Berezin-Toeplitz Quantization [3]

Consider a compact Kähler manifold \((M, g, I, \omega)\).

**Definition.** A quantum line bundle on \(M\) is a holomorphic bundle \(L \to M\) together with a hermitian metric \(h\) and a connection \(\nabla\) compatible with both \(h\) and the complex structure such that the prequantum condition

\[
curvature_L \nabla = -i \omega
\]

is fulfilled. \(M\) is called quantizable if there exists such a line bundle.

For a general tensor product \(L^m := L^{\otimes m}\), one denotes by \(h^{(m)}\) and \(\nabla^{(m)}\) the induced metric and connection.

**Berezin-Toeplitz Operators**

Consider \(L^2(M, L^m)\) the \(L^2\)-completion of the space of smooth sections of \(L^m\) and \(\Gamma(M, L^m)\) the space of holomorphic sections. Denote by

\[
\Pi^{(m)} : L^2(M, L^m) \to \Gamma(M, L^m)
\]

the orthogonal projection induced by the hermitian metric.

**Definition.** For \(f \in C^\infty(M)\), the Toeplitz operator \(T_f^{(m)}\) of level \(m\) is defined by

\[
T_f^{(m)} := \Pi^{(m)}(f \cdot) : \Gamma(M, L^m) \to \Gamma(M, L^m).
\]

K3 Surfaces

**Definition.** A K3 surface \(M\) is a simply connected complex surface admitting 3 Kähler structures \((g, I_1, \omega_1, J_1, \omega_1, K, \omega_K)\) sharing the same metric \(g\) such that \(IJ = K\).

I am assuming that all three Kähler structures are quantizable to study some relations between them. The existence of three quantizable Kähler structures implies the existence of an infinitude of quantum structures (see Twistor Space at [1]); each of them determining a quantum line bundle. One can easily see that such line bundles are not isomorphic. However:

**Lemma.** There exists a constant \(c \in \mathbb{Z}\) such that \(h^0(M, L) = c\) for any quantum line bundle \(L\) of \(M\).

Note that, while there always exists an isomorphism between two vector spaces of the same (finite) dimension, I am looking for a canonical relation.

The transcendental lattice is the orthogonal complement \(T_I(M)\) of the Neron-Severi group (group of line bundles on a variety) with respect to a fixed complex structure \(I\):

\[
T_I(M) := NS_I(M)^\perp \subset H^2(M, \mathbb{Z})
\]

**Lemma.** If all three Kähler structures are quantizable, then \(rk(T_I(M)) = 2\) and one of the generators can be assumed to be \(\omega_I\).

**Proposition.** If \(T_I(M)\) has a symmetry as a lattice, then there exist \(a, b, c \in \mathbb{Q}\) and an \(I\)-holomorphic automorphism of \(M\) such that

\[
L^a_K = L^b_I \otimes f^* L^c_J.
\]

To study the spaces of holomorphic sections, I am using the projection used in the construction of the Toeplitz Operators.

One can also see that \((T_I(M) \cap T_J(M)) \otimes_\mathbb{Z} \mathbb{Q} = \langle \omega_K \rangle \otimes_\mathbb{Z} \mathbb{Q}\). Currently I am studying if, under some symmetries on all three transcendental lattices the intersection is indeed generated by \(\omega_K\). In this case, one can find a quantum line bundle \(L\) with respect to a fourth Kähler structure \(I_0\), \(\omega_0\) such that all other quantum line bundles are of the form

\[
f_I(L^{\otimes a}) \otimes f_J(L^{\otimes b}) \otimes f_K(L^{\otimes c})
\]

for three automorphisms \(f_I\, f_J\) and \(f_K\) each one holomorphic with respect to a different complex structure.

References

