Reflecting Diffusion Semigroup on Manifolds carrying Geometric Flow

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Abstract. Let $L_t := \Delta_t + Z_t$ for a $C^{1,1}$-vector field Z on a differentiable manifold M with boundary ∂M, where Δ_t is the Laplacian operator, induced by a time dependent metric g_t differentiable in $t \in [0, T_c)$. We first establish the derivative formula for the associated reflecting diffusion semigroup generated by L_t; then construct the couplings for the reflecting L_t-diffusion processes by parallel displacement and reflection, which are applied to gradient estimates and Harnack inequalities of the associated heat semigroup; and finally, by using the derivative formula, we present a number of equivalent inequalities for a new curvature lower bound and the convexity of the boundary, including the gradient estimates, Harnack inequalities, transportation-cost inequalities and other functional inequalities for diffusion semigroups.

1 Introduction and main results

It is well known that, functional inequalities, for instance, gradient inequalities and dimension-free Harnack inequalities, are useful tools on stochastic analysis to investigate the behavior of the underlying processes on Riemannian manifolds, see, for example, [9, 10, 14, 18, 36]. Among all those work, one usually make the assumption that the metric is fixed. However, when it comes to the case that metric is time-varying, a question arises naturally: how about functional inequalities on these manifolds? In recent year, M. Arnaudon, K. Coulibaly and A. Thalmaier [1] constructed g_t-Brownian motions (i.e., the diffusion process generated by $L_t = \frac{1}{2}\Delta_t$) on manifolds without boundary carrying a geometric flow, and established the Bismut formula under the Ricci flow, which in particular implies the gradient estimates of the associated heat semigroup. In [12], the first author studied functional inequalities, including on manifolds carrying geometric flow for the diffusion semigroup. Motivated by the aforementioned results, this article aim to extends these results in [1, 12] to the case with boundary.

The setting for our work is a differentiable manifold with boundary equipped with a geometric flow. More precisely, let M be a d-dimensional differentiable manifold with boundary ∂M, which carries a one-parameter $C^{1,\infty}$-family of complete Riemannian metrics $\{g_t\}_{t\in[0,T_c)}$, where T_c is the time when the curvature may blow up. Consider the elliptic operator $L_t := \Delta_t + Z_t$, where Δ_t is the Laplacian operator associated with the metric g_t and $(Z_t)_{t\in[0,T_c)}$ is a $C^{1,\infty}$-family of vector fields. Let (X_t) be a reflecting diffusion process generated by L_t (called the reflecting L_t-diffusion process), which is assumed to be non-explosive. This assumption immediately implies that this process then corresponds in a natural way to a strongly continuous semigroup $P_{s,t}$, i.e.,

$$P_{s,t} f(x) = \mathbb{E}(f(X_t)|X_s = x), \quad 0 \le s \le t < T_c.$$

2010 *Mathematics Subject Classification*. Primary 58J65; Secondary 60J60, 53C44.

Key words and phrases. Geometric flow, Ricci flow, curvature, second fundamental form, transportation-cost inequality, Harnack inequality, coupling.

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In this article, we fix on extending the former discussions to our content for the semigroup $P_{s,t}$. Compared with F.-Y. Wang's work on functional inequalities over Riemannian manifolds with boundary (see for example [31, 32, 33, 34, 35, 36] and the reference therein), we need make some necessary modifications to our inhomogeneous context, since e.g. geometric quantities are time-dependent and the underlying process is time-inhomogeneous.

Before moving on, let us briefly recall some known results in the time-inhomogeneous Riemannian setting without boundary. K. Coulibaly [2] investigated the optimal transportation inequality by constructing horizontal diffusion processes. Then, K. Kuwada and R. Philipowski [22] studied the non-explosion of g_t-Brownian motions under the super Ricci flow, and K. Kuwada [23] developed the coupling method to estimate the gradient of the semigroup. Very recently, the author [12] has considered the construction of coupling processes and some important functional inequalities on manifolds without boundary carrying a geometric flow. All those works lay solid foundation for our study.

Let ∇^t be the Levi-Civita connection associated with the metric g_t. For simplicity, we introduce the notation: for $X,Y \in TM$,

$$\mathscr{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t - \frac{1}{2}\partial_t g_t(X,Y),$$

where Ric_t is the Ricci curvature tensor with respect to the metric g_t, and $\langle \cdot,\cdot \rangle_t := g_t(\cdot,\cdot)$. Define the second fundamental form of the boundary with respect to g_t by

$$\mathrm{II}_t(X,Y) = -\left\langle \nabla_X^t N_t, Y \right\rangle_t, \;\; X,Y \in T\partial M,$$

where N_t is the inward unit normal vector field of the boundary associated with the metric g_t. If $\mathrm{II}_t \geq 0$ for all $t \in [0,T_c)$, then the geometric flow $\{g_t\}_{t\in[0,T_c)}$ is called to be convex. In fixed metric case, functional inequalities are always deduced under the Bakry-Emery curvature condition. In this paper, we begin our discussion by using the following curvature constraints:

$$\mathscr{R}_t^Z \geq K(t,\cdot) \quad \text{and} \quad \mathrm{II}_t \geq \sigma(t,\cdot) \tag{1.1}$$

for some continuous functions $K,\sigma \in C([0,T_c)\times M)$. Here and in what follows, for any two-tensor $\mathbf{T}_t$ and any function f, we write $\mathbf{T}_t \geq f$ if $\mathbf{T}_t(X,X) \geq f\langle X,X\rangle_t$, for $X \in TM$ and $t \in [0,T_c)$. Compared with the usual Bakry-Emery curvature condition, the time derivative about the metric will become a new important term involved in the curvature condition.

Let ρ_t be the Riemannian distance and $|\cdot|_t$ be the norm associated with the metric g_t. When the geometric flow is convex, we have the first main result of this paper.

Theorem 1.1. *For any $K \in C([0,T_c))$, the following statements are equivalent to each other.*

(i) *The following curvature condition holds,*

$$\mathscr{R}_t^Z \geq K(t) \;\; \textit{and} \;\; \mathrm{II}_t \geq 0 \; (\partial M \neq \varnothing) \;\; \textit{for all} \;\; t \in [0,T_c). \tag{1.2}$$

(ii) *The gradient inequality*

$$|\nabla^s P_{s,t} f|_s \leq e^{-\int_s^t K(r)\mathrm{d}r} P_{s,t}|\nabla^t f|_t, \; 0 \leq s \leq t < T_c \tag{1.3}$$

holds for $f \in C^1(M)$ such that f is constant outside a compact set of M.

(*iii*) *For any* $p>1$, $0\le s<t<T_c$ *and* $f\in\mathscr{B}_b^+(M)$,

$$(P_{s,t}f)^p(x)\le P_{s,t}f^p(y)\exp\left[\frac{p}{4(p-1)}\left(\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right)^{-1}\rho_s^2(x,y)\right]. \tag{1.4}$$

We intend to use the coupling method to prove that (1.2) implies (1.3) and (1.4). It is well known that coupling method is a useful tool in stochastic analysis. It is remarkable that M.-F. Chen and F.-Y. Wang [9, 10, 11] gave subtle estimates about the first eigenvalue on Riemannian manifolds by constructing suitable coupling processes. Note that K. Kuwada [23] first constructed the coupling processes for L_t-diffusion processes on manifolds without boundary via discrete approximation. In our recent work [12], we gave a direct construction for general coupling processes on manifolds without boundary. Here, we modify this proof to our setting.

On the other hand, to prove that each of (1.3) and (1.4) implies the curvature condition (1.2), we need to use the derivative formula to characterize $\mathscr{R}_t^Z$ and II_t first. In the following section, we construct a series of Hsu's multiplicative functionals to establish the derivative formula (see Theorem 2.3 below). When the metric is independent of t, our construction is due to [36, Theorem 3.2.1] for the constant metric case.

In fact, it is more difficult for us to deal with the case carrying the non-convex flow, since it is hard to control the effect from the boundary by using the coupling method. A direct thought is to make a conformal change of the metrics such that the new flow becomes convex. When the metric is independent of t, this method is successfully applied to the non-convex manifold, see [30, 33, 35]. First, let us introduce an important set:

$$\mathscr{D}=\{\phi\in C^{1,\infty}([0,T_c)\times M):\inf\phi_t=1,\ \mathrm{II}_t\ge -N_t\log\phi_t\}. \tag{1.5}$$

Then, by [30, Lemma 2.1], for $\phi\in\mathscr{D}$, the new flow $\tilde g_t:=\phi_t^{-2}g_t$ is convex. Moreover, we are required to having the following assumption on ϕ, Ric_t^Z, and $\partial_t g_t$ to continue our discussion.

(H1) Let $d\ge 2$. There exist functions $K_1,K_2\in C([0,T_c))$ such that

$$\mathrm{Ric}_t^Z:=\mathrm{Ric}_t-\nabla^t Z_t\ge K_1(t),\qquad \partial_t g_t\le K_2(t), \tag{1.6}$$

and $\phi\in\mathscr{D}$ such that $\|\nabla^t\phi_t\|_\infty<\infty$, $\|\phi_t\|_\infty<\infty$ and

$$\begin{aligned}K_{\phi,1}(t):&=\inf_M\left\{\phi_t^2K_1(t)+\frac12 L_t\phi_t^2-|\nabla^t\phi_t^2|_t\cdot|Z_t|_t-(d-2)|\nabla^t\phi_t|_t^2\right\}>-\infty,\\ K_{\phi,2}(t):&=\sup_M\{2\partial_t\log\phi_t\}+K_2(t)<\infty,\end{aligned}$$

where $\|\nabla^t f\|_\infty:=\sup_{x\in M}|\nabla^t f|_t(x)$.

If this assumption holds, then by constructing suitable coupling processes, we have the second main result of this paper.

Theorem 1.2. *Suppose that* **(H1)** *holds and*

$$K_\phi(t):=K_{\phi,1}^-(t)+\frac12K_{\phi,2}(t)+[2\|\phi_tZ_t+(d-2)\nabla^t\phi_t\|_\infty+d\|\nabla^t\phi_t\|_\infty]\|\nabla^t\phi_t\|_\infty<\infty.$$

Then the following conclusions hold.

(*i*) *For any* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$|\nabla^s P_{s,t} f|_s \le \|\phi_t\|_\infty \|\nabla^t f\|_\infty e^{\int_s^t K_\phi(r)\mathrm{d}r}, \ \ 0 \le s \le t < T_c.$$

(*ii*) *For* $0 \le s < t < T_c$*, let* $\delta_{s,t} = 1 - \sup_{r\in[s,t]} \|\phi_r\|_\infty^{-1}$*,* $\lambda_{s,t} = \inf_{(r,x)\in[s,t]\times M} \phi^{-1}$ *and*

$$\delta_p = \max\left\{\delta_{s,t}, \frac{\lambda_{s,t}}{2}(\sqrt{p}-1)\right\}.$$

Then for $p > (1+\frac{\delta_{s,t}}{\lambda_{s,t}})^2$*,* $x, y \in M$ *and* $f \in C_b(M)$*, it holds*

$$(P_{s,t} f(y))^p \le P_{s,t} f^p(x) \exp\left\{\frac{\sqrt{p}(\sqrt{p}-1)\rho_s(x,y)}{8\delta_p[(\sqrt{p}-1)\lambda_{s,t} - \delta_p]\int_s^t e^{-2\int_s^r (K_\phi(u)+\|\nabla^u\phi_u\|_\infty^2)\mathrm{d}u}\mathrm{d}r}\right\}.$$

As an important application of the induced conclusions above for general geometric flow, we consider the Ricci flow with umblic boundary as follows: for $\lambda \ge 0$,

$$\begin{cases} \frac{\partial}{\partial t} g(x,\cdot)(t) = 2\mathrm{Ric}(x,t), & (x,t) \in M\times[0,T]; \\ \\ \mathrm{II}(x,t) = \lambda g(x,t), & x \in \partial M. \end{cases} \tag{1.7}$$

Shen [27] proved the short time existence of the solution to the above equation. We also refer the reader to [3] for more geometric explanation for this Ricci flow. To our knowledge, there are few references about gradient estimate and Harnack inequalities for the solution to the heat equation under the Ricci flow carrying non-convex umbilic boundary. In Section 3.3, we will apply Theorems 1.1 and 1.2 to establish these inequalities for this system; see Theorems 3.6 and 3.8 below.

The rest parts of the paper are organized as follows. In Section 2, we construct the reflecting L_t-diffusion processes, prove the Kolmogorov equations and then establish the derivative formula for the associated semigroup. In Sections 3, we turn to prove Theorems 1.1 and 1.2 by constructing the coupling processes, which are applied to the Ricci flow with umbilic boundary. In Section 4, some important inequalities including transportation-cost inequality, Harnack inequalities and other functional inequalities are proved to be equivalent to the lower bound of $\mathscr{R}_t^Z$ and the convexity of the boundary.

We end this section by making some conventions on the notations. Let $\mathscr{B}_b(M)$ be the set of all measurable functions and $C_0^p(M)$ the set of all C^p-smooth real functions with compact supports on M. For any function f and φ respectively defined on $[0,T_c)\times M$ and $[0,T_c)\times M\times M$, we simply write $f_t(x) := f(t,x)$ and $\varphi_t(x,y) := \varphi(t,x,y)$, $t \in [0,T_c), x, y \in M$. In addition, $\|f_t\|_\infty := \sup_{x\in M} f(t,x)$ and $\|f\|_\infty = \sup_{(t,x)\in[0,T_c)\times M} f(t,x)$. For any time-depending vector field V_t, we write $\|V_t\|_\infty := \||V_t|_t\|_\infty$ for simplicity.

2 Preliminaries

In Subsection 2.1, we briefly introduce the construction of reflecting L_t-diffusion processes. In Subsection 2.2, the forward and backward Kolmogorov equations are established for Neumann diffusion semigroup. In Subsection 2.3, a derivative formula is established, which is further applied to characterizing $\mathscr{R}_t^Z$ and II_t.

