# STRONGLY BARYCENTRICALLY ASSOCIATIVE AND PREASSOCIATIVE FUNCTIONS

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Abstract. We study the property of strong barycentric associativity, a stronger version of barycentric associativity for functions with indefinite arities. We introduce and discuss the more general property of strong barycentric preassociativity, a generalization of strong barycentric associativity which does not involve any composition of functions. We also provide a generalization of Kolmogoroff-Nagumo's characterization of the quasi-arithmetic mean functions to strongly barycentrically preassociative functions.

## 1. Introduction

Let $X$ and $Y$ be arbitrary nonempty sets. Throughout this paper we regard tuples $\mathbf{x}$ in $X^n$ as $n$-strings over $X$. We let $X^* = \bigcup_{n\geqslant 0} X^n$ be the set of all strings over $X$, with the convention that $X^0 = \{\varepsilon\}$ (i.e., $\varepsilon$ denotes the unique 0-string on $X$). We denote the elements of $X^*$ by bold roman letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, etc. If we want to stress that such an element is a letter of $X$, we use non-bold italic letters $x$, $y$, $z$, etc. The *length* of a string $\mathbf{x}$ is denoted by $|\mathbf{x}|$. For instance, $|\varepsilon| = 0$. We endow the set $X^*$ with the concatenation operation, for which $\varepsilon$ is the neutral element. For instance, if $\mathbf{x} \in X^m$ and $y \in X$, then $\varepsilon\mathbf{x}y = \mathbf{x}y \in X^{m+1}$. Moreover, for every string $\mathbf{x}$ and every integer $n \geqslant 0$, the power $\mathbf{x}^n$ stands for the string obtained by concatenating $n$ copies of $\mathbf{x}$. In particular we have $\mathbf{x}^0 = \varepsilon$.

As usual, a map $F\colon X^n \to Y$ is said to be an *n-ary function* (an *n-ary operation on* $X$ if $Y = X$). Also, a map $F\colon X^* \to Y$ is said to be a *variadic function* (a *variadic operation on* $X$ if $Y = X \cup \{\varepsilon\}$, a *string function on* $X$ if $Y = X^*$; see [4]). For every variadic function $F\colon X^* \to Y$ and every integer $n \geqslant 0$, we denote by $F_n$ the *n-ary part* $F|_{X^n}$ of $F$. Finally, a variadic function $F\colon X^n \to Y$ is said to be *$\varepsilon$-standard* [8] if $\varepsilon \in Y$ and

$$F(\mathbf{x}) = \varepsilon \quad \Leftrightarrow \quad \mathbf{x} = \varepsilon.$$

Recall that a variadic operation $F\colon X^* \to X \cup \{\varepsilon\}$ is said to be *barycentrically associative* (or *B-associative* for short) [9] if it satisfies the equation

$$F(\mathbf{xyz}) \;=\; F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}), \qquad \mathbf{xyz} \in X^*.$$

B-associativity (also known as *decomposability* [1, 2]) was essentially introduced in 1909 by Schimmack [12] and then used later by Kolmogoroff [3] and Nagumo [11] in a characterization of the class of quasi-arithmetic mean functions. For general background and historical notes on B-associativity, see [9].

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The following stronger version of B-associativity (also known as *strong decomposability*) was introduced in [5, 7]. For every $\mathbf{x} \in X^*$ and every $K \subseteq \{1, \ldots, |\mathbf{x}|\}$ (with $K = \varnothing$ if $\mathbf{x} = \varepsilon$), we denote by $\mathbf{x}|_K$ the string obtained from $\mathbf{x}$ by removing all the letters $x_i$ for which $i \in K^c = \{1, \ldots, |\mathbf{x}|\} \setminus K$. A variadic operation $F\colon X^* \to X \cup \{\varepsilon\}$ is said to be *strongly barycentrically associative* (or *strongly B-associative* for short) if for every $\mathbf{x} \in X^*$ and every $K \subseteq \{1, \ldots, |\mathbf{x}|\}$, we have $F(\mathbf{x}) = F(\mathbf{x}')$, where $\mathbf{x}' \in X^{|\mathbf{x}|}$ is defined by $\mathbf{x}'|_K = F(\mathbf{x}|_K)^{|\mathbf{x}|_K|}$ and $\mathbf{x}'|_{K^c} = \mathbf{x}|_{K^c}$.

For instance, if the operation $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then it satisfies the condition

$$F(\mathbf{xyz}) \;=\; F(F(\mathbf{xz})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{xz})^{|\mathbf{z}|}), \qquad \mathbf{xyz} \in X^*. \tag{1}$$

It is not difficult to see that any strongly B-associative operation $F\colon X^* \to X \cup \{\varepsilon\}$ is B-associative. The converse holds if $F$ is symmetric (i.e., $F_n$ is symmetric for every $n \geqslant 1$). However, it does not hold in general. For instance, the $\varepsilon$-standard operation $F\colon \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined as $F_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$ for every integer $n \geqslant 1$, is strongly B-associative and hence B-associative. However, the $\varepsilon$-standard operation $F\colon \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined by

$$F_n(\mathbf{x}) \;=\; \sum_{i=1}^{n} \frac{2^{i-1}}{2^n - 1}\, x_i\,, \qquad n \geqslant 1,$$

is B-associative but not strongly B-associative (see [2, p. 37]). It is also noteworthy that the strongly B-associative operations need not be symmetric. For instance the $\varepsilon$-standard operation $F\colon X^* \to X \cup \{\varepsilon\}$ defined by $F_n(\mathbf{x}) = x_1$ for every $n \geqslant 1$ is strongly B-associative, and similarly if $F_n(\mathbf{x}) = x_n$ for every $n \geqslant 1$.

Recall that a variadic function $F\colon X^* \to Y$ is said to be *barycentrically preassociative* (or *B-preassociative* for short) [9] if, for every $\mathbf{xyy'z} \in X^*$, we have

$$|\mathbf{y}| = |\mathbf{y}'| \quad \text{and} \quad F(\mathbf{y}) = F(\mathbf{y}') \quad \Rightarrow \quad F(\mathbf{xyz}) = F(\mathbf{xy'z}).$$

It is easy to see that any B-associative operation $F\colon X^* \to X \cup \{\varepsilon\}$ is necessarily B-preassociative [9]. This observation motivates the introduction of the following property, which generalizes strong B-associativity.

