{Z}^2\,\rangle = Q_1' \cap Q_2' \cap Q_3'$$

$$\operatorname{Ass}_R(M) = \big\{\, P_1' = \langle\, \bar{X},\bar{Y}\,\rangle \ , \ P_2' = \langle\, \bar{Y},\bar{Z}\,\rangle \ , \ P_3' = \langle\, \bar{X},\bar{Z}\,\rangle\,\big\}$$

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 = V(P_1) \cup V(P_2) \cup V(Q_3) \quad , \quad \dim \mathcal{X} = 2$$

$$\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 = V(P_1') \cup V(P_2') \cup V(Q_3') \quad , \quad \dim \mathcal{F} = 1$$

M is not torsion-free over R: $P_2' \nsubseteq P_i$ for all i.
The torsion submodule is $\mathcal{T}_R(M) = \langle\, [\bar{X}]\,\rangle$.
$\mathcal{F}$ is not torsion-free on $\mathcal{X}$: $P_2' \notin \operatorname{Ass}_R(R)$.
$\mathcal{T}(\mathcal{F})$ is supported on $\mathcal{Z}_2$.
$\mathcal{F}$ is pure of dimension 1: $\dim \mathcal{Z}_1 = \dim \mathcal{Z}_2 = \dim \mathcal{Z}_3$.

Appendix F

Example of a Simpson moduli space that is not fine

In this appendix we present the case of the moduli space M_{2m+2} of semistable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $2m+2$. Since $\gcd(2,2) \neq 1$, it is not fine and its closed points only represent s-equivalence classes of semistable sheaves (Theorem 4.2.14). In particular there exist properly semistable sheaves for which the isomorphism classes differ from the s-equivalence classes.
We want to illustrate how singular sheaves may look like if the moduli space is not fine. From Theorem 4.3.10 we know that M_{2m+2} is irreducible and of dimension 5. However we will see that properly semistable sheaves may give rise to certain problems, e.g. in Example F.1.7 we obtain an s-equivalence class that simultaneously contains singular and non-singular sheaves (compare Remark 4.4.11), thus there is no "subvariety" (in the usual sense) of singular sheaves in M_{2m+2}. In order to still make sense out of M'_{2m+2}, we therefore need a new definition.
The main results of this section, which have all been proven by Trautmann in [67], are that an s-equivalence class $[\mathcal{F}] \in M_{2m+2}$ may be identified with the support of $\mathcal{F}$ (Theorem F.1.14), hence that $M_{2m+2} \cong \mathbb{P}_5$ and that the singular sheaves corresponds to those whose support is reducible, hence that M'_{2m+2} is singular and of codimension 1 (Corollary F.2.6).

F.1 Description of sheaves in M_{2m+2}

Proposition F.1.1. [[67], p.76] and [[15], p.2]
The isomorphism classes of sheaves $\mathcal{F} \in M_{2m+2}$ are exactly those which are given by an exact sequence

$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2}(-1) \xrightarrow{A} 2\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F} \longrightarrow 0\ , \tag{F.1}$$

where

$$A = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}$$

for some linear forms $z_1, z_2, w_1, w_2 \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ such that $\langle \det A \rangle \neq 0$.

Remark F.1.2. By Example 4.3.1 one checks again that

$$P_{\mathcal{F}}(m) = 2 \cdot \frac{(m+2)(m+1)}{2} - 2 \cdot \frac{m(m+1)}{2} = 2m+2\ .$$

Here we don't need an assumption about $\langle z_1 \rangle$ and $\langle z_2 \rangle$ to be linearly independent. If we e.g. try a similar argument as for M_{3m+1} in Remark 4.6.6 we end up with a structure sheaf of a line L such that $\mathcal{O}_L \hookrightarrow \mathcal{F}$. But this does not contradict semistability since

$$p_{\mathcal{O}_L}(m) = m+1 \qquad \text{and} \qquad p_{\mathcal{F}}(m) = \frac{2m+2}{2} = m+1\ .$$

Moreover we see that for linearly dependent $\langle z_1 \rangle$ and $\langle z_2 \rangle$ we can obtain properly semistable sheaves.

Remark F.1.3. Proposition F.1.1 indeed describes the isomorphism classes of sheaves in M_{2m+2}. There exist for example non-isomorphic sheaves given as in (F.1) by non-similar matrices, but which are still s-equivalent (consider e.g. Example F.1.9).

