# On Integration on $\mathbb{Z}_2^n$-Manifolds

Norbert Poncin

*"Kto pyta, prawdy szuka."*

**Abstract**

The aim of the present text is to describe a generalization of Superalgebra and Supergeometry to $\mathbb{Z}_2^n$-gradings, $n > 1$. The corresponding sign rule is not given by the product of the parities, but by the scalar product of the involved $\mathbb{Z}_2^n$ - degrees. This $\mathbb{Z}_2^n$ - Supergeometry exhibits interesting differences with classical Supergeometry, provides a sharpened viewpoint, and has better categorical properties. Further, it is closely related to Clifford calculus: Clifford algebras have numerous applications in Physics, but the use of $\mathbb{Z}_2^n$ - gradings has never been investigated. More precisely, we discuss the geometry of $\mathbb{Z}_2^n$-supermanifolds, give examples of such colored supermanifolds beyond graded vector bundles, and study the generalized Batchelor-Gawedski theorem. However, the main focus is on the $\mathbb{Z}_2^n$-Berezinian and on first steps towards the corresponding integration theory, which is related to an algebraic variant of the multivariate residue theorem.

## 1 Introduction

This paper is based on talks given by the author at the conference 'Geometry of Jets and Fields', held from 10 to 16 May 2015 at the MRCC in Bedlewo, Poland, in honor of Janusz Grabowski, as well as at the conference 'Glances at Manifolds', held from 17 to 20 July 2015 at the Jagiellonian University of Krakow, Poland, in honor of Krzysztof M. Pawałowski, András Szücs, Július Korbaš, Józef H. Przytycki, Paweł Traczyk, and Robert Wolak. Both, the present text and the underlying lectures, report on joint works with Tiffany Covolo, Valentin Ovsienko, Janusz Grabowski, and, more recently, Stephen Kwok. The objective of our research is Generalized Supergeometry, i.e., Supergeometry over $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ .

## 2 Motivations and Applications

In standard Supergeometry, coordinates have degree 0 or 1; in $\mathbb{Z}_2^2$-Supergeometry (resp., $\mathbb{Z}_2^3$-Supergeometry), we consider coordinates of degree $(0,0),(1,1),(0,1)$, and $(1,0)$ (resp., $(0,0,0),(0,1,1)$, $(1,0,1),(1,1,0),(0,0,1),(0,1,0),(1,0,0)$, and $(1,1,1)$). The generalization to the $\mathbb{Z}_2^n$-case is obvious. Clearly, the first $2^{n-1}$ degrees are even and the second $2^{n-1}$ odd. However, the commutation rule for the coordinates is not, as usual, given by the product of the parities, but by the scalar product of the involved degrees. More precisely, if $y$ (resp., $\eta$) is of degree $(0,1,1)$ (resp., $(0,1,0)$), we set

$$y \cdot \eta = (-1)^{\langle (0,1,1),(0,1,0)\rangle} \eta \cdot y = -\eta \cdot y \ , \tag{1}$$

where $\langle -, - \rangle$ denotes the standard scalar product in $\mathbb{R}^3$. This 'scalar product commutation rule' implies significant differences with the classical theory: even coordinates may anticommute ($(-1)^{\langle (1,1,0),(1,0,1)\rangle} = -1$), odd coordinates may commute ($(-1)^{\langle (1,0,0),(0,1,0)\rangle} = +1$), and non-zero degree even coordinates are not nilpotent ($(-1)^{\langle (1,1,0),(1,1,0)\rangle} = +1$).

The study of $\mathbb{Z}_2^n$-gradings, $n \geq 0$, together with the commutation rule (1), is in some sense necessary and sufficient. Sufficient, since any sign rule, for any finite number $m$ of coordinates, is of the form (1), for some $n \leq 2m$ [CGP14a]; and necessary, in view of the needs of Physics, Algebra, and Geometry. In Physics, $\mathbb{Z}_2^n$-gradings, $n \geq 2$, are used in string theory and in parastatistical supersymmetry [AFT10], [YJ01]. In Mathematics, there exist good examples of $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative algebras: the algebra of Deligne differential superforms is $\mathbb{Z}_2^2$-commutative,

$$\alpha \wedge \beta = (-1)^{\deg(\alpha)\deg(\beta) + \mathrm{p}(\alpha)\,\mathrm{p}(\beta)} \beta \wedge \alpha \ ,$$

where deg (resp., p) denotes the cohomological degree (resp., the parity) of the superforms $\alpha$ and $\beta$, the algebra $\mathbb{H}$ of quaternions is $\mathbb{Z}_2^3$-commutative, and, more generally, any Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{R})$ is $\mathbb{Z}_2^{p+q+1}$-commutative [COP12], ... And there exist interesting examples of $\mathbb{Z}_2^n$-supermanifolds: the tangent and cotangent bundles $T\mathcal{M}$ and $T^\star\mathcal{M}$ of a standard supermanifold, the superization of double vector bundles such as, e.g., $TTM$ and $T^*TM$, where $M$ is a classical purely even manifold, and, more generally, the superization of $n$-vector bundles, ...

For instance, if $(x,\xi)$ are the coordinates of $\mathcal{M}$, the coordinates of $T\mathcal{M}$ are $(x,\xi,\mathrm{d}\,x,\mathrm{d}\,\xi)$. As concerns degrees, we have two possibilities. Either, we add the cohomological degree 1 of d and the parities 0 (resp., 1) of $x$ (resp., $\xi$), or, we keep them separated (richer information). In the first case, the coordinates $(x,\xi,\mathrm{d}\,x,\mathrm{d}\,\xi)$ have the parities $(0,1,1,0)$, we use the standard supercommutation rule and obtain a classical supermanifold; in the second, the coordinates $(x,\xi,\mathrm{d}\,x,\mathrm{d}\,\xi)$ have the degrees $((0,0),(0,1),(1,0),(1,1))$, we apply the $\mathbb{Z}_2^2$-commutation rule (1) and get a $\mathbb{Z}_2^2$-manifold. The local model of the supermanifold $T\mathcal{M}$ is of course made of the polynomials $C^\infty(x,\mathrm{d}\,\xi)[\xi,\mathrm{dx}]$ in the odd indeterminates with coefficients that are smooth with respect to the even variables. On the other hand, the base of the $\mathbb{Z}_2^2$-manifold $T\mathcal{M}$ is – exactly as in $\mathbb{Z}$-graded geometry – made only of the degree $(0,0)$ variables, whereas, with respect the other indeterminates, we consider not only polynomials, but all power series $C^\infty(x)[[\xi,\mathrm{d}\,x,\mathrm{d}\,\xi]]$.