**Definition 1.1.** We say that a variadic function $F\colon X^* \to Y$ is *strongly barycentrically preassociative* (or *strongly B-preassociative* for short) if for every $\mathbf{x} \in X^*$, every $\mathbf{x}' \in X^{|\mathbf{x}|}$, and every $K \subseteq \{1, \ldots, |\mathbf{x}|\}$, we have

$$F(\mathbf{x}|_K) = F(\mathbf{x}'|_K) \quad \text{and} \quad \mathbf{x}'|_{K^c} = \mathbf{x}|_{K^c} \quad \Rightarrow \quad F(\mathbf{x}) = F(\mathbf{x}').$$

Just as strong B-associativity is a stronger version of B-associativity, strong B-preassociativity is a stronger version of B-preassociativity. However, these latter two properties are equivalent under the symmetry assumption. Also, since none of these properties involve any composition of functions, they allow us to consider a codomain Y that may differ from the set $X \cup \{\varepsilon\}$. For instance, the length function $F\colon X^* \to \mathbb{R}$, defined as $F(\mathbf{x}) = |\mathbf{x}|$, is strongly B-preassociative.

In Section 2 of this paper we investigate both strong B-associativity and strong B-preassociativity. In particular, we provide equivalent formulations of these properties. For instance, we establish the surprising result that strong B-associativity is completely characterized by Eq. (1). We also provide factorization results for strongly B-preassociative functions. Finally, in Section 3 we recall a variant of Kolmogoroff-Nagumo's characterization of the class of quasi-arithmetic means based on the strong B-associativity property and we generalize this characterization to strongly B-preassociative functions.

The terminology used throughout this paper is the following. The domain and range of any function $f$ are denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. The identity operation on any nonempty set $E$ is denoted by $\mathrm{id}_E$. For every integer $n \geqslant 1$, the diagonal section $\delta_F: X \to Y$ of a function $F: X^n \to Y$ is defined as $\delta_F(x) = F(x^n)$.

*Remark* 1. As already observed in [9], if a B-associative operation $F: X^* \to X \cup \{\varepsilon\}$ is such that $\mathrm{ran}(F_n) \subseteq X$ for every $n \geqslant 1$, then the value of $F(\varepsilon)$ is unimportant in the sense that if we modify this value, then the resulting operation is still B-associative. Clearly, this observation also holds for strongly B-associative operations, B-preassociative functions, and strongly B-preassociative functions.

## 2. Strong barycentric associativity and preassociativity

In this section we investigate both strong B-associativity and strong B-preassociativity properties. We start our investigation by showing that, surprisingly, strong B-associativity can be characterized simply by condition (1), thus providing a very concise definition of this (equational) property by means of a single equation.

**Proposition 2.1.** *A variadic operation $F: X^* \to X \cup \{\varepsilon\}$ is strongly B-associative if and only if it satisfies Eq. (1). Moreover, we may assume that $|\mathbf{y}| \leqslant 1$ in (1).*

*Proof.* The condition is clearly necessary. Let us show that it is also sufficient. Assuming that $F$ satisfies (1), we have to prove that for every $\mathbf{x} \in X^*$ and every $K \subseteq \{1, \ldots, |\mathbf{x}|\}$, we have $F(\mathbf{x}) = F(\mathbf{x}')$, where $\mathbf{x}' \in X^{|\mathbf{x}|}$ is defined by $\mathbf{x}'|_K = F(\mathbf{x}|_K)^{|\mathbf{x}|_K|}$ and $\mathbf{x}'|_{K^c} = \mathbf{x}|_{K^c}$. Let us proceed by induction on $n = |\mathbf{x}|$. The result clearly holds for $n = 0$. It also holds for $n = 1$ since we have $F(x) = F(F(x))$ for any $x \in X$ (take $\mathbf{x} = x$ and $\mathbf{yz} = \varepsilon$ in (1)). It also holds for $n = 2$ since a similar argument gives $F(xy) = F(F(x)y) = F(xF(y)) = F(F(xy))$ for any $x, y \in X$. Now, suppose that the result holds for any $n \geqslant 2$ and let us show that it holds for $n + 1$. Let $\mathbf{x} \in X^{n+1}$, let $K \subseteq \{1, \ldots, n+1\}$, and let $\mathbf{x}' \in X^{n+1}$ be defined by $\mathbf{x}'|_K = F(\mathbf{x}|_K)^{|\mathbf{x}|_K|}$ and $\mathbf{x}'|_{K^c} = \mathbf{x}|_{K^c}$, where $K^c = \{1, \ldots, n+1\} \setminus K$. The result is trivial if $|K| = n + 1$ since we have $F(\mathbf{x}) = F(F(\mathbf{x})^{|\mathbf{x}|})$ (take $\mathbf{yz} = \varepsilon$ in (1)). So assume that $|K| \leqslant n$ and take $k \in K^c$. Then there exist $\mathbf{uv}, \mathbf{u}'\mathbf{v}' \in X^n$, with $|\mathbf{u}| = |\mathbf{u}'|$ and $|\mathbf{v}| = |\mathbf{v}'|$, such that $\mathbf{x} = \mathbf{u}x_k\mathbf{v}$ and $\mathbf{x}' = \mathbf{u}'x_k\mathbf{v}'$. We then have

$$F(\mathbf{x}) \ = \ F(F(\mathbf{uv})^{|\mathbf{u}|}x_kF(\mathbf{uv})^{|\mathbf{v}|}) \ = \ F(F(\mathbf{u}'\mathbf{v}')^{|\mathbf{u}'|}x_kF(\mathbf{u}'\mathbf{v}')^{|\mathbf{v}'|}) \ = \ F(\mathbf{x}'),$$

where the first and last equalities hold by (1) and the second equality by the induction hypothesis. This completes the proof of the proposition. □

The following proposition provides equivalent formulations of strong B-preassociativity.

**Proposition 2.2.** *Let $F: X^* \to Y$ be a variadic function. The following assertions are equivalent.*

(i) *$F$ is strongly B-preassociative.*
(ii) *For every $\mathbf{xx}' \in X^*$ such that $|\mathbf{x}| = |\mathbf{x}'|$ and every $K \subseteq \{1, \ldots, |\mathbf{x}|\}$ we have*

$$F(\mathbf{x}|_K) = F(\mathbf{x}'|_K) \quad \text{and} \quad F(\mathbf{x}|_{K^c}) = F(\mathbf{x}'|_{K^c}) \quad \Rightarrow \quad F(\mathbf{x}) = F(\mathbf{x}').$$

(iii) *For every* $\mathbf{x}\mathbf{x}'\mathbf{y}\mathbf{z}\mathbf{z}' \in X^*$ *we have*

$$|\mathbf{x}| = |\mathbf{x}'|, \quad |\mathbf{z}| = |\mathbf{z}'|, \quad \textit{and} \quad F(\mathbf{x}\mathbf{z}) = F(\mathbf{x}'\mathbf{z}') \quad \Rightarrow \quad F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}'\mathbf{y}\mathbf{z}').$$

*Moreover, we may assume that* $|\mathbf{y}| = 1$ *in assertion (iii).*

*Proof.* (i) ⇔ (ii) ⇒ (iii). Trivial or straightforward.

