A modal logic for games with lies

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The RÉNYI – ULAM game
A searching game with lies

1. ALICE chooses an element in \( \{1, \ldots, M\} \).

2. BOB tries to guess this number by asking Yes/No questions.

3. ALICE is allowed to lie \( n - 1 \) times in her answers.

BOB tries to guess ALICE’s number as fast as possible.
The game is around for more than 50 years

Finding an optimal strategy:


Using logic and algebras to model the states of the games:

Tools for the algebraic/logical approach

Static model
Łukasiewicz logic  MV-algebras

Dynamic model
Modal logic  Kripke semantics
Łukasiewicz \((n + 1)\)-valued logic and MV\(_n\)-algebras

\[
\mathcal{L} = \{\neg, \rightarrow, 1\}
\]

\(-x := 1 - x, \quad x \rightarrow y := \min(1, 1 - x + y)\)

\[
\begin{aligned}
\phi &\in \text{CPL} \\
\iff & \quad 2 \models \phi \\
\end{aligned}
\]

\[
\begin{aligned}
\text{A is a Boolean algebra} \\
\iff & \quad A \models \text{CPL}
\end{aligned}
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Łukasiewicz \((n+1)\)-valued logic and \(MV_n\)-algebras

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\[-x := 1 - x, \quad x \rightarrow y := \min(1, 1 - x + y)\]

\[\phi \in \text{CPL} \iff 2 \models \phi\]

\[\text{A is a Boolean algebra} \iff A \models \text{CPL}\]

\[\mathcal{L}_n \]

\[\text{A is a } MV_n\text{-algebra} \iff A \models \text{PŁ}\]
Łukasiewicz \((n + 1)\)-valued logic and MV\(_n\)-algebras

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\begin{align*}
\phi &\in \text{CPL} \\
\implies &
\begin{cases}
2 \models \phi \\
\end{cases}
\end{align*}
\]

\(A\) is a Boolean algebra

\[
\begin{align*}
\phi &\in \text{PL}_n \\
\iff &
\begin{cases}
\mathcal{L}_n \models \phi \\
\end{cases}
\end{align*}
\]

\(A\) is a MV\(_n\)-algebra

Static model
Łukasiewicz logic  MV-algebras

Dynamic model
Modal logic  Kripke semantics
An algebraic encoding of the states of knowledge

1. Knowledge space $K = \mathcal{L}_n^M$.

2. A state of knowledge $s \in \mathcal{L}_n^M : s(m)$ is the 'distance' between $m$ and the set of numbers that can be discarded.

3. A question $Q$ is a subset of $\{1, \ldots, M\}$.

4. The positive answer to $Q$ is the map $f_Q : M \to \{\frac{n-1}{n}, 1\}$ defined by

   $$f_Q(m) = 1 \iff m \in Q$$

The negative answer to $Q$ is the map $f_{M \setminus Q}$. 
\[ x \oplus y := (\neg x \rightarrow y) \]
\[ = \min(1, x + y) \quad \text{in } \mathcal{L}_n \]

\[ x \odot y := \neg(\neg x \oplus \neg y) \]
\[ = \max(0, x + y - 1) \quad \text{in } \mathcal{L}_n \]

**Proposition.** If ALICE answers positively to \( Q \) at state of knowledge \( s \) then the new state of knowledge is

\[ s' := s \odot f_Q. \]
Static model
Łukasiewicz logic  MV-algebras

Dynamic model
Modal logic  Kripke semantics
A dynamic model for every instance of the game

The model only talks about states of an instance of the game.

\[ S_0 \quad S_1 \quad \ldots \quad S_{k-1} \quad S_k \]
A dynamic model for every instance of the game

The model only talks about *states* of an instance of the game.