Let us comment on the latter local model. Consider an arbitrary $\mathbb{Z}_2^2$-manifold with coordinates $(x,y,\xi,\eta)$ of degrees $((0,0),(1,1),(0,1),(1,0))$, and let

$$\phi : \{x,y,\xi,\eta\} \mapsto \{x',y',\xi',\eta'\}$$

be the coordinate transformation

$$x' = x + y^2, \quad y' = y, \quad \xi' = \xi, \quad \eta' = \eta \,. \tag{2}$$

Note that (2) respects the $\mathbb{Z}_2^2$-degree and that $y$ is, as mentioned above, *not* nilpotent. If we now change coordinates in a target function of the type $F(x')$, we get, using as usual a formal Taylor expansion,

$$F(x') = F(x + y^2) = \sum_\alpha \frac{1}{\alpha!}(\partial_{x'}^\alpha F)(x) y^{2\alpha} \,,$$

where the RHS is really a series, precisely because $y$ is not nilpotent. However, the pullback of a target function must be a source function. The only way out is to decide that functions are formal series, thus opting for the aforechosen local model

$$(U,\, C^\infty(x)[[y, \xi, \eta]]) \,, \tag{3}$$

where $U$ is open in some $\mathbb{R}^p$ (of course, since $\xi$ and $\eta$ *are* nilpotent, they appear in the series with exponent 0 or 1). With this in mind, one easily sees that the most general coordinate transformation is

$$\begin{cases} x' = \sum_r f_r^{x'}(x) y^{2r} + \sum_r g_r^{x'}(x) y^{2r+1}\xi\eta \\ y' = \sum_r f_r^{y'}(x) y^{2r+1} + \sum_r g_r^{y'}(x) y^{2r}\xi\eta \\ \xi' = \sum_r f_r^{\xi'}(x) y^{2r}\xi + \sum_r g_r^{\xi'}(x) y^{2r+1}\eta \\ \eta' = \sum_r f_r^{\eta'}(x) y^{2r}\eta + \sum_r g_r^{\eta'}(x) y^{2r+1}\xi \end{cases} \,, \tag{4}$$

where $r \in \mathbb{N}$, so that all sums are series, and where the coefficients are smooth in $x$. Let us stress that if we perform this general coordinate transformation (with series) in a target function (that is itself a series), we might a priori obtain series of smooth coefficients, which would then lead to convergence conditions. Fortunately, one can show that this problem does not appear.

Note now that the Jacobian matrix that corresponds to (4) is of the type

$$\partial_{(x,y,\xi,\eta)}(x', y', \xi', \eta') = \left(\begin{array}{cc|cc} (0,0) & (1,1) & (0,1) & (1,0) \\ (1,1) & (0,0) & (1,0) & (0,1) \\ \hline (1,0) & (0,1) & (0,0) & (1,1) \\ (0,1) & (1,0) & (1,1) & (0,0) \end{array}\right) \,, \tag{5}$$

i.e., is a block matrix, where all the entries of a same block have the same $\mathbb{Z}_2^2$-degree. Since the classical determinant only works for matrices with commuting entries (to simplify we will speak about commutative (or noncommutative) matrices), and the entries of a Jacobian matrix as above do not necessarily commute, we have to look for an appropriate determinant, i.e., for the $\mathbb{Z}_2^2$-, or, more generally, the $\mathbb{Z}_2^n$-Berezinian. We will denote this generalized Berezinian by $\Gamma\mathrm{Ber}$. Before describing the construction of $\Gamma\mathrm{Ber}$, let us remember that:

**Remark 1.** *The big diagonal blocks of the Jacobian matrix (5) contain entries with even $\mathbb{Z}_2^2$-degrees, whereas the entries of the small diagonal blocks are all of degree $(0,0)$; further, even though objects with even $\mathbb{Z}_2^2$-degrees do commute, this is no longer true for even $\mathbb{Z}_2^n$-degrees, with $n \geq 3$.*

# 3 Higher Berezinian of matrices over a $\mathbb{Z}_2^n$-commutative algebra

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-commutative algebra (supercommutative, associative, unital $\mathbb{R}$-algebra). Recall that the classical Berezinian

$$\mathrm{Ber}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det(A - BD^{-1}C)\,\det{}^{-1}D \tag{6}$$

is the unique group morphism $\mathrm{Ber} : \mathrm{GL}^0(\mathcal{A}) \to (\mathcal{A}^0)^\times$ from invertible even matrices with entries in $\mathcal{A}$ to invertible even elements of $\mathcal{A}$, which takes value 1 on upper and lower unitriangular matrices and whose value on block diagonal matrices is given by

$$\mathrm{Ber}\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) = \det A \cdot \det{}^{-1}D\ . \tag{7}$$

When trying to extend these characterizing axioms to invertible even matrices with entries in a $\mathbb{Z}_2^n$-commutative algebra $\mathcal{A}$, we observe that the big diagonal blocks $A$ and $D$ (see (7) and (5)) are noncommutative matrices (see Remark 1) and that therefore $\det A$ and $\det D$ are meaningless. Hence, before being able to write axioms for the generalized Berezinian $\Gamma\mathrm{Ber}$, we have first to find a generalized determinant $\Gamma\mathrm{det}$ for even matrices with entries in an algebra that is graded by the even part $(\mathbb{Z}_2^n)_{\mathrm{ev}}$ of $\mathbb{Z}_2^n$ (Remark 1).