(iii) ⇒ (i). Follows from repeated applications of the stated condition. To illustrate, suppose that we have $F(x_1x_3) = F(x_1'x_3')$ for some $x_1x_3x_1'x_3' \in X^4$. Then for any $x_2, x_4 \in X$, we have $F(x_1x_2x_3) = F(x_1'x_2x_3')$, and then $F(x_1x_2x_3x_4) = F(x_1'x_2x_3'x_4)$. □

Recall that a variadic operation $F: X^* \to X \cup \{\varepsilon\}$ is said to be *arity-wise range-idempotent* [9] if $F(F(\mathbf{x})^{|\mathbf{x}|}) = F(\mathbf{x})$ for every $\mathbf{x} \in X^*$. Clearly, any B-associative or strongly B-associative variadic operation is arity-wise range-idempotent. Actually, it can be shown [9] that if an operation $F: X^* \to X \cup \{\varepsilon\}$ is B-associative then it is both B-preassociative and arity-wise range-idempotent. The converse result holds whenever $\mathrm{ran}(F_n) \subseteq X$ for every $n \geqslant 1$ (note that this latter condition was wrongly omitted in [9]). The following proposition shows that this result still holds if we replace B-associativity and B-preassociativity by their strong versions.

**Proposition 2.3.** *If a variadic operation* $F: X^* \to X \cup \{\varepsilon\}$ *is strongly B-associative, then it is both strongly B-preassociative and arity-wise range-idempotent. The converse result holds whenever* $\mathrm{ran}(F_n) \subseteq X$ *for every* $n \geqslant 1$.

*Proof.* The first result holds trivially by Proposition 2.1. We now prove the converse result by using Proposition 2.1 again. Let $\mathbf{x}\mathbf{y}\mathbf{z} \in X^*$. If $\mathbf{x}\mathbf{z} = \varepsilon$, then (1) holds trivially. If $\mathbf{x}\mathbf{z} \neq \varepsilon$, then we have $F(\mathbf{x}\mathbf{z}) = F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|})$ by arity-wise range-idempotence and then (1) holds by Proposition 2.2(iii). □

Various alternative formulations of B-associativity have been given in [9]. For instance, we can prove that an operation $F: X^* \to X \cup \{\varepsilon\}$ is B-associative if and only if we have $F(\mathbf{x}\mathbf{y}) = F(F(\mathbf{x})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|})$ for every $\mathbf{x}\mathbf{y} \in X^*$. The following proposition provides similar formulations for strong B-associativity.

**Proposition 2.4.** *Let* $F: X^* \to X \cup \{\varepsilon\}$ *be a variadic operation. The following assertions are equivalent.*

(i) $F$ *is strongly B-associative.*

(ii) *For every* $\mathbf{x} \in X^*$ *and every* $K \subseteq \{1, \ldots, |\mathbf{x}|\}$, *we have* $F(\mathbf{x}) = F(\mathbf{x}')$, *where* $\mathbf{x}' \in X^{|\mathbf{x}|}$ *is defined by* $\mathbf{x}'|_K = F(\mathbf{x}|_K)^{|\mathbf{x}|_K|}$ *and* $\mathbf{x}'|_{K^c} = F(\mathbf{x}|_{K^c})^{|\mathbf{x}|_{K^c}|}$.

(iii) *For every* $\mathbf{x}\mathbf{y}\mathbf{z} \in X^*$, *we have* $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|})$.

*Moreover, we may assume that* $|\mathbf{y}| \leqslant 1$ *in assertion (iii).*

*Proof.* (i) ⇒ (ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). First observe that $F$ is arity-wise range-idempotent (take $\mathbf{y}\mathbf{z} = \varepsilon$). Let $\mathbf{x}\mathbf{y}\mathbf{z} \in X^*$, with $|\mathbf{y}| \leqslant 1$. If $F(\mathbf{x}\mathbf{z}) = \varepsilon$, then we have

$$\begin{aligned} F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|}) &= F(\mathbf{y}) = F(F(\mathbf{y})^{|\mathbf{y}|}) = F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|}) \\ &= F(\mathbf{x}\mathbf{y}\mathbf{z}). \end{aligned}$$

Otherwise, if $F(\mathbf{x}\mathbf{z}) \in X$, then setting $\mathbf{x}' = F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}$ and $\mathbf{z}' = F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|}$, we have

$$\begin{aligned} F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}\mathbf{y}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|}) &= F(\mathbf{x}'\mathbf{y}\mathbf{z}') = F(F(\mathbf{x}'\mathbf{z}')^{|\mathbf{x}'|}F(\mathbf{y})^{|\mathbf{y}|}F(\mathbf{x}'\mathbf{z}')^{|\mathbf{z}'|}) \\ &= F(F(\mathbf{x}\mathbf{z})^{|\mathbf{x}|}F(\mathbf{y})^{|\mathbf{y}|}F(\mathbf{x}\mathbf{z})^{|\mathbf{z}|}) = F(\mathbf{x}\mathbf{y}\mathbf{z}). \end{aligned}$$

In both cases, we have shown that $F$ is strongly B-associative by Proposition 2.1. □

Proposition 2.1 states that an operation $F\colon X^* \to X\cup\{\varepsilon\}$ is strongly B-associative if and only if it satisfies the following two conditions:

(a) $F(\mathbf{y}) = F(F(\mathbf{y})^{|\mathbf{y}|})$ for every $\mathbf{y} \in X^*$ (arity-wise range-idempotence),
(b) $F(\mathbf{x}y\mathbf{z}) = F(F(\mathbf{xz})^{|\mathbf{x}|}yF(\mathbf{xz})^{|\mathbf{z}|})$ for every $\mathbf{x}y\mathbf{z} \in X^*$.

