Bull. Sci. math. 140 (2016) 541–561

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Transportation-cost inequalities on path spaces over manifolds carrying geometric flows

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ARTICLE INFO

Article history:
Received 3 February 2015
Available online 14 September 2015

MSC:
primary 60J60
secondary 58J65, 53C44

Keywords:
Geometric flow
Ricci flow
Transportation-cost inequality
Curvature
Path space
(Reflecting) L_t-diffusion processes

ABSTRACT

Let $L_t := \Delta_t + Z_t$ for a $C^{1,1}$-vector field Z on a differential manifold M possibly with a boundary ∂M, where Δ_t is the Laplacian operator induced by a time dependent metric g_t differentiable in $t \in [0, T_c)$. In this article, by constructing suitable coupling, transportation-cost inequalities on the path space of the (reflecting if $\partial M \neq \varnothing$) diffusion process generated by L_t are proved to be equivalent to a new curvature lower bound condition and the convexity of the geometric flow (i.e., the boundary keeps convex). Some of them are further extended to non-convex flows by using conformal changes of the flows. As an application, these results are applied to the Ricci flow with the umbilic boundary.

1. Introduction

In this study, we aim to establish transportation-cost inequalities associated with the uniform distance, which is on the path space of (reflecting) diffusion processes over manifolds carrying a complete geometric flow. More precisely, our base manifold is a d-dimensional differential manifold M possibly with boundary ∂M equipped with

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http://dx.doi.org/10.1016/j.bulsci.2015.08.001
0007-4497/

a family of complete Riemannian metrics $(g_t)_{t\in[0,T_c)}$ for some $T_c \in (0,\infty]$, which is C^1 in t. Let ∇^t and Ric_t be, respectively, the Levi-Civita connection and the Ricci curvature tensor associated with the metric g_t. For simplicity, we take the notation: for $X, Y \in TM$,

$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y\right\rangle_t - \frac{1}{2}\partial_t g_t(X,Y),$$

where Z_t is a $C^{1,1}$-vector field and $\langle\cdot,\cdot\rangle_t := g_t(\cdot,\cdot)$. Define the second fundamental form of the boundary with respect to g_t by

$$\mathrm{II}_t(X,Y) = -\left\langle \nabla_X^t N_t, Y\right\rangle_t, \quad \text{for all} \;\; X, Y \in T\partial M,$$

where N_t is the inward unit normal vector field of the boundary associated with the metric g_t and $T\partial M$ is the tangent space of ∂M. If $\mathrm{II}_t \geq 0$ for all $t \in [0,T_c)$, i.e., ∂M keeps convex for all $t \in [0,T_c)$, then we call $\{g_t\}$ a convex flow.

Consider the elliptic operator $L_t := \Delta_t + Z_t$ on $[0,T_c)\times M$, where Δ_t is the Laplacian operator with respect to the metric g_t and Z is a $C^{1,1}$-vector field. Let $\mu \in \mathscr{P}(M)$, where $\mathscr{P}(M)$ is the set including all probability measures on M. A (reflecting) diffusion process X^μ, generated by L_t with initial distribution μ, can be constructed as in [4]. Assume that X_t^μ is non-explosive before time T_c, by a similar discussion as in [5, Corollary 2.2], which is the case if

$$\mathcal{R}_t^Z \geq K(t), \text{ for some } K \in C([0,T_c)) \text{ and } \;\mathrm{II}_t \geq 0\,(\text{if } \partial M \neq \varnothing),\; t \in [0,T_c). \tag{1.1}$$

When $\mu = \delta_x$, we simply denote $X_t^{\delta_x} = X_t^x$. Moreover, by [4], we know that X_t^x solves the following equation

$$\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t,\;\; X_0 = x, \tag{1.2}$$

where $B_t := (B_t^1, B_t^2, \cdots, B_t^d)$ is a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$, u_t is the horizontal lift process of X_t and l_t is an increasing process supported on $\{t \geq 0 : X_t \in \partial M\}$. Note that if $\partial M = \varnothing$, then $l_t = 0$.

Given $\mu \in \mathscr{P}(M)$ and $0 \leq S < T < T_c$, let $\Pi_\mu^{[S,T]}$ be the distribution of $X_{[S,T]} := \{X_t : t \in [S,T]\}$ with initial law μ at time S. Then $\Pi_\mu^{[S,T]}$ is a probability measure on $W^{[S,T]} := C([S,T]; M)$ with σ-field $\mathcal{F}^{[S,T]}$ induced by cylindrically measurable functions. When $S = 0$, we simply denote $\Pi_\mu^T := \Pi_\mu^{[0,T]}$ and $W^T := W^{[0,T]}$. Our aim is to establish transportation-cost inequalities for $\Pi_\mu^{[S,T]}$ under some new curvature conditions, which may include the influence from the time changing of the metric.

Transportation-cost inequality was first introduced by Talagrand [15] in 1996 to bound from above the L^2-Wasserstein distance to the standard Gaussian measure on $\mathbb{R}^d$ by the relative entropy. This inequality has been extended to distributions on finite- and infinite-dimensional spaces. In particular, this inequality was established on the path space of diffusion processes with respect to several different distances (i.e., cost functions). See [7,19,20] on the path space of diffusions with the uniform distance; see [8] on the Wiener space with the Cameron–Martin distance; see [6,16] on the path space of diffusions with the L^2-distance and [17] on the Riemannian path space with intrinsic distance induced by the Malliavin gradient operator. In their previous works, the metric of the base space is fixed and the corresponding diffusion process is homogeneous. A natural question is how to extend these results to the time-inhomogeneous diffusion case on manifolds with time-changing metrics.

Indeed, since Arnaudon et al. [1] first construct the g_t-Brownian motion (i.e. the diffusion process generated by $\frac{1}{2}\Delta_t$) on manifolds with time-depending metrics, there has been tremendous interest in developing stochastic analysis on these manifolds. One of the results is the transportation-cost inequality with respect to the L^2-Wasserstein distance, which is induced by the g_t-distance (the Riemannian distance with respect to the metric g_t); see e.g. [2,4,5]. Let $\rho_t(x,y)$ be the g_t-Riemannian distance between x and y for $x, y \in M$. For $\nu, \mu \in \mathscr{P}(M)$, the L^2-Wasserstein distance of μ and ν, induced by the g_t-distance, is defined by

$$W_{2,t}(\nu,\mu) = \inf_{\eta\in\mathscr{C}(\nu,\mu)} \left\{ \int_{M\times M} \rho_t(x,y)^2 \mathrm{d}\eta(x,y) \right\}^{1/2},$$

where $\mathscr{C}(\nu,\mu)$ is the set of all couplings for ν and μ. In [4], the author has proved that the curvature condition (1.1) is equivalent to that for any $x \in M$, $0 \le s \le t < T_c$ and nonnegative f with $P_{s,t}f(x) = 1$,

$$W_{2,t}(P_{s,t}(x,\cdot), fP_{s,t}(x,\cdot))^2 \le \mathrm{e}^{-2\int_s^t K(r)\mathrm{d}r}\, W_{2,s}(\mu,\nu). \tag{1.3}$$

In this article, we extend this result to the path space $W^{[S,T]}$. More precisely, we aim to estimate some Wasserstein distance between two different probability measures on the path space $W^{[S,T]}$. The main idea is to modify the argument of [19] where fixed metric is considered.

For $0 \le S < T < T_c$, consider the following uniform norm on the path space $W^{[S,T]}$:

$$\rho_{[S,T]}(\gamma,\eta) := \sup_{t\in[S,T]} \rho_t(\gamma_t,\eta_t), \quad \gamma,\eta \in W^{[S,T]}.$$

Let $W_2^{\rho_{[S,T]}}$ be the L^2-Wasserstein distance (or L^2-transportation cost) associated with $\rho_{[S,T]}$. In general, for any $p \in [1,\infty)$ and two probability measures Π_1, Π_2 on $W^{[S,T]}$,

$$W_p^{\rho_{[S,T]}}(\Pi_1,\Pi_2) := \inf_{\pi\in\mathscr{C}(\Pi_1,\Pi_2)} \left\{ \int_{W^{[S,T]}\times W^{[S,T]}} \rho_{[S,T]}(\gamma,\eta)^p \pi(\mathrm{d}\gamma,\mathrm{d}\eta) \right\}^{1/p}$$

is the L^p-Wasserstein distance (or L^p-transportation cost) of Π_1 and Π_2, induced by the uniform norm, where $\mathscr{C}(\Pi_1,\Pi_2)$ is the set of all couplings for Π_1 and Π_2.