Let thus $\mathcal{A}$ be a $(\mathbb{Z}_2^n)_{\mathrm{ev}}$-commutative algebra. Our task is to show that there exists a unique algebra morphism $\Gamma\mathrm{det} : \mathrm{gl}^0(\mathcal{A}) \to \mathcal{A}^0$, from matrices of degree 0 with entries in $\mathcal{A}$ to degree 0 elements of $\mathcal{A}$, which takes value 1 on upper and lower unitriangular matrices and whose value on a block diagonal matrix is given by

$$\Gamma\mathrm{det}\left(\begin{array}{c|c|c|c} \star & 0 & 0 & 0 \\ \hline 0 & \star & 0 & 0 \\ \hline 0 & 0 & \star & 0 \\ \hline 0 & 0 & 0 & \star \end{array}\right) = \prod \det \star\ . \tag{8}$$

Since the small diagonal blocks $\star$ (see (5) and note that we passed from $n = 2$ in (5) to $n = 3$ in (8)) are commutative matrices (see Remark 1), their classical determinant det is defined and the preceding axioms make sense.

According to Gel'fand, any noncommutative determinant – in particular $\Gamma\mathrm{det}$, $\Gamma\mathrm{Ber}$, and Ber – must be constructed from quasideterminants. And indeed, the classical Berezinian Ber (see (6)) contains the difference $A - BD^{-1}C$, which is the quasideterminant

$$\left|\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right|_{11} := A - BD^{-1}C\ , \tag{9}$$

computed with respect to entry $(1, 1)$. In fact, for any square matrix $X$ with entries in a ring, one can define the quasideterminant $|X|_{ij}$ with respect to the entry $(i, j)$ by a similar formula. For instance, if

$$X = \begin{pmatrix} x & a & b \\ c & y & d \\ e & f & z \end{pmatrix} \quad ,$$

we get

$$|X|_{11} = x - bz^{-1}e - (a - bz^{-1}f)(y - dz^{-1}f)^{-1}(c - dz^{-1}e)$$

(quasideterminant with respect to the ordinary entry $(1, 1)$), and if

$$X = \left(\begin{array}{cc|c} x & a & b \\ c & y & d \\ \hline e & f & z \end{array}\right) \quad ,$$

we obtain

$$|X|_{11} = \begin{pmatrix} x - bz^{-1}e & a - bz^{-1}f \\ c - dz^{-1}e & y - dz^{-1}f \end{pmatrix}$$

(quasideterminant with respect to the block entry $(1, 1)$). The main observation to keep in mind is that quasideterminants are not polynomial, but rational with respect to their entries.

Since the axioms of $\Gamma$det require that $\Gamma$det be multiplicative and fix its value on any Upper unitriangular, Diagonal, and Lower unitriangular matrix, it seems natural to look for an UDL decomposition $X = UDL$ of the considered noncommutative matrix $X$. It turns out that such a decomposition exists and that its three factors are made of quasideterminants. For instance, if $X$ is a $p \times p$ block matrix ($p = 2^{n-1}$), the diagonal blocks of its central factor $D$ are $|X|_{11}$, $|X^{1:1}|_{22}$, $|X^{12:12}|_{33}$, ..., $X_{pp}$, where the superscripts in front of (resp., behind) ' : ' mean that the corresponding block rows (resp., columns) are omitted, and where subscripts indicate the original block entry of $X$ with respect to which the quasideterminant is computed. It follows that, if $\Gamma$det – satisfying the chosen axioms – does exist, it is necessarily given by

$$\Gamma\det X = \det |X|_{11} \cdot \det |X^{1:1}|_{22} \cdot \det |X^{12:12}|_{33} \cdot \ldots \cdot X_{pp} \, . \tag{10}$$

The delicate points are to check that the RHS actually satisfies the axioms, in particular the multiplicativity, and to see that the RHS rational expression in fact simplifies and is polynomial so that no existence conditions do appear [COP12].

We are now able to formulate the axioms for the generalized Berezinian $\Gamma$Ber and to study this new determinant. Let us emphasize once more that we consider here $2^n \times 2^n$ block matrices with a coarser $2 \times 2$ redivision (see (5)). The following fundamental theorem can be proven [COP12]:

**Theorem 1.** *Let $\mathcal{A}$ be a $\mathbb{Z}_2^n$-commutative associative unital $\mathbb{R}$-algebra ($n \geq 1$). There is a unique group homomorphism*

$$\Gamma\mathrm{Ber} : \mathrm{GL}^0(\mathcal{A}) \to (\mathcal{A}^0)^\times$$

*from invertible degree zero matrices with entries in $\mathcal{A}$ to invertible degree zero elements of $\mathcal{A}$, which takes value $1 \in (\mathcal{A}^0)^\times$ on upper and lower $2\times 2$ unitriangular block matrices and satisfies*

$$\Gamma\mathrm{Ber}\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) = \Gamma\det A \cdot \Gamma\det{}^{-1} D \in (\mathcal{A}^0)^\times \ .$$

*It is given by*

$$\Gamma\mathrm{Ber}\, X = \Gamma\mathrm{Ber}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \Gamma\det(A - BD^{-1}C)\cdot \Gamma\det{}^{-1} D \ . \tag{11}$$

In the case $n = 1$, we recover the classical Berezinian Ber. For $\mathcal{A} = \mathbb{H}$, the generalized Berezinian ΓBer coincides up to sign with the Dieudonné determinant. Eventually, the new determinant ΓBer admits an infinitesimal counterpart ΓTr – the generalized supertrace.

# 4 $\mathbb{Z}_2^n$-supermanifolds

There exist three equivalent definitions of $\mathbb{Z}_2^n$-supermanifolds ($\mathbb{Z}_2^n$-manifolds, for short), $n \in \mathbb{N}$. For further details about the first two, we refer the reader to [CGP14a].