Interestingly, this equivalence also shows how a strongly B-associative $\varepsilon$-standard operation $F\colon X^* \to X \cup \{\varepsilon\}$ can be constructed by choosing first $F_1$, then $F_2$, and so forth. In fact, $F_1$ can be chosen arbitrarily provided that it satisfies $F_1 \circ F_1 = F_1$. Then, if $F_k$ is already chosen for some $k \geqslant 1$, then $F_{k+1}$ can be chosen arbitrarily from among the solutions of the following equations

$$\begin{aligned} \delta_{F_{k+1}} \circ F_{k+1} &= F_{k+1}\,, \\ F_{k+1}(\mathbf{x}y\mathbf{z}) &= F_{k+1}(F_k(\mathbf{xz})^{|\mathbf{x}|}yF_k(\mathbf{xz})^{|\mathbf{z}|}), \qquad \mathbf{x}y\mathbf{z} \in X^{k+1}. \end{aligned}$$

In general, finding all the possible functions $F_{k+1}$ is not an easy task. However, we have the following two propositions, which hold for any strongly B-preassociative function and hence for any strongly B-associative operation. The proof of Proposition 2.5 is straightforward and thus omitted. Proposition 2.6 was established in [9].

**Proposition 2.5.** *Let $F\colon X^* \to Y$ be a B-preassociative (resp. strongly B-preassociative) function.*

(a) *If $F_k$ is symmetric for some $k \geqslant 2$, then so is $F_{k+1}$.*
(b) *If $F_k$ is constant for some $k \geqslant 1$, then so is $F_{k+1}$.*
(c) *For any sequence $(c_k)_{k\geqslant 1}$ in $Y$ and every $n \geqslant 1$, the function $G\colon X^* \to Y$ defined by $G_k = F_k$, if $k \leqslant n$, and $G_k = c_k$, if $k > n$, is B-preassociative (resp. strongly B-preassociative).*

**Proposition 2.6** ( [9])**.** *Let $F\colon X^* \to Y$ be a B-preassociative function and let $k \geqslant 2$ be an integer. If the function $\mathbf{y} \in X^k \mapsto F_{k+2}(x\mathbf{y}z)$ is symmetric for every $xz \in X^2$, then so is the function $\mathbf{y} \in X^{k+1} \mapsto F_{k+3}(x\mathbf{y}z)$ for every $xz \in X^2$.*

Proposition 2.6 motivates the question of finding necessary and sufficient conditions on a (strongly) B-preassociative function $F\colon X^* \to Y$ for the following condition to hold:

$$F(abcd) = F(acbd)\,, \qquad a,b,c,d \in X. \tag{2}$$

Not all strongly B-preassociative functions satisfy condition (2). To give a very simple example, just consider the identity function $F = \mathrm{id}_{X^*}$.

However, one can show that condition (2) holds for all strongly B-associative operations.

**Lemma 2.7.** *Any strongly B-associative operation $F\colon X^* \to X \cup \{\varepsilon\}$ satisfies condition* (2)*.*

*Proof.* Let $a,b,c,d \in X$ and set $x = F(ab)$ and $y = F(cd)$. Repeated applications of strong B-associativity give

$$F(abcd) = F(xxyy) = F(F(xy)^4) = F(xyxy) = F(acbd),$$

which shows that condition (2) holds. □

From Proposition 2.6 and Lemma 2.7 we immediately derive the following corollary, which states that, for every strongly B-associative operation $F\colon X^* \to X \cup \{\varepsilon\}$ and every integer $n \geqslant 4$, the $n$-ary part $F_n$ is invariant under any permutation of its arguments except the first and the last ones.

**Corollary 2.8.** *If $F\colon X^* \to X \cup \{\varepsilon\}$ is strongly B-associative, then, for every integer $k \geqslant 1$ and every $xz \in X^2$, the function $\mathbf{y} \in X^k \mapsto F_{k+2}(x\mathbf{y}z)$ is symmetric.*

In [9], the authors show how new B-preassociative functions can be constructed from given B-preassociative functions by compositions with unary maps. These results still hold for strongly B-preassociative functions. The proofs are straightforward.

**Proposition 2.9** (Right composition)**.** *If $F\colon X^* \to Y$ is strongly B-preassociative then, for every function $g\colon X' \to X$, any function $H\colon X'^* \to Y$ such that $H_n = F_n \circ (g, \ldots, g)$ for every $n \geqslant 1$ is strongly B-preassociative.*

**Proposition 2.10** (Left composition)**.** *Let $F\colon X^* \to Y$ be a strongly B-preassociative function and let $(g_n)_{n\geqslant 1}$ be a sequence of functions from $Y$ to $Y'$. If $g_n|_{\mathrm{ran}(F_n)}$ is one-to-one for every $n \geqslant 1$, then any function $H\colon X^* \to Y'$ such that $H_n = g_n \circ F_n$ for every $n \geqslant 1$ is strongly B-preassociative.*

We now give a factorization result for strongly B-preassociative functions. We first restrict ourselves to strongly B-preassociative functions $F\colon X^* \to Y$ which are *arity-wise quasi-range-idempotent* [9], i.e., such that $\mathrm{ran}(\delta_{F_n}) = \mathrm{ran}(F_n)$ for every $n \geqslant 1$. The following theorem gives a characterization of the B-preassociative and arity-wise quasi-range-idempotent functions $F\colon X^* \to Y$ as compositions of the form $F_n = f_n \circ H_n$, where $H\colon X^* \to X \cup \{\varepsilon\}$ is a B-associative $\varepsilon$-standard operation and $f_n\colon \mathrm{ran}(H_n) \to Y$ is one-to-one.

Recall that a function $g$ is a *quasi-inverse* [13, Sect. 2.1] of a function $f$ if

$$f \circ g|_{\mathrm{ran}(f)} = \mathrm{id}|_{\mathrm{ran}(f)} \quad \text{and} \quad \mathrm{ran}(g|_{\mathrm{ran}(f)}) = \mathrm{ran}(g).$$

Recall also that the statement "every function has a quasi-inverse" is equivalent to the Axiom of Choice (AC). Throughout this paper we denote the set of all quasi-inverses of $f$ by $Q(f)$.

**Theorem 2.11** ( [9])**.** *Assume AC and let $F\colon X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *$F$ is B-preassociative and arity-wise quasi-range-idempotent.*
(ii) *There exists a B-associative $\varepsilon$-standard operation $H\colon X^* \to X \cup \{\varepsilon\}$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n\colon \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.*

*If condition (ii) holds, then for every $n \geqslant 1$ we have $F_n = \delta_{F_n} \circ H_n$, $f_n = \delta_{F_n}|_{\mathrm{ran}(H_n)}$, $f_n^{-1} \in Q(\delta_{F_n})$, and we may choose $H_n = g_n \circ F_n$ for any $g_n \in Q(\delta_{F_n})$.*

Using Propositions 2.2 and 2.3, we can show that Theorem 2.11 can be easily adapted to the strong version of B-preassociativity.

