N-point Virasoro algebras considered as Krichever–Novikov type algebras

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Abstract. We explain how the recently again discussed N-point Witt, Virasoro, and affine Lie algebras are genus zero examples of the multi-point versions of Krichever–Novikov type algebras as introduced and studied by Schlichenmaier. Using this more general point of view, useful structural insights and an easier access to calculations can be obtained. As example, explicit expressions for the three-point situation are given.

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1. Introduction

In the context of genus zero conformal field theory (CFT) the Witt algebra and its universal central extension, the Virasoro algebra, play an important role by encoding conformal symmetry [1]. Krichever–Novikov algebras are higher genus and multi-point analogs of them. For higher genus, but still only for two points where poles are allowed, some of the algebras were generalised in 1986 by Krichever and Novikov [18], [19], [20]. In 1990 the author [22], [22], [24], [25] extended the approach further to the general multi-point case. These extensions were not straight-forward generalizations. The crucial point was to introduce a replacement of the graded algebra structure present in the "classical" case. Krichever and Novikov found that an almost-grading, see Definition 1, will be enough to allow constructions in representation theory, like triangular decompositions, highest weight modules, Verma modules and

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many more thing. In [24], [25] it was realized that a splitting of the set of points A where poles are allowed into two disjoint non-empty subsets $A = I \cup O$ is crucial for introducing an almost-grading. For every such splitting the corresponding almost-grading was given. Essentially different splittings (not just corresponding to interchanging I and O) will yield essentially different almost-gradings. For the general theory (including the classical case) see the recent monograph [31].

The genus zero but more than two point case was also addressed by Bremner [3], [4], [5]. Recently, there was again a revived interest in the genus zero situation multi-point situation. See e.g. work by Cox, Jurisisch, Martins, and collaborators [6], [7], [8], [9], [15]. In particular, this interest comes from representation theory and its interpretations in the context of quantization of (conformal) field theory. In some of these articles the vector field algebras were called N-Virasoro algebras, affine algebras N-point affine algebras, etc. Here we like to stress the fact, that these algebras are also examples of genus zero Krichever–Novikov (KN) type algebras in their multi-point version as introduced by the current author.

In a recent manuscript [32] I showed this in detail. Furthermore, I give a common treatment of all these kind of algebras. Taking this interpretation serious, gives a better understanding of the situation and an easier approach to calculations. Furthermore, it explains certain properties remarked by the authors of [7], [9], [15].

In this write-up of a talk presented at the Bialowieza meeting in 2015, I will report on the results obtained in [32] and add some additional comments. For all proofs and calculations I refer to [32]. For general background information on Krichever–Novikov type algebras see [31], or the review [30].

Here we will recall the geometric setup for KN type algebras, introduce the algebras including their almost-grading and triangular decomposition. Then we determine “all” their central extensions.

The outcome will be that all cocycle classes for the vector field algebra and the differential operator algebras are geometric and that their universal central extensions can be explicitly given. The same is done for the current algebra. In this way multi-point affine algebras are obtained. The Heisenberg algebra will be obtained from the function algebra by cocycles which are multiplicative or equivalently $\mathcal{L}$-invariant, see the definitions below. The presentation allows an easy access to calculations of structure constants and cocycle values for these algebras. As an illustration we give explicit results for the three point genus zero situation.

2. Classical Algebras

In purely algebraic terms the *Virasoro algebra* $\mathcal{V}$ can be defined in terms of generators $\{e_n (n \in \mathbb{Z}), t\}$ and (Lie algebra) relations [1]

$$[e_n, e_m] = (m-n)e_{n+m} + \frac{1}{12}(n^3 - n)\delta_n^{-m} \cdot t, \quad [t, e_n] = 0. \tag{1}$$

The element t is called the central element.

Without term coming with the central element the algebra is called the *Witt algebra* $\mathcal{W}$. With respect to $\mathcal{W}$, the algebra $\mathcal{V}$ is its universal central extension.