We want a language to talk about *whole instances* of the game.
A dynamic model for every instance of the game

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\[ S_0 \quad S_1 \quad \ldots \quad S_{k-1} \quad S_k \]

We want a language to talk about whole instances of the game.

\[ Q \quad \quad \quad \quad \quad Q' \]

\[ S_0 \quad S_1 \quad \ldots \quad S_{k-1} \quad S_k \]

We want a language to talk about all instances of any game.

\[ Q_2 \quad Q_1 \quad \quad \quad \quad \quad Q_3 \quad Q_4 \]
Static model
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We need KRIPKE models with many-valued worlds

\[ \Pi_0 = \{ \text{atomic programs} \} \]
\[ \text{Prop} = \{ \text{propositional variables} \} \]

**Definition.** A *(dynamic n + 1-valued) KRIPKE model* is

\[ \mathcal{M} = \langle W, R, \text{Val} \rangle \]

where

- \( W \not= \emptyset \),
- \( R \) maps any \( a \in \Pi_0 \) to \( R_a \subseteq W \times W \),
- \( \text{Val}(u, p) \in \mathcal{L}_n \) for any \( u \in W \) and any \( p \in \text{Prop} \).
The Rényi-Ulam game has a Kripke model

Language:

- \( \text{Prop} = \{ p_m \mid m \in M \} \) where
  \[
  p_m \equiv \text{how } m \text{ is far from the set of rejected elements.}
  \]

- \( \Pi_0 = \{ m \mid m \in M \} \).

Model:

- \( W = \mathcal{L}_n^M \) is the knowledge space.
- \((s, t) \in R_m \) if \( t = s \odot f_{\{m\}} \)
- \( \text{Val}(s, p_m) = s(m) \).
A modal language for the Kripke models

Programs $\alpha \in \Pi$ and formulas $\phi \in \text{Form}$ are defined by

Formulas $\phi ::= p \mid 0 \mid \phi \to \phi \mid \neg \phi \mid [\alpha] \phi$

Programs $\alpha ::= a \mid \phi? \mid \alpha; \alpha \mid \alpha \cup \alpha \mid \alpha^*$

where $p \in \text{Prop}$ and $a \in \Pi_0$.

<table>
<thead>
<tr>
<th>Word</th>
<th>Reading</th>
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<tbody>
<tr>
<td>$\alpha; \beta$</td>
<td>$\alpha$ followed by $\beta$</td>
</tr>
<tr>
<td>$\alpha \cup \beta$</td>
<td>$\alpha$ or $\beta$</td>
</tr>
<tr>
<td>$\alpha^*$</td>
<td>any number of execution of $\alpha$</td>
</tr>
<tr>
<td>$\phi?$</td>
<td>test $\phi$</td>
</tr>
<tr>
<td>$[\alpha]$</td>
<td>after any execution of $\alpha$</td>
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Interpreting formulas in Kripke Models

Val(·, ·) and $R$ are extended by induction:

- In a truth functional way for $\neg$ and $\rightarrow$,
- $\text{Val}(u, [\alpha]\psi) := \bigwedge\{\text{Val}(v, \psi) \mid (u, v) \in R_\alpha\}$,
- $R_{\alpha;\beta} := R_\alpha \circ R_\beta$ and $R_{\alpha \cup \beta} := R_\alpha \cup R_\beta$,
- $R_\phi := \{(u, u) \mid \text{Val}(u, \phi) = 1\}$,
- $R_\alpha^* := (R_\alpha)^* = \bigcup_{k \in \omega} R_\alpha^k$.

Definition. $\mathcal{M}, u \models \phi$ if $\text{Val}(u, \phi) = 1$ and $\mathcal{M} \models \phi$ if $\mathcal{M}, u \models \phi$ for every $u \in W$. 
Interpreting formulas in Kripke Models

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Definition. $\mathcal{M}, u \models \phi$ if $\text{Val}(u, \phi) = 1$ and $\mathcal{M} \models \phi$ if $\mathcal{M}, u \models \phi$ for every $u \in W$.

$T_n := \{\phi \mid \mathcal{M} \models \phi \text{ for every Kripke model } \mathcal{M}\}$

Find an axiomatization of $T_n$. 
Static model
Łukasiewicz logic    MV-algebras

Dynamic model
Modal logic    Kripke semantics
Modal extensions of Łukasiewicz \((n + 1)\)-valued logic

\[
\mathcal{L}_\Box = \{\Box, \neg, \to, 1\}, \quad \mathcal{M} = \langle W, R, \text{Val} \rangle
\]

**Theorem.** The set \(K_n := \{\phi \in \text{Form}_{\mathcal{L}_\Box} \mid \phi \text{ is a tautology} \} \) is the smallest subset of \(\text{Form}_{\mathcal{L}_\Box} \) that