The atlas definition reads roughly as follows: A **$\mathbb{Z}_2^n$-manifold** is a topological space that is covered by $\mathbb{Z}_2^n$-graded $\mathbb{Z}_2^n$-commutative coordinate systems $(x, y, \ldots, \xi, \eta, \ldots)$ and is endowed with $\mathbb{Z}_2^n$-degree preserving coordinate transformations

$$\phi_{\beta\alpha} : (x, y, \ldots, \xi, \eta, \ldots) \mapsto (x', y', \ldots, \xi', \eta', \ldots)$$

that satisfy the usual cocycle condition.

Also the local ringed space definition comes as no surprise. A **$\mathbb{Z}_2^n$-manifold** is a $\mathbb{Z}_2^n$ local ringed space $\mathcal{M} = (M, \mathcal{A}_M)$ (this means that the sheaf $\mathcal{A}_M$ is a sheaf of $\mathbb{Z}_2^n$-commutative algebras over the topological space $M$) that is locally modeled on the abovediscussed local model $(\mathbb{R}^p, C^\infty_{\mathbb{R}^p}(-)[[\xi^1, \ldots, \xi^q]])$, where the $\xi^a$-s are $\mathbb{Z}_2^n$-commutative ($p$ and $q$ depend of course on the dimension of $\mathcal{M}$).

As for the functor of points approach, start with $V = \{z \in \mathbb{C}^n : P(z) = 0\} \in \texttt{Aff}$, where $P$ denotes a polynomial in $n$ indeterminates with complex coefficients and `Aff` the category of affine varieties. Grothendieck insisted on solving the equation $P(z) = 0$ not only in $\mathbb{C}^n$, but in $A^n$, for any algebra $A$ in the category `CA` of commutative (associative unital) algebras (over $\mathbb{C}$). This leads to an arrow

$$\mathrm{Sol}_P : \texttt{CA} \ni A \mapsto \mathrm{Sol}_P(A) = \{a \in A^n : P(a) = 0\} \in \texttt{Set} \ ,$$

which turns out to be a representable set-valued functor,

$$\mathrm{Sol}_P = \mathrm{Hom}_{\texttt{CA}}(\mathbb{C}[V], -) \in [\texttt{CA}, \texttt{Set}] \ ,$$

with representing object the algebra of polynomial functions of $V$. The dual of this functor, whose value $\mathrm{Sol}_P(A)$ is the set of $A$-points of $V$, is the functor

$$\mathrm{Hom}_{\mathtt{Aff}}(-,V) \in [\mathtt{Aff}^{\mathrm{op}}, \mathtt{Set}] \;,$$

whose value $\mathrm{Hom}_{\mathtt{Aff}}(W,V)$ is the set of $W$-points of $V$. The latter functor of points can be considered not only in `Aff`, but in any category, say, `Legos`. We thus obtain an arrow

$$\underline{\bullet} : \mathtt{Legos} \ni c \mapsto \underline{c} := \mathrm{Hom}_{\mathtt{Legos}}(-,c) \in [\mathtt{Legos}^{\mathrm{op}}, \mathtt{Set}] \;,$$

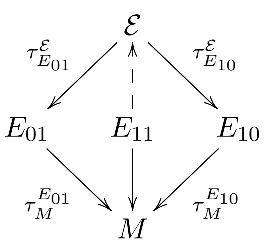
the so-called Yoneda embedding, which is fully faithful, so that, instead of studying $c$, we can just as well focus on $\underline{c}$. The RHS category has better properties, since it has in particular all limits and colimits. If the LHS category has also limits and colimits, the Yoneda embedding commutes with the first, i.e., $\mathrm{Lim}\,\underline{c} \simeq \underline{\mathrm{Lim}\,c}$, but not with the second, i.e.,

$$\mathrm{Colim}\,\underline{c} \to \underline{\mathrm{Colim}\,c} \tag{12}$$

is usually not an isomorphism. However, if we think about objects $c \in \mathtt{Legos}$ as trivial pieces of bigger spaces, the embedding $\underline{\bullet}$ should commute with gluings, i.e., with colimits. To achieve this, one defines on the source category `Legos` of the functors of points a (Grothendieck) topology such that all functors of points become sheaves. If we compute now the LHS colimit, not in the category $[\mathtt{Legos}^{\mathrm{op}}, \mathtt{Set}]$ of functors or presheaves, but in the just mentioned category $\mathtt{Sh}(\mathtt{Legos})$ of sheaves, then the arrow (12) turns into an isomorphism. In view of what has been said, a lego or trivial space corresponds to a representable sheaf, a space to a sheaf, and a manifold or variety, i.e., a locally trivial space, to a locally representable sheaf. We denote this category of 'categorical' varieties over `Legos` by $\mathtt{Var}(\mathtt{Legos})$. If `Legos` is the category `Aff` (resp., the category $\mathbb{Z}_2^{\mathtt{n}}$-`Dom` of $\mathbb{Z}_2^n$-domains), the category $\mathtt{Var}(\mathtt{Legos})$ of 'categorical' schemes (resp., 'categorical' $\mathbb{Z}_2^n$-manifolds) is equivalent to the category of 'ringed space' schemes (resp., the category of 'ringed space' $\mathbb{Z}_2^n$-manifolds).

We will not address in this text the technical issues whose solution allows to extend, despite the loss of nilpotency, most of the results of classical Supergeometry to the present $\mathbb{Z}_2^n$-setting. Let us however mention that the fundamental supergeometric morphism theorem [Var04, Subsection 4.3.], which makes Supergeometry a reasonable theory, can be generalized (a $\mathbb{Z}_2^n$-morphism is just a morphism of $\mathbb{Z}_2^n$ ringed spaces).

Before giving examples of $\mathbb{Z}_2^n$-manifolds, we briefly recall three definitions of the notion of double vector bundle. The standard one [Mac92] views a double vector bundle, roughly, as a manifold $\mathcal{E}$



endowed with two compatible (in the natural sense) vector bundle structures over two bases $E_{01}$ and $E_{10}$, which are themselves vector bundles over a same base $M$. A trivial example is the direct sum $E_{01} \oplus E_{10} \oplus E_{11}$ of three vector bundles over $M$. This sum carries indeed a vector bundle structure over $E_{01}$ and a vector bundle structure over $E_{10}$, which are compatible.

