# Torsion and purity on non-integral schemes & singular sheaves in the fine Simpson moduli spaces of one-dimensional sheaves on the projective plane

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# Motivation

Classification problem

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We are interested in pure 1-dimensional sheaves on $\mathbb{P}_2$.

• "most" of them can be shown to be locally free on their support

$\Rightarrow$ $M$ mostly consists of vector bundles on a curve

To prove this one needs to know that pure sheaves are torsion-free on their support.

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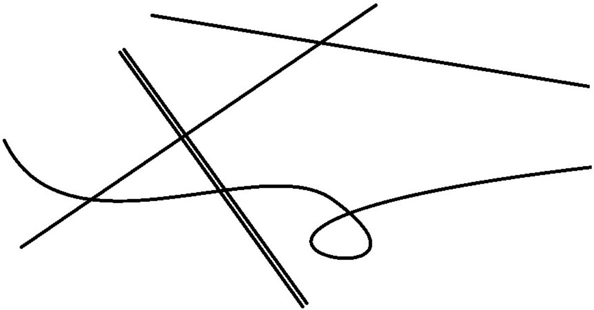

- What is the torsion of a sheaf on such a space?
- Prove that there is no torsion if the sheaf is pure.

# Motivation

$M$ also contains sheaves which are not locally free on their support.

- such sheaves are called singular
- they form a closed subvariety $M' \subset M$, in general non-empty

We want to find properties of $M'$ such as

− smoothness

− codimension in $M$

This gives information about the geometry of $M$.

# Part I

# Torsion on non-integral schemes and relations with purity

# Main result

**Reminder:**
On an affine Noetherian scheme $\operatorname{Spec} R$ there is a 1-to-1 correspondence between coherent sheaves and finitely generated $R$-modules.

$$\mathrm{Mod}^f(R) \xrightarrow{\sim} \mathtt{Coh}(\mathcal{O}_R) \ : \ M \longmapsto \widetilde{M}$$

**Theorem (L)**

*Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring $R$ and $M$ be a finitely generated module over $R$. Assume that the coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{F} = \widetilde{M}$ is pure and let $\mathcal{Z}$ be its support. Then $\mathcal{F}$ is a torsion-free $\mathcal{O}_{\mathcal{Z}}$–module.*

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- known for integral schemes
- generalization to the non-integral and non-reduced case

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**Important remark:**

$$\operatorname{supp} \mathcal{F} = \left\{ x \in \mathcal{X} \;\middle|\; \mathcal{F}_x \neq \{0\} \right\}$$

# Torsion of a module

Torsion submodule of a module $M$ over a ring $R$:

$$\mathcal{T}_R(M) = \left\{ m \in M \;\middle|\; \exists\, r \in R,\ r \neq 0 \text{ a NZD such that } r * m = 0 \right\}$$

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For a sheaf $\mathcal{F}$ on a scheme $\mathcal{X}$, we define the torsion subsheaf $\mathcal{T}(\mathcal{F})$.

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- the assignment $U \longmapsto \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big)$ is not a presheaf on non-integral schemes

$\Rightarrow$ define it on affines

# The torsion subsheaf: definition and properties

**Definition**

Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$, $U \subseteq \mathcal{X}$ be open and $s \in \mathcal{F}(U)$. $s$ is a torsion section of $\mathcal{F}$ if there exist an affine open covering $U = \bigcup_i U_i$ such that

$$s_{|U_i} \in \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U_i)}\big(\mathcal{F}(U_i)\big), \ \forall\, i$$

$\mathcal{T}(\mathcal{F})(U) =$ set of all torsion sections of $\mathcal{F}$ over $U$

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On locally Noetherian schemes we have:

- for $U \subseteq \mathcal{X}$ affine, $\mathcal{T}(\mathcal{F})(U) = \mathcal{T}_{\mathcal{O}_{\mathcal{X}}(U)}\big(\mathcal{F}(U)\big)$
- stalks: $\mathcal{T}(\mathcal{F})_x \cong \mathcal{T}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x)$, $\forall\, x \in \mathcal{X}$

**Definition**

$\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ is called torsion-free if $\mathcal{T}(\mathcal{F}) = 0$.

Torsion-free sheaves: stalks are torsion-free modules over local rings

# The torsion subsheaf: coherence

Is $\mathcal{T}(\mathcal{F})$ again coherent? In general: No!