- contains tautologies of Łukasiewicz \((n + 1)\)-valued logic
- contains \((K)\) \(\Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)\)
- contains
  \[
  \Box(\phi \star \phi) \leftrightarrow (\Box \phi) \star (\Box \phi)
  \]
  for \(\star \in \{\odot, \oplus\}\)
- is closed under MP, substitution and generalization.
Static model
Łukasiewicz logic MV-algebras

Dynamic model
Modal logic Kripke semantics

\[ T_n := \{ \phi \mid \mathcal{M} \models \phi \text{ for every Kripke model } \mathcal{M} \} \]
There are three ingredients in the axiomatization

**Definition.** PDL$_n$ is the smallest set of formulas that contains formulas in Ax$_1$, Ax$_2$, Ax$_3$ and closed for the rules in Ru$_1$, Ru$_2$.

<table>
<thead>
<tr>
<th>Łukasiewicz $n + 1$-valued logic</th>
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<tr>
<td><strong>Ax$_1$</strong></td>
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<td><strong>Ru$_1$</strong></td>
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<tr>
<th>Crisp modal $n + 1$-valued logic</th>
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<tbody>
<tr>
<td><strong>Ax$_2$</strong></td>
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<td><strong>Ru$_2$</strong></td>
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Program constructions

| $\alpha \cup \beta \ p \leftrightarrow [\alpha] p \land [\beta] p$ |
| $\alpha; \beta \ p \leftrightarrow [\alpha][\beta] p,$ |
| $[q?] p \leftrightarrow (\neg q^n \lor p)$ |
| $[\alpha^*] p \leftrightarrow (p \land [\alpha][\alpha^*] p)$, |
| $[\alpha^*] p \rightarrow [\alpha^*][\alpha^*] p,$ |
| $(p \land [\alpha^*](p \rightarrow [\alpha] p)^n) \rightarrow [\alpha^*] p.$ |

**Theorem** (Completeness).

$$T_n = \text{PDL}_n$$

**Remark.** If $n = 1$, it boils down to PDL (introduced by Fischer and Ladner in 1979).
Focus on the induction axiom

The formula

\[(p \land [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p\]

means

‘if after an undetermined number of executions of \(\alpha\) the truth value of \(p\) cannot decrease after a new execution of \(\alpha\), then the truth value of \(p\) cannot decrease after any undetermined number of executions of \(\alpha\).’
Focus on the homogeneity axioms

**Proposition.** The axioms

$$[\alpha](\phi \star \phi) \leftrightarrow ([\alpha]\phi \star [\alpha]\phi), \quad \star \in \{\odot, \oplus\},$$

can be replaced by $n$ axioms that state equivalence between

‘the truth value of $[\alpha]\phi$ is at least $\frac{i}{n}$’

and

‘after any execution of $\alpha$, the truth-value of $\phi$ is at least $\frac{i}{n}$’.
About the expressive power of the many-valued modal language.
The ability to distinguish between frames

\[ \mathcal{L}_{\square} = \{ \square, \neg, \rightarrow, 1 \} \]

Frame: \( \mathfrak{F} = \langle W, R \rangle \)

\[
\text{Mod}_n(\Phi) := \{ \mathfrak{F} \mid \mathcal{M} \models \Phi \text{ for any } (n+1)\text{-valued } \mathcal{M} \text{ based on } \mathfrak{F} \}
\]

**Definition.** A class \( \mathcal{C} \) of frames is \( \mathcal{L}_n \)-**definable** if there is \( \Phi \subseteq \text{Form}_{\mathcal{L}_{\square}} \) such that

\[ \mathcal{C} = \text{Mod}_n(\Phi). \]
The ability to distinguish between frames

\[ \mathcal{L}_{\Box} = \{ \Box, \neg, \rightarrow, 1 \} \]

Frame: \( \mathcal{F} = \langle W, R \rangle \)

\[ \text{Mod}_n(\Phi) := \{ \mathcal{F} | \mathcal{M} \models \Phi \text{ for any } (n + 1)\text{-valued } \mathcal{M} \text{ based on } \mathcal{F} \} \]

**Definition.** A class \( \mathcal{C} \) of frames is \( \mathcal{L}_n\)-definable if there is \( \Phi \subseteq \text{Form}_{\mathcal{L}_{\Box}} \) such that

\[ \mathcal{C} = \text{Mod}_n(\Phi). \]

**Proposition.** If \( \mathcal{C} \) is \( \mathcal{L}_1\)-definable then it is \( \mathcal{L}_n\)-definable for every \( n \geq 1 \).
Enriching the signature of frames

**Theorem.** If $C$ contains ultrapowers of its elements, then

$C$ is $Ł_n$-definable if and only if $C$ is $Ł_1$-definable.