**Remark 2.** *It must however be emphasized that the canonical graded vector bundle structure of this sum over $M$ is not part of its double vector bundle structure.*

A simpler but equivalent definition is due to [GR09]. In contrast to the more common extraction of the scalar multiplication from addition in a topological vector space, the authors construct the additions out of the multiplications by scalars, and, more precisely, out of an action on a manifold $E$ of the multiplicative monoid of nonnegative real numbers. This action can be replaced by its infinitesimal counterpart: the latter 'Euler' vector field thus encodes a vector bundle structure on $E$. In this language, a double vector bundle $\mathcal{F}$, i.e., two *compatible* vector bundle structures, is just a pair of *commuting* 'Euler' fields.

The third definition can be found in [Vor12]. It describes a double vector bundle $\mathcal{G}$ as a locally trivial fiber bundle (not vector bundle, see third transformation rule in (13)), whose standard fiber is a graded vector space $V_{01} \oplus V_{10} \oplus V_{11}$, and whose coordinate transformations are of the type

$$\begin{cases} \xi'^a &= f^a_u(x)\xi^u \\ \eta'^b &= g^b_v(x)\eta^v \\ y'^c &= h^c_w(x)y^w + k^c_{u,v}(x)\xi^u\eta^v \quad , \end{cases} \tag{13}$$

where $\xi, \xi'$ (resp., $\eta, \eta'$; $y, y'$) are coordinates in $V_{01}$ (resp., $V_{10}$, $V_{11}$), and where $x$ are coordinates in the base.

The first examples of $\mathbb{Z}_2^n$-manifolds ($n \geq 1$) are implemented by ordinary vector bundles. Indeed, it is well-known that, if we shift in a rank $r$ vector bundle $E \to M$ the parity of the fiber coordinates, i.e., if we consider $\Pi E := E[1]$, we get a supermanifold $\mathcal{M} = (M, \mathcal{A}(\Pi E))$ with function sheaf

$$\mathcal{A}(\Pi E) = \bigoplus_{k=0}^{r} \Gamma(\odot^k (\Pi E)^*) \ . \tag{14}$$

Similarly, when shifting in a graded vector bundle $E_{01} \oplus E_{10} \oplus E_{11} \to M$ the degree of the fiber coordinates, i.e., when considering

$$\Pi E := E_{01}[01] \oplus E_{10}[10] \oplus E_{11}[11] \ ,$$

we obtain – in view of the above ringed space definition – a $\mathbb{Z}_2^2$-manifold $\mathcal{M} = (M, \mathcal{A}(\Pi E))$, whose function sheaf is given by

$$\mathcal{A}(\Pi E) = \prod_{k \geq 0} \Gamma(\odot^k (\Pi E)^*) \ . \tag{15}$$

Such manifolds, implemented by vector bundles, are in some sense trivial: we refer to them as split $\mathbb{Z}_2^n$-manifolds.

Consider now a double vector bundle (see (13)) and superize its coordinates, assigning the degree $(0,0)$ (resp., $(0,1)$, $(1,0)$, $(1,1)$) to $x$ (resp., $\xi$, $\eta$, $y$). In light of the transformation rules (13), this can be done in a coherent way. Hence, we get – compare with the atlas definition – a $\mathbb{Z}_2^2$-manifold. The only interesting point is the cocycle condition. Indeed, it is satisfied for three systems of commuting coordinates of the considered fiber bundle. However, if, in a specific example of such a double bundle, we prove this compatibility condition, it might be necessary to commute coordinates. If we then repeat the computation with $\mathbb{Z}_2^2$-commuting coordinates, signs could appear and act as an obstruction to the validity of the cocycle condition. It can be checked that this problem does not arise in the considered situation [CGP14b]. The $\mathbb{Z}_2^2$-manifold we built that way is not induced by a graded vector bundle (see Remark 2), hence, it is not split.

However, in classical Supergeometry, any supermanifold is *noncanonically* diffeomorphic to a split one – at least in the categories of smooth and real analytic supermanifolds. This theorem is usually attributed to M. Batchelor [Bat79, Bat80], although it was discovered already some years earlier by K. Gawedski [Gaw77]. Further, D. Leites informed us that A. A. Kirillov and A. N. Rudakov convinced themselves independently of the correctness of this claim while using an elevator at Moscow State University. The result can be extended to $\mathbb{Z}_2^n$-manifolds.

**Theorem 2.** *Any smooth $\mathbb{Z}_2^n$-manifold ( $n \geq 1$ ) is noncanonically isomorphic to a split $\mathbb{Z}_2^n$-manifold, i.e., for any $\mathbb{Z}_2^n$-manifold $(M, \mathcal{A})$ there exists a $\mathbb{Z}_2^n$-graded vector bundle $E$, such that $\mathcal{A} \simeq \mathcal{A}(\Pi E)$, where $\simeq$ is noncanonical.*

For the quite technical proof, which is based on a Čech cohomology argument, we refer the reader to [CGP14b].

# 5 First steps towards integration on $\mathbb{Z}_2^n$-manifolds

## 5.1 Generalized Berezinian modules and sections

Consider a free module $M$ of rank $r$ over a commutative algebra $\mathcal{A}$. The top exterior power $\operatorname{Det} M := \bigwedge^r M$ is a rank 1 $\mathcal{A}$-module. Any basis $(e_i)_i$ of $M$ induces a basis $e_1 \wedge \ldots \wedge e_r$ of $\operatorname{Det} M = \bigwedge^r M$. Further, a base transformation $e'_j = e_i B^i_j$ in $M$, characterized by a transition matrix $B \in \mathrm{GL}(\mathcal{A})$, implements a base transformation

$$e'_1 \wedge \ldots \wedge e'_r = e_1 \wedge \ldots \wedge e_r \cdot \det B \ , \tag{16}$$

which is given by the determinant of the transition matrix $B$.