2.1 Reflecting L_t-diffusion processes

Let $\mathscr{F}(M)$ be the frame bundle over M and $\mathscr{O}_t(M)$ the orthonormal frame bundle over M with respect to the metric g_t. Set $\mathbf{p}: \mathscr{F}(M) \to M$ be the projection from $\mathscr{F}(M)$ onto M. Let $\{e_i\}_{i=1}^d$ be the canonical orthonormal basis of $\mathbb{R}^d$. For any $u \in \mathscr{O}_t(M)$, let $H_X^t(u)$ be the ∇^t-horizontal lift of $X \in T_{\mathbf{p}u}M$ and $H_i^t(u) = H_{ue_i}^t(u), i = 1, 2, \cdots, d$. For any $u \in \mathscr{F}(M)$, let $\{V_{\alpha,\beta}(u)\}_{\alpha,\beta=1}^d$ be the canonical basis of vertical fields over $\mathscr{F}(M)$.

Let $B_t := (B_t^1, B_t^2, \cdots, B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. As in the time-homogeneous case, to construct the reflecting L_t-diffusion process, we first construct the corresponding horizontal diffusion process by solving the Stratonovich stochastic diffusion equation (SDE):

$$\begin{cases} \mathrm{d}u_t = \sqrt{2}\sum_{i=1}^d H_i^t(u_t) \circ \mathrm{d}B_t^i + H_{Z_t}^t(u_t)\mathrm{d}t - \frac{1}{2}\sum_{i,j} \partial_t g_t(u_t e_i, u_t e_j) V_{i,j}(u_t)\mathrm{d}t + H_{N_t}^t(u_t)\mathrm{d}l_t, \\ u_0 \in \mathscr{O}_0(M),\ \mathbf{p}u_0 = x \in M, \end{cases}$$

where l_t is an increasing process supported on $\{t \in [0, \zeta) : X_t := \mathbf{p}u_t \in \partial M\}$, where $\zeta := \lim_{n\to\infty} \zeta_n$ and

$$\zeta_n := \inf\{t \in [0, T_c) : \rho_t(\mathbf{p}u_0, \mathbf{p}u_t) \geq n\},\ n \geq 1,\ \inf \varnothing = T_c.$$

Similarly as explained in [1], the last term is essential to ensure $u_t \in \mathscr{O}_t(M)$. Then, it is easy to see that $X_t := \mathbf{p}u_t$ solves the equation

$$\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t,\ X_0 = x$$

up to the life time ζ. By the Itô formula, for any $f \in C_0^{1,2}([0, T_c) \times M)$ with $N_t f_t := N_t f_t|_{\partial M} = 0$,

$$f(t, X_t) - f(0, x) - \int_0^t (\partial_s + L_s) f(s, X_s)\mathrm{d}s = \sqrt{2}\int_0^t \left\langle u_s^{-1}\nabla^s f(s, \cdot)(X_s), \mathrm{d}B_s\right\rangle_s$$

is a martingale up to the life time ζ. So, we call X_t the reflecting diffusion process generated by L_t.

Throughout this paper, we only consider the case where the reflecting L_t-diffusion process is non-explosive before T_c. In this case,

$$P_{s,t} f(x) := \mathbb{E}(f(X_t)|X_s = x),\ x \in M,\ 0 \leq s \leq t < T_c,\ f \in \mathscr{B}_b(M)$$

gives rise to a Markov semigroup $\{P_{s,t}\}_{0\leq s\leq t<T_c}$ on $\mathscr{B}_b(M)$, which is called the Neumann semigroup generated by L_t. Here and in what follows, $\mathbb{E}$ and $\mathbb{P}$ (resp. $\mathbb{E}^x$ and $\mathbb{P}^x$) stand for the expectation and probability taken for the underlying process (resp. the underlying process starting from $x \in M$).

2.2 Kolmogorov equations

Let

$$\mathscr{C}_N(L) = \{f \in C^{1,\infty}([0, T_c) \times M), N_t f_t|_{\partial M} = 0, (L_t + \partial_t) f \in \mathscr{B}_b(M), t \in [0, T_c)\}.$$

In this subsection, we now introduce the Kolmogorov equations for $P_{s,t}$ as follows.

Theorem 2.1. *For $f \in \mathscr{C}_N(L)$, the following forward Kolmogorov equation holds,*

$$\frac{\partial}{\partial t} P_{s,t} f(t,x) = P_{s,t}\left(L_t f + \partial_t f\right)(t,x), \quad 0 \le s < t < T_c. \tag{2.1}$$

Moreover, for $f \in \mathscr{B}_b(M)$, there hold

(i) for any $0 \le t < T_c$, $P_{\cdot,t} f \in C^{1,2}([0,t]\times M)$ and the backward Kolmogorov equation

$$\frac{\partial}{\partial s} P_{s,t} f = -L_s P_{s,t} f, \quad\quad 0 \le s < t < T_c, \tag{2.2}$$

moreover,

$$N_s P_{s,t} f = 0, \quad\quad 0 \le s < t < T_c;$$

(ii) if $|\nabla^{\cdot} P_{\cdot,t} f|_{\cdot}$ is bounded on $[r,t]\times M$ and $t \in (0,T_c]$, then

$$\frac{\partial}{\partial s} P_{r,s} \psi(P_{s,t} f) = P_{r,s}\left(\psi''(P_{s,t} f)|\nabla^s P_{s,t} f|_s^2\right), \;\; s \in [r,t],$$

where $\psi \in C^2(\mathbb{R})$ with compact support in $[\inf f, \sup f]$.

Proof. By using the Itô formula, the equality (2.1) follows directly. Moreover, (ii) can be calculated by combining (2.1) and (2.2). Thus it suffices for us to prove (i).

We first show that there exists a solution u to the following equation: for $0 \le s \le t < T_c$,

$$\begin{cases} \frac{\partial}{\partial s} u(\cdot,x)(s) = -L_s u(s,\cdot)(x), & x \in M; \\ N_s u(s,\cdot)(x) = 0, & x \in \partial M; \\ u(t,x) = f(x), & x \in M. \end{cases} \tag{2.3}$$

First, it is easy for us to see from [17] that by replacing the Laplacian operator Δ_t with $\Delta_t + Z_t$, and repeating the same argument as in the proof of [17, Theorem 2.1], there exists a fundamental solution $p(s,x;t,y)$ to the following equation

$$\begin{cases} \frac{\partial}{\partial s} p(\cdot,x;t,y)(s) = -L_s p(s,\cdot;t,y)(x); \\ \lim_{t\downarrow s} p(s,x;t,\cdot) = \delta_x(\cdot). \end{cases}$$

Then, by a similar discussion as in the proof of [16, Theorem 2 in Section 3], there exists a solution u to the Neumann problem with the form:

$$u(s,x) = -\int_s^t \int_{\partial M} p(r,x;t,y)\psi(t,y)\mu_{\partial,t}(\mathrm{d}y)dr - \int_M p(s,x;t,y) f(y)\mu_t(\mathrm{d}y),$$

where $\psi \in C_b^1([s,t]\times M)$ and $\mu_{\partial,t}$ is the area of ∂M induced by μ_t. Thus by the Feymann-Kac formula, we have

$$P_{s,t} f(x) = u(s,x) = \int_M p(s,x;t,y) f(y)\mu_t(\mathrm{d}y).$$

We then complete the proof. $\square$

Remark 2.2. For fixed $T \in (0,T_c)$, from Theorem 2.1, we see that $P_{t,T}f$ is a solution to the following heat equation with Neumann boundary condition,

$$\begin{cases} \partial_t u(\cdot,x)(t) = -L_t u(t,\cdot)(x), & (t,x) \in [0,T] \times M, \\ u(T,x) = f(x), & x \in M, \\ N_t u(t,\cdot)(x) = 0, & (x,t) \in \partial M \times (0,T]. \end{cases} \tag{2.4}$$

Then let $(X_t^T)_{t\in[0,T]}$ be the reflecting $L_{(T-t)}$-diffusion process with semigroup $\{\overline{P}_{s,t}\}_{0\le s\le t\le T}$. It is obvious that $\overline{P}_{T-t,T}f,\ t \in [0,T]$ solves the Neumann problem

$$\begin{cases} \partial_t u(\cdot,x)(t) = L_t u(t,\cdot)(x), & (t,x) \in [0,T] \times M, \\ u(0,x) = f(x), & x \in M, \\ N_t u(t,\cdot)(x) = 0, & x \in \partial M,\ t \in (0,T]. \end{cases} \tag{2.5}$$

Actually, the theory, presented in this paper, is meant to be applied to the solution of (2.5).

2.3 Derivative formula and applications to characterizing $\mathscr{R}_t^Z$ and II_t

This subsection is devoted to the derivative formula for the Neumann semigroup, which is further applied to characterizing $\mathscr{R}_t^Z$ and II_t.

Before moving on, let us introduce some basic notations first. For $u \in \mathscr{O}_t(M)$, the lift operators $\mathscr{R}_t^Z(u),\ \mathrm{II}_t(u) \in \mathbb{R}^d \otimes \mathbb{R}^d$ are defined by

$$\mathscr{R}_t^Z(u)(a,b) = \langle \mathscr{R}_t^Z(u)a, b\rangle = \mathscr{R}_t^Z(ua,ub),\ \ \mathrm{II}_t(u)(a,b) = \mathrm{II}_t(\mathbf{p}_\partial^t ua, \mathbf{p}_\partial^t ub),\ \ a,b \in \mathbb{R}^d,$$

where for $x \in \partial M$, $\mathbf{p}_\partial^t : T_xM \to T_x\partial M$ is the project operator on (M,g_t). We now introduce the derivative formula for the Neumann semigroup first.

Theorem 2.3. *Let $0 \le s < t < T_c$ and $x \in M$ be fixed. Let $K \in C([0,T_c)\times M)$ and $\sigma \in C([0,T_c)\times \partial M)$ be such that $\mathscr{R}_t^Z \ge K_t$ and $\mathrm{II}_t \ge \sigma_t$. Assume that*

$$\sup_{u\in[s,t]} \mathbb{E}\left(\exp\left\{-\int_s^u K(r,X_r)\mathrm{d}r - \int_s^u \sigma(r,X_r)\mathrm{d}l_r\right\}\middle| X_s = x\right) < \infty. \tag{2.6}$$

Then there exists a progressively measurable process $\{Q_{s,r}\}_{s\le r\le t}$ on $\mathbb{R}^d\otimes\mathbb{R}^d$ such that

$$Q_{s,s} = I,\ \ \|Q_{s,r}\| \le \exp\left[-\int_s^r K(u,X_u)\mathrm{d}u - \int_s^r \sigma(u,X_u)\mathrm{d}l_u\right],\ \ r\in[s,t].$$

Moreover, for any $f \in C_c^1(M)$ with $|\nabla^{\cdot} P_{\cdot,t}f|_{\cdot}$ being bounded on $[s,t]\times M$, and $h \in C^1([s,t])$ satisfying $h(s)=0, h(t)=1$, it holds

$$\begin{aligned}(u_s)^{-1}\nabla^s P_{s,t}f(x) &= \mathbb{E}\left\{Q_{s,t}^* u_t^{-1}\nabla^t f(X_t)\middle| X_s = x\right\} \\ &= \frac{1}{\sqrt{2}}\mathbb{E}\left\{f(X_t)\int_s^t h'(r)Q_{s,r}^*\mathrm{d}B_r\middle| X_s = x\right\}.\end{aligned} \tag{2.7}$$

To prove this theorem, we need the following two properties to investigate the short time behavior of the diffusion process first.

Proposition 2.4. *Let X_t be a reflecting L_t-diffusion process with $X_0 = x \in M$, Then,*

(a) *if $x \in M^\circ$, then for $t_0 \in [0, T_c)$, there exist constants $r_0 > 0$ and $c_1 > 0$ such that $B_{t_0}(x, r_0) \in M^\circ$ and*

$$\mathbb{P}^x(\sigma_r \le t) \le c_1 e^{-r^2/16t},\ r \in [0, r_0],\ t \in [0, 1 \wedge T_c]$$

holds, where $\sigma_r := \inf\{s : \rho_{t_0}(X_s, x) \ge r,\ s \in [0, T_c)\}$;

(b) *there exist constants $r_0 > 0$ and $c_2 > 0$ such that*

$$\mathbb{P}^x(\tilde{\sigma}_r \le t) \le c_2 e^{-r^2/16t},\ r \in [0, r_0],\ t \in [0, 1 \wedge T_c]$$

holds, where $\tilde{\sigma}_r := \inf\{s : \rho_s(X_s, x) \ge r,\ s \in [0, T_c)\}$.

Proof. First, we prove (a). Write $\rho_{t_0}(X_t) := \rho_{t_0}(x, X_t)$ for simplicity. By taking smaller r_0, we may and do assume that $B_{t_0}(x, r_0) \in M^\circ$ and $\rho_{t_0} \in C^\infty(M)$. By the Itô formula, we obtain

$$\mathrm{d}\rho_{t_0}^2(X_t) \le 2\sqrt{2}\rho_{t_0}(X_t)\mathrm{d}b_t + C_1 \mathrm{d}t,\ \ t \le \sigma_r$$

for some constant $C_1 > 0$, where b_t is a one-dimensional Brownian motion. Thus, for fixed $t > 0$ and $\delta > 0$,

$$Z_s := \exp\left(\frac{\delta}{t}\rho_{t_0}(X_s)^2 - \frac{\delta}{t}C_1 s - 4\frac{\delta^2}{t^2}\int_0^s \rho_{t_0}(X_u)^2 \mathrm{d}u\right), \qquad 0 \le s \le \sigma_r$$

is a supermartingale. Therefore,

$$\begin{aligned}\mathbb{P}^x(\sigma_r \le t) = \mathbb{P}^x\left\{\max_{s\in[0,t]} \rho_{t_0}(X_{s\wedge\sigma_r}) \ge r\right\} &\le \mathbb{P}^x\left\{\max_{s\in[0,t]} Z_{s\wedge\sigma_r} \ge e^{\delta r^2/t - \delta C_1 - 4\delta^2 r^2/t}\right\} \\ &\le \exp\left[C_1\delta - \frac{1}{t}(\delta r^2 - 4\delta^2 r^2)\right]. \end{aligned} \tag{2.8}$$

The proof of (a) is completed by taking $\delta := 1/8$.