There are other algebras which are relatives of the Virasoro algebra. We only recall the definition of the affine algebras [16], [21]. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and β the Cartan–Killing form. For $\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ we take the Lie algebra structure

$$[x \otimes z^n, y \otimes z^m] := [x, y] \otimes z^{n+m}, \quad x, y \in \mathfrak{g} \quad , n, m \in \mathbb{Z}. \tag{2}$$

Now we set $\widehat{\mathfrak{g}} = \mathbb{C} \otimes \overline{\mathfrak{g}}$ as vector space, denote by $\widehat{x \otimes z^n} := (0, x \otimes z^n)$ and $t := (1, 0)$, and take as Lie structure on $\widehat{\mathfrak{g}}$

$$[\widehat{x \otimes z^n}, \widehat{y \otimes z^m}] := \widehat{[x, y] \otimes z^{n+m}} - \beta(x, y) \cdot n\, \delta_m^{-n} \cdot t, \quad [t, \widehat{\mathfrak{g}}] = 0. \tag{3}$$

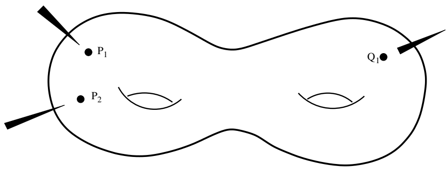
Indeed, this is a Lie algebra $\widehat{\mathfrak{g}}$. It is called the affine Lie algebra associated to $\mathfrak{g}$. Without central term, the algebra is called current or loop algebra.

We remark that all these Lie algebras are infinite dimensional graded Lie algebra. The grading is given by defining

$$\deg(e_n) = 0, \quad \deg(x \otimes z^n) = n, \quad \deg(t) = 0. \tag{4}$$

3. Geometric Set-up

Even if the results which we present here are dealing with genus zero, for a deeper understanding of the structure it will be helpful to consider Riemann surfaces of arbitrary genus. Hence, let Σ_g be a compact Riemann surface of genus $g = g(\Sigma_g)$ and A be a finite set of points of Σ_g (also called marked points). Let furthermore A be decomposed into $A = I \cup O$, $I = \{P_1, \ldots, P_K\}$ (in-points) and $O = \{Q_1, \ldots, Q_M\}$ (out-points), both non-empty and disjoint. All points should be pairwise distinct.



[1] δ_n^m is the Kronecker delta, which is 1 if $n = m$, otherwise 0.

For the case of genus zero with $A = \{P_1, P_2, \dots, P_N\}$, by a fractional linear transformation (i.e. a complex automorphism of the Riemann sphere). the point P_N can be brought to ∞. We obtain

$$P_i = a_i, \quad a_i \in \mathbb{C},\ i = 1, \dots, N-1, \qquad P_N = \infty, \tag{5}$$

with the local coordinates $z - a_i,\ i = 1, \dots, N-1,\ w = 1/z$, at the marked points. The *classical situation* is given by

$$\Sigma_0 = S^2, \qquad I = \{0\}, \quad O = \{\infty\}\ . \tag{6}$$

4. Geometric realizations of the Krichever–Novikov type algebras

Let $\mathcal{K}$ be the canonical line bundle, i.e. the line bundle over Σ whose local sections are the local holomorphic differentials. We consider the tensor power

$$\mathcal{K}^\lambda := \mathcal{K}^{\otimes \lambda} \quad \text{for } \lambda \in \mathbb{Z}\,. \tag{7}$$

Its sections are the forms of weight λ,. For example, for $\lambda = -1$ we obtain the local holomorphic vector fields and $\lambda = 0$ yields the functions. After fixing a square root L of $\mathcal{K}$ (also called theta characteristics, or spin structure) we can even consider half-integer λ powers. For higher genus g we have a finite number of choices. But for $g = 0$ there is only one square-root, the tautological bundle U. In this presentation we ignore the half-forms (e.g. the supercase).

Next we set

$$\mathcal{F}^\lambda := \mathcal{F}^\lambda(A) := \{f \text{is a global meromorphic section of } K^\lambda \mid \text{such that } f \text{ is holomorphic over } \Sigma \setminus A\}. \tag{8}$$

These are infinite dimensional vector spaces, their elements are called meromorphic forms of weight λ. We sum over all $\lambda \in \mathbb{Z}$ (respectively $\in 1/2\mathbb{Z}$)

$$\mathcal{F} := \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{F}^\lambda. \tag{9}$$

An associative structure

$$\cdot : \mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu}. \tag{10}$$

is defined in terms of local representing meromorphic functions

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto s\,dz^\lambda \cdot t\,dz^\nu = s \cdot t\ dz^{\lambda+\nu}. \tag{11}$$

This makes $\mathcal{F}$ to an associative and commutative graded algebra. Note that $\mathcal{A} := \mathcal{F}^0$ is a subalgebra and that the $\mathcal{F}^\lambda$ are modules over $\mathcal{A}$.