F.1.1 Stability and support

If $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ is given by a resolution (F.1), then $\det A = z_1 w_2 - w_1 z_2$ is a homogeneous polynomial of degree 2 and Proposition 4.5.9 implies that the support $C = \mathcal{Z}_f(\mathcal{F})$ is given by the quadratic curve $Z(\det A)$. By Proposition 4.4.16 we already know that $\mathcal{F}$ is non-singular if C is smooth.

In the following we are going to discuss all possible cases by distinguishing along the type of the conic C.

Proposition F.1.4. [[67], 12.2, p.76-77]
Let $\mathcal{F}$ be given as in (F.1) *with support $C = \mathcal{Z}_f(\mathcal{F})$. Then $\mathcal{F}$ is stable if and only if C is a smooth conic. In particular this implies that all stable sheaves are non-singular.*

Proof. $\underline{\Leftarrow}$: Assume that C is smooth. Then $\mathcal{F}$ is locally free on C by Proposition 4.4.16 and the structure sheaf $\mathcal{O}_C$ has Hilbert polynomial $2m+1$ because of (4.17). Since $P_{\mathcal{F}}(0) = 2$ we can choose a non-zero global section of $\mathcal{F}$ and construct an extension

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F} \longrightarrow \mathrm{Sky}_p(\mathbb{K}) \longrightarrow 0$$

for some $p \in C$ similarly as in Proposition 4.6.2. Using that $\mathcal{F}$ is locally free, one can show that it is isomorphic to the invertible sheaf $\mathcal{O}_C(p)$ given by the point divisor p. The latter is defined via the closed subscheme $\{p\} \hookrightarrow C$,

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{Sky}_p(\mathbb{K}) \longrightarrow 0$$

and can be considered as a hyperplane. Moreover $\mathcal{O}_C(p)$ is known to be stable, hence $\mathcal{F}$ is stable.

$\underline{\Rightarrow}$: Assume that C is not smooth. Then it is either a union of two lines $L_1 \cup L_2$ or a double line. In both cases, using coordinate changes and up to a constant, $\det A$ can then be written as a product $l_1 \cdot l_2$ and A is similar to a matrix of the form

$$A' = \begin{pmatrix} l_1 & w \\ 0 & l_2 \end{pmatrix} ,$$

where $l_1, l_2, w \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ are linear forms such that $\langle l_1 l_2 \rangle \neq 0$. Note that l_1 and l_2 may be equal if C is a double line. As already seen in Remark 4.6.6, the matrices $A \sim A'$ then induce an isomorphism of exact sequences in (F.1). Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{l_2} & \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{O}_{L_2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle (0,1)} & & \downarrow & & \\
0 & \longrightarrow & 2\,\mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{A'} & 2\,\mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_2}(-1) & \xrightarrow{l_1} & \mathcal{O}_{\mathbb{P}_2} & \longrightarrow & \mathcal{O}_{L_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{F.2}$$

The 9-Lemma (Lemma D.1.7) implies that the last column is exact as well. But then $\mathcal{O}_{L_2}$ is a non-zero proper coherent subsheaf of $\mathcal{F}$ with the same reduced Hilbert polynomial $m+1$. Hence $\mathcal{F}$ is not stable. $\square$

Remark F.1.5. We can construct such an extension for each $w \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$. For different w they will be non-isomorphic, but s-equivalent since $\mathcal{O}_{L_1}$ and $\mathcal{O}_{L_2}$ are stable, so we have an extension as in Example 4.1.23. For $w = 0$ we obtain the trivial extension $\mathcal{F} \cong \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$.

F.1.2 S-equivalence and counter-examples

Lemma F.1.6. *Let $z_1, z_2 \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ be two linear forms such that $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are linearly independent and let $p \in \mathbb{P}_2$ be the intersection point of the two lines $Z(z_1)$ and $Z(z_2)$.*
If $w \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$ is another linear form such that $w(p) = 0$, then w is a linear combination of z_1 and z_2, i.e. $w = \alpha_1 z_1 + \alpha_2 z_2$ for some $\alpha_1, \alpha_2 \in \mathbb{K}$.