**Corollary 2.12.** *Theorem 2.11 still holds if we replace B-preassociativity with strong B-preassociativity in assertion (i) and B-associativity with strong B-associativity in assertion (ii).*

*Proof.* (i) ⇒ (ii). Since $F$ satisfies condition (i) of Theorem 2.11, it also satisfies condition (ii). Since $H$ is B-associative, it is arity-wise range-idempotent. To see that $H$ is strongly B-associative, by Proposition 2.3 it suffices to show that it is strongly B-preassociative. Let $\mathbf{x}\mathbf{x}'\mathbf{y}\mathbf{z}\mathbf{z}' \in X^*$ such that $|\mathbf{x}| = |\mathbf{x}'|$, $|\mathbf{z}| = |\mathbf{z}'|$, $|\mathbf{x}\mathbf{z}| \geqslant 1$, and $H(\mathbf{x}\mathbf{z}) = H(\mathbf{x}'\mathbf{z}')$. Then, we have $f_{|\mathbf{x}\mathbf{z}|} \circ H(\mathbf{x}\mathbf{z}) = f_{|\mathbf{x}\mathbf{z}|} \circ H(\mathbf{x}'\mathbf{z}')$, that is, $F(\mathbf{x}\mathbf{z}) = F(\mathbf{x}'\mathbf{z}')$. By strong B-preassociativity of $F$, we then have $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}'\mathbf{y}\mathbf{z}')$ and hence

$$H(\mathbf{x}\mathbf{y}\mathbf{z}) = g_{|\mathbf{x}\mathbf{y}\mathbf{z}|} \circ F(\mathbf{x}\mathbf{y}\mathbf{z}) = g_{|\mathbf{x}\mathbf{y}\mathbf{z}|} \circ F(\mathbf{x}'\mathbf{y}\mathbf{z}') = H(\mathbf{x}'\mathbf{y}\mathbf{z}'),$$

which shows that $H$ is strongly B-preassociative by Proposition 2.2 (the case when $\mathbf{x}\mathbf{z} = \varepsilon$ is trivial).

(ii) ⇒ (i). $F$ is arity-wise quasi-range-idempotent by Theorem 2.11. It is also strongly B-preassociative by Proposition 2.10. □

We now provide a factorization result for the whole class of strongly B-preassociative functions. It is based on the following characterization of B-preassociative functions.

Recall that a string function $F: X^* \to X^*$ is said to be *associative* [4] if it satisfies the equation $F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z})$ for every $\mathbf{x}\mathbf{y}\mathbf{z} \in X^*$. It is said to be *length-preserving* [10] if $|F(\mathbf{x})| = |\mathbf{x}|$ for every $\mathbf{x} \in X^*$.

**Theorem 2.13** ( [10]). *Assume AC and let $F: X^* \to Y$ be a function. The following assertions are equivalent.*

(i) *$F$ is B-preassociative.*
(ii) *There exist an associative and length-preserving function $H: X^* \to X^*$ and a sequence $(f_n)_{n\geqslant 1}$ of one-to-one functions $f_n: \mathrm{ran}(H_n) \to Y$ such that $F_n = f_n \circ H_n$ for every $n \geqslant 1$.*

*If condition (ii) holds, then for every $n \geqslant 1$ we have $f_n = F|_{\mathrm{ran}(H_n)} = F_n|_{\mathrm{ran}(H_n)}$, $f_n^{-1} \in Q(F_n)$, and we may choose $H_n = g_n \circ F_n$ for any $g_n \in Q(F_n)$.*

Proceeding as in the proof of Corollary 2.12, from Theorem 2.13 we easily derive the following characterization of the class of strongly B-preassociative functions.

**Corollary 2.14.** *Theorem 2.13 still holds if we replace B-preassociativity with strong B-preassociativity in assertion (i) and add the condition that $H$ is strongly B-preassociative in assertion (ii).*

Clearly, Corollary 2.14 motivates the problem of characterizing the class of those string functions which are associative, length-preserving, and strongly B-preassociative.

We end this section by an investigation of those strong B-associative functions which are invariant by replication. Recall that a variadic function $F: X^* \to Y$ is *invariant by replication* [8] if for every $\mathbf{x} \in X^*$ and every $k \geqslant 1$ we have $F(\mathbf{x}^k) = F(\mathbf{x})$.

**Definition 2.15.** We say that a variadic function $F: X^* \to Y$ has a *multiplicatively growing range* if $\mathrm{ran}(F_n) \subseteq \mathrm{ran}(F_{kn})$ for any $k, n \geqslant 1$.

The following proposition is a simultaneous generalization of several results reported in [6] and [2, pp. 38-41].

**Proposition 2.16.** *Let $F: X^* \to X \cup \{\varepsilon\}$ be a strongly B-associative operation. The following assertions are equivalent.*

(i) *$F$ has a multiplicatively growing range.*

(ii) *$F$ is invariant by replication.*

(iii) *For any $k, n \geqslant 1$ and any $\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)} \in X^*$, we have*

$$F((\mathbf{x}^{(1)})^k \cdots (\mathbf{x}^{(n)})^k) = F(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}).$$

*Moreover, if any of these conditions hold, then*

(a) *For any $n \geqslant 1$ and any $\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)} \in X^*$ such that $|\mathbf{x}^{(1)}| = \cdots = |\mathbf{x}^{(n)}| \geqslant 1$, we have*

$$F(F(\mathbf{x}^{(1)}) \cdots F(\mathbf{x}^{(n)})) = F(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}).$$

(b) *For any $n \geqslant 1$ and any $\mathbf{x} = x_1 \cdots x_n \in X^n$, we have*

$$F(x_1 \cdots x_n) = F(x'_n \cdots x'_1),$$

*where $x'_k = F(\mathbf{x}_{-k})$ and $\mathbf{x}_{-k}$ is obtained from $\mathbf{x}$ by removing the letter $x_k$.*

(c) *$F$ is strongly bisymmetric, i.e., for every $p$-by-$n$ matrix whose entries are in $X$, we have*

$$F(F(\mathbf{r}_1) \cdots F(\mathbf{r}_p)) = F(F(\mathbf{c}_1) \cdots F(\mathbf{c}_n)),$$

*where $\mathbf{r}_1, \ldots, \mathbf{r}_p$ denote the rows and $\mathbf{c}_1, \ldots, \mathbf{c}_n$ denote the columns of the matrix.*

*Proof.* (iii) ⇒ (ii). Taking $n = 1$, we see that $F$ is invariant by replication.

(ii) ⇒ (i). Let $k, n \geqslant 1$ and $\mathbf{x} \in X^n$. Then $F(\mathbf{x}) = F(\mathbf{x}^k) \in \mathrm{ran}(F_{kn})$. Therefore, $F$ has a multiplicatively growing range.