Next we define a Lie algebra structure

$$\mathcal{F}^\lambda \times \mathcal{F}^\nu \to \mathcal{F}^{\lambda+\nu+1}, \qquad (s, t) \mapsto [s, t], \tag{12}$$

in local representatives of the sections as

$$(s\,dz^\lambda, t\,dz^\nu) \mapsto [s\,dz^\lambda, t\,dz^\nu] := \left((-\lambda) s \frac{dt}{dz} + \nu\, t \frac{ds}{dz}\right) dz^{\lambda+\nu+1}, \tag{13}$$

The space $\mathcal{F}$ with $[.,.]$ is a Lie algebra and with respect to $\cdot$ and $[.,.]$ it is a Poisson algebra. Obviously, $\mathcal{L} := \mathcal{F}^{-1}$ is a Lie subalgebra (the algebra of vector fields), and the $\mathcal{F}^\lambda$'s are Lie modules over $\mathcal{L}$.

The subspace $\mathcal{F}^0 \oplus \mathcal{F}^{-1} = \mathcal{A} \oplus \mathcal{L} =: \mathcal{D}^1$ is also a Lie subalgebra of $\mathcal{F}$. It is the Lie algebra of differential operators of degree ≤ 1

Finally, we define the (generalized) current algebra as follows. We fix an arbitrary finite-dimensional complex Lie algebra $\mathfrak{g}$. The generalized current algebra is defined as $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}$ with the Lie product

$$[x \otimes f, y \otimes g] = [x, y] \otimes f \cdot g, \qquad x, y \in \mathfrak{g}, \quad f, g \in \mathcal{A}. \tag{14}$$

All the above algebras consists of meromorphic objects defined over compact Riemann surfaces. We call them *Krichever–Novikov (KN) type algebras*. The classical algebras of Section 2 are obtained for the classical geometric situation (6)

5. Almost-graded structure

In the classical situation the introduced algebras are graded algebras. In the higher genus case and even in the genus zero case with more than two points where poles are allowed there is no non-trivial grading anymore. As realized by Krichever and Novikov [18] there is a weaker concept, an almost-grading, which to a large extend is a valuable replacement of a honest grading. As shown in [24] such an almost-grading is induced by a splitting of the set A into two non-empty and disjoint sets I and O.

Definition 1. Let $\mathcal{L}$ be a Lie or an associative algebra such that

$$\mathcal{L} = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n \tag{15}$$

is a vector space direct sum, then $\mathcal{L}$ is called an *almost-graded* (Lie-) algebra if

(i) $\dim \mathcal{L}_n < \infty$,
(ii) There exists constants $L_1, L_2 \in \mathbb{Z}$ such that

$$\mathcal{L}_n \cdot \mathcal{L}_m \subseteq \bigoplus_{h=n+m-L_1}^{n+m+L_2} \mathcal{L}_h, \qquad \forall n, m \in \mathbb{Z}.$$

The elements in $\mathcal{L}_n$ are called *homogeneous* elements of degree n, and $\mathcal{L}_n$ is called *homogeneous subspace* of degree n.

In a similar manner almost-graded modules over almost-graded algebras are defined. In [24], see also [31], an almost-grading for $\mathcal{F}^\lambda$ is introduced by exhibiting certain elements $f^\lambda_{n,p} \in \mathcal{F}^\lambda$, $p = 1, \ldots, K$ which constitute a basis of a subspace $\mathcal{F}^\lambda_n$ of homogeneous elements of degree n.

For the current presentation the details are not of importance. We only note that the basis element $f^\lambda_{n,p}$ of degree n fulfills

$$\operatorname{ord}_{P_i}(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta^p_i, \quad P_i \in I, \ i = 1, \ldots, K\,, \tag{16}$$

and that there are certain prescriptions at the points in O such that the element $f^\lambda_{n,p}$ is essentially unique. In the next section we will give the elements for genus zero explicitly.

But here a warning is in order: The decomposition (and hence the almost-grading) depends on the splitting of A into $I \cup O$.