Proof. Let us write

$$z_1 = a_0X_0 + a_1X_1 + a_2X_2 \quad , \quad z_2 = b_0X_0 + b_1X_1 + b_2X_2 \ ,$$
$$w = c_0X_0 + c_1X_1 + c_2X_2 \quad , \quad p = (p_0 : p_1 : p_2) \ .$$

for some coefficients $a_i, b_i, c_i, p_i \in \mathbb{K}$.

If $w(p) = 0$, we have the 3 equations

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ,$$

hence in order to get a non-trivial solution the matrix consisting of the coefficients must have determinant zero, i.e. the linear forms are linearly dependent, so w is a linear combination of z_1 and z_2. □

Example F.1.7. [[67], 12.2.1, p.77-78]
Using this we can now illustrate that it does not make sense to speak about singular sheaves when s-equivalence classes are involved. Consider the case where $C = L_1 \cup L_2$ is a union of two different lines, given by the determinant of A'. Thus it defines a sheaf $\mathcal{F} \in M_{2m+2} \setminus M^s_{2m+2}$ by Proposition F.1.4. Note that it depends on w, but for all choices of w the results will be s-equivalent. Let $p \in \mathbb{P}_2$ be the intersection point of the lines $Z(l_1)$ and $Z(l_2)$. This is the only singular point of C. In particular it implies that $\langle l_1 \rangle$ and $\langle l_2 \rangle$ are linearly independent. If $w(p) = 0$, then w is a linear combination of l_1 and l_2 because of Lemma F.1.6, hence A' is similar to a diagonal matrix and we get $\mathcal{F} \cong \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$. This is not a locally free $\mathcal{O}_{L_1 \cup L_2}$–module since the rank jumps at the intersection point p, thus $\mathcal{F}$ is singular. If $w(p) \neq 0$, a similar argument as in (4.40) gives

$$\mathbb{K}^2 \xrightarrow{A'(p)} \mathbb{K}^2 \longrightarrow \mathcal{F}(p) \longrightarrow 0 ,$$

where the rank of $A'(p)$ is 1, hence $\mathcal{F}_p \cong \mathcal{O}_{C,p}$. $\mathcal{F}$ being locally free on the smooth part $C \setminus \{p\}$, we conclude that this $\mathcal{F}$ is non-singular. Summarizing we obtained a non-singular sheaf $\mathcal{F}$ which is s-equivalent but non-isomorphic to the singular sheaf $\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$. So we see that an s-equivalence class can simultaneously contain singular and non-singular sheaves, i.e. the notion of being (non-)singular is not well-defined in $M_{2m+2} \setminus M^s_{2m+2}$.

Remark F.1.8. [[67], 12.2.2, p.77-78]
For completion let us also describe what happens in the case where C is a double line. The matrix is of the form

$$A' = \begin{pmatrix} l & w \\ 0 & l \end{pmatrix}$$

for some $l, w \in \Gamma\big(\mathbb{P}_2, \mathcal{O}_{\mathbb{P}_2}(1)\big)$. Denote $L = Z(l)$. If $w = \alpha l$ for some $\alpha \in \mathbb{K}$, then A' is again similar to a diagonal matrix and we get $\mathcal{F} \cong 2\,\mathcal{O}_L$, which is not a locally free $\mathcal{O}_C$–module. If l and w are linearly independent, let p denote the intersection point of L and $Z(w)$. Then $\mathcal{F}$ is locally free on $C \setminus \{p\}$, but not at p since for all $x \in C$ we obtain

$$\mathbb{K}^2 \xrightarrow{A'(x)} \mathbb{K}^2 \longrightarrow \mathcal{F}(x) \longrightarrow 0\ ,$$

where $A'(x)$ has rank 1 for $x \neq p$ and $A'(p) = 0$. So in this case all sheaves in the s-equivalence class are singular.

Example F.1.9. [[67], 12.3.3, p.80-81]
Let us give some more concrete examples of sheaves that are s-equivalent but non-isomorphic. Such sheaves usually arise as limits. By Proposition F.1.4 they must have singular support, otherwise they are stable, in which case isomorphism and s-equivalence classes coincide. For $t \in \mathbb{K}$ consider

$$A_t = \begin{pmatrix} z_1 & w_1 \\ t^2 z_2 & w_2 \end{pmatrix} \qquad , \qquad B_t = \begin{pmatrix} z_1 & t w_1 \\ t z_2 & w_2 \end{pmatrix}$$

and the sheaves $\mathcal{F}_t$ and $\mathcal{G}_t$ they define by taking cokernels as in (F.1). For $t \neq 0$ we have $A_t \sim B_t$ since