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**Theorem (L)**

*Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$ given by $\mathcal{F} \cong \widetilde{M}$ for some $R$-module $M$. Then*

$$\mathcal{T}(\mathcal{F}) \text{ is coherent } \Leftrightarrow \left(\mathcal{T}_R(M)\right)_P = \mathcal{T}_{R_P}(M_P)\ , \quad \forall\, P \in \operatorname{Spec} R\ .$$

If $\mathcal{T}(\mathcal{F})$ is coherent, then $\mathcal{T}(\mathcal{F}) \cong \widetilde{\mathcal{T}_R(M)}$.

- criterion always satisfied for integral and reduced schemes
- but not in general

# Example

$\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{F} = \widetilde{M}$ for

$$R = \mathbb{K}[X, Y, Z]/\langle XY, X^2, XZ \rangle \qquad , \qquad M = R/\langle \bar{Y}\bar{Z} \rangle$$

- $M$ is torsion-free: $\mathcal{T}_R(M) = \{0\}$
- but for $P = \langle \bar{X}, \bar{Y}, \bar{Z} - 1 \rangle$, we have $[\bar{Z}]_P \neq 0$ and $[\bar{Z}]_P \in \mathcal{T}_{R_P}(M_P)$

$\Rightarrow$ it cannot come from a global torsion element

$\Rightarrow$ $\mathcal{T}(\mathcal{F})$ is not coherent

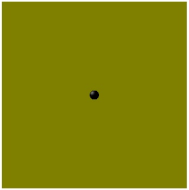
# Geometric interpretation

$$R = \mathbb{K}[X, Y, Z]/\langle\, XY, X^2, XZ \,\rangle \qquad , \qquad M = R/\langle\, \bar{Y}\bar{Z} \,\rangle$$

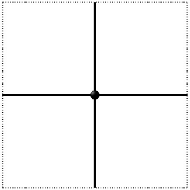
$$\langle\, XY, X^2, XZ \,\rangle = \langle\, X \,\rangle \cap \langle\, X^2, Y, Z \,\rangle$$

$$\langle\, XY, X^2, XZ, YZ \,\rangle = \langle\, X, Z \,\rangle \cap \langle\, X, Y \,\rangle \cap \langle\, X^2, Y, Z \,\rangle$$

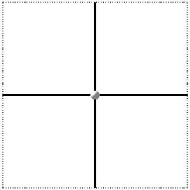
$\mathcal{X} = \operatorname{Spec} R$



$\operatorname{supp} M = \operatorname{supp} \mathcal{F}$



$\operatorname{supp} \mathcal{T}(\mathcal{F})$



The support of $\mathcal{T}(\mathcal{F})$ is not closed: $\mathcal{T}_{R_P}(M_P) \neq \{0\}, \forall\, P \in \operatorname{supp} M \setminus \{\mathfrak{M}\}$

# Primary Ideal Decomposition and associated primes

**Theorem (Lasker-Noether)**

*In a Noetherian ring, the zero ideal can be written as a finite intersection $\{0\} = Q_1 \cap \ldots \cap Q_\alpha$ of primary ideals $Q_i \trianglelefteq R$.*

The radicals $P_i = \mathrm{Rad}(Q_i)$ are prime ideals.

- called the associated primes of $R$: $\mathrm{Ass}_R(R) = \{P_1, \ldots, P_\alpha\}$
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Application: decomposition into irreducible components

$$\mathcal{X} = \operatorname{Spec} R = V\big(\{0\}\big) = V\big(\bigcap_i Q_i\big) = \bigcup_i V(Q_i) = \bigcup_i \mathcal{X}_i$$

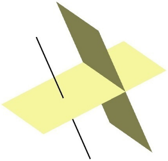
If $Q_i \subsetneq P_i$, the component $\mathcal{X}_i$ has a non-reduced structure.

# Examples

1) $R = \mathbb{K}[X, Y, Z]/\langle\, YZ(X-1), XZ(X-1)\,\rangle$

$$\{\bar{0}\} = \langle\, \bar{Z}\,\rangle \cap \langle\, \bar{X} - 1\,\rangle \cap \langle\, \bar{X}, \bar{Y}\,\rangle$$

two planes $\{Z = 0\}$, $\{X = 1\}$ and a line $\{X = Y = 0\}$



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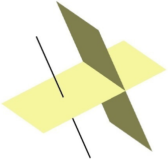

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2) $R = \mathbb{K}[X, Y, Z]/\langle\, XY, X^2, XZ\,\rangle$, $\bar{X}^2 = 0$

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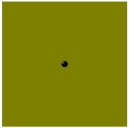

plane $\{X = 0\}$ with an embedded double point

$$\operatorname{Rad}(\langle\, \bar{Y}, \bar{Z}\,\rangle) = \langle\, \bar{X}, \bar{Y}, \bar{Z}\,\rangle \qquad , \qquad \langle\, \bar{X}\,\rangle \subsetneq \langle\, \bar{X}, \bar{Y}, \bar{Z}\,\rangle$$

The embedded prime $P_j \supsetneq P_i$ gives an embedded component $\mathcal{X}_j \subsetneq \mathcal{X}_i$.