In this article, we present two types of transportation-cost inequalities with respect to this distance, which are proved to be equivalent to the curvature condition (1.1); see Theorem 2.1 below. In particular, we prove that (1.1) is equivalent to the following Talagrand inequality: for $0 \le S < T < T_c$ and nonnegative F on $W^{[S,T]}$ with $\Pi_\mu^{[S,T]}(F) = 1$,

$$W_2^{\rho_{[S,T]}}(F\Pi_\mu^{[S,T]}, \Pi^{[S,T]}_{\mu_F^{[S,T]}})^2 \le 4\left(\sup_{t\in[S,T]} \int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r}\,\mathrm{d}u\right) \cdot \Pi_\mu^{[S,T]}(F\log F), \tag{1.4}$$

where

$$\mu_F^{[S,T]}(\mathrm{d}x) := \Pi_x^{[S,T]}(F)\mu(\mathrm{d}x) \in \mathscr{P}(M). \tag{1.5}$$

Moreover, (1.4) implies other types of Talagrand inequalities; see Corollary 2.2.

As in [19], we then extend these results to non-convex flow by using a conformal changing of metrics; see Theorems 3.4 and 3.5 below. We would like to indicate that when it reduces to the fixed metric case, i.e., $g_t \equiv g$, Theorem 3.4 simplifies the results in [19], see Remark 3.2 for details. This result is applied to the following Ricci flow with umbilic boundary: for $\lambda \in \mathbb{R}$,

$$\begin{cases} \frac{\partial}{\partial t} g_t = 2\mathrm{Ric}_t, & \text{in } M; \\ \mathrm{II}_t = \lambda, & \text{on } \partial M. \end{cases} \tag{1.6}$$

See [14] for the short time existence of the solution to this equation and [3] for more geometric explanation of this solution.

The rest parts of the paper are organized as follows. In Section 2, we prove the equivalence of some Wasserstein distance inequalities and the condition (1.1), and in Section 3, we extend part results to the case with non-convex setting.

2. Transportation-cost inequalities

The main result of this section is presented as follows.

Theorem 2.1. *For any $p \in [1,\infty)$ and $K \in C([0,T_c))$, the following statements are equivalent to each other.*

(i) (1.1) *holds.*
(ii) *For any $0 \le S \le T < T_c$, $\mu \in \mathscr{P}(M)$ and nonnegative F with $\Pi_\mu^{[S,T]}(F) = 1$,*

$$W_2^{\rho_\infty}(F\Pi_\mu^{[S,T]}, \Pi^{[S,T]}_{\mu_F^{[S,T]}})^2 \le 4\left(\sup_{t\in[S,T]} \int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r}\,\mathrm{d}u\right) \Pi_\mu^{[S,T]}(F\log F),$$

where $\mu_F^{[S,T]} \in \mathscr{P}(M)$ is defined as in (1.5).

(ii′) *For any $x \in M$, $0 \le S \le T < T_c$ and nonnegative F with $\Pi_x^{[S,T]}(F) = 1$,*

$$W_2^{\rho_\infty}(F\Pi_x^{[S,T]}, \Pi_x^{[S,T]})^2 \le 4 \left(\sup_{t\in[S,T]} \int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r} \,\mathrm{d}u \right) \Pi_x^{[S,T]}(F \log F).$$

(iii) *For any $0 \le S \le T < T_c$ and $\mu, \nu \in \mathscr{P}(M)$,*

$$W_p^{\rho_\infty}(\Pi_\mu^{[S,T]}, \Pi_\nu^{[S,T]}) \le \left(\sup_{t\in[S,T]} \mathrm{e}^{-\int_S^t K(r)\mathrm{d}r} \right) W_{p,S}(\mu, \nu).$$

(iv) *For any $x \in M$, $0 \le S \le T < T_c$ and nonnegative f with $P_{S,T}f(x) = 1$,*

$$W_{2,T}(P_{S,T}(x,\cdot), fP_{S,T}(x,\cdot))^2 \le 4 \left(\int_S^T \mathrm{e}^{-2\int_u^T K(r)\mathrm{d}r} \,\mathrm{d}u \right) P_{S,T}(f \log f)(x).$$

Proof. Firstly, we explain that (ii) and (ii′) are equivalent to each other. By taking $\mu = \delta_x$, we have $\mu_F^T = \Pi_x^T(F)\delta_x = \delta_x$, which implies that (ii′) follows from (ii) directly. To show "(ii′) ⇒ (ii)", we first observe that by (ii′), for each $x \in M$, there exists

$$\pi_x \in \mathscr{C}\left(\frac{F}{\Pi_x^T(F)} \Pi_x^T, \Pi_x^T \right)$$

such that

$$\int_{W^{[S,T]}\times W^{[S,T]}} \rho_\infty(\gamma,\eta)^2 \pi_x(\mathrm{d}\gamma, \mathrm{d}\eta) \le 4 \left(\sup_{t\in[S,T]} \int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r} \,\mathrm{d}u \right) \Pi_x^{[S,T]}(F \log F).$$

If $x \mapsto \pi_x(G)$ is measurable for bounded continuous function G on $W^{[S,T]} \times W^{[S,T]}$, then (ii) is derived by integrating both hand sides with respect to $\mu_F^T(\mathrm{d}x)$. The proof of measurability for $x \mapsto \pi_x$ is standard, see (b) in the proof of [7, Theorem 4.1]. Thus, (ii) and (ii′) are equivalent to each other.

Secondly, we need to show that "(i) ⇒ (ii′)". We only consider the case where ∂M is non-empty. For the case without boundary, the following argument works well by taking $l_t = 0$ and $N_t = 0$. We assume without loss of generality that $S = 0$. Simply denote $X_{[0,T]}^x = X_{[0,T]}$. Let F be a positive bounded measurable function on W^T such that $\inf F > 0$ and $\Pi_x^T(F) = 1$. Let $\mathrm{d}\mathbb{Q} = F(X_{[0,T]})\mathrm{d}\mathbb{P}$. Then $\mathbb{Q}$ is a probability measure on Ω due to the fact that $\mathbb{E}F(X_{[0,T]}) = \Pi_x^T(F) = 1$. Moreover, we need the following square-integrable $\mathscr{F}_t$-martingales

$$m_t := \mathbb{E}(F(X_{[0,T]})|\mathscr{F}_t), \quad L_t := \int_0^t \frac{\mathrm{d}m_s}{m_s}, \quad t \in [0,T].$$

It is easy to see that $m_t := \mathrm{e}^{L_t - \frac{1}{2}\langle L\rangle_t}$, $t \in [0,T]$. By the martingale representation, we conclude that there exists a unique $\mathscr{F}_t$-predict process β_t on $\mathbb{R}^d$ such that $L_t = \int_0^t \langle \beta_s, \mathrm{d}B_s\rangle$ and

$$F(X_{[0,T]}) = m_T = \mathrm{e}^{\int_0^T \langle \beta_s, \mathrm{d}B_s\rangle - \frac{1}{2}\int_0^T \|\beta_s\|^2 \mathrm{d}s},$$

where $\|\cdot\|$ is the norm on $\mathbb{R}^d$. Then by the Girsanov theorem, $\tilde{B}_t := B_t - \int_0^t \beta_s \mathrm{d}s$, $t \in [0,T]$ is a d-dimensional Brownian motion under $\mathbb{Q}$.

Let Y_t solve the following SDE

$$\mathrm{d}Y_t = \sqrt{2} P^t_{X_t,Y_t} u_t \circ \mathrm{d}\tilde{B}_t + Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = x, \tag{2.1}$$

where $P^t_{X_t,Y_t}$ is the g_t-parallel displacement along the minimal geodesic from X_t to Y_t, $\tilde{l}_t$ is the local time of Y_t on ∂M and u_t is the horizontal process of X_t given in (1.2). As explained, under $\mathbb{Q}$, $\tilde{B}_t$ is a d-dimensional Brownian motion and then Π_x^T is the distribution of $Y_{[0,T]}$.