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \cdot \begin{pmatrix} z_1 & w_1 \\ t^2 z_2 & w_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} z_1 & t w_1 \\ t z_2 & w_2 \end{pmatrix}$$

and hence $\mathcal{F}_t \cong \mathcal{G}_t$. But for $t = 0$ they will no longer be isomorphic. If we denote $L_1 = Z(z_1)$ and $L_2 = Z(w_2)$, then B_0 is a diagonal matrix and $\mathcal{G}_0$ will be isomorphic to the direct sum $\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$ whereas $\mathcal{F}_0$ is a non-trivial extension

$$0 \longrightarrow \mathcal{O}_{L_2} \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{O}_{L_1} \longrightarrow 0$$

if w_1 does not vanish at the intersection $\{p\} = L_1 \cap L_2$. On the other hand $\mathcal{O}_{L_1}$ and $\mathcal{O}_{L_2}$ are stable, so $\mathcal{F}_0$ and $\mathcal{G}_0$ are still s-equivalent by Example 4.1.23. This shows again why it is necessary to consider s-equivalence classes for the points in M_{2m+2} otherwise we could define two "sequences" $t \mapsto [\mathcal{F}_t]$ and $t \mapsto [\mathcal{G}_t]$ that are pointwise equal but with different limits in the moduli space.

Remark F.1.10. Other examples of s-equivalent but non-isomorphic sheaves can e.g. be found in [[67], 12.4.2, p.82] and [[23], 2.5, p.33]. All of them arise by similar limit processes.

F.1.3 Description of the moduli space

Now we want to find an "easy" space that parametrizes elements from M_{2m+2} and a suitable isomorphism which gives a concrete description. The goal is to show that s-equivalence classes of sheaves in M_{2m+2} can be identified with their supports. The first result is the following.

Proposition F.1.11. [[67], 12.4.1, p.81-82]
Let

$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2}(-1) \stackrel{A}{\longrightarrow} 2\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F}_A \longrightarrow 0\ ,$$
$$0 \longrightarrow 2\,\mathcal{O}_{\mathbb{P}_2}(-1) \stackrel{B}{\longrightarrow} 2\,\mathcal{O}_{\mathbb{P}_2} \longrightarrow \mathcal{F}_B \longrightarrow 0$$

be two exact sequences as in (F.1) *which define sheaves $\mathcal{F}_A, \mathcal{F}_B \in M_{2m+2}$ with supports C_A and C_B. Assume that the supports are smooth. Then we have*

$$A \sim B \ \Leftrightarrow\ \mathcal{F}_A \cong \mathcal{F}_B \ \Leftrightarrow\ C_A = C_B\ .$$

Proof. The implications $\Rightarrow$ are clear. We prove that there exist $g, h \in \mathrm{GL}_2(\mathbb{K})$ such that $B = g \cdot A \cdot h$ if the smooth conics C_A and C_B are equal. This means that $\det A = \lambda \cdot \det B$ for some $\lambda \in \mathbb{K}^*$. Write

$$A = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} \qquad , \qquad B = \begin{pmatrix} l_1 & v_1 \\ l_2 & v_2 \end{pmatrix}$$

and let $p \in \mathbb{P}_2$ be the point given by $z_1(p) = z_2(p) = 0$. This is possible since C_A is smooth, so $\langle z_1 \rangle$ and $\langle z_2 \rangle$ are linearly independent (otherwise $\det A$ is reducible). In particular $p \in C_A \cap C_B$ since $(\det A)(p) = 0$. Next we want to determine the rank of the matrix $B(p)$. It cannot be 2, otherwise $B(p)$ is invertible and this contradicts $\det B(p) = (\det B)(p) = 0$. Moreover it cannot be 0 since C_B is smooth. Indeed

$$\frac{\partial \det B}{\partial X_i} = \frac{\partial}{\partial X_i}\big(l_1 v_2 - v_1 l_2\big) = \partial_i l_1 \cdot v_2 + l_1 \cdot \partial_i v_2 - \partial_i v_1 \cdot l_2 - v_1 \cdot \partial_i l_2$$

and this vanishes at p if $l_1(p) = l_2(p) = v_1(p) = v_2(p) = 0$. Hence $\operatorname{rk} B(p) = 1$, which means that there is a non- trivial relation between the columns

$$\begin{pmatrix} l_1(p) & v_1(p) \\ l_2(p) & v_2(p) \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Assume e.g. that $\alpha \neq 0$ and set

$$k = \begin{pmatrix} \alpha & 0 \\ \beta & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{K}) .$$