# Absence of embedded primes

**Theorem (L)**

*Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_{\mathcal{X}}$–module. If $R$ has no embedded primes, then the torsion subsheaf $\mathcal{T}(\mathcal{F}) \subseteq \mathcal{F}$ is coherent.*

Proof uses a result from Epstein & Yao which allows to construct global NZDs from local ones

$\Rightarrow$ we obtain $\big(\mathcal{T}_R(M)\big)_P = \mathcal{T}_{R_P}(M_P)$

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Examples of rings with no embedded primes:

– integral domains and reduced rings

– spectrum defining an irreducible scheme

– quotients of polynomial rings by principal ideals

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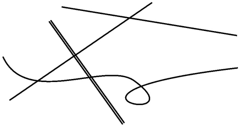

# Geometric interpretation of torsion

Is torsion supported in smaller dimension, in the sense that

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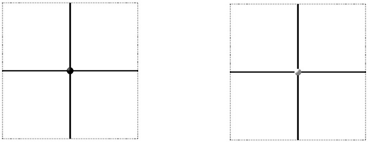

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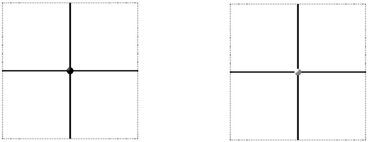

**Theorem**

*Let $M$ be a finitely generated module over a Noetherian ring $R$. Denote $\mathcal{F} = \widetilde{M}$, $\mathcal{X} = \operatorname{Spec} R$ and $\mathcal{X}_i = V(P_i)$ for all $i$, where $P_1, \ldots, P_\alpha$ are the associated primes of $R$. Then $M$ is a torsion module if and only if the codimension of $\operatorname{supp}\mathcal{F}$ is positive along each irreducible component:*

$$\operatorname{codim}_{\mathcal{X}_i}\big((\operatorname{supp}\mathcal{F}) \cap \mathcal{X}_i\big) \geq 1\,, \quad \forall\, i \in \{1, \ldots, \alpha\}\,.$$

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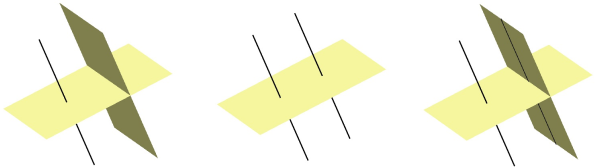
$M$ not torsion-free with $\mathcal{T}_R(M) = \langle\, [\bar{X}\bar{Z}]\,\rangle$

$R$ reduced $\;\Rightarrow\;$ $\mathcal{T}(\mathcal{F})$ is coherent and $\mathcal{T}(\mathcal{F}) = \widetilde{\mathcal{T}_R(M)}$



$\operatorname{supp}\mathcal{F}$ consists of a plane and 2 lines; $\mathcal{T}(\mathcal{F})$ is supported on a line $\operatorname{supp}\mathcal{T}(\mathcal{F}) = V(\bar{X}-1, \bar{Y})$, the dimension drops in each component of $\mathcal{X}$.

# Purity and torsion-freeness

**Definition**

Let $\mathcal{X}$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. $\mathcal{F}$ is pure of dimension $d \leq \dim \mathcal{X}$ if $\operatorname{supp} \mathcal{F}$ has dimension $d$ and every non-zero proper coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ is also supported in dimension $d$.

$\mathcal{X} = \bigcup_i \mathcal{X}_i$ has equidimensional components if $\dim \mathcal{X}_i = \dim \mathcal{X}_j$, $\forall\, i, j$
In particular there are no embedded components.

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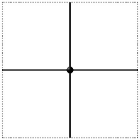
## Theorem (L)

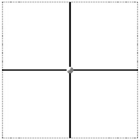
*Let $\mathcal{X} = \operatorname{Spec} R$ be an affine Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$. Assume that $\dim \mathcal{F} = \dim \mathcal{X} = d$ and that $\mathcal{X}$ has equidimensional components. Then $\mathcal{F}$ is pure of dimension $d$ if and only if $\mathcal{F}$ is torsion-free on $\mathcal{X}$.*

Moreover $\mathcal{T}(\mathcal{F}) = T_{d-1}(\mathcal{F})$:
Torsion sections are exactly those that are supported in dimension $< d$.