Next, we show (b). Let $\phi \in C^{1,\infty}([0,1] \times M)$ be constant outside $\mathbf{B} = \{(t, y) \in [0,1] \times M : \rho_t(x, y) \le r_0\}$ such that $\phi \ge 1$ in $\mathbf{B}$, and the boundary ∂M in $B_t(x, r_0)$ is convex under $\tilde{g}_t := \phi_t^{-2} g_t$ (see [30] for the existence of ϕ). Let $\tilde{\Delta}_t$ and $\tilde{\nabla}^t$ be respectively the Laplacian and the gradient operators induced by the metric $\tilde{g}_t$. Then, we have

$$\phi_t^2 L_t = \tilde{\Delta}_t + (d-1)\phi_t \nabla^t \phi_t + \phi_t^2 Z_t =: \tilde{\Delta}_t + \tilde{Z}_t,$$

and X_t solves the SDE:

$$\mathrm{d}_I X_t = \sqrt{2}\phi_t^{-1} u_t \mathrm{d}B_t + \phi_t^{-2}\tilde{Z}_t(X_t)\mathrm{d}t + \tilde{N}_t(X_t)\mathrm{d}l_t, \tag{2.9}$$

where $\tilde{N}_t$ is the inward unit normal vector field of the boundary associated with the metric $\tilde{g}_t$ and d_I denotes the Itô differential[1] on M. Let $\tilde{\rho}_t$ be the Riemannian distance under the metric $\tilde{g}_t$. By taking

[1] In local coordinates, the Itô differential for a continuous semi-martingale X_t on M is given by (see e.g. [13])

$$(\mathrm{d}_I X_t)^k = \mathrm{d}X_t^k + \frac{1}{2}\sum_{i,j=1}^d \Gamma_{i,j}^k(t, X_t)\mathrm{d}\left\langle X^i, X^j\right\rangle_t,\ \ 1 \le k \le d,$$

where $\Gamma_{ij}^k(t,x)$ are the Christoffel symbols with respect to the metric g_t.

smaller r_0, we may and do assume that $\tilde{\rho}^2 \in C^{1,\infty}(\mathbf{B})$. Then, we have that there exists a constant $C_2 > 0$ such that

$$(\partial_t + L_t)\tilde{\rho}_t^2(x,\cdot)(y) = 2\tilde{\rho}_t(x,y)L_t\tilde{\rho}_t(x,\cdot)(y) + 2\tilde{\rho}_t(x,y)\partial_t\tilde{\rho}_t(x,y) + 2|\nabla^t\tilde{\rho}_t|_t^2 \le C_2$$

holds on $\mathbf{B}$. By the Itô formula, we further obtain

$$\mathrm{d}\tilde{\rho}_t^2(x,X_t) \le 2\sqrt{2}\phi_t^{-1}\tilde{\rho}_t(x,X_t)\mathrm{d}b_t + C_2\mathrm{d}t, \ \ 0 \le t \le \sigma_{r_0}.$$

The remainder of the proof is similar to the proof of (a). □

Proposition 2.5. *Let $x \in \partial M$ and $\tilde{\sigma}_r$ be the same as in Proposition 2.4 for a fixed constant $r > 0$. Then,*

(a) $\mathbb{E}^x e^{\lambda l_{t\wedge\tilde{\sigma}_r}} < \infty$ *for any $\lambda > 0$;*

(b) $\mathbb{E}^x l_{t\wedge\tilde{\sigma}_r} = \frac{2\sqrt{t}}{\sqrt{\pi}} + O(t^{3/2})$ *holds for small $t > 0$.*

Proof. Due to Proposition 2.4 (b), the proof is similar to that of [31, Theorem 2.1] for constant manifolds, we omit it here. □

Proof of Theorem 2.3. Without loss of generality, we assume $s = 0$, and simply denote $Q_{0,t}$ by Q_t.

Following the idea of [18, Theorem 4.2], we need to construct the multiplicative functional Q_s first. For any $n \ge 1$, let $Q_s^{(n)}$ solve the equation

$$\begin{cases} \mathrm{d}Q_s^{(n)} = -\mathscr{R}_s^Z(u_s)Q_s^{(n)}\mathrm{d}s - \mathrm{II}_s(u_s)Q_s^{(n)}\mathrm{d}l_s \\ \qquad\qquad -\frac{1}{2}(n+2\sigma(s,X_s)^+)\left[(Q_s^{(n)})^*u_s^{-1}N_s\right] \otimes \left(u_s^{-1}N_s\right)\mathrm{d}l_s, \\ Q_0 = I. \end{cases}$$

It is easy to see that for any $a \in \mathbb{R}^d$,

$$\begin{aligned} \mathrm{d}\|Q_s^{(n)}a\|^2 &= 2\left\langle \mathrm{d}Q_s^{(n)}a, Q_s^{(n)}a \right\rangle \\ &= -2\mathscr{R}_s^Z(u_sQ_s^{(n)}a, u_sQ_s^{(n)}a)\mathrm{d}s - 2\mathrm{II}_s(\mathbf{p}_\partial^s u_sQ_s^{(n)}a, \mathbf{p}_\partial^s u_sQ_s^{(n)}a)\mathrm{d}l_s \\ &\quad - [n+2\sigma(s,X_s)^+]\left\langle u_sQ_s^{(n)}a, N_s\right\rangle_s^2 \mathrm{d}l_s \\ &\le -2\|Q_s^{(n)}a\|^2\left[K(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}l_s\right] - n\left\langle u_sQ_s^{(n)}a, N_s\right\rangle_s^2 \mathrm{d}l_s, \end{aligned}$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^d$. Therefore,

$$\|Q_s^{(n)}\|^2 \le \exp\left[-2\int_0^s K(r,X_r)\mathrm{d}r - 2\int_0^s \sigma(r,X_r)\mathrm{d}l_r\right] < \infty, \tag{2.10}$$

and for any $m \ge 1$,

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{E}^x\int_0^{t\wedge\zeta_m}\|(Q_s^{(n)})^*u_s^{-1}N_s\|^2\mathrm{d}l_s \\ &\quad \le \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n}\mathbb{E}^x\int_0^{t\wedge\zeta_m} 2\|Q_s^{(n)}\|^2\left[|K|(s,X_s)\mathrm{d}s + |\sigma|(s,X_s)\mathrm{d}l_s\right]\right) = 0, \end{aligned} \tag{2.11}$$

where the second equality follows from Proposition 2.4 (b), (2.10) and the boundedness of K and σ on $\{(s,y) : \rho_s(x,y) \le m,\ 0 \le s \le t\}$. Combining this with (2.10) and (2.6), we see that

$$\mathbb{E}^x \int_0^t \sup_{n\ge 1} \|Q_s^{(n)}\| \mathrm{d}s + \mathbb{E}^x \sup_{n\ge 1} \|Q_t^{(n)}\| < \infty.$$

Thus, there exists a subsequence $\{Q^{(n_k)}\}$ and a progressively measurable process Q such that for any bounded measurable process $(\varphi_s)_{s\in[0,t]}$ on $\mathbb{R}^d$ and any $\mathbb{R}^d$-valued random variable η, it holds

$$\lim_{k\to\infty} \left\{ \mathbb{E}^x \int_0^t (Q_s^{(n_k)} - Q_s)\varphi_s \mathrm{d}s + \mathbb{E}^x (Q_t^{(n_k)} - Q_t)\eta \right\} = 0.$$

Next, we turn to prove the first equality in (2.7). By observing $\mathrm{d}P_{s,t}f(X_t)$ as a vector

$$(u_s e_1(P_{s,t}f), u_s e_2(P_{s,t}f), \cdots, u_s e_d(P_{s,t}f)),$$

and using the Itô formula, we have

$$\begin{aligned} \mathrm{d}(\mathbf{d}P_{s,t}f)(X_s) =& \nabla^s_{u_s \mathbf{d}B_s}(\mathbf{d}P_{s,t}f)(X_s) + \mathrm{Ric}_s^Z(\cdot,\ \nabla^s P_{s,t}f)(X_s)\mathrm{d}s \\ &+ \nabla^s_{N_s}(\mathbf{d}P_{s,t}f)(X_s)\mathrm{d}l_s, \end{aligned} \tag{2.12}$$

where $\mathbf{d}$ is the exterior differential [2] and

$$\mathrm{Ric}_s^Z(X,Y) := \mathrm{Ric}_s(X,Y) - \langle \nabla^s_X Z_s, Y\rangle_s, \quad X,Y \in TM.$$

Now for any $a \in \mathbb{R}^d$,

$$\begin{aligned} \mathrm{d}u_s Q_s^{(n)} a =& -\mathrm{Ric}_s^Z(u_s Q_s^{(n)} a, \cdot)\mathrm{d}s - \mathrm{II}_s(\mathbf{p}^s_\partial u_s Q_s^{(n)} a, \cdot)\mathrm{d}l_s \\ &- \frac{1}{2}(n + 2\sigma(s,X_s))^+ \left\langle N_s, u_s Q_s^{(n)} a \right\rangle_s \langle N_s, \cdot\rangle_s \mathrm{d}l_s. \end{aligned}$$

By this and (2.12), we have

$$\begin{aligned} \mathrm{d}\left\langle \nabla^s P_{s,t}f(X_s), u_s Q_s^{(n)} a \right\rangle_s =& \mathrm{Hess}^s_{P_{s,t}f}(u_s Q_s^{(n)} a, u_s \mathrm{d}B_s) + \mathrm{Hess}^s_{P_{s,t}f}(u_s Q_s^{(n)} a, N_s)\mathrm{d}l_s \\ &- \mathrm{II}_s(\mathbf{p}^s_\partial u_s Q_s^{(n)} a, \nabla^s P_{s,t}f(X_s))\mathrm{d}l_s. \end{aligned} \tag{2.13}$$

Moreover, since for any $v \in T_y\partial M$, $y \in \partial M$, we have

$$0 = v\langle N_s, \nabla^s P_{s,t}f\rangle_s(y) = \langle \nabla^s_v N_s, \nabla^s P_{s,t}f\rangle_s(y) + \mathrm{Hess}^s_{P_{s,t}f}(v, N_s),$$

which implies

$$\mathrm{Hess}^s_{P_{s,t}f}(v, N_s) = \mathrm{II}_s(v, \nabla^s P_{s,t}f)(y).$$

Combining this with (2.13), we arrive at

$$\mathrm{d}\left\langle \nabla^s P_{s,t}f(X_s), u_s Q_s^{(n)} a \right\rangle_s$$

[2] For 0-form f, its exterior differential $\mathbf{d}f$ is defined by

$$\mathbf{d}f(X) := X(f) = \langle \nabla^t f, X\rangle_t, \ \text{ for } X \in TM.$$

$$= \mathrm{Hess}^s_{P_{s,t}f}(u_sQ^{(n)}_s a, u_s\mathrm{d}B_s) + \mathrm{Hess}^s_{P_{s,t}f}(N_s,N_s)\left\langle u_sQ^{(n)}_s a, N_s\right\rangle_s \mathrm{d}l_s. \tag{2.14}$$

It follows from (2.14), (2.11) and the boundedness of $|\nabla^{\cdot} P_{\cdot,t}f|_{\cdot}$ on $[0,t]\times M$ that

$$\begin{aligned}\left\langle \nabla^0 P_{0,t}f, u_0a\right\rangle_0 &= \lim_{m\to\infty}\lim_{k\to\infty}\mathbb{E}^x\left\langle \nabla^{t\wedge\zeta_m}P_{t\wedge\zeta_m,t}f(X_{t\wedge\zeta_m}), u_{t\wedge\zeta_m}Q^{(n_k)}_{t\wedge\zeta_m}a\right\rangle_{t\wedge\zeta_m}\\ &= \lim_{m\to\infty}\lim_{k\to\infty}\mathbb{E}^x\left\{\mathbf{1}_{\{t\le\zeta_m\}}\left\langle\nabla^t f(X_t), u_tQ^{(n_k)}_t a\right\rangle_t\right\}\\ &= \mathbb{E}^x\left\langle\nabla^t f(X_t), u_tQ_ta\right\rangle_t.\end{aligned}$$

This implies the first equality.

Finally, it only leaves us to show the second equality. Since by the Itô formula, we obtain

$$\mathrm{d}P_{s,t}f(X_s) = \sqrt{2}\left\langle\nabla^s f(X_s), u_s\mathrm{d}B_s\right\rangle_s.$$

Therefore, we have

$$f(X_t) = P_{0,t}f(x) + \sqrt{2}\int_0^t \left\langle\nabla^s P_{s,t}f(X_s), u_s\mathrm{d}B_s\right\rangle_s.$$

So, for any $a\in\mathbb{R}^d$ and $m\ge 1$, it follows from (2.10), (2.11) and the boundedness of $\{|\nabla^s P_{s,t}f|_s\}_{s\in[0,t]}$ that

$$\begin{aligned}&\frac{1}{\sqrt{2}}\mathbb{E}^x\left\{f(X_t)\int_0^t h'(s)\left\langle Q_sa,\mathrm{d}B_s\right\rangle\right\} = \mathbb{E}^x\left\{\int_0^t h'(s)\left\langle u_sQ_sa, \nabla^sP_{s,t}f\right\rangle_s(X_s)\mathrm{d}s\right\}\\ &= \lim_{k\to\infty}\mathbb{E}^x\left\{\int_0^t h'(s)\left\langle u_sQ^{(n_k)}_s a, \nabla^sP_{s,t}f\right\rangle_s(X_s)\mathrm{d}s\right\}\\ &= \lim_{m\to\infty}\lim_{k\to\infty}\int_0^t h'(s)\mathbb{E}^x\left\{\left\langle u_{s\wedge\zeta_m}Q^{(n_k)}_{s\wedge\zeta_m}a, \nabla^{s\wedge\zeta_m}P_{s\wedge\zeta_m,t}f\right\rangle_{s\wedge\zeta_m}(X_{s\wedge\zeta_m})\right\}\mathrm{d}s\\ &= \int_0^t h'(s)\left\langle u_0a,\nabla^0P_{0,t}f\right\rangle_0(x)\mathrm{d}s\\ &= \left\langle\nabla^0P_{0,t}f(x),u_0a\right\rangle_0.\end{aligned}$$

We complete the proof. □

By localizing the process on a fixed domain, we obtain the following local version of the derivative formula directly.