On the other hand, since $\tilde{B}_t = B_t - \int_0^t \beta_s \mathrm{d}s$, (1.2) implies

$$\mathrm{d}X_t = \sqrt{2} u_t \circ \mathrm{d}\tilde{B}_t + Z_t(X_t)\mathrm{d}t + \sqrt{2} u_t \beta_t \mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \quad X_0 = x. \tag{2.2}$$

Moreover, for any bounded measurable function G on W^T,

$$\mathbb{E}_{\mathbb{Q}} G(X_{[0,T]}) := \mathbb{E}(FG)(X_{[0,T]}) = \Pi_x^T(FG).$$

Thus, we conclude that the distribution of $X_{[0,T]}$ under $\mathbb{Q}$ coincides with $F\Pi_x^T$. Therefore,

$$W_2^{\rho_\infty}(F\Pi_x^T, \Pi_x^T)^2 \le \mathbb{E}_{\mathbb{Q}} \rho_{[0,T]}(X_{[0,T]}, Y_{[0,T]})^2 = \mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho_t(X_t, Y_t)^2. \tag{2.3}$$

Then it suffices for us to estimate $\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho_t(X_t, Y_t)^2$. To this end, we first observe that by the convexity of $(\partial M, g_t)$, we have

$$\left\langle N_t(x), \nabla^t \rho_t(\cdot, y)(x)\right\rangle_t = \left\langle N_t(x), \nabla^t \rho_t(y, \cdot)(x)\right\rangle_t \le 0, \quad x \in \partial M.$$

Combining this with the Itô formula (see [11]) and using the index lemma, we obtain from the condition $\mathcal{R}_t^Z \ge K(t)$ that

$$\begin{aligned}
&\mathrm{d}\rho_t(X_t, Y_t) \\
&\le \left\{ \int_0^{\rho_t(X_t,Y_t)} \left[-\mathrm{Ric}_t(\dot\gamma(s), \dot\gamma(s)) + \left\langle \nabla^t_{\dot\gamma(s)} Z_t, \dot\gamma(s)\right\rangle_t \right] \mathrm{d}s + (\partial_t \rho_t)(X_t, Y_t) \right\} \mathrm{d}t \\
&\quad + \sqrt{2} \left\langle u_t \beta_t, \nabla^t \rho_t(\cdot, Y_t)(X_t)\right\rangle_t \mathrm{d}t
\end{aligned}$$

$$= \left\{ \int_0^{\rho_t(X_t,Y_t)} \left[-\mathrm{Ric}_t(\dot\gamma(s),\dot\gamma(s)) + \left\langle \nabla^t_{\dot\gamma(s)} Z_t, \dot\gamma(s) \right\rangle_t + \frac{1}{2}\partial_t g_t(\dot\gamma(s),\dot\gamma(s)) \right] \mathrm{d}s \right\} \mathrm{d}t$$
$$+ \sqrt{2}\left\langle u_t\beta_t, \nabla^t \rho_t(\cdot, Y_t)(X_t)\right\rangle_t \mathrm{d}t$$
$$\le -K(t)\rho_t(X_t,Y_t)\mathrm{d}t + \sqrt{2}\left\langle u_t\beta_t, \nabla^t \rho_t(\cdot, Y_t)(X_t)\right\rangle_t \mathrm{d}t$$
$$\le (-K(t)\rho_t(X_t,Y_t) + \sqrt{2}\|\beta_t\|)\mathrm{d}t, \tag{2.4}$$

where γ is the minimal geodesic connecting X_t and Y_t associated with the metric g_t and the second equality holds true due to the following equality (see [12, Lemma 5 and Remark 6])

$$(\partial_t\rho_t)(X_t,Y_t) = \int_0^{\rho_t(X_t,Y_t)} \frac{1}{2}\partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s.$$

Since $X_0 = Y_0 = x$, (2.4) implies that for any $t \in [0,T]$,

$$\rho_t(X_t,Y_t)^2 \le \mathrm{e}^{-2\int_0^t K(r)\mathrm{d}r} \left(\sqrt{2} \int_0^t \mathrm{e}^{\int_0^s K(r)\mathrm{d}r} \|\beta_s\| \mathrm{d}s \right)^2$$
$$\le 2\,\mathrm{e}^{-2\int_0^t K(r)\mathrm{d}r} \int_0^t \mathrm{e}^{2\int_0^s K(r)\mathrm{d}r}\,\mathrm{d}s \cdot \int_0^t \|\beta_s\|^2 \mathrm{d}s.$$

Taking the maximum on both hand sides over $t \in [0,T]$, and then taking the expectation under $\mathbb{Q}$, we have

$$\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho_t(X_t,Y_t)^2 \le 2 \max_{t\in[0,T]} \int_0^t \mathrm{e}^{-2\int_s^t K(r)\mathrm{d}r}\,\mathrm{d}s \cdot \int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2 \mathrm{d}s. \tag{2.5}$$

To estimate $\int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s$, we first observe that

$$\mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2 = \mathbb{E}(m_T\|\beta_s\|^2) = \mathbb{E}(\|\beta_s\|^2\mathbb{E}(m_T|\mathscr{F}_s)) = \mathbb{E}(m_s\|\beta_s\|^2),\ \ s\in[0,T].$$

Then, by the Itô formula, we have

$$\mathrm{d}(m_t\log m_t) = (1+\log m_t)\mathrm{d}m_t + \frac{\mathrm{d}\langle m\rangle_t}{2m_t}$$
$$= (1+\log m_t)\mathrm{d}m_t + \frac{m_t^2\mathrm{d}\langle L\rangle_t}{2m_t}$$
$$= (1+\log m_t)\mathrm{d}m_t + \frac{m_t}{2}\|\beta_t\|^2\mathrm{d}t,$$

which implies

$$\int_0^T \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2 \mathrm{d}s = 2\mathbb{E}F(X_{[0,T]})\log F(X_{[0,T]}). \tag{2.6}$$

Therefore, (ii′) follows from (2.5) and (2.6).

Thirdly, we turn to prove "(i) ⇔ (iv)". If (1.1) holds, then by "(i) ⇒ (ii′)" and taking $\mu = \delta_x$ and $F(X_{[0,T]}) = f(X_T)$ into the inequality in (ii′), we obtain (iv) directly.

To prove "(iv) ⇒ (i)", let $f \in C_c^2(M)$ such that $P_{S,T}f(x) = 0$, where $C_c^2(M) := \{f \in C^2(M) : f \text{ is constant outside some compact set}\}$. Then, for small $\varepsilon > 0$ such that $f_\varepsilon := 1 + \varepsilon f \geq 0$, we obtain from [13] that

$$\begin{aligned}(P_{S,T}f^2)^2(x) \leq \Bigg[&\frac{1}{\varepsilon}\sqrt{P_{S,T}|\nabla^T f|_T^2}\,W_{2,T}(f_\varepsilon P_{S,T}(x,\cdot), P_{S,T}(x,\cdot)) \\ &+ \frac{\|\mathrm{Hess}_f^T\|_\infty}{2\varepsilon} W_{2,T}(f_\varepsilon P_{S,T}(x,\cdot), P_{S,T}(x,\cdot))^2\Bigg]^2,\end{aligned} \tag{2.7}$$

where $\|\mathrm{Hess}_f^s\|_\infty := \sup_M \|\mathrm{Hess}_f^s\|_{HS}$, $\mathrm{Hess}_f^s(X,Y) := \langle \nabla_X^s \nabla^s f, Y\rangle_s$, $X, Y \in TM$ and $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm. To estimate the term $W_{2,T}(f_\varepsilon P_{S,T}(x,\cdot), P_{S,T}(x,\cdot))$, using the condition (iv), we have

$$W_{2,T}(P_{S,T}(x,\cdot), f_\varepsilon P_{S,T}(x,\cdot))^2 \leq 4\left(\int_S^T \mathrm{e}^{-2\int_u^T K(r)\mathrm{d}r}\,\mathrm{d}u\right) P_{S,T}(f_\varepsilon \log f_\varepsilon)(x). \tag{2.8}$$

Using the Taylor expansion of $\log f_\varepsilon$ at x, we further obtain

$$\begin{aligned}P_{S,T}(f_\varepsilon \log f_\varepsilon)(x) &= P_{S,T}\left\{(1+\varepsilon f)\left(\varepsilon f - \frac{1}{2}(\varepsilon f)^2 + o(\varepsilon^2)\right)\right\}(x) \\ &= \frac{\varepsilon^2}{2} P_{S,T}f^2(x) + o(\varepsilon^2).\end{aligned}$$

Combining this with (2.7) and (2.8), and letting $\varepsilon \to 0$, we conclude that for $0 \leq S \leq T < T_c$,

$$\begin{aligned}(P_{S,T}f^2)^2(x) &\leq 4\left(\int_S^T \mathrm{e}^{-2\int_u^T K(r)\mathrm{d}r}\,\mathrm{d}u\right) P_{S,T}|\nabla^T f|_T^2(x) \cdot \lim_{\varepsilon\to 0}\frac{P_{S,T}f_\varepsilon \log f_\varepsilon(x)}{\varepsilon^2} \\ &\leq 2\left(\int_S^T \mathrm{e}^{-2\int_u^T K(r)\mathrm{d}r}\,\mathrm{d}u\right) P_{S,T}|\nabla^T f|_T^2(x) \cdot P_{S,T}f^2(x).\end{aligned}$$

This is equivalent to [4, Theorem 5.3] (ix) for $\sigma = 0$, $p = 2$ and continuous function K. Therefore, by [4, Theorem 5.3] "(ix) ⇔ (i)", (iv) implies (i).