# Application

- counter-example to $\mathcal{T}(\mathcal{F}) = T_{d-1}(\mathcal{F})$ if $\mathcal{X}$ is not equidimensional: $\dim \mathcal{F} = 1$ and $\operatorname{supp} \mathcal{T}(\mathcal{F})$ dense, but $T_0(\mathcal{F})$ is supported at the origin





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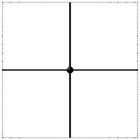
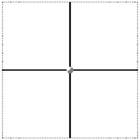
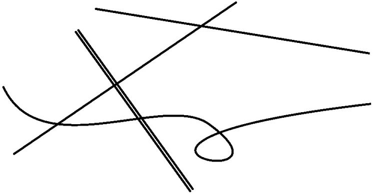




- important example of a scheme with equidimensional components:



# Fitting and annihilator support

- $\mathcal{F} \in \texttt{Coh}(\mathcal{O}_{\mathcal{X}}) \;\Rightarrow\;$ supp $\mathcal{F} \subseteq \mathcal{X}$ is a closed algebraic subset

find a subscheme structure, locally given by a quotient $R \to R/I$

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- $\mathcal{X} = \operatorname{Spec} R$, $\mathcal{F} = \widetilde{M}$; let $I = \mathrm{Ann}_R(M)$ and $I' = \mathrm{Fitt}_0(M)$
Finite presentation

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

$I'$ is generated by the $n \times n$–minors of the matrix of relations $A$; $I' \subseteq I$

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**Definition**

$\mathcal{Z}_a(\mathcal{F}) := V(I) \cong \operatorname{Spec}(R/I)$ and $\mathcal{Z}_f(\mathcal{F}) := V(I') \cong \operatorname{Spec}(R/I')$.

$\mathcal{Z}_a(\mathcal{F}) \subsetneq \mathcal{Z}_f(\mathcal{F})$ is in general a proper subscheme (richer structure)

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Finite presentation

$$R^m \xrightarrow{A} R^n \longrightarrow M \longrightarrow 0$$

$I'$ is generated by the $n \times n$–minors of the matrix of relations $A$; $I' \subseteq I$

**Definition**

$\mathcal{Z}_a(\mathcal{F}) := V(I) \cong \operatorname{Spec}(R/I)$ and $\mathcal{Z}_f(\mathcal{F}) := V(I') \cong \operatorname{Spec}(R/I')$.

$\mathcal{Z}_a(\mathcal{F}) \subsetneq \mathcal{Z}_f(\mathcal{F})$ is in general a proper subscheme (richer structure)

- General fact: $J \subseteq \operatorname{Ann}_R(M) \;\Rightarrow\; M$ is also a module over $R/J$
Hence $\mathcal{F} = \widetilde{M}$ can be seen as a sheaf on the schemes $\mathcal{Z}_a(\mathcal{F})$ and $\mathcal{Z}_f(\mathcal{F})$.
$\to$ study torsion-freeness of pure sheaves on their support

# Criterion for purity

**Associated points:**

- for an $R$-module $M$: set of associated primes

$$\mathrm{Ass}_R(M) = \left\{ P = \mathrm{Ann}_R(m) \text{ prime} \;\middle|\; m \in M \right\}$$

- for $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathcal{X}})$ on a scheme $\mathcal{X}$:

$$\mathrm{Ass}(\mathcal{F}) = \left\{ x \in \mathcal{X} \;\middle|\; \mathfrak{M}_x \in \mathrm{Ass}_{\mathcal{O}_{\mathcal{X},x}}(\mathcal{F}_x) \right\}$$

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**Theorem (Huybrechts-Lehn)**

*Let $\mathcal{X}$ be a Noetherian scheme and $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_\mathcal{X})$ with $d = \dim \mathcal{F}$. Then $\mathcal{F}$ is pure of dimension $d$ if and only if all points in $\mathrm{Ass}(\mathcal{F})$ are of dimension $d$.*

- affine case: $\mathcal{X} = \mathrm{Spec}\, R$, $\mathcal{F} = \widetilde{M}$

all primes $P_i \in \mathrm{Ass}_R(M)$ define components $\mathcal{X}_i = V(P_i)$ of dimension $d$

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$\mathrm{Ass}_R(M)$ has no information about the embedded components.

**Proposition (L)**

*Let $\mathcal{X} = \operatorname{Spec} R$ be affine and $\mathcal{F} \cong \widetilde{M}$ be coherent with $d = \dim\mathcal{F}$. If the annihilator support $\mathcal{Z}_a(\mathcal{F})$ of $\mathcal{F}$ has a component of dimension $< d$, then $\mathcal{F}$ is not pure.*

The annihilator support of a pure sheaf has equidimensional components.
$\Rightarrow$ purity and torsion-freeness on $\mathcal{Z}_a(\mathcal{F})$ are equivalent

# Proof of the main result

However...
The Fitting support of a pure sheaf may have embedded components!