Corollary 2.6. *Assume $\mathscr{R}^Z_r \ge K_r$ and $\mathrm{II}_r\ge\sigma_r$ for some $K\in C([0,T_c)\times M)$ and $\sigma\in C([0,T_c)\times\partial M)$. Let $0\le s\le t<T_c$, $x\in M$ and D be a compact domain of M such that $x\in D^\circ$, the inner set of D. Let X_t be a reflecting L_t-diffusion process starting from x at time s and $\tau_D=\inf\{t\in[s,T_c), X_t\in\partial D, X_s=x\}$. Then for all $0\le s\le r\le t$, there exists a progressively measurable process $\{Q_{s,r}\}_{r\in[s,t]}$ on $\mathbb{R}^d\otimes\mathbb{R}^d$ such that*

$$Q_{s,s}=I,\ \|Q_{s,r}\|\le\exp\left[-\int_s^{r\wedge\tau_D}K(u,X_u)\mathrm{d}u-\int_s^{r\wedge\tau_D}\sigma(u,X_u)\mathrm{d}l_u\right].$$

In addition, for any $\mathbb{R}_+$-valued process h satisfying $h(s)=0$, $h(r)=1$ for $r>t\wedge\tau_D$ and

$$\mathbb{E}\left(\int_s^t h'(r)^2\mathrm{d}r\right)^\alpha<\infty$$

for some $\alpha > 1/2$*, it holds*

$$u_s^{-1}\nabla^s P_{s,t} f(x) = \frac{1}{\sqrt{2}}\mathbb{E}\left\{ f(X_{t\wedge\tau_D}) \int_s^t h'(r) Q_{s,r}^* \mathrm{d}B_r \middle| X_s = x \right\}, \quad f \in \mathscr{B}_b(M).$$

By using the derivative formula established above, we have the following formulae to characterize $\mathscr{R}_t^Z$ and II_t, respectively. When the metric is fixed, the formulae for Ric were established in [5] and [4, Propositions 2.1 and 2.6], and the formulae for second fundamental form were proved by F.-Y. Wang [32]. There formulae are always applied to proving that some functional inequalities imply corresponding curvature conditions.

Theorem 2.7. *For each* $s \in [0, T_c)$*, let* $x \in M^\circ$ *(the inner set of* M*) and* $X \in T_x M$ *with* $|X|_s = 1$*. Let* $f \in C_0^\infty(M)$ *such that* $f = 0$ *around the boundary,* $\mathrm{Hess}_f^s(x) = 0$ *and* $\nabla^s f = X$*. Set* $f_n = f + n$ *for* $n \geq 1$*. Then,*
(i) *for any* $p > 0$*,*

$$\mathscr{R}_s^Z(X,X) = \lim_{t\downarrow s} \frac{P_{s,t}|\nabla^t f|_t^p(x) - |\nabla^s P_{s,t} f|_s^p(x)}{p(t-s)}; \tag{2.15}$$

(ii) *for any* $p > 1$*,*

$$\begin{aligned}
\mathscr{R}_s^Z(X,X) &= \lim_{n\to\infty}\lim_{t\downarrow s} \frac{1}{t-s}\left\{ \frac{p[P_{s,t} f_n^2 - (P_{s,t} f_n^{\frac{2}{p}})^p]}{4(p-1)(t-s)} - |\nabla^s P_{s,t} f_n|_s^2 \right\}(x) \\
&= \lim_{n\to\infty}\lim_{t\downarrow s} \frac{1}{t-s}\left\{ P_{s,t}|\nabla^t f|_t^2 - \frac{p[P_{s,t} f_n^2 - (P_{s,t} f_n^{\frac{2}{p}})^p]}{4(p-1)(t-s)} \right\}(x);
\end{aligned}$$

(iii) $\mathscr{R}_s^Z(X,X)$ *is equal to each of the following limits:*

$$\lim_{n\to\infty}\lim_{t\downarrow s} \frac{1}{(t-s)^2}\left\{ (P_{s,t} f_n)\left[P_{s,t}(f_n \log f_n) - (P_{s,t} f_n)\log P_{s,t} f_n\right] - (t-s)|\nabla^s P_{s,t} f|_s^2 \right\}(x);$$

$$\lim_{n\to\infty}\lim_{t\downarrow s} \frac{1}{4(t-s)^2}\left\{ 4(t-s) P_{s,t}|\nabla^t f|_t^2 + (P_{s,t} f_n^2)\log P_{s,t} f_n^2 - P_{s,t} f_n^2 \log f_n^2 \right\}(x).$$

Proof. Without loss of generality, we only consider $s = 0$. Let $r > 0$ and $t_0 \in (0, T_c)$ be such that $B_t(x,r) \subset M^\circ,\ t \in [0, t_0]$ and $|\nabla^t f|_t \geq \frac{1}{2}$ on $\{(t,x) : t \in [0,t_0], x \in B_t(x,r) \subset M^\circ\}$. Due to Proposition 2.4 (a), the proof of [12, Theorem 4.1] works for the present setting by replacing s with $s \wedge \tilde{\sigma}_r$, where $\tilde{\sigma}_r := \inf\{s : X_s \notin B_s(x,r),\ X_0 = x,\ s \in [0,t_0]\}$ and set $t_0 = \inf\varnothing$ by convention, so that the boundary condition needs not to be considered.

To avoid redundancy, we only prove (i) to explain the idea. By Proposition 2.4 (a) and $\mathrm{Hess}_f^0(x) = 0$, we have

$$\begin{aligned}
P_{0,t}|\nabla^t f|_t^p &= \mathbb{E}^x\{|\nabla^t f|_t^p(X_{t\wedge\sigma_r})\} + \mathrm{o}(t) \\
&= |\nabla^0 f|_0^p + \left[\frac{p}{2}|\nabla^0 f|_0^{p-2} L_0 |\nabla^0 f|_0^2 - \frac{p}{2}|\nabla^0 f|_0^{p-2}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\right] t + \mathrm{o}(t),
\end{aligned} \tag{2.16}$$

where the second equality comes from the following formula,

$$\partial_t |\nabla^t f|_t^2 = -\partial_t g_t(\nabla^t f, \nabla^t f).$$

Moreover, since $f \in C_0^\infty(M)$ and $f = 0$ around the boundary, by the Kolmogorov equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla^0 P_{0,t} f|_0^p|_{t=0} = p|\nabla^0 f|_0^{p-2}\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0,$$

we have

$$|\nabla^0 P_{0,t} f|_0^p = |\nabla^0 f|_0^p + p|\nabla^0 f|_0^{p-2}\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0 t + \mathrm{o}(t).$$

Combining this with (2.16) yields (2.15) for $s = 0$. □

Theorem 2.8. *For each $s \in [0, T_c)$, let $x \in \partial M$ and $X \in T_x M$ with $|X|_s = 1$. Then for any constant $p > 0$ and $f \in C_0^\infty(M)$ such that $\nabla^s f(x) = X$, it holds*

$$\begin{aligned}\mathrm{II}_s(X,X) &= \lim_{t\downarrow s}\frac{\pi}{2p\sqrt{t-s}}\left\{P_{s,t}|\nabla^t f|_t^p - |\nabla^s f|_s^p\right\}(x)\\ &= \lim_{t\downarrow s}\frac{\pi}{2p\sqrt{t-s}}\left\{P_{s,t}|\nabla^t f|_t^p - |\nabla^s P_{s,t} f|_s^p\right\}(x).\end{aligned} \tag{2.17}$$

If moreover $f > 0$, then for any $p \in [1,2]$,

$$\begin{aligned}\mathrm{II}_s(X,X) &= -\lim_{t\downarrow s}\frac{3}{8}\sqrt{\frac{\pi}{t-s}}\left\{|\nabla^s f|_s^2 + \frac{p[(P_{s,t}f^{2/p})^p - P_{s,t}f^2]}{4(p-1)(t-s)}\right\}(x)\\ &= -\lim_{t\downarrow s}\frac{3}{8}\sqrt{\frac{\pi}{t-s}}\left\{|\nabla^s P_{s,t} f|_s^2 + \frac{p[(P_{s,t}f^{2/p})^p - P_{s,t}f^2]}{4(p-1)(t-s)}\right\}(x),\end{aligned}$$

where when $p = 1$, we set $\frac{(P_{s,t}f^{2/p})^p - P_{s,t}f^2}{p-1}$ as the following limit

$$\lim_{p\downarrow 1}\frac{(P_{s,t}f^{2/p})^p - P_{s,t}f^2}{p-1} = (P_{s,t}f^2)\log P_{s,t}f^2 - P_{s,t}(f^2\log f^2).$$

Proof. Due to Propositions 2.4 and 2.5, the proof is straightforward. For readers' convenience, we include the proof of the first equality in (2.17).

Let $r > 0$ and $t_0 \in (0, T_c)$ such that $|\nabla^t f|_t \geq \frac{1}{2}$ holds on $\{(x,t) : x \in B_t(x,r), t \in [0,t_0]\}$. Let $\tilde{\sigma}_r := \inf\{t \in [0,t_0] : X_t \notin B_t(x,r)\}$ and $t_0 := \inf\varnothing$. As $N_s|\nabla^s f|_s^2 = 2\mathrm{II}_s(\nabla^s f, \nabla^s f)$ holds on ∂M. So, by using the Itô formula and Propositions 2.4 and 2.5,

$$\begin{aligned}P_{0,t}|\nabla^t f|_t^p(x) =& \mathbb{E}^x|\nabla^t f|_t^p(X_{t\wedge\sigma_r}) + \mathrm{o}(t)\\ =& |\nabla^0 f|_0^p(x) + \mathbb{E}^x\int_0^{t\wedge\sigma_r}(L_s + \partial_s)|\nabla^s f|_s^p(X_s)\mathrm{d}s\\ &+ p\{|\nabla^s f|_s^{p-2}\mathrm{II}_s(\nabla^s f, \nabla^s f)\}(X_s)\mathrm{d}l_s + \mathrm{o}(t)\\ =& |\nabla^0 f|_0^p(x) + \frac{2p\sqrt{t}}{\sqrt{\pi}}\mathrm{II}_0(X,X) + \mathrm{o}(\sqrt{t})\end{aligned}$$

holds for small $t > 0$. This proves the first equality in (2.17).

Note that the additional terms, derived from the time derivative of the metric, have the order $\mathrm{o}(t)$. Here, from the discussion above, we find that it does not need to take care of these terms larger than order $\mathrm{o}(\sqrt{t})$. Thus, the calculation is similar to that in the fixed metric case. For the rest of the proof, we refer the reader to [32, Theorem 1.2] and [36, Theorem 3.2.4] for details.

□

3 Proof of main results

In Subsection 3.1, we construct the coupling processes under convex flows by parallel displacement and reflection first, then using coupling method, we give the proof of Theorem 1.1. In Subsection 3.2, we complete the proof of Theorem 1.2 by conformal change of the metrics and also the coupling method. In Section 3.3, we applied Theorems 1.1 and 1.2 to the forward Ricci flow with umbilic boundary.

3.1 Proof of Theorem 1.1 (Convex flow)

We first introduce the coupling method for the reflecting L_t-diffusion processes. Let $\mathrm{Cut}_t(x)$ be the set of the g_t cut-locus of x on M. Then, the g_t cut-locus Cut_t and the space time cut-locus $\mathrm{Cut}_{\mathrm{ST}}$ are respectively defined by

$$\begin{aligned}\mathrm{Cut}_t &= \{(x,y)\in M\times M \mid y\in \mathrm{Cut}_t(x)\};\\ \mathrm{Cut}_{\mathrm{ST}} &= \{(t,x,y)\in[0,T_c)\times M\times M \mid (x,y)\in \mathrm{Cut}_t\}.\end{aligned}$$

Set $D(M)=\{(x,x)|x\in M\}$. For $(x,y)\notin \mathrm{Cut}_t$, let $\{J_i^t\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic γ from x to y with respect to the metric g_t such that at points x and y, $\{J_i^t,\dot\gamma : 1\le i\le d-1\}$ is an orthonormal basis. Let

$$\begin{aligned}I_Z(t,x,y)=&\sum_{i=1}^{d-1}\int_\gamma\Big(\big\langle\nabla^t_{\dot\gamma}J_i^t,\nabla^t_{\dot\gamma}J_i^t\big\rangle_t-\big\langle R_t(J_i^t,\dot\gamma)\dot\gamma,J_i^t\big\rangle_t\Big)(\gamma(s))\mathrm{d}s+\frac12\int_\gamma\partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s\\&+Z_t\rho_t(\cdot,y)(x)+Z_t\rho_t(x,\cdot)(y),\end{aligned}\tag{3.1}$$

where R_t is the Ricci tensor with respect to the metric g_t. Moreover, let $P^t_{x,y}:T_xM\to T_yM$ be the g_t-parallel transform along the geodesic γ, and let

$$M^t_{x,y}:T_xM\to T_yM;\ v\mapsto P^t_{x,y}v-2\langle v,\dot\gamma\rangle_t(x)\dot\gamma(y)$$

be the mirror reflection associated with the metric g_t. Then $P^t_{x,y}$ and $M^t_{x,y}$ are smooth outside $\mathrm{Cut}_t\cup D(M)$. For convenience, set $P^t_{x,x}$ and $M^t_{x,x}$ be the identity for $x\in M$.

Lemma 3.1. *Let $x\neq y$ and $0<T<T_c$ be fixed. Let $U:[0,T)\times M\times M\to TM$ be C^1-smooth in $(\mathrm{Cut}_{\mathrm{ST}}\cup[0,T]\times D(M))^c$ such that $U(t,x_1,x_2)\in T_{x_2}M$ for $(t,x_1,x_2)\in[0,T]\times M\times M$.*

(a) There exist two Brownian motions B_t and $\tilde B_t$ on the probability space $(\Omega,\{\mathscr{F}_t\}_{t\ge0},\mathbb{P})$ such that

$$\mathbf{1}_{\{(X_t,\tilde X_t)\notin\mathrm{Cut}_t\}}\mathrm{d}\tilde B_t=\mathbf{1}_{\{(X_t,\tilde X_t)\notin\mathrm{Cut}_t\}}\tilde u_t^{-1}P^t_{X_t,\tilde X_t}u_t\mathrm{d}B_t$$

holds, where X_t with lift u_t and local time l_t, and $\tilde X_t$ with lift $\tilde u_t$ and and local time $\tilde l_t$ solve the equation

$$\begin{cases}\mathrm{d}X_t=\sqrt2u_t\circ\mathrm{d}B_t+Z_t(X_t)\mathrm{d}t+N_t(X_t)\mathrm{d}l_t, & X_0=x,\\ \mathrm{d}\tilde X_t=\sqrt2\tilde u_t\circ\mathrm{d}\tilde B_t+\Big\{Z_t(\tilde X_t)+U(t,X_t,\tilde X_t)\mathbf{1}_{\{X_t\neq\tilde X_t\}}\Big\}\mathrm{d}t+N_t(\tilde X_t)\mathrm{d}\tilde l_t, & \tilde X_0=y.\end{cases}\tag{3.2}$$

Moreover, for any $J\in C([0,T]\times M\times M)$ such that $J\ge I_Z$ on $(\mathrm{Cut}_{\mathrm{ST}}\cup[0,T]\times D(M))^c$,

$$\mathrm{d}\rho_t(X_t,\tilde X_t)\le\Big\{J(t,X_t,\tilde X_t)+\big\langle U(t,X_t,\tilde X_t),\nabla^t\rho_t(X_t,\cdot)(\tilde X_t)\big\rangle_t\mathbf{1}_{\{X_t\neq\tilde X_t\}}\Big\}\mathrm{d}t\tag{3.3}$$

holds up to the coupling time $T_0:=\inf\{t\in[0,T]:X_t=\tilde X_t\}$, $\inf\varnothing=T$.