(i) ⇒ (iii). Let us first show that

$$F(F(\mathbf{x})^{k|\mathbf{x}|}) = F(\mathbf{x}), \qquad \mathbf{x} \in X^*, \ k \geqslant 1. \tag{3}$$

Let $m \geqslant 0$, $k \geqslant 1$, and take $\mathbf{x} \in X^m$ and $\mathbf{z} \in X^{km}$ such that $F(\mathbf{x}) = F(\mathbf{z})$. Since $F$ is arity-wise range-idempotent by Proposition 2.3, it follows that $F(F(\mathbf{x})^{km}) = F(F(\mathbf{z})^{km}) = F(\mathbf{z}) = F(\mathbf{x})$, which proves (3).

Then, for any $n \geqslant 1$ and any $\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)} \in X^*$ we have

$$\begin{aligned} F((\mathbf{x}^{(1)})^k \cdots (\mathbf{x}^{(n)})^k) &= F(F(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)})^{k|\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}|}) && \text{(strong B-associativity)} \\ &= F(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}) && \text{(by condition (3))}. \end{aligned}$$

Let us now show that conditions (a), (b), and (c) hold.

(a) Setting $k = |\mathbf{x}^{(1)}| = \cdots = |\mathbf{x}^{(n)}| \geqslant 1$ we have

$$\begin{aligned} F(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}) &= F(F(\mathbf{x}^{(1)})^k \cdots F(\mathbf{x}^{(n)})^k) && \text{(B-associativity)} \\ &= F(F(\mathbf{x}^{(1)}) \cdots F(\mathbf{x}^{(n)})) && \text{(by (iii))}. \end{aligned}$$

(b) We have

$$\begin{aligned} F(x_1 \cdots x_n) &= F(x_1^{n-1} \cdots x_n^{n-1}) && \text{(by (iii))} \\ &= F((x'_n \cdots x'_1)^{n-1}) && \text{(strong B-associativity)} \\ &= F(x'_n \cdots x'_1) && \text{(by (ii))}. \end{aligned}$$

(c) We have

$$\begin{aligned} F(F(\mathbf{r}_1) \cdots F(\mathbf{r}_p)) &= F(\mathbf{r}_1 \cdots \mathbf{r}_p) && \text{(by (a))} \\ &= F((F(\mathbf{c}_1) \cdots F(\mathbf{c}_n))^p) && \text{(strong B-associativity)} \\ &= F(F(\mathbf{c}_1) \cdots F(\mathbf{c}_n)) && \text{(by (ii))}. \end{aligned}$$

The proof is now complete. □

*Remark* 2. Let $F: X^* \to X \cup \{\varepsilon\}$ be a strongly B-associative operation having a multiplicative growing range and such that $\mathrm{ran}(F_n) \subseteq X$ for every $n \geqslant 1$. Proposition 2.16(a) enables us to translate certain functional conditions involving $F$ and letters in $X$ into similar functional conditions involving $F$ and strings of the same length. To illustrate, starting from the condition

$$F(xyz) = F(F(xz)yF(xz)), \qquad xyz \in X^3, \tag{4}$$

which holds by Eq. (1), we derive the condition

$$F(\mathbf{xyz}) = F(F(\mathbf{xz})F(\mathbf{y})F(\mathbf{xz})), \qquad \mathbf{xyz} \in X^*, \quad |\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| \geqslant 1.$$

Indeed, it suffices to set $x = F(\mathbf{x})$, $y = F(\mathbf{y})$, and $z = F(\mathbf{z})$ in (4) and apply Proposition 2.16(a) to the resulting condition.

## 3. Strongly B-preassociative mean functions

In this final section we recall a variant of Kolmogoroff-Nagumo's characterization of the class of quasi-arithmetic means based on the strong B-associativity property. We also generalize this characterization to strongly B-preassociative functions.

Let $\mathbb{I}$ be a nontrivial real interval (i.e., nonempty and not a singleton), possibly unbounded. Recall that a function $F: \mathbb{I}^* \to \mathbb{R}$ is said to be a *quasi-arithmetic pre-mean function* [9] if there exist continuous and strictly increasing functions $f: \mathbb{I} \to \mathbb{R}$ and $f_n: \mathbb{R} \to \mathbb{R}$ $(n \geqslant 1)$ such that

$$F_n(\mathbf{x}) = f_n\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right), \qquad n \geqslant 1.$$

This function is said to be a *quasi-arithmetic mean function* (see, e.g., [2, Sect. 4.2]) if $f_n = f^{-1}$ for every $n \geqslant 1$. In this case we have $\mathrm{ran}(F_n) \subseteq \mathbb{I}$ for every $n \geqslant 1$.

Thus defined, the class of quasi-arithmetic pre-mean functions includes all the quasi-arithmetic mean functions. Actually the quasi-arithmetic mean functions are exactly those quasi-arithmetic pre-mean functions which are *idempotent*, that is, such that $\delta_{F_n} = \mathrm{id}_{\mathbb{I}}$ for every $n \geqslant 1$. However, there are also many non-idempotent quasi-arithmetic pre-mean functions. Taking for instance $f_n(x) = nx$ and $f(x) = x$ over the reals $\mathbb{I} = \mathbb{R}$, we obtain the sum function. Taking $f_n(x) = \exp(nx)$ and $f(x) = \ln(x)$ over $\mathbb{I} = ]0, \infty[$, we obtain the product function.

The following proposition [9] shows that the generators $f_n$ and $f$ of any quasi-arithmetic pre-mean function are defined up to an affine transformation.

**Proposition 3.1** ( [9]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. Let $f, g: \mathbb{I} \to \mathbb{R}$ and $f_n, g_n: \mathbb{R} \to \mathbb{R}$ $(n \geqslant 1)$ be continuous and strictly monotonic functions. Then the functions $f_n(\frac{1}{n}\sum_{i=1}^{n} f(x_i))$ and $g_n(\frac{1}{n}\sum_{i=1}^{n} g(x_i))$ coincide on $\mathbb{I}^n$ if and only if there exist $r, s \in \mathbb{R}$, $r \neq 0$, such that $g_n^{-1} \circ f_n = g \circ f^{-1} = r\,\mathrm{id} + s$ for every $n \geqslant 1$.*

We now recall the characterization of the class of quasi-arithmetic mean functions as given by Kolmogoroff [3] and Nagumo [11]. The following theorem gives the characterization following Kolmogoroff (we set $F(\varepsilon)$ to an arbitrary value in $\mathbb{I}$, see Remark 1). Nagumo's characterization is the same except that the strict increasing monotonicity of each function $F_n$ is replaced with the strict internality of $F_2$ (i.e., $x < y$ implies $x < F_2(x, y) < y$).