**Proposition (L)**

*Let $\mathcal{F} = \widetilde{M}$ for some finitely generated module $M$ over a Noetherian ring $R$ and $I, I' \subseteq \operatorname{Ann}_R(M)$ be two ideals defining different subscheme structures on $\operatorname{supp} \mathcal{F}$. Assume that $\mathcal{F}$ is torsion-free on $V(I)$ which has no embedded components. Then $\mathcal{F}$ is also torsion-free on $V(I')$.*

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"Pure sheaves are torsion-free on their Fitting support."

*Proof.* $\mathcal{F}$ pure
$\Rightarrow$ $\mathcal{Z}_a(\mathcal{F})$ has equidimensional components
$\Rightarrow$ $\mathcal{F}$ is torsion-free on $\mathcal{Z}_a(\mathcal{F})$
$\Rightarrow$ $\mathcal{F}$ is torsion-free on $\mathcal{Z}_f(\mathcal{F})$

# Conclusion

## Theorem (L)

*Let $\mathcal{X} = \operatorname{Spec} R$ for some Noetherian ring $R$ and $M$ be a finitely generated module over $R$. Assume that the coherent $\mathcal{O}_{\mathcal{X}}$–module $\mathcal{F} = \widetilde{M}$ is pure of dimension $d \leq \dim \mathcal{X}$. We denote $I = \mathrm{Fitt}_0(M)$ and $\mathcal{Z} = V(I) \cong \operatorname{Spec}(R/I)$. Then $\mathcal{F}$ is a torsion-free $\mathcal{O}_{\mathcal{Z}}$–module.*

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The behaviour of torsion can be very counter-intuitive when there are embedded primes. But

## Purity always implies torsion-freeness of a sheaf on its support.

## Part II

## Singular sheaves in the fine Simpson moduli spaces of one-dimensional sheaves on the projective plane

# Hilbert polynomial of a sheaf

Projective plane $\mathbb{P}_2$ with structure sheaf $\mathcal{O}_{\mathbb{P}_2}$ of regular functions

- $\mathcal{O}_{\mathbb{P}_2}(1) =$ very ample invertible sheaf, called Serre's twisting sheaf:

$$\mathcal{O}_{\mathbb{P}_2}(1)|_{U_i} \cong \mathcal{O}_{\mathbb{P}_2}|_{U_i} \text{ with cocycles } u_{ij}(x) = \frac{x_j}{x_i}$$

alternative definition: dual of the tautological bundle, denoted $\mathcal{O}_{\mathbb{P}_2}(-1)$
$\mathcal{O}_{\mathbb{P}_2}(k) = \mathcal{O}_{\mathbb{P}_2}(1) \otimes \ldots \otimes \mathcal{O}_{\mathbb{P}_2}(1)$, $k \in \mathbb{Z}$

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- Euler characteristic of a sheaf $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$: $h^i(\mathcal{F}) = \dim_{\mathbb{K}} H^i(\mathcal{F})$,

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \cdot h^i(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F})$$

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- twisted sheaf $\mathcal{F}(k) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_2}(k)$

- Hilbert polynomial of $\mathcal{F}$: $P_{\mathcal{F}}(m) = \chi\big(\mathcal{F}(m)\big) \in \mathbb{Q}[m]$
$\dim \mathcal{F} = d \;\; \Rightarrow \;\;$ polynomial expression in $m$ of degree $d$

# Semistability

Write the Hilbert polynomial of $\mathcal{F}$ as

$$P_{\mathcal{F}}(m) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \cdot \frac{m^i}{i!}$$

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**Definition**

Let $\mathcal{F} \in \mathtt{Coh}(\mathcal{O}_{\mathbb{P}_2})$ with $d = \dim \mathcal{F}$. $\mathcal{F}$ is semistable if
1) $\mathcal{F}$ is of pure dimension $d$, i.e. $\dim \mathcal{F}' = d$ for any proper non-zero coherent subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$.
2) Any proper non-zero coherent subsheaf $\mathcal{F}' \subsetneq \mathcal{F}$ satisfies $p_{\mathcal{F}'} \leq p_{\mathcal{F}}$.

$\mathcal{F}$ is stable if this inequality is strict: $p_{\mathcal{F}'} < p_{\mathcal{F}}$.