6. Genus zero – standard splitting

Now we return to the genus zero case. We take the standard splitting:

$$I = \{P_1, P_2, \dots, P_{N-1}\}, \qquad O = \{\infty\}, \tag{17}$$

and have $K = N-1$. It is enough to construct a basis $\{A_{n,p}\}$ of $\mathcal{A}$, as then $\mathcal{F}^\lambda_n = \mathcal{A}_{n-\lambda} dz^\lambda$, with $f^\lambda_{n,p} = A_{n-\lambda,p} dz^\lambda$. We set for $n \in \mathbb{Z}$

$$A_{n,p}(z) := (z-a_p)^n \cdot \prod_{\substack{i=1\\ i\neq p}}^{K} (z-a_i)^{n+1} \cdot \alpha(p)^{n+1}, \quad p = 1, \dots, K\,. \tag{18}$$

Here $\alpha(p)$ is a normalization factor such that in the local coordinate $(z-a_p)$

$$A_{n,p}(z) = (z-a_p)^n (1 + O(z-a_p))\,. \tag{19}$$

The order at ∞ is automatically fixed as $-(Kn+K-1)$. For the vector fields we take

$$e_{n,p} := f^{-1}_{n,p} = A_{n+1,p} \frac{d}{dz}, \quad p = 1, \dots, K\,. \tag{20}$$

The above algebras are almost-graded algebras. In fact,

$$\mathcal{F}^\lambda = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^\lambda_m, \qquad \text{with} \quad \dim \mathcal{F}^\lambda_m = K, \tag{21}$$

and there exist R_1, R_2 (independent of n and m) such that

$$\mathcal{A}_n \cdot A_m \subseteq \bigoplus_{h=n+m}^{n+m+R_1} \mathcal{A}_h\ , \qquad [\mathcal{L}_n, \mathcal{L}_m] \subseteq \bigoplus_{h=n+m}^{n+m+R_2} \mathcal{L}_h\ . \tag{22}$$

For genus zero and the standard splitting we have

$$R_1 = \begin{cases} 0, & N = 2, \\ 1, & N > 2, \end{cases} \qquad\qquad R_2 = \begin{cases} 0, & N = 2, \\ 1, & N = 3, \\ 2, & N > 3\,. \end{cases} \tag{23}$$

An important consequence of the almost-grading (not only in genus zero) is the existence of a triangular decomposition $\mathcal{U} = \mathcal{U}_{[-]} \oplus \mathcal{U}_{[0]} \oplus \mathcal{U}_{[+]}$ with

$$\mathcal{U}_{[+]} := \bigoplus_{m>0} \mathcal{U}_m, \quad \mathcal{U}_{[0]} = \bigoplus_{m=-R_i}^{m=0} \mathcal{U}_m, \quad \mathcal{U}_{[-]} := \bigoplus_{m<-R_i} \mathcal{U}_m. \tag{24}$$

Here $\mathcal{U}_{[+]}$ and $\mathcal{U}_{[-]}$ are subalgebras, whereas $\mathcal{U}_{[0]}$ is only a subspace. Such a triangular decomposition is of relevance for the construction of highest weight representation.

Another basis. Our algebra $\mathcal{A}$ can also be given as the algebra

$$\mathcal{A} = \mathbb{C}[(z-a_1), (z-a_1)^{-1}, (z-a_2)^{-1}, \ldots, (z-a_{N-1})^{-1}], \qquad (25)$$

with the obvious relations. If we set $A_n^{(i)} := (z-a_i)^n$, then

$$A_n^{(i)}, \quad n \in \mathbb{Z},\ i = 1, \ldots, N-1 \qquad (26)$$

is a generating set of $\mathcal{A}$. A basis is given e.g. by

$$A_n^{(1)},\ n \in \mathbb{Z}, \quad A_{-n}^{(i)},\ n \in \mathbb{N},\ i = 2, \ldots, N-1. \qquad (27)$$

But this basis does not define an almost-graded structure.

7. Central extensions

Next we want to introduce central extensions of our algebras. The following is also valid in arbitrary genus.

Let C_i be positively oriented (deformed) circles on the Riemann surface Σ_g around the points P_i in I, $i = 1, \ldots, K$, and C_j^* positively oriented circles around the points Q_j in O, $j = 1, \ldots, M$. A cycle C_S on Σ_g is called a separating cycle if it is smooth, positively oriented of multiplicity one and if it separates the in-points from the out-points.