Fourthly, we turn to show that "(ii′) ⇒ (i)". By taking $\mu = \delta_x$ and $F(X_{[S,T]}) = f(X_T)$, we obtain (iv) from (ii′) directly. Then by "(iv) ⇔ (i)", we conclude that (ii′) implies (i).

Finally, it leaves us to show that "(i) ⇔ (iii)". By taking $\mu = \delta_x$ and $F(X_{[S,T]}) = f(X_T)$, we have that $\mu_F^T = \Pi_x^T(F)\delta_x = \delta_x$ and then (iii) implies

$$W_{p,S}(\delta_x P_{S,T}, \delta_y P_{S,T}) \leq \mathrm{e}^{-\int_S^T K(r)\mathrm{d}r} \rho_S(x,y), \quad 0 \leq S < T < T_c,$$

where $P_{S,T}(x,\cdot)$ is the distribution of X_T with conditional $X_S = x$. This further implies (i) by [4, Theorem 4.2].

As the proof of "(i) ⇒ (iii)" is similar to that of Theorem 3.4, we skip it here. □

The following result is a direct consequence of Theorem 2.1.

Corollary 2.2. *For any $p \in [1,\infty)$ and $K \in C([0,T_c))$, the following statements are equivalent to each other.*

(i) (1.1) *holds.*
(ii) *For any $0 \leq S \leq T < T_c$, $\mu \in \mathscr{P}(M)$ and $F \geq 0$ with $\Pi_\mu^{[S,T]}(F) = 1$,*

$$W_2^{\rho_{[S,T]}}(F\Pi_\mu^{[S,T]}, \Pi_\mu^{[S,T]}) \leq 2\left\{\left(\sup_{t\in[S,T]}\int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r}\,\mathrm{d}u\right)\Pi_\mu^{[S,T]}(F\log F)\right\}^{1/2} + \left(\max_{t\in[S,T]} \mathrm{e}^{-\int_S^t K(r)\mathrm{d}r}\right) W_{2,S}(\mu_F^{[S,T]}, \mu).$$

(iii) *For any $\mu \in \mathscr{P}(M)$ and nonnegative function $G \in C([0,T_c))$ such that*

$$W_{2,S}(f\mu,\mu)^2 \leq G(S)\mu(f\log f), \; f \geq 0, \; \mu(f) = 1, \tag{2.9}$$

it holds

$$W_2^{\rho_{[S,T]}}(F\Pi_\mu^{[S,T]}, \Pi_\mu^{[S,T]}) \leq \left[2\sqrt{\sup_{t\in[S,T]}\int_S^t \mathrm{e}^{-2\int_u^t K(r)\mathrm{d}r}\,\mathrm{d}u} + \sqrt{G(S)}\left(\max_{t\in[S,T]} \mathrm{e}^{-\int_S^t K(r)\mathrm{d}r}\right)\right]^2 \times \Pi_\mu^{[S,T]}(F\log F)$$

for $F \geq 0$ and $\Pi_\mu^{[S,T]}(F) = 1$.

Proof. It is clear that (iii) follows from the inequality in (ii) and the condition (2.9). On the other hand, as

$$W_2^{\rho[S,T]}(F\Pi_\mu^{[S,T]},\Pi_\mu^{[S,T]}) \le W_2^{\rho[S,T]}\left(F\Pi_\mu^{[S,T]},\Pi_{\mu_F^{[S,T]}}^{[S,T]}\right) + W_2^{\rho[S,T]}\left(\Pi_{\mu_F^{[S,T]}}^{[S,T]},\Pi_\mu^{[S,T]}\right),$$

then (ii) is derived from (i) by combining Theorem 2.1(ii), (iii). By taking $\mu = \delta_x$, it is easy to see that (i) follows from each of (ii) and (iii). We then complete the proof. □

Remark 2.3. When the metric is constant, i.e., $g_t \equiv g$, the curvature condition (1.1) becomes

$$\mathrm{Ric} - \nabla Z \ge K \quad \text{and} \quad \mathrm{II} \ge 0 \ (\partial M \neq \varnothing)$$

for some constant K. Then, under this curvature condition, the inequalities in Theorem 2.1 and Corollary 2.2 are reduced to that in [19, Theorem 1.1].

3. Extension to non-convex flow and Ricci flow

As in [19], we first consider $L_t = \psi_t^2(\Delta_t + Z_t)$ with diffusion coefficient ψ_t on manifolds with convex flows; then extend to the case with non-convex flow. Finally, we apply these results into the Ricci flow with umbilic boundary.

3.1. The case with a diffusion coefficient

Let $\psi_t(\cdot) = \psi(t,\cdot) > 0$ be a smooth function on (M, g_t) and constant outside a compact set $K \subset M$. Let $\Pi_{\mu,\psi}^T$ be the distribution of the (reflecting if $\partial M \neq \varnothing$) diffusion process generated by $L_t = \psi_t^2(\Delta_t + Z_t)$ on time interval $[0,T] \subset [0,T_c)$ with initial distribution μ. Write $\Pi_{x,\psi}^T = \Pi_{\delta_x,\psi}^T$, $x \in M$ for simplicity. Moreover, for any positive function F with $\Pi_{\mu,\psi}^T(F) = 1$, let

$$\mu_{F,\psi}^T(\mathrm{d}x) = \Pi_{x,\psi}^T(F)\mu(\mathrm{d}x).$$

Write $\|\nabla^t f\|_\infty := \sup_{x\in M} |\nabla^t f(x)|_t$ for simplicity.

Theorem 3.1. *Assume that $\mathrm{II}_t \ge 0$, $\mathrm{Ric}_t^Z \ge K_1(t)$ and $\partial_t g_t \le K_2(t)$ for all $t \in [0,T_c)$ and some continuous functions K_1, K_2 on $[0,T_c)$. Let*

$$K_\psi(t) = d\|\nabla^t\psi_t\|_\infty^2 + K_1^-(t)\|\psi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t\psi_t\|_\infty + \frac{1}{2}K_2^+(t).$$

Then for some positive function F with $\Pi_{\mu,\psi}^T(F) = 1$, and $\mu \in \mathscr{P}(M)$,

$$W_2^{\rho[0,T]}(F\Pi_{\mu,\psi}^T, \Pi_{\mu_{F,\psi}^T,\psi}^T)^2$$
$$\le 4(1 + C(T,\psi))\mathrm{e}^{C(T,\psi)}\left(\int_0^T \|\psi_s\|_\infty^2\, \mathrm{e}^{2\int_s^T K_\psi(r)\mathrm{d}r}\,\mathrm{d}s\right)\Pi_{\mu,\psi}^T(F\log F)$$

holds for

$$C(T,\psi) := 4T \sup_{s\in[0,T]} \|\nabla^s \psi_s\|_\infty^2 \left(\sqrt{1 + \left(2T \sup_{s\in[0,T]} \|\nabla^s \psi_s\|_\infty^2\right)^{-1}} + 1 \right).$$

Proof. We shall only consider the case that ∂M is non-empty. As explained in the proof of Theorem 2.1 above, it suffices for us to prove this for $\mu = \delta_x$, $x \in M$. In this case, the desired inequality reduces to

$$\begin{aligned} &W_2^{\rho_{[0,T]}}(F\Pi^T_{x,\psi}, \Pi^T_{x,\psi})^2 \\ &\quad \le 4(1 + C(T,\psi))\mathrm{e}^{C(T,\psi)} \left(\int_0^T \|\psi_s\|_\infty^2 \, \mathrm{e}^{2\int_s^T K_\psi(r)\mathrm{d}r} \, \mathrm{d}s \right) \Pi^T_{x,\psi}(F \log F), \\ &F \ge 0, \ \Pi^T_{x,\psi}(F) = 1. \end{aligned}$$