(b) *The first assertion in (a) holds with $M^t_{X_t,\tilde{X}_t}$ in place of $P^t_{X_t,\tilde{X}_t}$. In this case, for any $J\in C([0,T]\times M\times M)$ such that $J\geq I_Z$ on $(\mathrm{Cut}_{\mathrm{ST}}\cup[0,T]\times D(M))^c$,*

$$\mathrm{d}\rho_t(X_t,\tilde{X}_t)\leq 2\sqrt{2}\mathrm{d}b_t+\left\{J(t,X_t,\tilde{X}_t)+\left\langle U(t,X_t,\tilde{X}_t),\nabla^t\rho_t(X_t,\cdot)(\tilde{X}_t)\right\rangle_t\mathbf{1}_{\{X_t\neq\tilde{X}_t\}}\right\}\mathrm{d}t \tag{3.4}$$

holds up to the coupling time T_0, where b_t is a one-dimensional Brownian motion.

Proof. We follow the argument in the proof of [12, Theorem 3.4], but construct the coupling processes $(X_t,Y^{n,\varepsilon}_t)$ with reflecting boundary. Then we should add more argument for one more term caused by the local time on the boundary. More precisely, when applying the Itô formula to the radial process $\rho_t(X_t,Y^{n,\varepsilon}_t)$, we have the additional term

$$\mathbf{1}_{(M\times M)\setminus\mathrm{Cut}_t}(X_t,Y^{n,\varepsilon}_t)(N_t(X_t)+N_t(Y^{n,\varepsilon}_t))\rho_t(X_t,Y^{n,\varepsilon}_t)\mathrm{d}l^{n,\varepsilon}_t,$$

where $l^{n,\varepsilon}_t$ is an increasing process which increasing only when $(X_t,Y^{n,\varepsilon}_t)\in\partial(M\times M)\setminus\mathrm{Cut}_t$. Thus to pass through the proof for the present case, we only need to show that $N_t\rho_t(x,\cdot)(y)\leq 0$ for any $y\in\partial M, x\in M, (x,y)\in(M\times M)\setminus\mathrm{Cut}_t$ and $t\in[0,T_c)$. This is ensured by the convexity of the geometric flow. Therefore, the proof of [12, Theorem 3.4] also works for the reflecting L_t-diffusion case. □

By using the parallel coupling process, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. First, by Theorem 2.7 (i)(ii) and Theorem 2.8 (i.e. the characterizations for $\mathscr{R}^Z_t$ and II_t), each of (1.3) and (1.4) implies (1.2) directly.

Now, suppose the curvature condition (1.2) holds. We prove (1.3). We first observe from the index lemma that

$$\begin{aligned}I_Z(t,x,y)\leq&\frac{1}{2}\int_0^{\rho_t(x,y)}\partial_t g_t(\dot\gamma,\dot\gamma)(\gamma(s))\mathrm{d}s-\int_0^{\rho_t(x,y)}\mathrm{Ric}^Z_t(\dot\gamma,\dot\gamma)(\gamma(s))\mathrm{d}s\\ =&-\int_0^{\rho_t(x,y)}\mathscr{R}^Z_t(\dot\gamma,\dot\gamma)(\gamma(s))\mathrm{d}s\\ \leq&-K(t)\rho_t(x,y),\end{aligned} \tag{3.5}$$

where $\gamma:[0,\rho_t(x,y)]\to M$ is the minimal geodesic from x and y associated with g_t. Now let $U=0$ and $(X_t,\tilde{X}_t)$ be the coupling by parallel displacement for $X_0=x,\tilde{X}_0=y$. By Lemma 3.1 for $U=0$, (3.5) and $N_t\rho_t(x,\cdot)(y)\leq 0$ for $y\in\partial M$ and $x\in M$, we obtain

$$\mathrm{d}\rho_t(X_t,\tilde{X}_t)\leq -K(t)\rho_t(X_t,\tilde{X}_t)\mathrm{d}t.$$

Thus, $\rho_t(X_t,\tilde{X}_t)\leq e^{-\int_s^t K(u)\mathrm{d}u}\rho_s(X_s,\tilde{X}_s)$, which together with the dominated convergence theorem, we have

$$\begin{aligned}|\nabla^s P_{s,t}f(x)|_s\leq&\limsup_{y\to x}\frac{\mathbb{E}(|f(X_t)-f(\tilde{X}_t)|\,|\,(X_s,\tilde{X}_s)=(x,y))}{\rho_s(x,y)}\\ \leq&e^{-\int_s^t K(u)\mathrm{d}u}\limsup_{y\to x}\mathbb{E}\left(\frac{|f(X_t)-f(\tilde{X}_t)|}{\rho_t(X_t,\tilde{X}_t)}\,\middle|\,(X_s,\tilde{X}_s)=(x,y)\right)\\ \leq&e^{-\int_s^t K(u)\mathrm{d}u}P_{s,t}|\nabla^t f|_t(x).\end{aligned}$$

Finally, we prove the Harnack inequality. Let $f \in C_0^\infty(M)$ be such that $f \geq 1$ and f is constant outside a compact set. Given $x \neq y$ and $t > 0$, let $\gamma : [0,t] \to M$ be the g_0-geodesic from x to y with length $\rho_0(x,y)$. Let $v_s = \frac{\mathrm{d}\gamma_s}{\mathrm{d}s}$. Then we have $|v_s|_0 = \rho_0(x,y)/t$. Let

$$h(s) = \frac{t\int_0^s e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}{\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}.$$

Then $h(0) = 0$ and $h(t) = t$. Set $y_s = \gamma_{h(s)}$ and

$$\varphi(s) = \log P_{0,s}(P_{s,t}f)^p(y_s),\ s \in [0,t].$$

To get the derivative of φ, by using the Itô formula, we first have

$$\mathrm{d}(P_{s,t}f)^p(X_s) = \mathrm{d}M_s + p(p-1)(P_{s,t}f)^{p-2}(X_s)|\nabla^s P_{s,t}f|_s^2(X_s)\mathrm{d}s, \quad 0 < s < \zeta_n,$$

where M_s is a local martingale. As explained above, $|\nabla^s P_{s,t}f|_s \leq e^{-\int_s^t K(r)\mathrm{d}r} P_{s,t}|\nabla^t f|_t$ and $(P_{s,t}f)^{p-2}$ is bounded, it is easy to deduce that

$$P_{0,s}(P_{s,t}f)^p(x) - (P_{0,t}f)^p(x) = p(p-1)\int_0^s P_{0,r}[(P_{r,t}f)^{p-2}|\nabla^r P_{r,t}f|_r^2](x)\mathrm{d}r.$$

That is

$$\frac{\mathrm{d}P_{0,s}(P_{s,t}f)^p(x)}{\mathrm{d}s} = p(p-1)P_{0,s}[(P_{s,t}f)^{p-2}|\nabla^s P_{s,t}f|_s^2](x),$$

which implies that for any $s \in [0,t]$,

$$\begin{aligned}
\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} =& \frac{1}{P_{0,s}(P_{s,t}f)^p}\Big\{P_{0,s}\Big(p(p-1)(P_{s,t}f)^p|\nabla^s \log P_{s,t}f|_s^2 + h'(s)\left\langle \nabla^0 P_{0,s}(P_{s,t}f)^p, v_s\right\rangle_0\Big\}\\
\geq& \frac{p}{P_{0,s}(P_{s,t}f)^p}P_{0,s}\Big\{(P_{s,t}f)^p\Big((p-1)|\nabla^s \log P_{s,t}f|_s^2\\
&\qquad - \frac{\rho_0(x,y)}{t}h'(s)e^{-\int_0^s K(u)\mathrm{d}u}|\nabla^s \log P_{s,t}f|_s\Big)\Big\}\\
\geq& \frac{-p\rho_0^2(x,y)h'(s)^2 e^{-2\int_0^s K(u)\mathrm{d}u}}{4(p-1)t^2}.
\end{aligned}$$

Since $h'(s) = \frac{te^{\int_0^s 2K(u)\mathrm{d}u}}{\int_0^t e^{\int_0^r 2K(u)\mathrm{d}u}\mathrm{d}r}$, we arrive at

$$\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} \geq \frac{-p\rho_0(x,y)^2 e^{\int_0^s 2K(u)\mathrm{d}u}}{4(p-1)(\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r)^2}, \quad s \in [0,t].$$

By integrating over s from 0 and t, we complete the proof of (1.4) for $s = 0$. □

Remark 3.2. We point out that by letting $\phi_t \equiv 1$ in the proof of Theorem 1.2 (ii), the Harnack inequality can be deduced by using coupling method directly.

3.2 Proof of Theorem 1.2 (Non-convex flow)

The proof of Theorem 1.2 is divided into two parts. First, we prove that the curvature condition **(H1)** implies the gradient inequality.

Proof of Theorem 1.2. (Gradient inequality) We also use coupling method to prove the gradient inequality. To this end, we need make a conformal change of the geometric flow g_t first. Let $\phi \in \mathscr{D}$. As announced, the new flow $\tilde{g}_t := \phi_t^{-2} g_t$ is convex flow. Let $\tilde{\Delta}_t$ and $\tilde{\nabla}^t$ be the Laplacian and gradient operator associated with the metric $\tilde{g}_t$. According to [28, (2.2)],

$$L_t = \phi_t^{-2}(\tilde{\Delta}_t + \tilde{Z}_t) \text{ and } \tilde{Z}_t = \phi_t Z_t + \frac{d-2}{2}\nabla^t \phi_t^2. \tag{3.6}$$

To simplify the discussion, we consider the process generated by $L_t' = \varphi_t^2(\Delta_t + Z_t)$ on the manifold carrying convex flow $\{g_t\}_{t\in[0,T_c)}$ first, where $\varphi \in C^{1,\infty}([0,T_c)\times M)$ and $0 < \varphi \le 1$. Moreover, suppose

$$\mathrm{Ric}_t^Z \ge k_1(t) \quad \text{and} \quad \partial_t g_t \le k_2(t)$$

for some functions $k_1, k_2 \in C([0,T_c))$. Let X_t solve

$$\mathrm{d}_I X_t = \sqrt{2}\varphi_t(X_t)u_t \mathrm{d}B_t + \varphi_t^2(X_t)Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \quad X_0 = x. \tag{3.7}$$

Let Y_t solve

$$\mathrm{d}_I Y_t = \sqrt{2}\varphi_t(Y_t)P^t_{X_t,Y_t}u_t \mathrm{d}B_t + \varphi_t^2(X_t)Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = y. \tag{3.8}$$

As the boundary $(\partial M, g_t)$ is convex for all $t \in [0,T_c)$, by the Itô formula, we have

$$\begin{aligned}\mathrm{d}\rho_t(X_t,Y_t) \le& \sqrt{2}(\varphi_t(X_t) - \varphi_t(Y_t))\mathrm{d}b_t + \Bigg\{\sum_{i=1}^n (U_i^t)^2\rho_t(X_t,Y_t) + \partial_t\rho_t(X_t,Y_t)\\ &+ \left\langle \varphi_t^2 Z_t(Y_t), \nabla^t\rho_t(X_t,\cdot)(Y_t)\right\rangle_t + \left\langle \varphi_t^2 Z_t(X_t), \nabla^t\rho_t(\cdot,Y_t)(X_t)\right\rangle_t\Bigg\}\mathrm{d}t,\end{aligned}$$

where b_t is a one-dimensional Brownian motion, $\{U_i^t\}_{i=1}^n$ are vector fields on $M\times M$ such that $\nabla^t U_i^t(X_t,Y_t) = 0$ and

$$U_i^t(X_t,Y_t) = \varphi_t(X_t)V_i^t + \varphi_t(Y_t)P^t_{X_t,Y_t}V_i^t, \ \ 1 \le i \le n$$

for $\{V_i^t\}_{i=1}^n$ a g_t-orthonormal basis of $T_{X_t}M$. Let $\rho_t = \rho_t(X_t,Y_t)$. Define

$$J_i^t(s) = \left(\frac{s}{\rho_t}\varphi_t(Y_t) + \frac{\rho_t - s}{\rho_t}\varphi_t(X_t)\right)P^t_{\gamma(0),\gamma(s)}V_i^t, \ \ 1 \le i \le n,$$

where $J_i^t(0) = \varphi_t(X_t)V_i^t$ and $J_i^t(\rho_t) = \varphi_t(Y_t)P^t_{X_t,Y_t}V_i^t$. Note that $P^t_{\gamma(0),\gamma(s)}V_i^t$ are parallel vector fields along γ_t,

$$\begin{aligned}&\sum_{i=1}^d (U_i^t)^2\rho_t(X_t,Y_t)\\ &\le \sum_{i=1}^d \int_0^{\rho_t} \left\{|\nabla^t_{\dot\gamma}J_i^t|_t^2 - \left\langle R_t(\dot\gamma, J_i^t)J_i^t, \dot\gamma\right\rangle_t\right\}(\gamma(s))\mathrm{d}s\end{aligned}$$

$$\leq d\|\nabla^t \varphi_t\|_\infty^2 \rho_t - \frac{1}{\rho_t^2}\int_0^{\rho_t} \{s\varphi_t(Y_t) + (\rho_t - s)\varphi_t(X_t)\}^2 \mathrm{Ric}_t(\dot\gamma(s), \dot\gamma(s))\mathrm{d}s. \tag{3.9}$$