In the following we will integrate over closed curves C meromorphic differentials on Σ_g without poles in $\Sigma_g \setminus A$. In this context integration over C and C' give the same value if $[C] = [C']$ in $\mathrm{H}_1(\Sigma_g \setminus A, \mathbb{Z})$. Moreover,

$$[C_S] = \sum_{i=1}^{K} [C_i] = -\sum_{j=1}^{M} [C_j^*] \qquad (28)$$

Given such a separating cycle C_S (respectively cycle class) we define the linear form

$$\mathcal{F}^1 \to \mathbb{C}, \qquad \omega \mapsto \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega\,. \qquad (29)$$

This integration corresponds to calculating residues

$$\omega \quad \mapsto \quad \frac{1}{2\pi \mathrm{i}} \int_{C_S} \omega \;=\; \sum_{i=1}^{K} \mathrm{res}_{P_i}(\omega) \;=\; -\sum_{l=1}^{M} \mathrm{res}_{Q_l}(\omega). \qquad (30)$$

A central extension of a Lie algebra $\mathcal{U}$ is defined on the vector space direct sum $\widehat{\mathcal{U}} = \mathbb{C} \oplus U$ ($\hat{x} := (0, x)$, $t := (1, 0)$)

$$[\hat{x}, \hat{y}] = \widehat{[x, y]} + \Phi(x, y) \cdot t, \quad [t, \widehat{U}] = 0, \quad x, y \in U, \qquad (31)$$

with a bilinear form Φ on $\mathcal{U}$. Recall that $\widehat{U}$ will be a Lie algebra, if and only if Φ is antisymmetric and fulfills the Lie algebra 2-cocycle condition for all $x, y, z \in \mathcal{U}$

$$0 = d_2\Phi(x, y, z) := \Phi([x, y], z) + \Phi([y, z], x) + \Phi([z, x], y). \qquad (32)$$

A 2-cocycles Φ is a coboundary if there exists a $\phi : \mathcal{U} \to \mathbb{C}$ such that

$$\Phi(x, y) = d_1\phi(x, y) = \phi([x, y]). \qquad (33)$$

It is well-known that the second Lie algebra cohomology $\mathrm{H}^2(\mathcal{U},\mathbb{C})$ of $\mathcal{U}$ with values in the trivial module $\mathbb{C}$ classifies equivalence classes of central extensions.

A Lie algebra $\mathcal{U}$ is called perfect if $[\mathcal{U},\mathcal{U}] = \mathcal{U}$. Perfect Lie algebras admit universal central extensions.

8. Local and bounded cocycles

In the previous section we considered all central extensions. Now we are heading towards central extensions which are "compatible" with the almost-grading.

Definition 2. (a) Let γ be a 2-cocycle for the almost-graded Lie algebra $\mathcal{U}$, then γ is called a *local cocycle* if $\exists T_1, T_2$ such that

$$\gamma(\mathcal{U}_n,\mathcal{U}_m) \neq 0 \implies T_2 \leq n+m \leq T_1. \tag{34}$$

(b) A 2-cocycle γ is called *bounded* (from above) if $\exists T_1$ such that

$$\gamma(\mathcal{U}_n,\mathcal{U}_m) \neq 0 \implies n+m \leq T_1. \tag{35}$$

(c) A cocycle class $[\gamma]$ is called a *local (bounded) cohomology class* if and only if it admits a representing cocycle which is local (respectively bounded).

Note that e.g. in a local cocycle class not all representing cocycles are local. Obviously, the set of local (or bounded) cocycles is a subspace of all cocycles. Moreover, the set $\mathrm{H}^2_{loc}(\mathcal{U},\mathbb{C})$ (respectively $\mathrm{H}^2_b(\mathcal{U},\mathbb{C})$) of local (respectively bounded) cohomology classes is a subspace of the full cohomology space.

In [27] and [28] I classified all bounded and local cocycles for the KN type algebras.