**Theorem 3.2** (Kolmogoroff-Nagumo)**.** *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A variadic function $F\colon \mathbb{I}^* \to \mathbb{I}$ is B-associative and, for every $n \geqslant 1$, the $n$-ary part $F_n$ is symmetric, continuous, idempotent, and strictly increasing in each argument if and only if $F$ is a quasi-arithmetic mean function.*

As recently observed by the authors [9], idempotence can be removed from the assumptions of Theorem 3.2. Indeed, if a B-associative function $F\colon \mathbb{I}^* \to \mathbb{I}$ is such that $\delta_{F_n}$ is one-to-one for some $n \geqslant 1$, then necessarily $\delta_{F_n} = \mathrm{id}_{\mathbb{I}}$. This observation immediately follows from the identity $\delta_{F_n} = \delta_{F_n} \circ \delta_{F_n}$, which holds whenever $F$ is B-associative.

In the following theorem, we show that Kolmogoroff-Nagumo's characterization still holds if we replace both B-associativity and symmetry with strong B-associativity. This result was already established in [6]. However, here we provide an alternative proof based on Kolmogoroff's ideas. Here again, idempotence is redundant.

We first consider a lemma which generalizes the result reported in [2, Lemma 4.9].

**Lemma 3.3.** *Let $F\colon X^* \to X \cup \{\varepsilon\}$ be a strongly B-associative operation having a multiplicatively growing range. Then, for any $\mathbf{a}, \mathbf{b} \in X^* \setminus \{\varepsilon\}$, there exists a function $\psi\colon [0,1] \cap \mathbb{Q} \to X \cup \{\varepsilon\}$, namely*

$$\psi(p/q) = F(\mathbf{b}^p \mathbf{a}^{q-p}), \qquad p/q \in [0,1] \cap \mathbb{Q},$$

*with $\psi(0) = F(\mathbf{a})$ and $\psi(1) = F(\mathbf{b})$, such that for every $n \geqslant 1$ and every $\mathbf{z} \in [0,1]^n \cap \mathbb{Q}^n$ such that $z_1 \neq 0$ and $z_n \neq 1$, we have*

$$F(\psi(z_1) \cdots \psi(z_n)) = \psi\left(\frac{1}{n} \sum_{i=1}^{n} z_i\right).$$

*Proof.* We first observe that $\psi$ is a well-defined function. Indeed, if $p/q = p'/q'$ are two representations of the same rational, then we have

$$\begin{aligned} F(\mathbf{b}^p \mathbf{a}^{q-p}) &= F(\mathbf{b}^{p'p} \mathbf{a}^{p'(q-p)}) \qquad \text{(by Proposition 2.16(iii))} \\ &= F(\mathbf{b}^{pp'} \mathbf{a}^{p(q'-p')}) \\ &= F(\mathbf{b}^{p'} \mathbf{a}^{q'-p'}) \qquad \text{(by Proposition 2.16(iii))}. \end{aligned}$$

Now, for any $z_1 = p_1/q, \ldots, z_n = p_n/q$, with $p_i \leqslant q$, $p_1 \neq 0$ and $p_n \neq q$, we have

$$\begin{aligned} F(\psi(z_1) \cdots \psi(z_n)) &= F(F(\mathbf{b}^{p_1} \mathbf{a}^{q-p_1}) \cdots F(\mathbf{b}^{p_n} \mathbf{a}^{q-p_n})) \\ &= F(\mathbf{b}^{p_1} \mathbf{a}^{q-p_1} \cdots \mathbf{b}^{p_n} \mathbf{a}^{q-p_n}) \qquad \text{(by Proposition 2.16(a))} \\ &= F(\mathbf{b}^{\sum p_i} \mathbf{a}^{nq - \sum p_i}) \qquad \text{(by Corollary 2.8)} \\ &= \psi\left(\frac{1}{nq} \sum_{i=1}^{n} p_i\right) = \psi\left(\frac{1}{n} \sum_{i=1}^{n} z_i\right). \end{aligned}$$

This completes the proof of the lemma. □

**Theorem 3.4.** *Theorem 3.2 still holds if we replace B-associativity and symmetry with strong B-associativity. Also, idempotence can be removed.*

*Proof.* (Necessity) Let $F\colon \mathbb{I}^* \to \mathbb{I}$ be a strongly B-associative function such that, for every $n \geqslant 1$, the function $F_n$ is continuous, idempotent, and strictly increasing in each argument.

We first assume that $\mathbb{I}$ is a closed interval $[a,b]$, with $b > a$. Since $F_n$ is idempotent for every $n \geqslant 1$, $F$ has a multiplicative growing range. By Lemma 3.3 the function $\psi\colon[0,1]\cap\mathbb{Q}\to[a,b]$ defined by $\psi(p/q)=F(b^pa^{q-p})$ is well defined and such that for every $n\geqslant 1$ and every $\mathbf{z}\in[0,1]^n\cap\mathbb{Q}^n$ such that $z_1\neq 0$ and $z_n\neq 1$, we have

$$F(\psi(z_1)\cdots\psi(z_n)) = \psi\bigg(\frac{1}{n}\sum_{i=1}^n z_i\bigg). \tag{5}$$

Moreover, it is easy to see that the function $\psi$ is strictly increasing.

Let us now show that the restriction of $\psi$ to $]0,1[\cap\mathbb{Q}$ can be extended to a continuous function $\overline{\psi}\colon]0,1[\to[a,b]$.

Let $x\in]0,1[$. Since $\psi$ is nondecreasing we can define

$$u = \lim_{z\to x^-}\psi(z)\quad\text{and}\quad v = \lim_{z\to x^+}\psi(z).$$

Let us show that $u=v$. For contradiction, assume that $u<v$ and consider two sequences $(z_m)_{m\geqslant 1}$ and $(z'_m)_{m\geqslant 1}$ in $[0,1]\cap\mathbb{Q}$ such that $z_m\to x^-$, $z'_m\to x^+$, and $(z_m+z'_m)/2<x$. Using (5) and the continuity of $F$, we then have

$$u = \lim_{m\to\infty}\psi\Big(\frac{z_m+z'_m}{2}\Big) = \lim_{m\to\infty}F(\psi(z_m),\psi(z'_m)) = F(u,v) < F(u,u) = u,$$

a contradiction.