On the other hand,

$$\begin{aligned}
&\varphi_t^2(X_t)\left\langle Z_t(X_t), \nabla^t \rho_t(\cdot, Y_t)(X_t)\right\rangle_t + \varphi_t^2(Y_t)\left\langle Z_t(Y_t), \nabla^t \rho_t(X_t, \cdot)(Y_t)\right\rangle_t \\
&= \frac{1}{\rho_t^2}\int_0^{\rho_t} \frac{\mathrm{d}}{\mathrm{d}s}\{(s\varphi_t(Y_t) + (\rho_t - s)\varphi_t(X_t))^2 \langle Z_t(\gamma(s)), \dot\gamma(s)\rangle_t\}\mathrm{d}s \\
&\leq \frac{1}{\rho_t^2}\int_0^{\rho_t} (s\varphi_t(Y_t) + (\rho_t - s)\varphi_t(X_t))^2 \left\langle (\nabla^t_{\dot\gamma} Z_t)\circ\gamma, \dot\gamma\right\rangle_t (\gamma(s))\mathrm{d}s + 2\|Z_t\|\|\nabla^t\varphi_t\|_\infty \rho_t. \qquad (3.10)
\end{aligned}$$

Moreover,

$$\partial_t \rho_t(X_t, Y_t) = \frac{1}{2}\int_0^{\rho_t} \partial_t g_t(\dot\gamma(s), \dot\gamma(s))\mathrm{d}s \leq \frac{1}{2}k_2(t)\rho_t.$$

Combining this with (3.9) and (3.10), we have

$$\begin{aligned}
\mathrm{d}\rho_t(X_t, Y_t) \leq& \sqrt{2}(\varphi_t(X_t) - \varphi_t(Y_t))\mathrm{d}b_t - k_1(t)\left\{\frac{1}{\rho_t^2}\int_0^{\rho_t} [s\varphi_t(Y_t) + (\rho_t - s)\varphi_t(X_t)]^2 \mathrm{d}s\right\}\mathrm{d}t \\
&+ \left\{d\|\nabla^t\varphi_t\|_\infty^2 \rho_t + 2\|Z_t\|_\infty \|\nabla^t\varphi_t\|_\infty \rho_t + \frac{1}{2}k_2(t)\rho_t\right\}\mathrm{d}t \\
\leq& \sqrt{2}(\varphi_t(X_t) - \varphi_t(Y_t))\mathrm{d}b_t + K_\varphi(t)\rho_t(X_t, Y_t)\mathrm{d}t, \qquad (3.11)
\end{aligned}$$

where

$$K_\varphi(t) := d\|\nabla^t\varphi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\nabla^t\varphi_t\|_\infty + k_1^-(t) + \frac{1}{2}k_2(t). \tag{3.12}$$

Now we return to the diffusion processes generated by $L_t = \phi_t^{-2}(\tilde\Delta_t + \tilde Z_t)$ (see (3.6)). Let $\varphi_t = \phi_t^{-1}$ and $\widetilde{\mathrm{Ric}}_t$ be the new Ricci curvature tensor with respect to the metric $\tilde g_t$. By [6, Theorem 1.129] and [15, (3.2)], for any $X \in TM$ such that $\tilde g_t(X, X) = 1$, i.e. $|X|_t = \phi_t$, we have

$$\widetilde{\mathrm{Ric}}_t(X, X) = \mathrm{Ric}_t(X, X) + (d-2)\phi_t^{-1}\mathrm{Hess}^t_{\phi_t}(X, X) + \frac{1}{2}\nabla^t\phi_t^2 - (d-2)|\nabla^t\phi_t|_t^2,$$

and

$$\begin{aligned}
\tilde g_t(\tilde\nabla^t_X \tilde Z_t, X) =& \left\langle \nabla^t_X Z_t, X\right\rangle_t + 2\left\langle \nabla^t \log\phi_t, X\right\rangle_t \langle Z_t, X\rangle_t \\
&+ (d-2)\phi_t^{-1}\mathrm{Hess}^t_{\phi_t}(X, X) + (d-2)\left\langle X, \nabla^t\log\phi_t\right\rangle_t^2 \\
&- \phi_t\left\langle Z_t, \nabla^t\phi_t\right\rangle_t - (d-2)|\nabla^t\phi_t|_t^2.
\end{aligned}$$

Therefore, noting that $|X|_t = \phi_t$, we have

$$\begin{aligned}
\widetilde{\mathrm{Ric}}_t^{\tilde Z}(X, X) :=& \widetilde{\mathrm{Ric}}_t(X, X) - \tilde g_t(\tilde\nabla^t_X \tilde Z_t, X) \\
=& \mathrm{Ric}_t^Z(X, X) + \frac{1}{2}L_t\phi_t^2 - 2\left\langle \nabla^t\log\phi_t, X\right\rangle_t \langle Z_t, X\rangle_t - (d-2)\left\langle X, \nabla^t\log\phi_t\right\rangle_t^2 \\
\geq& K_1(t)\phi_t^2 + \frac{1}{2}L_t\phi_t^2 - |\nabla^t\phi_t^2|_t \cdot |Z_t|_t - (d-2)|\nabla^t\phi_t|_t^2 \\
\geq& K_{\phi,1}(t),
\end{aligned}$$

and

$$\partial_t \tilde{g}_t(X,X) = \partial_t[\phi_t^{-2} g_t(X,X)] = (\partial_t \phi_t^{-2})\phi_t^2 + \phi_t^{-2}\partial_t g_t(X,X)$$
$$\leq -2\partial_t \log \phi_t + K_2(t) \leq K_{\phi,2}(t).$$

Moreover, let $|\cdot|_t'$ be the norm with respect to the metric $\tilde{g}_t$. Then,

$$|\tilde{\nabla}^t \phi_t^{-1}|_t' \leq |\nabla^t \phi_t|_t \quad \text{and} \quad |\tilde{Z}_t|_t' \leq \left|\phi_t Z_t + \frac{d-2}{2}\nabla^t \phi_t^2\right|_t,$$

which, together with (3.11), yields

$$\mathrm{d}\tilde{\rho}_t(X_t,Y_t) \leq \sqrt{2}(\phi_t^{-1}(X_t) - \phi_t^{-1}(Y_t))\mathrm{d}b_t + K_\phi(t)\tilde{\rho}_t(X_t,Y_t)\mathrm{d}t,$$

where

$$K_\phi(t) := K_{\phi,1}^-(t) + \frac{1}{2}K_{\phi,2}(t) + 2\|\phi_t Z_t + (d-2)\nabla^t\phi_t\|_\infty \|\nabla^t \phi_t\|_\infty + d\|\nabla^t\phi_t\|_\infty^2.$$

In addition, $\phi_t \geq 1$, we therefore have $\tilde{\rho}_t \leq \rho_t \leq \|\phi_t\|_\infty \tilde{\rho}_t$, which implies

$$\rho_t(X_t,Y_t) \leq \|\phi_t\|_\infty e^{\int_s^t K_\phi(r)\mathrm{d}r}\tilde{\rho}_s(x,y) \leq \|\phi_t\|_\infty e^{\int_s^t K_\phi(r)\mathrm{d}r}\rho_s(x,y), \ s \leq t < T_c.$$

Then,

$$|\nabla^s P_{s,t} f|(x) = \lim_{y\to x}\left|\frac{P_{s,t}f(x) - P_{s,t}f(y)}{\rho_s(x,y)}\right| = \left|\mathbb{E}^{(x,y)}\left[\frac{f(X_t)-f(Y_t)}{\rho_t(X_t,Y_t)}\frac{\rho_t(X_t,Y_t)}{\rho_s(x,y)}\right]\right|$$
$$\leq \|\nabla^t f\|_\infty \|\phi_t\|_\infty e^{\int_s^t K_\phi(r)\mathrm{d}r},$$

which leads to complete the proof directly. □

The following result is derived from Theorem 1.2 (i) and Theorem 2.3.

Corollary 3.3. *Assume* (**H1**) *holds. If there exists* $\phi \in \mathscr{D}$ *such that* $K_\phi(t) < \infty$ *for all* $0 \leq t < T_c$*, then for* $p \in [1,\infty)$ *and* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$|\nabla^s P_{s,t} f|_s \leq \|\phi_t\|_\infty (P_{s,t}|\nabla^t f|_t^{p/(p-1)})^{(p-1)/p} e^{-\int_s^t K_\phi^{(p)}(r)\mathrm{d}r}, \ 0 \leq s \leq t < T_c$$

holds for $K_\phi^{(p)}(r) := \inf\{\phi_r^{-1}(L_r + \partial_r)\phi_r - (p+1)|\nabla^r \log \phi_r|_r^2\}$*. Moreover, for* $f \in \mathscr{B}_b(M)$*,*

$$|\nabla^s P_{s,t} f|_s^2 \leq \frac{1}{2}\left[\int_s^t \|\phi_u\|_\infty^{-2} e^{2\int_s^u K_\phi^{(2)}(r)\mathrm{d}r}\mathrm{d}u\right]^{-1} P_{s,t} f^2, \ \ 0 \leq s < t < T_c. \tag{3.13}$$

Proof. The proof is due to [36, Corollary 3.2.8]. We include it in Appendix for readers' convenience. □

We apply the coupling method to the proof of the Harnack inequality (See Theorem 1.2 (ii)). In [35], F.-Y.Wang constructed a proper coupling process to get the Harnack inequalities on manifolds with fixed metric. Here, we should modify the idea to our setting, where the main difficulty is to construct the coupling process such that it does not miss the information from the Ricci curvature.

Proof of Theorem 1.2. (Part I: Harnack inequality) Without loss of generality, we assume $s=0$ and $t=T$. Now, let $x,y\in M$ and $T\in(0,T_c)$ be fixed. To simplify the discussion, we also consider the process generated by $L_t'=\varphi_t^2(\Delta_t+Z_t)$ on a manifold carrying convex flow and suppose

$$\mathrm{Ric}_t^Z\geq k_1(t)\quad\text{and}\quad\partial_t g_t\leq k_2(t)$$

for some $k_1,k_2\in C([0,T_c))$.

Let X_t solve (3.7) with $X_0=x$. For some strictly positive function $\xi_t\in C([0,T))$, let Y_t solve

$$\mathrm{d}_I Y_t=\sqrt{2}\varphi_t(Y_t)P^t_{X_t,Y_t}u_t\mathrm{d}B_t+\varphi_t^2 Z_t(Y_t)\mathrm{d}t-\frac{\varphi_t(Y_t)\rho_t(X_t,Y_t)}{\varphi_t(X_t)\xi_t}\nabla^t\rho_t(X_t,\cdot)(Y_t)\mathrm{d}t+N_t(Y_t)\mathrm{d}\tilde{l}_t,$$
$$Y_0=y, \tag{3.14}$$

where $\tilde{l}_t$ is the local time of Y_t on ∂M. In the spirit of Lemma 3.1, we may assume that the cut-locus of M is empty such that the parallel displacement is smooth. Let

$$\mathrm{d}\tilde{B}_t=\mathrm{d}B_t+\frac{\rho_t(X_t,Y_t)}{\sqrt{2}\xi_t\varphi_t(X_t)}u_t^{-1}\nabla^t\rho_t(\cdot,Y_t)(X_t)\mathrm{d}t,\ \ 0\leq t<T. \tag{3.15}$$

By a similar calculation as in (3.11), we have

$$\begin{aligned}\mathrm{d}\rho_t(X_t,Y_t)\leq&\sqrt{2}(\varphi_t(X_t)-\varphi_t(Y_t))\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde{B}_t\right\rangle_t+K_\varphi(t)\rho_t(X_t,Y_t)\mathrm{d}t\\&-\frac{\rho_t(X_t,Y_t)}{\xi_t}\mathrm{d}t,\quad 0\leq t<T,\end{aligned} \tag{3.16}$$

which implies

$$\begin{aligned}\mathrm{d}\frac{\rho_t(X_t,Y_t)^2}{\xi_t}\leq&\frac{2\sqrt{2}}{\xi_t}\rho_t(X_t,Y_t)(\varphi_t(X_t)-\varphi_t(Y_t))\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde{B}_t\right\rangle_t\\&-\frac{\rho_t(X_t,Y_t)^2}{\xi_t^2}\left[\xi_t'-(2\|\nabla^t\varphi_t\|_\infty^2+2K_\varphi(t))\xi_t+2\right]\mathrm{d}t,\end{aligned} \tag{3.17}$$

where K_φ is defined as in (3.12). Therefore, by letting

$$\xi_t=(2-\theta)\int_t^T e^{-2\int_t^s(K_\varphi(r)+\|\nabla^r\varphi_r\|_\infty)\mathrm{d}r}\mathrm{d}s,\ \ t\in[0,T),\ \theta\in(0,2),$$

we know that $\xi_t,t\in[0,T)$ solves the following equation,

$$2-(2\|\nabla^t\varphi_t\|_\infty^2+2K_\varphi(t))\xi_t+\xi_t'=\theta.$$

Combining this with (3.17), we obtain

$$\mathrm{d}\frac{\rho_t(X_t,Y_t)^2}{\xi_t}\leq\frac{2\sqrt{2}}{\xi_t}\rho_t(X_t,Y_t)(\varphi_t(X_t)-\varphi_t(Y_t))\left\langle\nabla^t\rho_t(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde{B}_t\right\rangle_t-\frac{\rho_t(X_t,Y_t)^2}{\xi_t^2}\theta\mathrm{d}t. \tag{3.18}$$

Then, the following discussion is similar to that of [35, Theorem 1.1], we omit it here. □

3.3 Application to Ricci flow

Now, we turn to consider the Ricci flow (1.7). Assume that $\{g_t\}_{t\in[0,T]}$, $T\in(0,\infty)$ is a complete solution to the equation (1.7). Let $\{P_{s,t}\}_{0\le s\le t\le T}$ be the Neumann diffusion semigroup generated by Δ_t. Then, it is obvious to see that $P_{s,T}f$ is a solution to the following heat equation

$$\begin{cases} \frac{\partial}{\partial t}u(x,\cdot)(t) = -\Delta_t u(\cdot,t)(x), & (x,t)\in M\times[0,T];\\ \\ N_t u(\cdot,t)(x) = 0, & x\in\partial M,\ t\in[0,T). \end{cases} \tag{3.19}$$

When $\lambda\ge 0$, the corresponding gradient estimate and Harnack inequality can be derived from Theorem 1.1 directly.