A cocycle $\gamma : \mathcal{U}\times\mathcal{U} \to \mathbb{C}$ is called a *geometric cocycle* if there is a bilinear map $\widehat{\gamma} : \mathcal{U}\times\mathcal{U} \to \mathcal{F}^1$, such that γ is the composition of $\widehat{\gamma}$ with an integration, i.e.

$$\gamma = \gamma_C := \frac{1}{2\pi\mathrm{i}}\int_C \widehat{\gamma} \tag{36}$$

with C a curve on $\Sigma_g \setminus A$.

Given $\widehat{\gamma}$ only the class of C in $\mathrm{H}_1(\Sigma_g\setminus A,\mathbb{C})$ plays a role. Recall that

$$\dim \mathrm{H}_1(\Sigma_g\setminus A,\mathbb{C}) = \begin{cases} 2g, & \#A = N = 0,1,\\ 2g+(N-1), & \#A = N \geq 2\,. \end{cases} \tag{37}$$

In particular, for genus zero and $N \geq 1$ we have

$$\dim \mathrm{H}_1(\Sigma_0\setminus A,\mathbb{C}) = (N-1). \tag{38}$$

In this case a basis is e.g. given by circles C_i around the points P_i, where we leave out one of them. For example $[C_i]$, $i=1,\ldots,N-1$ will do. But there might be a more convenient choice, e.g. for the standard splitting we take $[C_S] = -[C_\infty]$ and $[C_i]$, $i=1,\ldots,N-2$.

9. Main results

The results presented in this section are valid for genus zero and the multi-point situation. In this situation the algebras are sometimes called N-Virasoro algebra or N-point γ-algebras.

The results presented here (and some more) are obtained in [32]. There also the proofs can be found. Here I only give the results and the basic strategy employed.

1. We show that all cocycle classes are bounded cocycle classes with respect to the almost-grading induced by the standard splitting,

2. Next, the classification result of bounded cocycle classes [27], [28] of the author is used which gives a complete classification and explicit expressions by integrals over curves

3. In particular, in genus zero our cocycles classes are geometric cocycles classes with respect to certain explicitly given one-forms

4. In genus zero the geometric cocycles can be obtained via integration over circles around the points in I, or alternatively around ∞ and hence can be calculate via residues

5. In case that the Lie algebra is perfect the universal central extension can directly be given.

9.1. Function algebra – Heisenberg algebra

The function algebra is abelian, hence there are too many Lie algebra cocycles. For the above classification we have to restrict ourselves to the following naturally be given cocycle classes

- A cocycle γ is called $\mathcal{L}$-invariant if and only if

$$\gamma(e\,.\,f,g)+\gamma(f,e\,.\,g)=0,\qquad f,g\in\mathcal{A},\quad e\in\mathcal{L}. \tag{39}$$

- A cocycle γ is called multiplicative if

$$\gamma(fg,h)+\gamma(gh,f)+\gamma(hf,g)=0,\qquad f,g,h\in\mathcal{A}. \tag{40}$$

Theorem 3. [32] *If one of the above properties is fulfilled then it is a geometric cocycle. It is linear combination of*

$$\gamma_i^{\mathcal{A}}(f,g)=\frac{1}{2\pi \mathrm{i}}\int_{C_i} fdg=\mathrm{res}_{a_i}(fdg),\quad i=1,\ldots,N-1. \tag{41}$$

The cocycle is bounded from above with respect to the almost-grading given by the standard splitting.

As one can show that the cocycles of the type (41) are both $\mathcal{L}$-invariant and multiplicative, we obtain that every $\mathcal{L}$-invariant cocycle is multiplicative and vice versa.

The unique cocycle (up to scaling) of this type which is local with respect to the standard splitting is obtained as the sum of the $\gamma_i^{\mathcal{A}}$, $i=1,\ldots,N-1$, or alternatively as $\gamma_\infty^{\mathcal{A}}$ [27].

In the two point situation we obtain

$$\gamma(A_n,A_m)=\alpha\cdot(-n)\cdot\delta_m^{-n}. \tag{42}$$

The Heisenberg algebra is a central extension of the function algebra obtained via such a cocycle. This could be either the local one or the "full" one (depending on the convention one is using). For the full one the center is $(N-1)$-dimensional. Of course, the function algebra does not have a universal central extension, but the full Heisenberg algebra might be some kind of substitute.