Then

$$B \cdot k = \begin{pmatrix} y_1 & v_1 \\ y_2 & v_2 \end{pmatrix} \qquad , \qquad \det B \cdot \det k = y_1 v_2 - v_1 y_2$$

with $y_i = \alpha l_i + \beta v_i$ for $i = 1, 2$, so $y_i(p) = 0$ and hence it is a linear combination of z_1 and z_2. Write e.g.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = g \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ,$$

where $g \in \mathrm{GL}_2(\mathbb{K})$. In particular $\langle y_1 \rangle$ and $\langle y_2 \rangle$ are still linearly independent. Now we have

$$g \cdot A = \begin{pmatrix} y_1 & w_1' \\ y_2 & w_2' \end{pmatrix} \qquad , \qquad \det g \cdot \det A = y_1 w_2' - w_1' y_2 .$$

Combining with $y_1 v_2 - v_1 y_2 = \det k \cdot \det B = \det k \cdot \frac{\det A}{\lambda}$, this gives

$$y_1 v_2 - v_1 y_2 = \det k \cdot \frac{y_1 w_2' - w_1' y_2}{\lambda \cdot \det g} = \mu \cdot (y_1 w_2' - w_1' y_2)$$
$$\Rightarrow \; y_1 \cdot (v_2 - \mu w_2') = y_2 \cdot (v_1 - \mu w_1') .$$

By linear dependence we hence need that $v_2 = \mu w_2'$ and $v_1 = \mu w_1'$. Finally

$$B \cdot k \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix} = g \cdot A \quad \Rightarrow \quad g \cdot A \cdot \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \cdot k^{-1} = B .$$

□

Remark F.1.12. Hence the stable sheaves from M_{2m+2}^s are in 1-to-1 correspondence with smooth conics in $\mathbb{P}_2$. From Example F.1.9 we see that this no longer holds true when the support is singular ; there exist non-isomorphic (but s-equivalent) sheaves having the same singular support.

On the other hand a similar bijection holds true for properly semistable sheaves and singular supports. Indeed

Proposition F.1.13. [[67], 13.5.2, p.87]
There is a 1-to-1 correspondence between s-equivalent classes of properly semistable sheaves and reducible conics in $\mathbb{P}_2$.

Proof. Assume that A is in the orbit of the matrix

$$\begin{pmatrix} l_1 & w \\ 0 & l_2 \end{pmatrix} ,$$

then the sheaf $\mathcal{F}_A$ defined by (F.1) has determinant $l_1 \cdot l_2$ and is thus properly semistable as one can always construct an extension as in (F.2). All these extensions will be s-equivalent for different w, so they represent the same point in M_{2m+2}. Thus det ignores the type of the extension and we always get the s-equivalence class of $\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$. So the only way to get different points in M_{2m+2} is by changing the linear forms l_1 and l_2 which define the class of $\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$. Vice-versa given a properly semistable sheaf $\mathcal{F}$, we know from Proposition F.1.4 that it must have a singular support and we can recover $\mathcal{F}$ by uniqueness of cokernels as some $\mathcal{F}_A$ by the construction above.
Corollary D.1.15, which is a consequence of Bézout's Theorem on $\mathbb{P}_2$, moreover says that all irreducible curves in $\mathbb{P}_2$ of degree 2 can be written as $X_0^2 - X_1 X_2$ up to a change of variables and are hence smooth (i.e. being irreducible and being smooth is equivalent for conics in $\mathbb{P}_2$). □

Summarizing we have proven

Theorem F.1.14. [[67], 13.5, p.85]
There is an isomorphism of projective varieties

$$M_{2m+2} \xrightarrow{\sim} \mathcal{C}_2(\mathbb{P}_2) \cong \mathbb{P}_5 \ : \ [\mathcal{F}] \longmapsto \mathcal{Z}_f(\mathcal{F}) ,$$

which identifies the open subvariety M^s_{2m+2} *of isomorphism classes of stable sheaves with the open subvariety of smooth conics and the closed complement* $M_{2m+2} \setminus M^s_{2m+2}$ *of s-equivalence classes of properly semistable sheaves with the closed subset of singular quadratic curves from* (4.21).