# Theorems of Simpson and Le Potier

Classification of semistable sheaves with fixed Hilbert polynomial

**Theorem (Simpson)**

*Let $P \in \mathbb{Q}[m]$ be a fixed numerical polynomial of degree $d \leq 2$. There exists a moduli space $M_P(\mathbb{P}_2)$ of semistable sheaves on $\mathbb{P}_2$ of pure dimension $d$ and Hilbert polynomial $P$. Moreover $M_P(\mathbb{P}_2)$ is a projective variety itself.*

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We are interested in linear Hilbert polynomials $P(m) = am + b$, $a \geq 1$.

**Theorem (Le Potier)**

*If $\gcd(a, b) = 1$, then the closed points of $M_{am+b}(\mathbb{P}_2)$ parametrize isomorphism classes of stable sheaves on $\mathbb{P}_2$ of pure dimension $d$ and Hilbert polynomial $am + b$.*

# 1-dimensional sheaves on their support

**Proposition**

*There is a morphism of projective varieties*

$$\sigma \ : \ M_{am+b}(\mathbb{P}_2) \longrightarrow \mathcal{C}_a(\mathbb{P}_2) \ : \ [\mathcal{F}] \longmapsto \mathcal{Z}_f(\mathcal{F})$$

- Part I implies:

$\mathcal{F} \in M_{am+b}(\mathbb{P}_2)$, $C = \mathcal{Z}_f(\mathcal{F}) \ \Rightarrow \ \mathcal{F}$ is a torsion-free $\mathcal{O}_C$–module

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• Bertini's Theorem:
The set of smooth curves of degree $a$ is open and dense in $\mathcal{C}_a(\mathbb{P}_2)$.
$\Rightarrow$ "almost all" sheaves in $M_{am+b}(\mathbb{P}_2)$ are vector bundles on curves

# Singular sheaves

**Definition**

A stable sheaf $\mathcal{F} \in M_{am+b}$ is called singular if it is not locally free on its support.

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$M'_{am+b}$ = closed subset of non-singular sheaves
We are interested in studying properties of $M' = M'_{am+1}$.
− irreducibility
− smoothness
− codimension in $M = M_{am+b}$

First examples [Le Potier, Trautmann, Freiermuth, Iena]

- $M_{m+1}$ and $M_{2m+1} \Rightarrow M' = \emptyset$
- $M_{3m+1} \Rightarrow \operatorname{codim}_M M' = 2$, $M'$ smooth and irreducible
- $M_{4m+1} \Rightarrow \operatorname{codim}_M M' = 2$, $M'$ singular and connected

# Main result

Duality Theorem of Maican: $M_{am+b} \cong M_{am-b}$

**Theorem (Iena-Leytem)**

*For any integer $d \geq 4$, let $M = M_{dm-1}(\mathbb{P}_2)$ be the Simpson moduli space of stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm - 1$. If $M' \subset M$ denotes the closed subvariety of singular sheaves in $M$, then $M'$ is singular and of codimension 2.*

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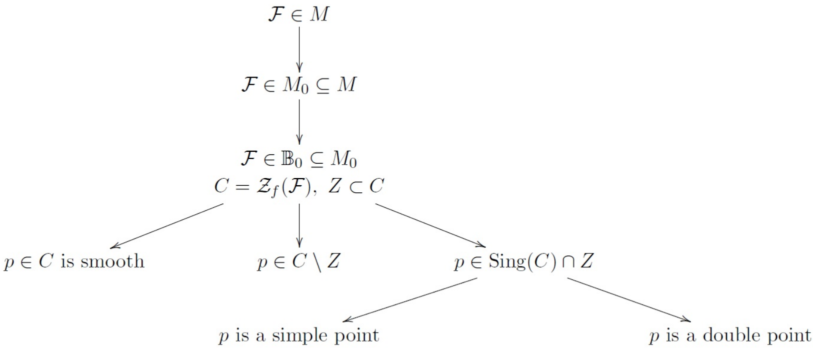
- article available as arXiv preprint
- submitted and accepted;

going to be published in the *Canadian Mathematical Bulletin*

# Diagram: proof by reduction

Let $C = \mathcal{Z}_f(\mathcal{F})$; $\mathcal{F}$ is singular if $\exists\, p \in C$ such that $\mathcal{F}_p \ncong \mathcal{O}_{C,p}$.