The many-valued modal language is **not adapted** for the signature of frames.
Enriching the signature of frames

**Theorem.** If $\mathcal{C}$ contains ultrapowers of its elements, then $\mathcal{C}$ is $\mathcal{L}_n$-definable if and only if $\mathcal{C}$ is $\mathcal{L}_1$-definable.

The many-valued modal language is not adapted for the signature of frames.

**Definition.** An $\mathcal{L}_n$-**frame** is a structure

$$\mathfrak{S} = \langle W, R, \{r_m \mid m \text{ is a divisor of } n \rangle, \rangle,$$

where $r_m \subseteq W$ for any $m$. A model $\mathcal{M} = \langle W, R, \text{Val} \rangle$ is based on $\mathfrak{S}$ if

$$u \in r_m \implies \text{Val}(u, \phi) \in \mathcal{L}_m$$
Ł_n-frames

\[ \mathcal{F} = \langle W, R, \{r_m \mid m \text{ is a divisor of } n \} \rangle, \]

\[ u \in r_m \implies \text{Val}(u, \phi) \in \mathcal{L}_m \]

**Example.** (Forbidden situation)

If \( \text{Val}(u, p) = 1 \) and \( \text{Val}(v, p) = 1/3 \) then \( \text{Val}(u, \Box p) = 1/3 \notin \mathcal{L}_2 \)
\( \mathfrak{F} = \langle W, R, \{ r_m \mid m \text{ is a divisor of } n \} \rangle, \)

\[ u \in r_m \implies \text{Val}(u, \phi) \in \mathfrak{L}_m \]

**Example.** (Forbidden situation)

\[ \text{If } \text{Val}(u, p) = 1 \text{ and } \text{Val}(v, p) = 1/3 \text{ then } \text{Val}(u, \Box p) = 1/3 \not\in \mathfrak{L}_2 \]

**Additional conditions on \( \mathfrak{L}_n \)-frames:**

- \( r_m \cap r_k = r_{\gcd(m,k)} \)
- \( u \in r_m \implies Ru \subseteq r_m \)
Goldblatt - Thomason theorem

\[ C \cup \{ \mathfrak{F} \} = \text{class of } \mathcal{L}_n\text{-frames} \]

\[ \text{Mod}(\Phi) := \{ \mathfrak{F} \mid \mathcal{M} \models \Phi \text{ for any } (n+1)\text{-valued } \mathcal{M} \text{ based on } \mathfrak{F} \} \]

**Definition.** A class \( C \) of \( \mathcal{L}_n \)-frames is *definable* if there is \( \Phi \subseteq \text{Form}_{\mathcal{L}_n} \) such that

\[ C = \text{Mod}(\Phi). \]
Goldblatt - Thomason theorem

\[ \mathcal{C} \cup \{ \mathfrak{F} \} = \text{class of } \mathcal{L}_n\text{-frames} \]

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**Definition.** A class \( \mathcal{C} \) of \( \mathcal{L}_n \)-frames is *definable* if there is \( \Phi \subseteq \text{Form}_{\mathcal{L}_\Box} \) such that

\[ \mathcal{C} = \text{Mod}(\Phi). \]

**Theorem.** If \( \mathcal{C} \) is closed under ultrapowers, then the following conditions are equivalent.

- \( \mathcal{C} \) is definable
- \( \mathcal{C} \) is closed under \( \mathcal{L}_n \)-generated subframes, \( \mathcal{L}_n \)-bounded morphisms, disjoint unions and reflects canonical extensions.
What to do next?

1. Shows that $\text{PDL}_n$ can actually help in stating many-valued program specifications.

2. There is an epistemic interpretation of PDL. Can it be generalized to the $n + 1$-valued realm?

3. What happens if Kripke models are not crisp.

4. Can coalgebras explain why PDL and $\text{PDL}_n$ are so related?