Theorem 3.4. *Suppose $\{g_t\}_{t\in[0,T]}$ is a complete solution to* (1.7) *with $\lambda\ge 0$. Then for $f\in C^1(M)$ such that f is constant outside a compact set, $P_{s,T}f,\ s\in[0,T]$ is a solution to* (3.19) *and*

$$|\nabla^s P_{s,T}f|_s \le P_{s,T}|\nabla^T f|_T,\ 0\le s\le T. \tag{3.20}$$

Moreover, for $f\in\mathscr{B}_b(M)$ and $s\in[0,T]$,

$$(P_{s,T}f)^p(x)\le P_{s,T}f^p(y)\exp\left[\frac{p}{4(p-1)(T-s)}\rho_s^2(x,y)\right]. \tag{3.21}$$

Remark 3.5. It is easy to see that these results above are similar to that for the Ricci flat manifold. Indeed, (3.20) and (3.21) also can be derived when $\{g_t\}$ is a convex Ricci flow. We would like to indicate that Pulemotov [25] gave the proof of the short time existence of the convex Ricci flow.

When $\lambda<0$, we need more curvature information around the boundary to deal with this case. Let Sect_t be the section curvature of M and $\rho_t^\partial(x)$ be the distance between x and ∂M associated with the metric g_t. The required assumption is presented as follows.

(H2) There exist positive constants k, r_0, k_1 such that $|\mathrm{Ric}_t|\le k$ and on the set $\partial_{r_0}^t M := \{x\in M : \rho_t^\partial(x)\le r_0\}$, ρ_t^∂ is smooth and $\mathrm{Sect}_t\le k_1$.

If this assumption holds for $r_0<\frac{\pi}{2\sqrt{k_1}}$, then by constructing explicit ϕ_t, the constants in terms of ϕ in Theorem 1.2 can be estimated. Thus, we have the gradient estimates for the solution to (3.19) by using Theorem 1.2 (i) as follows.

Theorem 3.6. *Suppose that $\{g_t\}_{t\in[0,T]}$ is a complete solution to* (1.7) *with $\lambda<0$. Assume that the assumption* (**H2**) *holds for $0<r_0\le\frac{\pi}{2\sqrt{k_1}}$. Then, for $f\in C^1(M)$ such that f is constant outside a compact set, $P_{s,T}f$ is a solution to* (3.19) *and*

$$|\nabla^s P_{s,T}f|_s\le\left(1-\frac{\lambda r_0 d}{2}\right)\|\nabla^T f\|_\infty\exp\left\{(T-s)\left[-\frac{\lambda d}{r_0}+\left(4d-\frac{11}{2}\right)\lambda^2 d^2-\lambda d r_0 k+2k\right]\right\}. \tag{3.22}$$

Proof. From the assumption **(H2)**, we deduce that $\mathrm{Ric}_t\le -k$ and $\partial_t g_t = 2\mathrm{Ric}_t\le 2k$, which leads to the following estimate

$$K_\phi(t)\le\inf\{\phi_t\Delta_t\phi_t\}^- + \inf\{\partial_t\log\phi_t\}^- + 2k + (4d-6)\|\nabla^t\phi_t\|_\infty^2<\infty.$$

We now turn to construct an explicit $\phi \in C^{1,2}([0,T]\times M)$. Let

$$h(s)=\cos(\sqrt{k_1}\,s),\quad \text{for all } \ s\geq 0. \tag{3.23}$$

Then $0\leq h(s)\leq 1$ for $s\in[0,\frac{\pi}{2\sqrt{k_1}}]$. Moreover, let

$$\delta=\delta(r_0,\lambda,k_1)=\frac{-\lambda(1-h(r_0))^{d-1}}{\int_0^{r_0}(h(s)-h(r_0))^{d-1}\mathrm{d}s}. \tag{3.24}$$

Consider

$$\phi_t:=\varphi\circ\rho_t^{\partial},\quad \text{for all } \ t\in[0,T],$$

where

$$\varphi(r)=1+\delta\int_0^r(h(s)-h(r_0))^{1-d}\mathrm{d}s\int_{s\wedge r_0}^{r_0}(h(u)-h(r_0))^{d-1}\mathrm{d}u.$$

By an approximation argument, we may regard ϕ as $C^\infty([0,T]\times M)$-smooth. Obviously, $\phi\geq 1$ and $N_s\log\phi_s=-\lambda=-\mathrm{II}_s$, for all $s\in[0,T]$. So, $\phi\in\mathscr{D}$.

Next, we need to estimate $\inf\{\phi_t\Delta_t\phi_t\}^-$, $\inf\{\partial_t\log\phi_t\}^-$, $\|\nabla^t\phi_t\|_\infty^2$ and $\|\phi_t\|_\infty$ in terms of λ,d,r_0 and k. As h is decreasing on $[0,r_0]$, we conclude that

$$|\partial_t\log\phi_t|=\left|\frac{\delta(h(\rho_t^{\partial}\wedge r_0)-h(r_0))^{1-d}\int_{\rho_t^{\partial}\wedge r_0}^{r_0}(h(u)-h(r_0))^{d-1}\mathrm{d}u}{\phi_t}\partial_t\rho_t^{\partial}\right|\leq\frac{\delta r_0}{\phi_t}|\partial_t\rho_t^{\partial}|,\quad \rho_t^{\partial}\leq r_0.$$

Moreover, using the following formula,

$$\partial_t\rho_t^{\partial}=\frac{1}{2}\int_0^{\rho_t^{\partial}}\partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s,\ \ \rho_t^{\partial}\leq r_0,$$

where γ is the minimal curvature from x to ∂M, we obtain

$$|\partial_t\log\phi_t|\leq\delta r_0^2k,\ \ \rho_t^{\partial}\leq r_0. \tag{3.25}$$

Similarly, it holds

$$|\nabla^t\phi_t|_t^2\leq\delta^2r_0^2. \tag{3.26}$$

In addition,

$$\int_0^{r_0}(h(s)-h(r_0))^{1-d}\mathrm{d}s\int_s^{r_0}(h(u)-h(r_0))^{d-1}\mathrm{d}u\leq\int_0^{r_0}(r_0-s)\mathrm{d}s=\frac{r_0^2}{2},$$

which implies

$$\|\phi_t\|_\infty=1+\delta\int_0^{r_0}(h(s)-h(r_0))^{1-d}\mathrm{d}s\int_s^{r_0}(h(u)-h(r_0))^{d-1}\mathrm{d}u\leq 1+\frac{\delta r_0^2}{2}. \tag{3.27}$$

Moreover, since $\mathrm{II}_t=\lambda\leq 0$ and $\mathrm{Sect}_t\leq k_1$ on $\partial_{r_0}^t(M)$, according to the Laplacian comparison theorem for ρ_s^{∂} (see [20, 21]), we have

$$\Delta_t\phi_t\geq\left(\frac{(d-1)\varphi'h'}{h}+\varphi''\right)(\rho_t^{\partial})\geq-\delta,\ 0<\rho_t^{\partial}\leq r_0\left(\leq\frac{\sqrt{k_1}\pi}{2}\right),\ t\in[0,T].$$

Combining this with (3.27) implies

$$\inf\{\phi_t\Delta_t\phi_t\} \geq -\|\phi_t\|_\infty\|\Delta_t\phi_t\|_\infty \geq -\left(1+\frac{\delta r_0^2}{2}\right)\delta^2 r_0^2. \tag{3.28}$$

Concluding from (3.25), (3.26), (3.27) and (3.28), it suffices for us to estimate δ. Since $-h'$ is increasing and h is decreasing, by the FKG inequality, we have

$$\int_0^{r_0}(h(s)-h(r_0))^{d-1}\mathrm{d}s \geq \frac{-r_0\int_0^{r_0}(h(s)-h(r_0))^{d-1}h'(s)\mathrm{d}s}{-\int_0^{r_0}h'(s)\mathrm{d}s} = \frac{r_0}{d}(1-h(r_0))^{d-1}.$$

By this and (3.24), we obtain $\delta \leq -\lambda d/r_0$. Concluding all these estimates above, we have

$$K_{\phi,1}(t)^- \leq k - \frac{\lambda d}{r_0} + \left(d-\frac{3}{2}\right)\lambda^2 d^2;\ \ K_{\phi,2}(t) \leq -2\lambda dkr_0 + 2k. \tag{3.29}$$

Then,

$$K_\phi(t) \leq 2k - \frac{\lambda d}{r_0} + \left(4d-\frac{11}{2}\right)\lambda^2 d^2 - \lambda dr_0 k, \tag{3.30}$$

which leads to complete the proof. □

In addition,

$$\begin{aligned} K_\phi^{(p)}(r) &= \inf\{\phi_r^{-1}(\Delta_r\phi_r) + \partial_r\log\phi_r - (p+1)|\nabla^r\log\phi_r|_r^2\} \\ &\geq -\delta - \delta r_0 k - \delta^2 r_0^2(p+1) \\ &\geq \frac{\lambda d}{r_0} + \lambda dr_0 k - \lambda^2 r_0^2(p+1). \end{aligned} \tag{3.31}$$

Using this estimate and Corollary 3.3, we have the following result directly.

Corollary 3.7. *Under the some conditions of Theorem 3.6, we have that for $f \in C^1(M)$ such that f is constant outside a compact set and $0 \leq s < T$,*

$$|\nabla^s P_{s,T}f|_s \leq \left(1-\frac{\lambda r_0 d}{2}\right)e^{(T-s)K_p}(P_{s,t}|\nabla^T f|_T^{p/(p-1)})^{(p-1)/p},$$

where

$$K_p = -\frac{\lambda d}{r_0} - \lambda dr_0 k + \lambda^2 r_0^2(p+1). \tag{3.32}$$

Moreover, we have

$$|\nabla^s P_{s,T}f|_s^2 \leq \left(1-\frac{\lambda r_0 d}{2}\right)^2 \frac{K_2}{1-e^{2K_2(s-T)}}P_{s,T}f^2,\ \ s\in[0,T),\ f\in\mathscr{B}_b(M).$$

Now, we turn to consider system (1.7)-(3.19). By Theorem 1.2, we have

Theorem 3.8. *Under same condition of Theorem 3.6, for* $p > (\frac{2-2\lambda r_0 d}{2-\lambda r_0 d})^2$ *and* $0 \le s < T$*, the Harnack inequality*

$$(P_{s,T} f(y))^p \le P_{s,T} f^p(x) \exp\left\{ \frac{\sqrt{p}(\sqrt{p}-1)\tilde{K}\rho_s(x,y)}{8\tilde{\delta}_p[2(\sqrt{p}-1)/(2-\lambda r_0 d) - \tilde{\delta}_p](1-e^{2\tilde{K}(s-T)})} \right\}$$

holds for $\tilde{\delta}_p := \max\left\{ \frac{-\lambda r_0 d}{2-\lambda r_0 d}, \frac{\sqrt{p}-1}{2-\lambda r_0 d} \right\}$ *and*

$$\tilde{K} := -\frac{\lambda d}{r_0} + \left(4d - \frac{9}{2}\right)\lambda^2 d^2 + 2k - \lambda d k r_0.$$

Proof. It is easy to see from (3.27) that

$$\delta_T = \sup_{t\in[0,T]} (\sup \phi_t^{-1} - \inf \phi_t^{-1}) \le \frac{-\lambda r_0 d}{2-\lambda r_0 d}; \;\; \lambda_T = \inf_{[0,T]\times M} \phi^{-1} \ge \frac{2}{2-\lambda r_0 d}.$$

Combining this with the estimates obtained in the proof of Theorem 1.2, we complete the proof. □

4 Equivalent functional inequalities for curvature conditions

In this section, we present the gradient estimates for the curvature conditions (1.1), which is an extension of [12, Theorem 4.3] for the time-inhomogeneous manifold without boundary. This part is mainly based on [32, Theorem 1.1] for the case when the metric is independent of time.

Theorem 4.1. *Let* $p \in [1,\infty)$ *and* $\tilde{p} = p \wedge 2$*. Then for any* $[s,t] \subset [0,T_c)$*,* $K \in C_b([s,t]\times M)$ *and* $\sigma \in C_b([s,t]\times \partial M)$*, the following statements are equivalent to each other.*

(*i*) $\mathscr{R}_t^Z \ge K_t$ *and* $\mathrm{II}_t \ge \sigma_t$ *hold for any* $0 \le t < T_c$*.*

(*ii*) $|\nabla^s P_{s,t} f(x)|_s^p \le \mathbb{E}\{|\nabla^t f|_t^p(X_t)\exp[-p\int_s^t K(r,X_r)\mathrm{d}r - p\int_s^t \sigma(r,X_r)\mathrm{d}l_r]|X_s = x\}$ *holds for* $x \in M$*,* $0 \le s \le t < T_c$*, and* $f \in C^1(M)$ *such that* f *is constant outside a compact set.*

(*iii*) *For any* $0 \le s \le t < T_c$*,* $x \in M$ *and positive* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$\frac{\tilde{p}[P_{s,t}f^2 - (P_{s,t}f^{1/\tilde{p}})^{\tilde{p}}]}{4(\tilde{p}-1)} \le \mathbb{E}\left\{ |\nabla^t f|_t^2(X_t)\int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r}\mathrm{d}u \,\middle|\, X_s = x \right\},$$

where when $p = 1$*, the inequality is understood as its limit as* $p \downarrow 1$*:*

$$\begin{aligned} &P_{s,t}(f^2\log f^2)(x) - (P_{s,t}f^2\log P_{s,t}f^2)(x) \\ &\le 4\mathbb{E}\left\{ |\nabla^t f|_t(X_t)\int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r}\mathrm{d}u \,\middle|\, X_s = x \right\}. \end{aligned}$$

(*iv*) *For any* $0 \le s < t < T_c$*,* $x \in M$ *and positive function* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$|\nabla^s P_{s,t} f|_s^2(x)$$

$$\leq \frac{[P_{s,t}f^{\tilde{p}} - (P_{s,t}f)^{\tilde{p}}](x)}{\tilde{p}(\tilde{p}-1)\int_s^t \left(\mathbb{E}\{(P_{u,t}f)^{2-\tilde{p}}(X_u)e^{-2\int_s^u K(r,X_r)\mathrm{d}r - 2\int_s^u \sigma(r,X_r)\mathrm{d}l_r}|X_s = x\}\right)^{-1}\mathrm{d}u},$$

where when $p = 1$, the inequality is understood as its limit as $p \downarrow 1$:

$$|\nabla^s P_{s,t}f|_s^2(x) \leq \frac{[P_{s,t}(f\log f) - (P_{s,t}f)\log P_{s,t}f](x)}{\int_s^t \left(\mathbb{E}\left\{P_{u,t}f(X_u)e^{-2\int_s^u K(r,X_r)\mathrm{d}r - 2\int_s^u \sigma(r,X_r)\mathrm{d}l_r}\middle|X_s = x\right\}\right)^{-1}\mathrm{d}u}.$$

Proof. By the derivative formula established in Theorem 2.3, it is easy to derive (ii) from (i); then according to Theorem 2.7, we see that (ii)–(iv) implies (i); and finally, taking $f \in C^\infty(M)$ and f is constant outside a compact set, we derive (iii), (iv) from (ii) by a similarly discussion as in the proof of [36, Theorem 2.3.1] for the case with constant metric. We just take the proof of "(ii) $\Rightarrow$ (iii)" for example. A similar argument leads to "(ii)$\Rightarrow$ (iv)".