9.2. Vector field algebra

Theorem 4. [32] *Every cocycle class is geometric and given by*

$$\gamma^{\mathcal{L}}_{C,R}(e,f) = \frac{1}{2\pi\mathrm{i}}\int_C (\frac{1}{2}(ef''' - e'''f) - R(ef' - e'f)dz, \tag{43}$$

where R is a projective connection.

We do no repeat the definition of a projective connection here, as for our coordinates we can take $R = 0$. The strategy explained above yields that $\mathrm{H}^2(\mathcal{L},\mathbb{C})$ is $(N-1)$-dimensional and is generated by integrating (43) over the C_i. Furthermore, these cocycles generate the universal central extension.

By different techniques Skryabin [33] has shown the existence of a universal central extension for arbitrary genus.

9.3. Differential operator algebra

Also here the main result is that all cocycle classes are geometric. The $\mathcal{L}$-invariant cocycles for $\mathcal{A}$ and arbitrary cocycles for $\mathcal{L}$ define two cocycle types for $\mathcal{D}^1$. But there is a another type, called mixing cocycles

$$\gamma^{(m)}_{C,T}(e,g) := \frac{1}{2\pi\mathrm{i}}\int_C (eg'' + Teg')dz, \qquad e \in \mathcal{L}, g \in \mathcal{A}, \tag{44}$$

Here T is an affine connection. As it can be taken to be zero on the affine part we do not repeat its definition here.

Theorem 5. [32] *All cocycle classes are geometric and are linear combinations of the introduced three types. The Lie algebra $\mathcal{D}^1$ is perfect and the universal central extension has a $3\cdot(N-1)$ dimensional center.*

9.4. Current algebra

Recall that the current algebra $\overline{\mathfrak{g}}$ is defined with respect to a finite dimensional Lie algebra $\mathfrak{g}$. For the classification results we assume that $\mathfrak{g}$ is simple. Let β be the Cartan–Killing form, then we show [32] that all cocycles are geometric and cohomologous to (with C an arbitrary curve)

$$\gamma^{\overline{\mathfrak{g}}}_{\beta,C}(x\otimes f, y\otimes g) = \beta(x,y)\cdot\gamma^{\mathcal{A}}_C(f,g) = \beta(x,y)\cdot\frac{1}{2\pi\mathrm{i}}\int_C f dg\,. \tag{45}$$

As $\overline{\mathfrak{g}}$ is perfect, it admits a universal central extension which has a $(N-1)$-dimensional center which can be explicitly given. If we consider only central extensions which admit an extension of the almost-grading (e.g. with respect to the standard splitting) we obtain that this central extension class is unique [28].

The author has also corresponding results for the general reductive case. Furthermore, the superalgebra could be treated in the same manner [29].

10. Three-point algebras

The case of only three points where poles are allowed is to a certain extend special as we have additional symmetries. These symmetries can be used to simplify the calculations of structure constants even further. Additionally, the three-point case plays a role in quite a number of applications. See e.g. the tetrahedron algebra appearing in statistical mechanics, in particular the work of Terwilliger and collaborators [13], [2], [14]. See also Kazhdan and Lusztig [17]. For applications to deformations of Lie algebras see also [26], [10], [11], [12].

By a fractional linear transformation, respectively by a PGL(2) action, the three points can be brought to the points $0, 1$ and ∞. After having fixed $A = \{0, 1, \infty\}$, by applying an automorphism from the remaining symmetry group, we obtain the situation

$$A = I \cup O, \qquad I := \{0, 1\}, \quad \text{and} \quad O := \{\infty\}. \tag{46}$$

This we will consider here. We will "symmetrize" and "anti-symmetrize" our basis elements (18)

$$A_n(z) := z^n (z-1)^n, \quad B_n(z) := z^n (z-1)^n (2z-1)\,. \tag{47}$$

The structure equations read as

$$\begin{aligned} A_n \cdot A_m &= A_{n+m}, \\ A_n \cdot B_m &= B_{n+m}, \\ B_n \cdot B_m &= A_{n+m} + 4A_{n+m+1}. \end{aligned} \tag{48}$$

The space of cocycles is two-dimensional. First we take the residues around ∞ and get for the cocycle values

$$\begin{aligned} \gamma_\infty^{\mathcal{A}}(A_n, A_m) &= 2n\, \delta_m^{-n}, \\ \gamma_\infty^{\mathcal{A}}(A_n, B_m) &= 0, \\ \gamma_\infty^{\mathcal{A}}(B_n, B_m) &= 2n\delta_m^{-n} + 4(2n+1)\, \delta_m^{-n-1}\,. \end{aligned} \tag{49}$$