Since the diffusion coefficient is non-constant, it is convenient to adopt the Itô differential d_I for the Girsanov transformation. So the reflecting L_t-diffusion process can be constructed by solving the Itô SDE

$$\mathrm{d}_I X_t = \sqrt{2}\psi_t(X_t)u_t \mathrm{d}B_t + \psi_t^2(X_t)Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \quad X_0 = x,$$

where B_t is the d-dimensional Brownian motion with natural filtration $\mathscr{F}_t$. Let β_t, $\mathbb{Q}$ and $\tilde{B}_t$ be the same as in the proof of Theorem 2.1. Then

$$\begin{aligned} &\mathrm{d}_I X_t = \sqrt{2}\psi_t(X_t)u_t \mathrm{d}\tilde{B}_t + \{\psi_t^2(X_t)Z_t(X_t) + \sqrt{2}\psi_t(X_t)u_t\beta_t\}\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \\ &X_0 = x. \end{aligned} \tag{3.1}$$

Let Y_t solve

$$\mathrm{d}_I Y_t = \sqrt{2}\psi_t(Y_t)P^t_{X_t,Y_t}u_t \mathrm{d}\tilde{B}_t + \psi_t^2(Y_t)Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = x, \tag{3.2}$$

where $\tilde{l}_t$ is the local time of Y_t on ∂M. As explained in the proof of Theorem 2.1, under $\mathbb{Q}$, the distribution of $Y_{[0,T]}$ and $X_{[0,T]}$ are $\Pi^T_{x,\psi}$ and $F\Pi^T_{x,\psi}$, respectively. Thus,

$$W_2^{\rho_{[0,T]}}(F\Pi^T_{x,\psi}, \Pi^T_{x,\psi})^2 \le \mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho_t(X_t, Y_t)^2. \tag{3.3}$$

We now turn to estimate $\mathbb{E}_{\mathbb{Q}} \max_{t\in[0,T]} \rho_t(X_t, Y_t)^2$. Note that due to the convexity of the boundary,

$$\left\langle N_t(x), \nabla^t \rho_t(\cdot, y)(x) \right\rangle_t = \left\langle N_t(x), \nabla^t \rho_t(y, \cdot)(x) \right\rangle_t \le 0, \quad x \in \partial M.$$

Combining this with (3.1), (3.2) and the Itô formula, we obtain

$$\begin{aligned}\mathrm{d}\rho_t(X_t,Y_t) \le & \sqrt{2}(\psi_t(X_t)-\psi_t(Y_t))\left\langle \nabla^t\rho_t(\cdot,Y_t)(X_t),u_t\mathrm{d}\tilde{B}_t\right\rangle_t + \Bigg\{\sum_{i=1}^d (U_i^t)^2\rho_t(X_t,Y_t) \\ & + \left\langle \psi_t^2 Z_t(Y_t), \nabla^t\rho_t(X_t,\cdot)(Y_t)\right\rangle_t + \left\langle \psi_t^2 Z_t(X_t), \nabla^t\rho_t(\cdot,Y_t)(X_t)\right\rangle_t\Bigg\}\mathrm{d}t \\ & + (\partial_t\rho_t)(X_t,Y_t)\mathrm{d}t + \left\langle \nabla^t\rho_t(\cdot,Y_t)(X_t),\sqrt{2}\psi_t(X_t)u_t\beta_t\right\rangle_t \mathrm{d}t,\end{aligned}$$

where b_t is a one-dimensional Brownian motion, $\{U_i^t\}_{i=1}^d$ are vector fields on $M\times M$ such that $\nabla^t U_i^t=0$ and

$$U_i^t(X_t,Y_t)=\psi_t(X_t)V_i^t+\psi_t(Y_t)P_{X_t,Y_t}^t V_i^t,\quad 1\le i\le d$$

for $\{V_i^t\}_{i=1}^d$ a g_t-orthonormal basis of $T_{X_t}M$ with $V_n^t=\nabla^t\rho_t(\cdot,Y_t)(X_t)$. Let $\rho_t=\rho_t(X_t,Y_t)$. Define

$$J_i^t(s)=\left(\frac{s}{\rho_t}\psi_t(Y_t)+\frac{\rho_t-s}{\rho_t}\psi_t(X_t)\right)P_{\gamma(0),\gamma(s)}^t V_i^t,\quad 1\le i\le d,$$

where $J_i^t(0)=\psi_t(X_t)V_i^t$ and $J_i^t(\rho_t)=\psi_t(Y_t)P_{X_t,Y_t}^t V_i^t$. Note that $P_{\gamma(0),\gamma(s)}^t V_i^t$ are parallel vector fields along γ.

$$\begin{aligned}&\sum_{i=1}^d (U_i^t)^2\rho_t(X_t,Y_t) \\ &\quad \le \sum_{i=1}^d\int_0^{\rho_t}\left\{|\nabla_{\dot\gamma}^t J_i^t|_t^2-\left\langle R_t(\dot\gamma,J_i^t)J_i^t,\dot\gamma\right\rangle_t\right\}(\gamma(s))\mathrm{d}s \\ &\quad \le d\|\nabla^t\psi_t\|_\infty^2\rho_t-\frac{1}{\rho_t^2}\int_0^{\rho_t}\{s\psi_t(Y_t)+(\rho_t-s)\psi_t(X_t)\}^2\mathrm{Ric}_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s. \qquad (3.4)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\psi_t^2(X_t)\left\langle Z_t(X_t),\nabla^t\rho_t(\cdot,Y_t)(X_t)\right\rangle_t+\psi_t^2(Y_t)\left\langle Z_t(Y_t),\nabla^t\rho_t(X_t,\cdot)(Y_t)\right\rangle_t \\ &\quad =\frac{1}{\rho_t^2}\int_0^{\rho_t}\frac{\mathrm{d}}{\mathrm{d}s}\{[s\psi_t(Y_t)+(\rho_t-s)\psi_t(X_t)]^2\left\langle Z_t(\gamma(s)),\dot\gamma(s)\right\rangle_t\}\mathrm{d}s \\ &\quad \le \frac{1}{\rho_t^2}\int_0^{\rho_t}[s\psi_t(Y_t)+(\rho_t-s)\psi_t(X_t)]^2\left\langle(\nabla_{\dot\gamma}^t Z_t)\circ\gamma,\dot\gamma\right\rangle_t(s)\mathrm{d}s \\ &\qquad +2\|Z_t\|_\infty\|\nabla^t\psi_t\|_\infty\|\psi_t\|_\infty\rho_t. \qquad (3.5)\end{aligned}$$

Moreover,

$$(\partial_t \rho_t)(X_t, Y_t) = \frac{1}{2}\int_0^{\rho_t} \partial_t g_t(\dot\gamma(s), \dot\gamma(s))\mathrm{d}s \le \frac{1}{2}K_2(t)\rho_t.$$

Combining this with (3.4) and (3.5), we have

$$\begin{aligned}
\mathrm{d}\rho_t(X_t, Y_t) \le{}& \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t \rho_t(\cdot, Y_t)(X_t), u_t \mathrm{d}\tilde B_t\right\rangle_t \\
&- K_1(t)\left\{\frac{1}{\rho_t^2}\int_0^{\rho_t} [s\psi_t(Y_t) + (\rho_t - s)\psi_t(X_t)]^2 \mathrm{d}s\right\}\mathrm{d}t \\
&+ \left\{d\|\nabla^t\psi_t\|_\infty^2\rho_t + 2\|Z_t\|_\infty\|\nabla^t\psi_t\|_\infty\|\psi_t\|_\infty\rho_t + \frac{1}{2}K_2(t)\rho_t\right\}\mathrm{d}t \\
&+ \sqrt{2}\|\psi_t\|_\infty\|\beta_t\|\mathrm{d}t \\
\le{}& \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t \rho_t(\cdot, Y_t)(X_t), u_t \mathrm{d}\tilde B_t\right\rangle_t \\
&+ K_\psi(t)\rho_t(X_t, Y_t)\mathrm{d}t + \sqrt{2}\|\psi_t\|_\infty\|\beta_t\|\mathrm{d}t, \qquad (3.6)
\end{aligned}$$

where

$$K_\psi(t) := d\|\nabla^t\psi_t\|_\infty^2 + K_1^-(t)\|\psi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\nabla^t\psi_t\|_\infty\|\psi_t\|_\infty + \frac{1}{2}K_2^+(t) > 0.$$