The interest of the result is clear (modulo replacement of (16) by its covariant version, i.e., replacement of $M$ by a module of covariant vectors): if $M = \Omega^1_N(U)$ is the $C^\infty_N(U)$-module of differential 1-forms of a classical $r$-dimensional smooth manifold $N$ over a coordinate patch $(U, (x^1, \ldots, x^r))$, and if we pass to new coordinates $(x'^1, \ldots, x'^r)$ in $U$, the basis $(\mathrm{d}\, x^i)_i$ of $\Omega^1_N(U)$ changes to $(\mathrm{d}\, x'^i)_i$, with $\mathrm{d}\, x'^j = \mathrm{d}\, x^i \ \partial_{x^i} x'^j$, and the basis of differential top forms changes according to

$$\mathrm{d}\, x'^1 \wedge \ldots \wedge \mathrm{d}\, x'^r = \mathrm{d}\, x^1 \wedge \ldots \wedge \mathrm{d}\, x^r \cdot \det(\partial_x x') \tag{17}$$

– which is the transformation rule of the standard volume element and the key of integral calculus on smooth manifolds.

Therefore, it is natural to consider now a free $\mathbb{Z}_2^n$-graded module $M$ of total rank $r$ over a $\mathbb{Z}_2^n$-commutative algebra $\mathcal{A}$, and to look for a rank 1 $\mathcal{A}$-module $\Gamma\mathrm{Ber}\, M$, such that a base transformation in $M$, characterized by a matrix $B \in \mathrm{GL}^0(\mathcal{A})$, induces a base transformation in $\Gamma\mathrm{Ber}\, M$, characterized by $\Gamma\mathrm{Ber}\, B$. This module $\Gamma\mathrm{Ber}\, M$ is constructed as follows. First, one defines a kind of Koszul cochain complex

$$\mathcal{K} := \odot\, \Pi M \otimes \odot\, M^\star\ ,$$

where $\odot$ denotes the $\mathbb{Z}_2^n$-graded symmetric tensor product over $\mathcal{A}$, where $\Pi M$ refers to the shift of the $\mathbb{Z}_2^n$-degrees in $M$ by any fixed odd $\mathbb{Z}_2^n$-degree (our result will in fact be independent of this choice), and where $M^\star$ denotes the dual $\mathrm{Hom}_\mathcal{A}(M, \mathcal{A})$ of $M$. The differential $d := \sum_i \Pi e_i \otimes \varepsilon^i$, where $(e_i)_i$ is a basis of $M$ and $(\varepsilon^i)_i$ its dual basis in $M^\star$, acts on cochains by left multiplication. The cohomological degree of a cochain is the number of odd tensor factors. Note that each term of $d$ contains exactly one odd factor, so that $d$ is actually a cohomology operator. It can be shown [Cov12] that

$$H(\mathcal{K}, d) = H^r(\mathcal{K}, d) = [\omega] \cdot \mathcal{A}\ .$$

The basis $[\omega]$ of this rank 1 cohomology $\mathcal{A}$-module is the class of the product of all odd vectors in $(\Pi e_i)_i$ and $(\varepsilon^i)_i$. Now, a base transformation in $M$, characterized by a matrix $B \in \mathrm{GL}^0(\mathcal{A})$, induces a base transformation in $M^\star$, characterized by the inverse ${}^{\Gamma t}B^{-1} \in \mathrm{GL}^0(\mathcal{A})$ of the generalized supertranspose $\Gamma t$ of $B$. These automorphisms extend to a degree 0 algebra automorphism $\phi_B$ of $\mathcal{K}$. The latter descends to the rank 1 quotient $H(\mathcal{K}, d)$ thus providing a degree 0 automorphism $\Phi_B \in (\mathcal{A}^0)^\times$. The arrow $\Phi : \mathrm{GL}^0(\mathcal{A}) \to (\mathcal{A}^0)^\times$ not only acts between the same groups as $\Gamma\mathrm{Ber}$, but it further satisfies the characterizing properties of the latter, so that $\Phi_B = - \cdot \Gamma\mathrm{Ber}\, B$. Eventually, we may think about $\omega$ as an algebraic (we work with an arbitrary module $M$) generalized (our setting is $\mathbb{Z}_2^n$-graded) Berezinian volume, which, in an underlying base transformation $B$, transforms according to

$$\omega' = \omega \cdot \Gamma\mathrm{Ber}\, B\ . \tag{18}$$

**Remark 3.** *To simplify, we consider in the following $\mathbb{Z}_2^2$-manifolds of dimension* $(1|1|1|1)$*, meaning that their coordinate systems contain 1 coordinate of each degree* $(0,0)$*,* $(1,1)$*,* $(0,1)$*, and* $(1,0)$*.*

Let now $(U, (x, y, \xi, \eta))$ be a coordinate chart of a $\mathbb{Z}_2^2$-manifold $\mathcal{N} = (N, \mathcal{A}_N)$. The $\mathcal{A}_N(U)$-module $M := \Omega^1_N(U)$ admits the basis $(\mathrm{d}\,x, \mathrm{d}\,y, \mathrm{d}\,\xi, \mathrm{d}\,\eta)$, whose dual is the basis $(\partial_x, \partial_y, \partial_\xi, \partial_\eta)$. Observe that the spirit and purpose of the shift $\Pi$ in [Cov12] is to ensure that the factors $\Pi e_i$ and $\varepsilon^i$ in $d$ have opposite parity. However, the vectors of the preceding dual bases have the degrees

$$(1,0,0),\ (1,1,1),\ (1,0,1),\ (1,1,0) \quad \text{and} \quad (0,0,0),\ (0,1,1),\ (0,0,1),\ (0,1,0)\ ,$$

respectively. Hence, due to the conventions of the present paper, any two dual vectors have already opposite parity, so that the shift is redundant here. If follows that the generalized Berezinian volume is given by

$$\omega = \mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta \;, \tag{19}$$

where we omitted – as usually done in classical Supergeometry – the graded symmetric tensor product and interpreted the partial derivatives as composed graded derivations. If we change coordinates

$$(x, y, \xi, \eta) \mapsto (x', y', \xi', \eta')$$

in $U$, the basis of $M = \Omega^1_N(U)$ changes as well and the corresponding transition matrix is the Jacobian matrix $\big(\partial_{(x,y,\xi,\eta)}(x', y', \xi', \eta')\big)$. Hence, the generalized Berezinian volume transforms according to

$$\mathrm{d}\,x'\,\mathrm{d}\,y' \otimes \partial_{\xi'} \partial_{\eta'} = \mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta \cdot \Gamma\mathrm{Ber}(\partial_{(x,y,\xi,\eta)}(x', y', \xi', \eta')) \;. \tag{20}$$