Remark F.1.15. In particular we see that a point in M_{2m+2} is completely determined by its support, in contrast to the moduli space M_{3m+1} where we identified a sheaf with its support and a point lying on that curve in order to obtain $\mathfrak{U}(3)$, see Theorem 4.6.17. Moreover we saw in Proposition F.1.4 that all sheaves in M^s_{2m+2} are non-singular and that it does not make sense to speak about (non-)singular sheaves for s-equivalence classes in $M_{2m+2} \setminus M^s_{2m+2}$.

Corollary F.1.16. *The closed subvariety of s-equivalence classes of properly semistable sheaves in M_{2m+2} is of codimension* 1*.*

Proof. As the moduli space M_{2m+2} is isomorphic to the space of conics on $\mathbb{P}_2$ (Theorem F.1.14), we conclude that $M_{2m+2} \setminus M^s_{2m+2}$ is given by the subset of singular conics, i.e. reducible homogeneous polynomials of degree 2. These can be described as follows ; a quadratic form Q is identified with an element $(a_0 : \ldots : a_5) \in \mathbb{P}_5$ and can be written as

$$\begin{aligned} Q(X_0, X_1, X_2) &= a_0 X_0^2 + a_1 X_0 X_1 + a_2 X_0 X_2 + a_3 X_1^2 + a_4 X_1 X_2 + a_5 X_2^2 \\ &= \begin{pmatrix} X_0 & X_1 & X_2 \end{pmatrix} \cdot \begin{pmatrix} a_0 & \frac{a_1}{2} & \frac{a_2}{2} \\ \frac{a_1}{2} & a_3 & \frac{a_4}{2} \\ \frac{a_2}{2} & \frac{a_4}{2} & a_5 \end{pmatrix} \cdot \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} . \end{aligned}$$

The polynomial Q is reducible if and only if the determinant of this matrix is zero. Hence we get

$$\begin{aligned} (a_0 : \ldots : a_5) \in \mathbb{P}_5 \text{ defines a singular sheaf } &\Leftrightarrow \det \begin{pmatrix} a_0 & \frac{a_1}{2} & \frac{a_2}{2} \\ \frac{a_1}{2} & a_3 & \frac{a_4}{2} \\ \frac{a_2}{2} & \frac{a_4}{2} & a_5 \end{pmatrix} = 0 \\ &\Leftrightarrow \frac{1}{4} \cdot \left(4a_0 a_3 a_5 - a_0 a_4^2 - a_1^2 a_5 + a_1 a_2 a_4 - a_2^2 a_3 \right) = 0 . \end{aligned}$$

This is just one homogeneous equation, so we obtain that the subset of singular conics is of codimension 1 in $\mathbb{P}_5 \cong M_{2m+2}$. □

Remark F.1.17. This is compatible with Theorem 4.3.11, which states that the closed subvariety $M_{2m+2} \setminus M^s_{2m+2}$ has codimension at least $2 \cdot 2 - 3 = 1$. In Corollary F.1.16 we showed that here it is indeed equal to 1.

F.2 Subvariety of singular sheaves

Nevertheless there are solutions out of the problem that (non-)singular sheaves are not defined for s-equivalence classes. The idea is to make sense of the notion of a (non-)singular equivalence class. As we know that all stable sheaves in M_{2m+2} are non-singular, we only have to look for singular sheaves among the properly semistable ones. Several attempts are possible.

Definition F.2.1. Let $[\mathcal{F}] \in M_{2m+2} \setminus M^s_{2m+2}$ be an s-equivalence class of a properly semistable sheaf. We say that $[\mathcal{F}]$ is singular if and only if it contains a singular representative.

Another definition, which actually comes down to the same, is the following.

Definition F.2.2. We say that a semistable sheaf is *polystable* if it is a direct sum of stable sheaves. It is shown in [[38], 1.5.4, p.24] that every s-equivalence class of a semistable sheaf contains exactly one polystable sheaf up to isomorphism. Then we say that a properly semistable sheaf $[\mathcal{F}] \in M_{2m+2} \setminus M^s_{2m+2}$ is singular if and only if its unique polystable representative is singular.