Then

$$M_t := \sqrt{2}\int_0^t \mathrm{e}^{-\int_0^s K_\psi(r)\mathrm{d}r}(\psi_s(X_s) - \psi_s(Y_s))\left\langle \nabla^s\rho_s(\cdot, Y_s)(X_s), u_s\mathrm{d}\tilde B_s\right\rangle_s$$

is a $\mathbb{Q}$-martingale such that

$$\rho_t(X_t, Y_t) \le \mathrm{e}^{\int_0^t K_\psi(r)\mathrm{d}r}\left(M_t + \sqrt{2}\int_0^t \mathrm{e}^{-\int_0^s K_\psi(r)\mathrm{d}r}\|\psi_s\|_\infty\|\beta_s\|\mathrm{d}s\right), \quad t \in [0, T].$$

Thus, by the Doob inequality, we obtain

$$\begin{aligned}
h_t &:= \mathrm{e}^{-2\int_0^t K_\psi(s)\mathrm{d}s}\,\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}\rho_s(X_s, Y_s)^2 \\
&\le (1+R)\mathbb{E}_{\mathbb{Q}}\max_{s\in[0,t]}M_s^2 + 2(1+R^{-1})\mathbb{E}_{\mathbb{Q}}\left(\int_0^t \mathrm{e}^{-\int_0^s K_\psi(r)\mathrm{d}r}\|\psi_s\|_\infty\|\beta_s\|\mathrm{d}s\right)^2 \\
&\le 4(1+R)\mathbb{E}_{\mathbb{Q}}M_t^2 + 2(1+R^{-1})\int_0^t \mathrm{e}^{-2\int_0^s K_\psi(r)\mathrm{d}r}\|\psi_s\|_\infty^2\mathrm{d}s\int_0^t \mathbb{E}_{\mathbb{Q}}\|\beta_s\|^2\mathrm{d}s
\end{aligned}$$

$$\leq 8(1+R)\sup_{s\in[0,T]}\|\nabla^s\psi_s\|_\infty^2\int_0^t h_s\mathrm{d}s$$
$$+2(1+R^{-1})\int_0^T\|\psi_s\|_\infty^2\,\mathrm{e}^{-2\int_0^s K_\psi(r)\mathrm{d}r}\,\mathrm{d}s\cdot\int_0^T\mathbb{E}_\mathbb{Q}\|\beta_s\|^2\mathrm{d}s,\quad t\in[0,T]. \tag{3.7}$$

Since $h_0=0$, by the Gronwall inequality, this implies

$$h_T=\mathrm{e}^{-2\int_0^T K_\psi(s)\mathrm{d}s}\,\mathbb{E}_\mathbb{Q}\max_{s\in[0,T]}\rho_s(X_s,Y_s)^2$$
$$\leq 2(1+R^{-1})\int_0^T\|\psi_s\|_\infty^2\,\mathrm{e}^{-2\int_0^s K_\psi(r)\mathrm{d}r}\,\mathrm{d}s$$
$$\times\exp\left[8(1+R)T\sup_{s\in[0,T]}\|\nabla^s\psi_s\|_\infty\right]\cdot\int_0^T\mathbb{E}_\mathbb{Q}\|\beta_s\|^2\mathrm{d}s. \tag{3.8}$$

Moreover, as explained in (2.6), it holds

$$\int_0^T\mathbb{E}_\mathbb{Q}\|\beta_s\|^2\mathrm{d}s=2\mathbb{E}F(X_{[0,T]})\log F(X_{[0,T]}),$$

which, together with (3.8), implies

$$\mathbb{E}_\mathbb{Q}\max_{s\in[0,T]}\rho_s(X_s,Y_s)^2$$
$$\leq 4(1+R^{-1})$$
$$\cdot\exp\left[8(1+R)T\sup_{s\in[0,T]}\|\nabla^s\psi_s\|_\infty^2\right]\int_0^T\|\psi_s\|_\infty^2\,\mathrm{e}^{2\int_s^T K_\psi(r)\mathrm{d}r}\,\mathrm{d}s\cdot\Pi_{x,\psi}^T(F\log F).$$

Combining this with (3.3), and taking

$$R=\frac{1}{2}\left[\sqrt{1+\left(2T\sup_{s\in[0,T]}\|\nabla^s\psi_s\|_\infty^2\right)^{-1}}-1\right]$$

into the term on right hand of the above inequality, we complete the proof. □

Remark 3.2.

(1) We would like to indicate that in [19], the author used the FKG inequality to deal with the term $\mathbb{E}_\mathbb{Q}\max_{s\in[0,t]}\rho^2(X_s,Y_s)$. Here, we apply the Gronwall inequality directly, which leads to the wanted estimates in a simple way.

(2) Using $\mathcal{R}_t^Z \geq K_1(t)$ in place of $\mathrm{Ric}_t^Z \geq K_1(t)$ for some continuous function K_1 on $[0, T_c)$, the assertion of Theorem 3.1 still holds with the following $\tilde{K}_\psi$:

$$\tilde{K}_\psi(t) = d\|\nabla^t\psi_t\|_\infty^2 + K_1^-(t)\|\psi_t\|_\infty^2 + 2\|Z_t\|_\infty\|\psi_t\|_\infty\|\nabla^t\psi_t\|_\infty + \frac{1}{2}(1 + \|\psi_t\|_\infty^2)K_2^+(t).$$

The coupling method also implies the following result.

Theorem 3.3. *In the situation of Theorem 3.1, it holds*

$$W_2^{\rho_{[0,T]}}(\Pi_{\nu,\psi}^T, \Pi_{\mu,\psi}^T) \leq 2\,\mathrm{e}^{\int_0^T (K_\psi(t) + \|\nabla^t\psi_t\|_\infty^2)\mathrm{d}t}\, W_{2,0}(\nu, \mu).$$

Proof. As explained in the proof of [19, Theorem 1.1] "(6) ⇒ (5)", we only consider $\nu = \delta_x$, and $\nu = \delta_y$. Let X_t and Y_t solve the following SDEs respectively.

$$\begin{aligned}
\mathrm{d}_I X_t &= \sqrt{2}\psi_t(X_t)u_t\mathrm{d}B_t + \psi_t^2(X_t)Z_t(X_t)\mathrm{d}t + N_t(X_t)\mathrm{d}l_t, \quad X_0 = x;\\
\mathrm{d}_I Y_t &= \sqrt{2}\psi_t(Y_t)P_{X_t,Y_t}^t u_t\mathrm{d}B_t + \psi_t^2(Y_t)Z_t(Y_t)\mathrm{d}t + N_t(Y_t)\mathrm{d}\tilde{l}_t, \quad Y_0 = y.
\end{aligned}$$

Then, as explained in Theorem 3.1, by the Itô formula,

$$\mathrm{d}\rho_t(X_t, Y_t) \leq \sqrt{2}(\psi_t(X_t) - \psi_t(Y_t))\left\langle \nabla^t\rho_t(\cdot, Y_t)(X_t), u_t\mathrm{d}B_t\right\rangle_t + K_\psi(t)\rho_t(X_t, Y_t)\mathrm{d}t.$$

Therefore,

$$\rho_t(X_t, Y_t) \leq \mathrm{e}^{\int_0^t K_\psi(s)\mathrm{d}s}(\hat{M}_t + \rho_0(x, y)),\ t \geq 0 \tag{3.9}$$

for $\hat{M}_t := \sqrt{2}\int_0^t \mathrm{e}^{-\int_0^s K_\psi(u)\mathrm{d}u}(\psi_s(X_s) - \psi_s(Y_s))\left\langle \nabla^s\rho_s(\cdot, Y_s)(X_s), u_s\mathrm{d}B_s\right\rangle_s$. By this, we arrive at

$$\begin{aligned}
&W_2^{\rho_{[0,T]}}(\Pi_{x,\psi}^T, \Pi_{y,\psi}^T)^2\\
&\leq \mathbb{E}\max_{t\in[0,T]}\rho_t(X_t, Y_t)^2\\
&\leq \mathrm{e}^{2\int_0^T K_\psi(t)\mathrm{d}t}\,\mathbb{E}\max_{t\in[0,T]}(\hat{M}_t + \rho_0(x, y))^2\\
&\leq 4\,\mathrm{e}^{2\int_0^T K_\psi(t)\mathrm{d}t}\,\mathbb{E}(\hat{M}_T + \rho_0(x, y))^2 = 4\,\mathrm{e}^{2\int_0^T K_\psi(t)\mathrm{d}t}\,\mathbb{E}(\hat{M}_T^2 + \rho_0^2(x, y))\\
&\leq 4\,\mathrm{e}^{2\int_0^T K_\psi(t)\mathrm{d}t}\left(\rho_0(x, y)^2 + 2\int_0^T \mathrm{e}^{-2\int_0^t K_\psi(s)\mathrm{d}s}\,\|\nabla^t\psi_t\|_\infty^2\mathbb{E}\rho_t(X_t, Y_t)^2\mathrm{d}t\right)\\
&\leq 4\,\mathrm{e}^{2\int_0^T K_\psi(t)\mathrm{d}t}\left(\rho_0(x, y)^2 + 2\int_0^T \mathrm{e}^{-2\int_0^t K_\psi(s)\mathrm{d}s}\,\|\nabla^t\psi_t\|_\infty^2\mathbb{E}\max_{s\in[0,t]}\rho_s(X_s, Y_s)^2\mathrm{d}t\right)\\
&\leq 4\,\mathrm{e}^{2\int_0^T (K_\psi(t) + \|\nabla^t\psi_t\|_\infty^2)\mathrm{d}t}\cdot\rho_0(x, y)^2,
\end{aligned}$$

where the third inequality is due to the Doob inequality and the last inequality is due to the Gronwall inequality. This implies the desired inequality for $\mu = \delta_x$ and $\nu = \delta_y$. $\square$