# Open stratum of $M$

**Theorem (Maican)**

*There exists an open subset $M_0 \subseteq M$ of sheaves $\mathcal{F}$ which have a resolution of the type*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_2}(-3) \oplus (n-1)\,\mathcal{O}_{\mathbb{P}_2}(-2) \xrightarrow{A} n\,\mathcal{O}_{\mathbb{P}_2}(-1) \longrightarrow \mathcal{F} \longrightarrow 0$$

*where $A = \binom{Q}{\Phi}$ is such that $\Phi \in \mathbb{V}^s$ and $\det A \neq 0$.*

$$A = \begin{pmatrix} Q \\ \Phi \end{pmatrix} = \left(\begin{array}{cccc} q_1 & q_2 & \dots & q_n \\ \hline z_{11} & z_{12} & \dots & z_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \dots & z_{n-1,n} \end{array}\right)$$

$\Phi$ = stable Kronecker module

Such matrices parametrize the sheaves in $M_0 : \mathcal{F} = [A]$.

# Maximal minors of a Kronecker module

$$\Phi = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n} \end{pmatrix}$$

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- assume they are coprime: $\gcd(d_1, \ldots, d_n) = 1$

$\Rightarrow$ they define a 0-dimensional subscheme

$$Z = Z(d_1, \ldots, d_n) \quad \text{with} \quad Z \subseteq C = Z(\det A)$$

of length $\binom{n}{2} = \frac{n^2-n}{2}$; the points in $Z$ may have multiplicities.

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- $\mathbb{V}_0 \subseteq \mathbb{V}^s$: subset of Kronecker modules with coprime maximal minors
- $N_0 = \mathbb{V}_0 / G$ where $G = \mathrm{GL}_{n-1}(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K})$, $(g, h) \,.\, \Phi = g \cdot \Phi \cdot h^{-1}$

# Description of sheaves in $\mathbb{B}_0$

$\mathbb{B}_0$ = open subset of sheaves in $M_0$ given by $A = \binom{Q}{\Phi}$ and $\Phi \in \mathbb{V}_0$

## Proposition

*The sheaves $\mathcal{F}$ in $\mathbb{B}_0$ are exactly the twisted ideal sheaves $\mathcal{I}_{Z \subseteq C}(d-3)$ given by a short exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(d-3) \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

*where $Z$ is a 0-dimensional subscheme of length $l = \binom{n}{2}$ lying on a curve $C$ of degree $d$ such that $Z$ is not contained in a curve of degree $d-3$.*

$\mathcal{F} \in \mathbb{B}_0$ given by some $A = \binom{Q}{\Phi}$ with $\Phi \in \mathbb{V}_0$
$C$ = support of $\mathcal{F}$, curve of degree $d$ defined by $Z(\det A)$
$Z$ = zero set defined by the coprime maximal minors of $\Phi$, $Z \subseteq C$

# Projective bundle $\mathbb{B}_0 \to N_0$

Consider

$$\nu \;:\; \mathbb{B}_0 \longrightarrow N_0 \quad : \quad \begin{array}{r} \mathcal{F} \longmapsto Z \\ [A] \longmapsto [\Phi] \end{array}$$

$\nu : \mathbb{B}_0 \to N_0$ is a projective bundle with fiber $\mathbb{P}_{3d-1}$.

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## Proposition

*A fiber of $\nu : \mathbb{B}_0 \to N_0$ corresponds to the space of curves of degree $d$ passing through the corresponding subscheme of $l = \binom{n}{2}$ points. The identification is given by the map*

$$\nu^{-1}([\Phi]) \longrightarrow \mathcal{C}_d(\mathbb{P}_2) \ : \ [A] \longmapsto \langle \det A \rangle$$

# Final reduction step

Take stalks at $p \in C$ of the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(d-3) \longrightarrow \mathcal{O}_Z \longrightarrow 0$$
$$\Rightarrow \; 0 \longrightarrow \mathcal{F}_p \longrightarrow \mathcal{O}_{C,p} \longrightarrow \mathcal{O}_{Z,p} \longrightarrow 0$$

3 cases:

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3) $p \in \mathrm{Sing}(C) \cap Z$
Singularities can only appear at singular points of $C$ which belong to $Z$.

$\rightarrow$ distinguish according to the multiplicity of $p$ in $Z$

# Ideals of simple points on planar curves

## Lemma

*Let* $R = \mathcal{O}_{C,p}$ *be the local Noetherian ring of a curve* $C \subset \mathbb{P}_2$ *at a point* $p \in C$ *with unique maximal ideal* $\mathfrak{M}$. *Consider the exact sequence of R-modules*

$$0 \longrightarrow \mathfrak{M} \longrightarrow R \longrightarrow \Bbbk_p \longrightarrow 0 \ .$$

*Then* $\mathfrak{M}$ *is free (of rank* $1$*) if and only if* $R$ *is regular, i.e. if and only if* $p$ *is a smooth point of* $C$.