Remark F.2.3. In our case we know that a properly semistable sheaf $\mathcal{F}$ has singular support and its s-equivalence class always contains the direct sum of structure sheaves of lines $\mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2}$, where $L_1 = L_2$ gives the double line. Moreover this direct sum is the polystable representative of the class since each $\mathcal{O}_{L_i}$ is stable by Proposition 4.3.9. But we saw all direct sums to be singular in Example F.1.7 and Remark F.1.8. Hence every s-equivalence class of a properly semistable sheaf is singular.

Using this new definition one can again study the question of the codimension of the subvariety of singular sheaves $M'_{2m+2} \subset M_{2m+2}$. Collecting all the results from Section F.1.3, we have the following criterion.

Proposition F.2.4. *Let $[\mathcal{F}] \in M_{2m+2}$ be an s-equivalence class of a semistable sheaf. Then the following conditions are equivalent:*

1) *$[\mathcal{F}]$ is singular.*
2) *$[\mathcal{F}]$ is the s-equivalence class of a properly semistable sheaf.*
3) *$\mathcal{Z}_f(\mathcal{F})$ is a singular conic (or equivalently, a reducible conic).*

Proof. Being singular and being reducible are equivalent for conics in $\mathbb{P}_2$ by Corollary D.1.15.
The equivalence of the first 2 assertions is proven in Remark F.2.3. Moreover we know by Proposition 4.4.16 that singular sheaves necessarily have singular support, so it remains to show the converse. If the support of the sheaf is singular, hence reducible, the sheaf is properly semistable by Proposition F.1.4 and the unique polystable representative of its s-equivalence class is a direct sum of structure sheaves of lines, which is not locally free on its support. Thus sheaves in M_{2m+2} with singular support are singular. $\square$

Remark F.2.5. Do not forget that we have shown in Example F.1.7 that there do exist non-singular sheaves with singular support (and which are thus properly semistable). The criterion of Proposition F.2.4 only looks at the polystable representative.

Corollary F.2.6. *The closed subvariety $M'_{2m+2} \subset M_{2m+2}$ is singular and of codimension* 1.

Proof. Since all properly semistable sheaves are singular we conclude from Corollary F.1.16 that M'_{2m+2} is of codimension 1 in M_{2m+2} and hence of dimension 4. Moreover it is not smooth: let us write

$$F(a_0, \ldots, a_5) = 4a_0a_3a_5 - a_0a_4^2 - a_1^2a_5 + a_1a_2a_4 - a_2^2a_3 \ ,$$

$$\begin{aligned} \partial_0 F(a_0, \ldots, a_5) &= 4a_3a_5 - a_4^2 \quad , \quad & \partial_1 F(a_0, \ldots, a_5) &= a_2a_4 - 2a_1a_5 \ , \\ \partial_2 F(a_0, \ldots, a_5) &= a_1a_4 - 2a_2a_3 \quad , \quad & \partial_3 F(a_0, \ldots, a_5) &= 4a_0a_5 - a_2^2 \ , \\ \partial_4 F(a_0, \ldots, a_5) &= a_1a_2 - 2a_4a_0 \quad , \quad & \partial_5 F(a_0, \ldots, a_5) &= 4a_0a_3 - a_1^2 \ . \end{aligned}$$

Then we see e.g. that $(1:0:0:0:0:0)$ is a singular point of this variety. $\square$

Remark F.2.7. In particular this shows that the inequality $\mathrm{codim}_\Omega(\Omega^{sing}) \geq 2$ of Proposition 4.4.12 may no longer hold true in the moduli space (i.e. after dividing out the $\mathrm{SL}(V)$-action on Ω).

Remark F.2.8. Another attempt is to define an s-equivalence class $[\mathcal{F}] \in M_{2m+2}$ to be non-singular if and only if it contains a non-singular representative. In this case we have seen in Example F.1.7 and Example F.1.8 that the only s-equivalence classes in which all representatives are singular are those whose support is given by a double line. The latter can be be represented by the form

$$\begin{aligned} &(aX_0 + bX_1 + cX_2)^2 \\ &= a^2X_0^2 + b^2X_1^2 + c^2X_2^2 + 2abX_0X_1 + 2acX_0X_2 + 2bcX_1X_2 \end{aligned}$$

with $(a, b, c) \neq (0, 0, 0)$, so in this case the subvariety of singular sheaves M'_{2m+2} would be of codimension 3.

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