3.2. Non-convex flow

By using a proper conformal change of the geometric flow, we are able to establish the following transportation-cost inequality on manifolds carrying a non-convex geometric flow. Let

$$\mathscr{D} = \{\phi \in C_b^2([0, T_c) \times M) : \inf \phi_t = 1,\ \mathrm{II}_t \geq -N_t \log \phi_t\}.$$

Assume that $\mathscr{D} \neq \varnothing$ and for some $K_1, K_2 \in C([0, T_c))$,

$$\mathrm{Ric}_t^Z \geq K_1(t) \ \text{ and } \ \partial_t g_t \leq K_2(t) \tag{3.10}$$

hold. Let $\phi_t \in \mathscr{D}$. Then, by [18, Lemma 2.1], ∂M becomes convex under $\tilde{g}_t = \phi_t^{-2} g_t$. Note that we do not need the condition "$\nabla^t \phi_t \parallel N_t$", which is included in [18, Lemma 2.1], since we find that it is not used in this proof. Let $\tilde{\Delta}_t$ and $\tilde{\nabla}^t$ be, respectively, the Laplacian and gradient operator induced by the new metric $\tilde{g}_t$. As $\phi_t \geq 1$, $\rho_t(x, y)$ is larger than $\tilde{\rho}_t(x, y)$, which is denoted as the Riemannian $\tilde{g}_t$-distance between x and y.

Theorem 3.4. *Let $\partial M \neq \varnothing$ and $\mathrm{II}_t \geq -\sigma(t)$ for some positive function $\sigma \in C([0, T_c))$. Assume (3.10) holds. Let $\phi \in \mathscr{D}$ such that*

$$K_\phi(t) := d\|\nabla^t \phi_t\|_\infty^2 + K_{\phi,1}^-(t) + 2\|\phi_t Z_t + (d-2)\nabla^t \phi_t\|_\infty \|\nabla^t \phi_t\|_\infty + \frac{1}{2} K_{\phi,2}^+(t) < \infty,$$

where

$$K_{\phi,1}(t) := \inf_M \left\{ \phi_t^2 K_1(t) + \frac{1}{2} L_t \phi_t^2 - |\nabla^t \phi_t^2|_t |Z_t|_t - (d-2)|\nabla^t \phi_t|_t^2 \right\},$$
$$K_{\phi,2}(t) := \sup_M \{-2\partial_t \log \phi_t\} + K_2(t).$$

Then for any $\mu \in \mathscr{P}(M)$ and nonnegative function F with $\Pi_\mu^T(F) = 1$,

$$W_2^{\rho_{[0,T]}}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T) \leq 4(1 + C(T,\phi))\, \mathrm{e}^{C(T,\phi)} \left(\sup_{t\in[0,T]} \|\phi_t\|_\infty^2 \right) \int_0^T \mathrm{e}^{2\int_s^T K_\phi(r)\mathrm{d}r}\, \mathrm{d}s \cdot \Pi_\mu^T(F \log F)$$

holds for

$$C(T,\phi) = 4T \sup_{s\in[0,T]} \|\nabla^s \phi_s\|_\infty^2 \left(\sqrt{1 + \left(2T \sup_{s\in[0,T]} \|\nabla^s \phi_s\|_\infty^2 \right)^{-1}} + 1 \right).$$

Proof. From the proof of [4, Theorem 1.2], we know that $L_t = \phi_t^{-2}(\tilde{\Delta}_t + \tilde{Z}_t)$, where $\tilde{Z}_t = \phi_t^2 Z_t + \frac{d-2}{2}\nabla^t \phi_t^2$, and

$$\widetilde{\mathrm{Ric}}_t^{\tilde{Z}} \geq K_{\phi,1}(t), \quad \partial_t \tilde{g}_t \leq K_{\phi,2}(t).$$

Let K_ψ be the same as in Theorem 3.4 for the manifold equipped with $\{\tilde{g}_t\}_{t\in[0,T_c)}$. Then for $L_t = \psi_t^2(\tilde{\Delta}_t + \tilde{Z}_t)$, where $\psi_t = \phi_t^{-1}$, it is clear from $\phi_t \geq 1$ that $K_\psi(t) \leq K_\phi(t)$ and $C(T,\psi) \leq C(T,\phi)$, where K_ψ and $C(T,\psi)$ are defined as in Theorem 3.1 with $\psi_t = \phi_t^{-1}$. Hence, it follows from Theorem 3.1 that for any $F \geq 0$ with $\Pi_\mu^T(F) = 1$,

$$W_2^{\tilde{\rho}_{[0,T]}}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T)^2 \leq 4(1 + c(T,\phi))\,\mathrm{e}^{c(T,\phi)} \int_0^T \mathrm{e}^{2\int_s^T K_\phi(r)\mathrm{d}r}\,\mathrm{d}s \cdot \Pi_\mu^T(F\log F),$$

where $\tilde{\rho}_\infty$ is the uniform distance on W^T induced by the metric $\tilde{g}_t$. The proof is completed by using the fact that $\rho_{[0,T]} \leq \sup_{t\in[0,T]} \|\phi_t\|_\infty \tilde{\rho}_{[0,T]}$. $\square$

As explained in the proof above, $K_\psi(t) \leq K_\phi(t)$ and $\tilde{\rho}_t \leq \rho_t \leq \|\phi_t\|_\infty \tilde{\rho}_t$, which imply the following result by using Theorem 3.3 with $\psi = \phi^{-1}$.

Theorem 3.5. *In the situation of Theorem 3.4, it holds*

$$W_2^{\rho_{[0,T]}}(\Pi_\mu^T, \Pi_\nu^T) \leq 2\left(\sup_{t\in[0,T]} \|\phi_t\|_\infty\right) \mathrm{e}^{\int_0^T (K_\phi(t) + \|\nabla^t \phi_t\|_\infty^2)\mathrm{d}t}\, W_{2,0}(\mu,\nu),$$

for $\mu,\nu \in \mathscr{P}(M)$ and $T \in [0,T_c)$.

3.3. Applications to Ricci flow with umbilic boundary

As an application of Theorems 3.4 and 3.5, we now turn to consider the Ricci flow with umbilic boundary (see (1.6)).

Suppose $\{g_t\}_{t\in[0,T_c)}$ is a complete solution to the equation (1.6). When $\mathrm{II}_t = \lambda \geq 0$ in (1.6), by Theorem 2.1 "(i) implies (ii), (iii)", we obtain that for $0 \leq S \leq T < T_c$, $\mu \in \mathscr{P}(M)$ and nonnegative F with $\Pi_\mu^T(F) = 1$,

$$W_2^{\rho_{[S,T]}}\left(F\Pi_\mu^{[S,T]}, \Pi_{\mu_F^{[S,T]}}^{[S,T]}\right)^2 \leq 2\left[1 - \mathrm{e}^{-2(T-S)}\right] \Pi_\mu^{[S,T]}(F\log F). \tag{3.11}$$

Moreover, for $\mu,\nu \in \mathscr{P}(M)$, it holds

$$W_p^{\rho_{[S,T]}}(\Pi_\mu^{[S,T]}, \Pi_\nu^{[S,T]}) \leq W_{p,S}(\mu,\nu).$$

From this and (3.11), it is easy to see that these results look like those on Ricci flat manifolds.