As usual, a (generalized) Berezinian section of a $\mathbb{Z}_2^2$-manifold $\mathcal{N}$ is a product $(\mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta)\, f(x, y, \xi, \eta)$, where $f$ is a local $\mathbb{Z}_2^2$-function of $\mathcal{N}$. More precisely, we give the following

**Definition 1.** *A* **Berezinian section** *of a $\mathbb{Z}_2^2$-manifold $\mathcal{N}$ with an oriented base is a family*

$$(\mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta)\, f(x, y, \xi, \eta),\; (\mathrm{d}\,x'\,\mathrm{d}\,y' \otimes \partial_{\xi'} \partial_{\eta'})\, f'(x', y', \xi', \eta'), \ldots \;,$$

*indexed by the coordinate systems of an atlas of $\mathcal{N}$, whose components transform according to the rule*

$$f(x, y, \xi, \eta) =$$
$$\Gamma\mathrm{Ber}(\mathcal{J}(x, y, \xi, \eta))\; f'(x'(x, y, \xi, \eta), y'(x, y, \xi, \eta), \xi'(x, y, \xi, \eta), \eta'(x, y, \xi, \eta)) \;. \tag{21}$$

In (21), which is natural due to (20), the matrix $\mathcal{J}(x, y, \xi, \eta)$ is the (modified) Jacobian matrix $\partial_{(x,y,\xi,\eta)}(x', y', \xi', \eta')$. The definition makes sense since the usual cocycle condition is satisfied in view of the multiplicativity of $\Gamma\mathrm{Ber}$ and of $\mathcal{J}$.

## 5.2 Propagation of monomials

Compactly supported Berezinian sections can be integrated over oriented standard supermanifolds. The present paragraph contains results that are needed to investigate integration of compactly supported Berezinian sections over oriented $\mathbb{Z}_2^2$-manifolds.

Let $\psi : (u, v, \varsigma, \theta) \rightleftarrows (x, y, \xi, \eta)$ be a coordinate transformation (4) in a $\mathbb{Z}_2^2$-domain $\mathcal{U}$ over an open subset $U$ (notation has been changed to avoid prime symbols). We denote by

$$\mathcal{J} = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

the Jacobian matrix of $\psi$ with its $2 \times 2$ redivision. If follows from the inverse function theorem [CKP16] that $\mathcal{J}$ is invertible, if and only if the square matrices $A$ and $D$ are invertible modulo the ideal $J$ generated by the nonzero degree coordinates. Hence, the invertibility of $\mathcal{J}$ implies that (see (4))

$$d_x u := d_x f_0^u,\; v(x) := f_0^v(x),\; \varsigma(x) := f_0^\varsigma(x), \;\text{ and }\; \theta(x) := f_0^\theta(x)$$

are invertible in $C^\infty(U)$.

As from now, 'monomial of the type $y^\kappa \xi^\alpha \eta^\beta$' means a monomial of the type $y^k \xi^a \eta^b$ with $k \geq \kappa$, $a \geq \alpha$, and $b \geq \beta$.

**Theorem 3.** *A monomial of the type $y^\kappa \xi^\alpha \eta^\beta$ in an entry of $\mathcal{J}$ induces in* $\Gamma\mathrm{Ber}\,\mathcal{J}$ *only terms of the same type.*

*Proof.* Since $\Gamma\mathrm{det}$ is polynomial in its entries, the claim is obvious, except as concerns the inverses.

Recall now [CGP14a] that a series $w \in C^\infty(U)[[y,\xi,\eta]]$ is invertible in $C^\infty(U)[[y,\xi,\eta]]$ if and only if its base term $w_0$ is invertible in $C^\infty(U)$, and that, in this case, the inverse of $w = w_0(1-v)$ is given by $w^{-1} = w_0^{-1}\sum_{a\geq 0} v^a$. It follows that the claim is true for $w^{-1} = \Gamma\mathrm{det}^{-1} D$.

As for $D^{-1}$, note that

$$D := \begin{pmatrix} d & e \\ f & g \end{pmatrix} = \begin{pmatrix} \sum_r f_r^\varsigma(x) y^{2r} & -\sum_r g_r^\varsigma(x) y^{2r+1} \\ -\sum_r g_r^\theta(x) y^{2r+1} & \sum_r f_r^\theta(x) y^{2r} \end{pmatrix},$$

and that the base term of the series $dg - fe$, i.e., $f_0^\varsigma(x) f_0^\theta(x) = \varsigma(x)\theta(x)$, is invertible in $C^\infty(U)$, so that the series can itself be inverted. Therefore,

$$D^{-1} = \begin{pmatrix} d & e \\ f & g \end{pmatrix}^{-1} = (dg - fe)^{-1} \begin{pmatrix} g & -e \\ -f & d \end{pmatrix}.$$

Eventually, the claim is also true for $D^{-1}$, what completes the proof. □

**Theorem 4.** *If the component $f'(x', y', \xi', \eta')$ of a Berezinian section in some coordinate system does not contain the monomial $f'(x')\,y'$, then its component $f(x,y,\xi,\eta)$ in any other coordinate system – over the same open (base) subset – does not contain the monomial $f(x)\,y$.*

*Proof.* To simplify notation, we write, as above, $u, v, \varsigma, \theta$ (resp., $g$) instead of $x', y', \xi', \eta'$ (resp., $f'$). Change now coordinates in

$$g = \sum g(u) v^k \varsigma^a \theta^b \tag{22}$$

($k \in \mathbb{N}$, $a, b \in \{0,1\}$). The coordinate transformation

$$g(u) = g(u(x) + j) = \sum_{c\geq 0} \frac{1}{c!} (d_u^c g)(u(x)) j^c, \tag{23}$$

where $j = j(x,y,\xi,\eta)$ contains only terms with at least 2 parameters, generates a term $f(x)$, but no term $f(x)\,y$. As for $v^k \varsigma^a \theta^b$, note first that, since $g$ does not contain the monomial

$g(u)v$, the term of degree $(k,a,b)=(1,0,0)$ does not exist. However, all the other terms do not induce a term $f(x)\,y$.