Then around 0 and get

$$\begin{aligned} \gamma_0^{\mathcal{A}}(A_n, A_m) &= -n\, \delta_m^{-n}, \\ \gamma_0^{\mathcal{A}}(A_n, B_m) &= n\, \delta_m^{-n} + 2n\, \delta_m^{-n-1} \\ &\qquad + \sum_{k=2}^{\infty} n\, (-1)^{k-1} 2^k \frac{(2k-3)!!}{k!} \delta_m^{-n-k}\,, \\ \gamma_0^{\mathcal{A}}(B_n, B_m) &= -n\delta_m^{-n} - 2(2n+1)\, \delta_m^{-n-1}\,. \end{aligned} \tag{50}$$

We see that the cocycle $\gamma_\infty^{\mathcal{A}}$ is local but $\gamma_0^{\mathcal{A}}$ is not. This is in accordance to the uniqueness result of [27].

Next we come to the vector field algebra. The basis is given by

$$e_n := A_{n+1}\frac{d}{dz}, \qquad f_n := B_{n+1}\frac{d}{dz}, \quad n \in \mathbb{Z}. \tag{51}$$

This yields the structure equations

$$\begin{aligned} [e_n, e_m] &= (m-n)\, f_{m+n}, \\ [e_n, f_m] &= (m-n)\, e_{m+n} + (4(m-n)+2)\, e_{n+m+1}, \\ [f_n, f_m] &= (m-n)\, f_{m+n} + 4(m-n)\, f_{n+m+1}. \end{aligned} \tag{52}$$

Its universal central extension is two-dimensional, and as above obtained by calculating residues of (43) at ∞ and 0:

$$\begin{aligned} \gamma_0^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_0(e\cdot f''' - f\cdot e''')dz \\ \gamma_\infty^{\mathcal{L}}(e,f) &= 1/2\,\mathrm{res}_\infty(e\cdot f''' - f\cdot e'''). \end{aligned} \tag{53}$$

We get at ∞:

$$\begin{aligned} \gamma_\infty^{\mathcal{L}}(e_n, e_m) &= 2(n^3-n)\,\delta_m^{-n} + 4n(n+1)(2n+1)\delta_m^{-n-1} \\ \gamma_\infty^{\mathcal{L}}(e_n, f_m) &= 0, \\ \gamma_\infty^{\mathcal{L}}(f_n, f_m) &= 2(n^3-n)\,\delta_m^{-n} + 8n(n+1)(2n+1)\delta_m^{-n-1} \\ &\qquad + 8(n+1)(2n+1)(2n+3)\delta_m^{-n-2}, \end{aligned} \tag{54}$$

and at 0:

$$\begin{aligned} \gamma_0^{\mathcal{L}}(e_n, e_m) &= -(n^3-n)\,\delta_n^{-m} - 2n(n+1)(2n+1)\delta_m^{-n-1} \\ \gamma_0^{\mathcal{L}}(e_n, f_m) &= (n^3-n)\,\delta_m^{-n} + 6n^2(n+1)\delta_m^{-n-1} + 6n(n+1)^2\delta_m^{-n-2} \\ &\qquad + \sum_{k\ge 3} n(n+1)(n+k-1)(-1)^k 2^k\cdot 3\cdot\frac{(2k-5)!!}{k!}\delta_m^{-n-k} \\ \gamma_0^{\mathcal{L}}(f_n, f_m) &= -(n^3-n)\,\delta_m^{-n} - 4n(n+1)(2n+1)\delta_m^{-n-1} \\ &\qquad - 4(n+1)(2n+1)(2n+3)\delta_m^{-n-2}. \end{aligned} \tag{55}$$

In accordance with the results in [27] $\gamma_\infty^{\mathcal{L}}$ is local, but $\gamma_0^{\mathcal{L}}$ is not.

The principal picture should be clear now. For the corresponding results for the differential operator algebra and the Lie superalgebra I refer to [32]. Also there (in Appendix B.), the universal central extension for the $\mathfrak{sl}(2,\mathbb{C})$ current algebra is given.

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