We again assume $s = 0$. As the boundedness of $|\nabla^{\cdot} P_{\cdot,t}f|_{\cdot}$ on $[0,t]\times M$ is verified above, by using the derivative formula in Theorem 2.3,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}P_{0,u}(P_{u,t}f^{2/p})^p(x)\\
&= p(p-1)P_{0,u}\{(P_{u,t}f^{2/p})^{p-2}|\nabla^u P_{u,t}f^{2/p}|_u^2\}\\
&\leq \frac{4(p-1)}{p}\mathbb{E}^x\Big\{(P_{u,t}f^{2/p})^{p-2}(X_u)(P_{u,t}f^{\frac{2(2-p)}{p}})(X_u)\\
&\qquad\qquad \times \mathbb{E}\left(|\nabla^t f|_t^2(X_t)e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r}\middle|\mathscr{F}_u\right)\Big\}
\end{aligned}$$

holds for $x \in M$, $0 \leq s \leq t < T_c$ and $f \in C^1(M)$ such that f is constant outside a compact set. Since $2-p \in [0,1]$, by the Jensen inequality and the Markov property, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}u}P_{0,u}(P_{u,t}f^{2/p})^p(x) \leq \frac{4(p-1)}{p}\mathbb{E}^x\left\{|\nabla^t f|_t^2(X_t)e^{-2\int_u^t K(r,X_r)\mathrm{d}r - 2\int_u^t \sigma(r,X_r)\mathrm{d}l_r}\right\}.$$

Integrating with respect to u over $[0,t]$ yields (iii) for $s = 0$. □

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, we define a cost function $C_t(x,y) = \varphi(\rho_t(x,y))$. To this cost function, we associate the Monge-Kantorovich minimization between two probability measures on M,

$$W_{C_t}(\mu,\nu) = \inf_{\eta\in\mathscr{C}(\mu,\nu)}\int_{M\times M} C_t(x,y)\mathrm{d}\eta(x,y), \tag{4.1}$$

where $\mathscr{C}(\mu,\nu)$ is the set of all probability measures on $M\times M$ with marginal $\mu,\nu \in \mathscr{P}(M)$ and $\mathscr{P}(M)$ being the space of all probability measure on M. We denote

$$W_{p,t}(\mu,\nu) = (W_{\rho_t^p}(\mu,\nu))^{1/p}$$

the Wasserstein distance associated to $p > 0$.

We now list more equivalent statements for (1.2), which is an extension of [12, Theorem 4.3] to manifolds with boundary carrying convex flows. See also [31, 33, 34] for the corresponding conclusions on the constant manifold with boundary.

Theorem 4.2. *Let $p \in [1,\infty)$, $K \in C([0,T_c))$ and $\{p_{s,t}\}_{0\le s\le t<T_c}$ be the heat kernel of $\{P_{s,t}\}_{0\le s\le t<T_c}$ associated with the volume measure μ_t with respect to the metric g_t. Then the following assertions are equivalent to each other.*

(i) (1.2) *holds.*

(ii) *For any $x,y\in M$ and $0\le s<t<T_c$,*

$$W_{p,t}(\delta_x P_{s,t},\delta_y P_{s,t}) \le \rho_s(x,y)e^{-\int_s^t K(r)\mathrm{d}r}.$$

(ii') *For any $\mu_1,\mu_2\in\mathscr{P}(M)$ and $0\le s<t<T_c$,*

$$W_{p,t}(\nu_1 P_{s,t},\nu_2 P_{s,t}) \le W_{p,s}(\nu_1,\nu_2)e^{-\int_s^t K(r)\mathrm{d}r}.$$

(iii) *When $p>1$, for any $f\in\mathscr{B}_b^+(M)$ and $0\le s<t<T_c$,*

$$(P_{s,t}f)^p(x)\le P_{s,t}f^p(y)\exp\left[\frac{p}{4(p-1)}\left(\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right)^{-1}\rho_s^2(x,y)\right].$$

(iv) *For any $f\in\mathscr{B}_b^+(M)$ with $f\ge 1$ and $0\le s\le t<T_c$,*

$$P_{s,t}\log f(x)\le \log P_{s,t}f(y)+\left(4\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right)^{-1}\rho_s^2(x,y).$$

(v) *When $p>1$, for any $0\le s\le t<T_c$ and $x,y\in M$,*

$$\int_M p_{s,t}(x,y)\left(\frac{p_{s,t}(x,y)}{p_{s,t}(y,z)}\right)^{\frac{1}{p-1}}\mu_t(\mathrm{d}z)\le \exp\left[\frac{p}{4(p-1)^2}\left(\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right)^{-1}\rho_s^2(x,y)\right].$$

(vi) *For any $0\le s<t<T_c$ and $x,y\in M$,*

$$\int_M p_{s,t}(x,y)\log\frac{p_{s,t}(x,y)}{p_{s,t}(y,z)}\mu_t(\mathrm{d}z)\le \rho_s^2(x,y)\left(4\int_s^t e^{2\int_s^r K(u)\mathrm{d}u}\mathrm{d}r\right)^{-1}.$$

(vii) *For any $0\le s<u\le t<T_c$ and $1<q_1\le q_2$ such that*

$$\frac{q_2-1}{q_1-1}=\frac{\int_s^t e^{2\int_s^\tau K(r)\mathrm{d}r}\mathrm{d}\tau}{\int_s^u e^{2\int_s^\tau K(r)\mathrm{d}r}\mathrm{d}\tau}, \tag{4.2}$$

it holds

$$\{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}}\le (P_{s,t}f^{q_1})^{\frac{1}{q_1}},\ f\in\mathscr{B}_b^+(M).$$

$(viii)$ *For any $0\le s\le u\le t<T_c$ and $0<q_2\le q_1$ or $q_2\le q_1<0$ such that* (4.2) *holds,*

$$(P_{s,t}f^{q_1})^{\frac{1}{q_1}}\le \{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}},\ \ f\in\mathscr{B}_b^+(M).$$

(*ix*) *For any* $0 \le s < u \le t < T_c$ *and* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$|\nabla^s P_{s,t} f|_s^p \le e^{-p\int_s^t K(r)\mathrm{d}r} P_{s,t} |\nabla^t f|_t^p.$$

(*x*) *For any* $0 \le s < u \le t < T_c$ *and positive function* $f \in C^1(M)$ *such that* f *is constant outside a compact set,*

$$\frac{(p\wedge 2)[P_{s,t} f^2 - (P_{s,t} f^{2/(p\wedge 2)})^{p\wedge 2}]}{4(p\wedge 2-1)} \le \int_s^t e^{-2\int_u^t K(r)\mathrm{d}r} \mathrm{d}u \cdot P_{s,t} |\nabla^t f|_t^2.$$

When $p = 1$*, the inequality reduces to the log-Sobolev inequality*

$$P_{s,t}(f^2 \log f^2) - (P_{s,t} f^2) \log P_{s,t} f^2 \le 4 \int_s^t e^{-2\int_u^t K(r)\mathrm{d}r} \mathrm{d}u \cdot P_{s,t} |\nabla^t f|_t^2.$$

Proof. First, by Theorems 2.1, 3.1 and 4.1, the inequalities (ii)–(x) can be derived from (i) by a similar discussion as in [12, Theorem 4.3] for the case without boundary.

Then, we assume (iv) and prove (i). For a fixed point $x \in M^\circ$, $t \in [0, T_c)$ and $X \in T_x M$, take $f \in C_0^\infty(M)$ such that $\nabla^t f = X$, $\mathrm{Hess}_f^t(x) = 0$ and $f = 0$ in a neighborhood of ∂M. The argument in [12, Theorem 4.4] works well for this case, i.e. $\mathscr{R}_t^Z \ge K(t)$ can be induced from (iv).

So, it only leaves for us to derive $\mathrm{II}_t \ge 0$. By Theorem 2.8, it is obvious to see that we only need to consider the term with order $\sqrt{t}$. So we do not need to care about the terms, which come from the time derivative about the metric, since they at least have order t. Therefore, by a similar procedure as in time-homogeneous case (see [32]). We conclude that ∂M is convex under the metric g_t for all $t \in [0, T_c)$. □

5 Appendix

Proof of Theorem 3.3. Without loss of generality, we also consider $s = 0$ for simplicity.

(a) By the Itô formula, we have

$$\begin{aligned} \mathrm{d}\phi_t^{-p}(X_t) &= \left\langle \nabla^t \phi_t^{-p}(X_t), u_t \mathrm{d}B_t \right\rangle_t + (L_t \phi_t^{-p}(X_t) + \partial_t \phi_t^{-p}(X_t))\mathrm{d}t + N_t \phi_t^{-p}(X_t)\mathrm{d}l_t \\ &\le \left\langle \nabla^t \phi_t^{-p}(X_t), u_t \mathrm{d}B_t \right\rangle_t - p\phi_t^{-p}(X_t)\{K_\phi^{(p)}(t)\mathrm{d}t + N_t \log \phi_t(X_t)\mathrm{d}l_t\}. \end{aligned}$$

So, $M_t := \phi_t^{-p}(X_t) \exp\left[p\int_0^t K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^t N_s \log\phi_s(X_s)\mathrm{d}l_s\right]$ is a local martingale. Thus, using the Fatou lemma and noting that $\phi_t \ge 1$, we have

$$\begin{aligned} &\mathbb{E}\left\{\phi_t^{-p}(X_t) \exp\left[p\int_0^t K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^t N_s \log\phi_s(X_s)\mathrm{d}l_s\right]\right\} \\ &\le \liminf_{n\to\infty} \mathbb{E}^x \left\{\phi_t^{-p}(X_{t\wedge\zeta_n}) \exp\left[p\int_0^{t\wedge\zeta_n} K_\phi^{(p)}(s)\mathrm{d}s + p\int_0^{t\wedge\zeta_n} N_s \log\phi_s(X_s)\mathrm{d}l_s\right]\right\} \\ &\le \phi_0^{-p}(x) \le 1. \end{aligned}$$

Therefore,

$$\mathbb{E}^x \exp\left[p\int_0^t N_s \log\phi_s(X_s)\mathrm{d}l_s\right] \le \|\phi_t\|_\infty^p e^{-p\int_0^t K_\phi^p(s)\mathrm{d}s}, \quad t \ge 0. \tag{5.1}$$

Since $\mathrm{II}_t \geq -N_t \log \phi_t$, by combining this with Theorem 2.3 for $\sigma(t,\cdot) = -N_t \log \phi_t$ and Theorem 1.2 (i), we obtain

$$\begin{aligned}
|\nabla^0 P_{0,t} f|_0^p(x) \leq & (P_{0,t} |\nabla^t f|_t^{p/(p-1)})^{(p-1)}(x) \mathbb{E}^x \|Q_t\|^p \\
\leq & (P_{0,t} |\nabla^t f|_t^{p/(p-1)}(x))^{(p-1)} \mathbb{E}^x \exp\left[p \int_0^t N_s \log \phi_s(X_s) \mathrm{d}s\right] \\
\leq & \|\phi_t\|_\infty^p (P_{0,t} |\nabla^t f|_t^{p/(p-1)})^{(p-1)}(x) e^{-p \int_0^t K_\phi^p(s) \mathrm{d}s}.
\end{aligned}$$

Therefore, the first inequality holds.

(b) Let

$$h(s) = \frac{\int_0^s \|\phi_u\|_\infty^{-2} e^{2\int_0^u K_\phi^{(2)}(r)\mathrm{d}r} \mathrm{d}u}{\int_0^t \|\phi_u\|_\infty^{-2} e^{2\int_0^u K_\phi^{(2)}(r)\mathrm{d}r} \mathrm{d}u}, \quad s \in [0,t].$$

Then the following inequality follows from the second formula in (2.7) and (5.1) for $p = 2$,

$$\begin{aligned}
|\nabla^0 P_{0,t} f|_0^2 \leq & \frac{P_{0,t} f^2}{2} \mathbb{E} \int_0^t h'(s)^2 \|Q_s\|^2 \mathrm{d}s \\
\leq & \frac{P_{0,t} f^2}{2} \mathbb{E} \int_0^t h'(s)^2 \|\phi_s\|_\infty^2 \exp\left[-2 \int_0^s K_\phi^{(2)}(r) \mathrm{d}r\right] \mathrm{d}s \\
\leq & \frac{1}{2} \left[\int_0^t \|\phi_u\|_\infty^{-2} e^{2\int_0^u K_\phi^{(2)}(r)\mathrm{d}r} \mathrm{d}u\right]^{-1} P_{0,t} f^2.
\end{aligned}$$

We complete the proof of (3.13). □

Acknowledgements The first author was supported in part by the starting-up research fund supplied by Zhejiang University of Technology (Grant No.109007329) and the Natural Science Foundation of Zhejiang University of Technology (Grant No. 2014XZ011).

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