For the case $\lambda < 0$, i.e., the boundary is non-convex, we need more information about the boundary. Let $\rho_t^\partial(x)$ be the distance between x and ∂M with respect to the metric g_t. Our following discussion needs the following curvature condition:

(H) There exist positive constants r_0, k and k_1 such that $|\mathrm{Ric}_t| \le k$ and on the set $\partial_{r_0}^t M := \{x \in M : \rho_t^\partial(x) \le r_0\}$, ρ_t^∂ is smooth and $\mathrm{Sect}_t \le k_1$.

Within this condition, Theorems 3.4 and 3.5 imply the following result.

Theorem 3.6. *Let $d \ge 2$. Suppose $\{g_t\}_{t\in[0,T_c)}$ is a complete solution to (1.6) with $\lambda < 0$. For $T \in (0, T_c)$, let Π_μ^T be the distribution of $X_{[0,T]}^\mu$, where X_t^μ is a Brownian motion generated by Δ_t with initial law μ. Assume* **(H)** *holds for $t \in [0, T]$ and $r_0 \le \frac{\pi}{2\sqrt{k_1}}$. Let*

$$K := -\left(\frac{1}{r_0} + \frac{3}{2} r_0 k\right)\lambda d + \left(4d - \frac{11}{2}\right)\lambda^2 d^2 + 2k;$$
$$C(T, \lambda, d) := 2\sqrt{4T^2\lambda^4 d^4 + 2T\lambda^2 d^2} + 4T\lambda^2 d^2.$$

Then for any nonnegative function F on W^T with $\Pi_\mu^T(F) = 1$ and $\mu \in \mathscr{P}(M)$, it holds

$$\begin{aligned} & W_2^{\rho_{[0,T]}}(F\Pi_\mu^T, \Pi_{\mu_F^T}^T) \\ & \le (C(T, \lambda, d) + 1)\, \mathrm{e}^{C(T,\lambda,d)} (2 - \lambda d r_0)^2 \frac{\mathrm{e}^{2KT} - 1}{2K} \Pi_\mu^T(F \log F). \end{aligned}$$

Moreover, for any $\mu, \nu \in \mathscr{P}(M)$, we have

$$W_2^{\rho_{[0,T]}}(\Pi_\mu^T, \Pi_\nu^T) \le (2 - \lambda d r_0)\, \mathrm{e}^{(K+\lambda^2 d^2)T}\, W_{2,0}(\mu, \nu).$$

If the condition **(H)** holds, then $\mathrm{Ric}_t \le k$ for some $k \in \mathbb{R}$, which implies that $K_1(t) = -k$ and $K_2(t) = 2k$ in (3.10). Thus, if there exists $\phi \in \mathscr{D}$ such that

$$\tilde{K}_\phi(t) := \inf\{\phi_t \Delta_t \phi_t\}^- + \|\partial_t \log \phi_t\|_\infty + k(1 + \|\phi_t\|_\infty^2) + (4d - 6)\|\nabla^t \phi_t\|_\infty^2 < \infty,$$

then Theorems 3.4 and 3.5 hold by replacing $K_\phi(t)$ with $\tilde{K}_\phi(t)$. Now, it leaves us to estimate $\tilde{K}_\phi(t)$ to complete the proof of Theorem 3.6.

Proof of Theorem 3.6. Under assumption **(H)**, to estimate $\tilde{K}_\phi(t)$, we construct a proper $\phi \in C^{1,2}([0, T] \times M)$ such that $\phi \in \mathscr{D}$ first. Let

$$h(s) = \cos(\sqrt{k_1}\, s), \quad s \ge 0. \tag{3.12}$$

Then $0 \le h(s) \le 1$ for $s \in [0, \frac{\pi}{2\sqrt{k_1}}]$. Moreover, let

$$\delta = \delta(r_0, \lambda, k_1) = \frac{-\lambda(1 - h(r_0))^{d-1}}{\int_0^{r_0} (h(s) - h(r_0))^{d-1} \mathrm{d}s}. \tag{3.13}$$

Consider $\phi_t := \varphi \circ \rho_t^\partial$, $t \in [0, T]$, where

$$\varphi(r) = 1 + \delta \int_0^r (h(s) - h(r_0))^{1-d} \mathrm{d}s \int_{s\wedge r_0}^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u.$$

By an approximation argument we may regard ϕ as $C^\infty([0,T]\times M)$-smooth. Obviously, $\phi \geq 1$ and $N_s \log \phi_s = -\lambda = -\mathrm{II}_s$ for all $s \in [0,T]$. Since $\mathrm{II}_t = \lambda \leq 0$ and $\mathrm{Sect}_t \leq k_1$ on $\partial_{r_0}^t(M)$, according to the Laplacian comparison theorem for ρ_s^∂ (see [9,10]), we have

$$\Delta_t \phi_t \geq \left(\frac{d\varphi' h'}{h} + \varphi''\right)(\rho_t^\partial) \geq -\delta, \quad t \in [0,T], \quad \rho_t^\partial \leq r_0.$$

As h is decreasing on $[0, r_0]$, we conclude that

$$\begin{aligned} |\partial_t \log \phi_t| = \left|\frac{\partial_t \phi_t}{\phi_t}\right| &= \left|\frac{\delta \left[h(\rho_t^\partial \wedge r_0) - h(r_0)\right]^{1-d} \int_{\rho_t^\partial \wedge r_0}^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u}{\phi_t} \partial_t \rho_t^\partial\right| \\ &\leq \frac{\delta r_0}{\phi_t} |\partial_t \rho_t^\partial|, \quad \rho_t^\partial \leq r_0. \end{aligned} \tag{3.14}$$

Moreover, taking the following formula into the above inequality, we have

$$\partial_t \rho_t^\partial = \frac{1}{2} \int_0^{\rho_t^\partial} \partial_t g_t(\dot\gamma(s), \dot\gamma(s)) \mathrm{d}s = \frac{1}{2} \int_0^{\rho_t^\partial} \mathrm{Ric}_t(\dot\gamma(s), \dot\gamma(s)) \mathrm{d}s, \quad \rho_t^\partial \leq r_0,$$

where γ is the minimal curvature from x to ∂M. Combining this with (3.14), we obtain

$$|\partial_t \log \phi_t| \leq \delta r_0^2 k, \quad \rho_t^\partial \leq r_0.$$

Similarly, we have $|\nabla^t \phi_t|_t^2 \leq \delta^2 r_0^2$. In addition,

$$\int_0^{r_0} (h(s) - h(r_0))^{1-d} \mathrm{d}s \int_s^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u \leq \int_0^{r_0} (r_0 - s) \mathrm{d}s = \frac{r_0^2}{2}, \tag{3.15}$$

which implies

$$\|\phi_t\|_\infty = 1 + \delta \int_0^{r_0} (h(s) - h(r_0))^{1-d} \mathrm{d}s \int_s^{r_0} (h(u) - h(r_0))^{d-1} \mathrm{d}u \leq 1 + \frac{\delta r_0^2}{2}.$$

Thus, we conclude that

$$\tilde K_\phi(t) \leq \delta + \delta k r_0^2 + \left(2 + \frac{\delta r_0^2}{2}\right) k + \left(4d - \frac{11}{2}\right) \delta^2 r_0^2. \tag{3.16}$$

Now, it leaves us to estimate δ. Since $-h'$ is increasing and h is decreasing, by the FKG inequality, we have

$$\int_0^{r_0} (h(s) - h(r_0))^{d-1} \mathrm{d}s \geq \frac{-r_0 \int_0^{r_0} (h(s) - h(r_0))^{d-1} h'(s) \mathrm{d}s}{-\int_0^{r_0} h'(s) \mathrm{d}s} = \frac{r_0}{d} (1 - h(r_0))^{d-1}.$$

From this and (3.15), we deduce that $\delta \leq -\lambda d / r_0$, which, together with (3.16), implies

$$\tilde{K}_\phi(t) \leq -\left(\frac{1}{r_0} + \frac{3}{2} r_0 k\right) \lambda d + \left(4d - \frac{11}{2}\right) \lambda^2 d^2 + 2k;$$
$$C(T, \phi) \leq 2\sqrt{4T^2 \lambda^4 d^4 + 2T\lambda^2 d^2} + 4T\lambda^2 d^2.$$

Combining this with Theorems 3.4 and 3.5, we complete the proof. □

Conflict of interest statement

The author declares that she has no conflict of interests.

Acknowledgements

The author would like to thank the anonymous reviewers for their valuable comments and suggestions. The author was supported in part by the starting-up research fund supplied by Zhejiang University of Technology (Grant No. 109007329), NSFC (Grant No. 11501508) and the Natural Science Foundation of Zhejiang University of Technology (Grant No. 2014XZ011).

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