Let $\overline{\psi}\colon]0,1[\to]\alpha,\beta[$ be the continuous extension of $\psi$, where $]\alpha,\beta[=\mathrm{ran}(\overline{\psi})$. Let us show that $\alpha=a$. Due to the uniqueness of the limit, we have $\lim_{z\to 0^+}\psi(z)=\lim_{t\to 0^+}\overline{\psi}(t)=\alpha$. Then, using (5) and the continuity of $F$, we have

$$\begin{aligned}F(b,\alpha) &= \lim_{z\to 0^+}F(\psi(1),\psi(z)) = \lim_{z\to 0^+}\psi\Big(\frac{1+z}{2}\Big) = \psi\Big(\frac{1}{2}\Big) = F(\psi(1),\psi(0))\\ &= F(b,a).\end{aligned}$$

Since $F_2$ is one-to-one in its second argument, we must have $\alpha=a$. We prove similarly that $\beta=b$. Thus, $\overline{\psi}$ can be further extended to a continuous and strictly increasing function from $[0,1]$ onto $[a,b]$. Denoting by $f\colon[a,b]\to[0,1]$ the inverse of this continuous extension, from (5) and continuity we derive the identity

$$F_n(\mathbf{x}) = f^{-1}\bigg(\frac{1}{n}\sum_{i=1}^n f(x_i)\bigg),\qquad \mathbf{x}\in[a,b]^n,\ n\geqslant 1,$$

which proves the result when $\mathbb{I}=[a,b]$.

Let us now prove the result for a general nontrivial interval $\mathbb{I}$. Here we use arguments sketched in [2, Theorem 4.10]. Let $M_f\colon\mathbb{I}^*\to\mathbb{I}$ denote the quasi-arithmetic mean function generated by $f\colon\mathbb{I}\to\mathbb{R}$. Let $a=\inf\mathbb{I}$ and $b=\sup\mathbb{I}$. Let also $(a_m)_{m\geqslant 1}$ (resp. $(b_m)_{m\geqslant 1}$) be a strictly decreasing (resp. strictly increasing) sequence in $\mathbb{I}$ converging to $a$ (resp. $b$). From the previous result it follows that there exist continuous and strictly increasing functions $f_m\colon[a_m,b_m]\to\mathbb{R}$ and $f_{m+1}\colon[a_{m+1},b_{m+1}]\to\mathbb{R}$ such that $F=M_{f_m}$ on $[a_m,b_m]^*$ and $F=M_{f_{m+1}}$ on $[a_{m+1},b_{m+1}]^*$. By Proposition 3.1, we have $M_{f_{m+1}}=M_{rf_{m+1}+s}$ for all $r,s\in\mathbb{R}$, with $r\neq 0$. It follows that $f_{m+1}$ can be chosen so that $f_{m+1}(a_m)=f_m(a_m)$ and $f_{m+1}(b_m)=f_m(b_m)$. Since $M_{f_{m+1}}=F=M_{f_m}$ on $[a_m,b_m]^*$ from Proposition 3.1 it follows that there exist $c,d\in\mathbb{R}$, with $c\neq 0$ such that $f_m=cf_{m+1}+d$. Due to the definition of $f_{m+1}$, we must have $c=1$ and $d=0$, that is, $f_{m+1}=f_m$ on $[a_m,b_m]$.

Define $f\colon\mathbb{I}\to\mathbb{R}$ by

$$f(x) = \begin{cases}\lim_{m\to\infty}f_m(x), & \text{if } x\in]a,b[,\\ \inf_{m\geqslant 1}f_m(a_m), & \text{if } x=a\in\mathbb{I},\\ \sup_{m\geqslant 1}f_m(b_m), & \text{if } x=b\in\mathbb{I}.\end{cases}$$

It is clear that $f$ is continuous and strictly increasing. Moreover, we have $F = M_f$ on $\bigcup_m [a_m, b_m]^*$ and even on $\mathbb{I}^*$ by continuity.

(Sufficiency) Straightforward. □

In [9] the authors established a generalization of Kolmogoroff-Nagumo's characterization to quasi-arithmetic pre-mean functions. In the next two theorems we state this result and show that its assumptions can be weakened by replacing both B-preassociativity and symmetry with strong B-preassociativity.

**Theorem 3.5** ( [9]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A function $F\colon \mathbb{I}^* \to \mathbb{R}$ is B-preassociative and, for every $n \geqslant 1$, the function $F_n$ is symmetric, continuous, and strictly increasing in each argument if and only if $F$ is a quasi-arithmetic pre-mean function.*

**Theorem 3.6.** *Theorem 3.5 still holds if we replace B-preassociativity and symmetry with strong B-preassociativity.*

*Proof.* We note that the proof is very similar to that of Theorem 3.5 (see [9]).

(Necessity) Since $F_n$ is increasing and continuous for every $n \geqslant 1$, it follows that $F$ is arity-wise quasi-range-idempotent. Let $H\colon \mathbb{I}^* \to \mathbb{I} \cup \{\varepsilon\}$ be the $\varepsilon$-standard operation defined by $H_n = \delta_{F_n}^{-1} \circ F_n$ for every $n \geqslant 1$. It is clear that every $H_n$ is continuous, idempotent, and strictly increasing. By Corollary 2.12 (here AC is not needed since $\delta_{F_n}^{-1}$ is an inverse), $H$ is strongly B-associative (and remains so if we modify the value of $H(\varepsilon)$ into any element of $\mathbb{I}$; see Remark 1). By Theorem 3.4 it follows that $H$ is a quasi-arithmetic mean function. This completes the proof.

(Sufficiency) Straightforward. □

## 4. Concluding remarks and open problems

We have investigated the strong B-associativity property for variadic operations and introduced a relaxation of this property, namely strong B-preassociativity. In particular, we have presented a characterization of the class of strongly B-preassociative functions in terms of associative string functions.

We end this paper with the following questions:

(a) Find necessary and sufficient conditions on a (strongly) B-preassociative function $F\colon X^* \to Y$ for condition (2) to hold.

(b) Find necessary and sufficient conditions on a B-associative operation $F\colon X^* \to X \cup \{\varepsilon\}$ satisfying $F(xyz) = F(F(xz)yF(xz))$ for every $xyz \in X^3$ to be strongly B-associative. What if $F$ satisfies the symmetry condition stated in Corollary 2.8?

(c) Similarly, find necessary and sufficient conditions on a B-preassociative function $F\colon X^* \to Y$ satisfying the condition

$$F(xz) = F(x'z') \quad \Rightarrow \quad F(xyz) = F(x'yz'), \qquad xx'yzz' \in X^5$$

to be strongly B-preassociative.

(d) Find a characterization of the class of those string functions which are associative, length-preserving, and strongly B-preassociative (cf. Corollary 2.14).

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