- $N_c \subseteq N_0$: open subset that corresponds to Kronecker modules which define a configuration; set $\mathbb{B}_c = \mathbb{B}|_{N_c}$

## Corollary

*Let* $\mathcal{F} \in \mathbb{B}_c$ *be a sheaf over* $[\Phi] \in N_c$ *with* $C = \operatorname{supp} \mathcal{F}$. *Then* $\mathcal{F}$ *is singular if and only if* $Z$ *contains a singular point of* $C$*, i.e. if and only if* $\operatorname{Sing}(C) \cap Z \neq \emptyset$.

# Proof in the case of simple points

- Intuitive proof: let $p = (0,0)$ and $C = Z(f)$.

$$f(X,Y) = a_0 + a_1X + a_2Y + a_3X^2 + a_4XY + a_5Y^2 + \dots$$

$p \in C \ \Rightarrow \ f(0,0) = 0 \ \Rightarrow \ f$ has no constant term: $a_0 = 0$ (given)
ideal is not free if and only if $p$ is a singular point of $C$
$\Rightarrow \ f$ has no terms in $X$ and $Y$: absence of 2 monomials

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- Denote $M'_0 = M' \cap M_0$ and study the fibers of $\nu : M'_0 \to N_c$.

**Proposition**

*The fibers of $M'_0$ over $N_c$ are unions of $l = \binom{n}{2}$ different projective subspaces of $\mathbb{P}_{3d-1}$ of codimension 2. In particular they are singular at the intersection points.*

# Ideals of double points on planar curves

Let $f \in \mathbb{K}[X, Y]$ be non-constant and $C = Z(f)$.
Assume that $p = (0, 0)$ is a singular point of $C$ and let $I = \langle x, y^2 \rangle$ be the ideal defining a double point in the local ring $R = \mathcal{O}_{C,p}$, where $x, y$ are the classes of $X, Y$.

**Proposition (L)**

*The following conditions are equivalent:*
*1) $I$ is a free $R$-module.*
*2) $I$ is generated by $x$.*
*3) $f$ contains the monomial $Y^2$.*
*4) The tangent cone of $C$ at $p$ consists of 2 lines (with multiplicities) not containing the line $X = 0$.*

# Proof in the case of a double point

- Intuitive proof: double point $p$ given by $\langle X, Y^2 \rangle$

$$f(X,Y) = a_0 + a_1 X + a_2 Y + a_3 X^2 + a_4 XY + a_5 Y^2 + \dots$$

$p \in C \;\Rightarrow\; \langle f \rangle \subseteq \langle X, Y^2 \rangle$
$\Rightarrow$ $f$ has no constant term and does not contain $Y$: $a_0 = a_2 = 0$ (given)
ideal is not free if and only if $p$ is singular and $f$ does not contain $Y^2$
$\Rightarrow$ $f$ has no terms in $X$ and $Y^2$: absence of 2 monomials

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$$a_1 = a_5 = 0$$

- $N_1 \subseteq N_0 \setminus N_c$: open subset that corresponds to $l - 2$ different simple points and one double point; set $\mathbb{B}_1 = \mathbb{B}|_{N_1}$

**Proposition**

*Let $[\Phi] \in N_1$. The sheaves over $[\Phi]$ that are singular at a double point, resp. singular at a simple point both form a closed linear projective subspace of codimension 2 in the fiber $\mathbb{P}_{3d-1}$ of $\mathbb{B}_0$.*

# Conclusion

## Corollary

*The fibers of $M_0'$ over $N_1$ are unions of $l-1$ different linear subspaces of $\mathbb{P}_{3d-1}$ of codimension 2. In particular they are singular at the intersection points.*

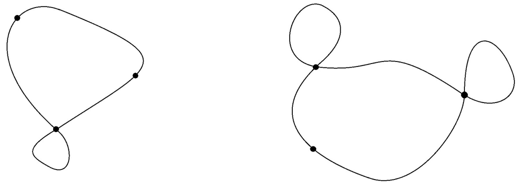
- sufficient since the complement of $N_c \cup N_1$ in $N_0$ is of codimension 2

$M'$ is singular since a generic fiber of $M_0'$ is singular.
What are its smooth points?

# Smooth points in $M'$

**Proposition**

*The smooth locus of $M'$ over $N_c$ consists of sheaves corresponding to $Z \subseteq C$ such that only one of the points in $Z$ is a singular point of $C$.*



# Open questions, Future research

1) Is $M'$ irreducible / connected?

2) Study other moduli spaces $M_{am+b}(\mathbb{P}_2)$.

3) Does the Fitting support of a pure sheaf on a reduced scheme have equidimensional components?

**Thanks for your attention!**