Hence, such a term can only originate from $\Gamma\mathrm{Ber}(\mathcal{J}(x,y,\xi,\eta))$. Looking for a term $f(x)\,y$ in the Berezinian, means that we can discard in the Berezinian all the terms of the type $y^2$ (in the above defined sense, i.e., all the terms $y^k\xi^a\eta^b$, $k\geq 2$, $a\geq 0$, $b\geq 0$), as well as all the terms of the type $\xi$ and of the type $\eta$. In view of Theorem 3, we can thus compute the Berezinian discarding all the terms of the type $y^2$, the type $\xi$, and the type $\eta$ in the Jacobian. In other words, in the coordinate transformation (4), we ignore all the terms that generate in the Jacobian exclusively terms of the types $y^2$, $\xi$, and $\eta$. Hence, it suffices to consider the coordinate transformation

$$\begin{cases} u = u(x) + f(x)\,y^2 \\ v = v(x)\,y \\ \varsigma = \varsigma(x)\,\xi + g(x)\,y\eta \\ \theta = \theta(x)\,\eta + h(x)\,y\xi \quad , \end{cases}$$

with Jacobian

$$\mathcal{J} = \left(\begin{array}{cc|cc} d_x u + y^2\,d_x f & 2f(x)\,y & 0 & 0 \\ y\,d_x v & v(x) & 0 & 0 \\ \hline \xi\,d_x\varsigma + y\eta\,d_x g & g(x)\,\eta & \varsigma(x) & -g(x)\,y \\ \eta\,d_x\theta + y\xi\,d_x h & h(x)\,\xi & -h(x)\,y & \theta(x) \end{array}\right) .$$

Due to the above remarks on ignorable terms, it even suffices to compute the Berezinian of

$$\mathfrak{J} = \left(\begin{array}{cc|cc} d_x u & 2f(x)\,y & 0 & 0 \\ y\,d_x v & v(x) & 0 & 0 \\ \hline 0 & 0 & \varsigma(x) & -g(x)\,y \\ 0 & 0 & -h(x)\,y & \theta(x) \end{array}\right) .$$

The latter is equal to (see Equations (11) and (10))

$$\Gamma\mathrm{Ber}\,\mathfrak{J} = \left(d_x u - 2\frac{f(x)d_x v}{v(x)}y^2\right)\,v(x)\,\left(\left(\varsigma(x) - \frac{g(x)h(x)}{\theta(x)}y^2\right)\,\theta(x)\right)^{-1} .$$

It is now clear that there is no term $f(x)\,y$ in $f(x,y,\xi,\eta)$. □

### 5.3 Integrable Berezinian sections

Let $\beta$ be a Berezinian section of a $\mathbb{Z}_2^2$-domain $\mathcal{U}$. In any coordinate system $(x,y,\xi,\eta)$, it reads

$$\beta = (\mathrm{d}\,x\,\mathrm{d}\,y\otimes\partial_\xi\partial_\eta)\,f(x,y,\xi,\eta) = (\mathrm{d}\,x\,\mathrm{d}\,y\otimes\partial_\xi\partial_\eta)\sum f(x)y^k\xi^a\eta^b .$$

We say that $\beta$ is **compactly supported** if and only if its components $f(x,y,\xi,\eta)$ are compactly supported. Note first that a component $f(x,y,\xi,\eta)$ vanishes identically at some point $x_0$ if and only if all its coefficients $f(x)$ vanish identically at $x_0$. Hence, the support of the component is the union of the supports of the coefficients. It follows that the component is compactly supported if and only if the supports of the coefficients are contained in a fixed compact subset of $U$. Observe moreover that, if the component in one coordinate system is compactly supported, then the component in any other system is automatically also compactly supported. This result is essentially due to the fact that the support of a derivative of a function (resp., sum, product) is contained in the support of this function (resp., in the union, intersection of the supports).

In view of the preceding remark and Theorem 4, we can give the following

**Definition 2.** *An* **integrable Berezinian section** *of a $\mathbb{Z}_2^2$-domain $\mathcal{U}$ is a compactly supported Berezinian section whose components do not contain the monomial $f(x)\,y$.*

**Definition 3.** *Consider an integrable Berezinian section $\beta$ of a $\mathbb{Z}_2^2$-domain $\mathcal{U} = (U, C_U^\infty[[v,\varsigma,\theta]])$ with an oriented base $U$, and let*

$$\beta = (\mathrm{d}\,u\,\mathrm{d}\,v \otimes \partial_\varsigma \partial_\theta)\, g(u,v,\varsigma,\theta) = (\mathrm{d}\,u\,\mathrm{d}\,v \otimes \partial_\varsigma \partial_\theta) \sum_{k=0}^{\infty}\sum_{a=0}^{1}\sum_{b=0}^{1} g_{kab}(u) v^k \varsigma^a \theta^b \ .$$

*The integral of $\beta$ is defined by*

$$\int_{\mathcal{U}} \beta := \int \mathrm{d}\,u \int \mathrm{d}\,v\ \partial_\varsigma \partial_\theta\ g(u,v,\varsigma,\theta) = \int \mathrm{d}\,u \int \mathrm{d}\,v\ \sum_{k=0}^{\infty} g_{k11}(u) v^k :=$$

$$\int_U g_{011}(u)\ \mathrm{d}\,u \int_0^1 \mathrm{d}\,v := \int_U g_{011}(u)\,\mathrm{d}\,u \in \mathbb{R}\ . \tag{24}$$

In this definition, the integral over $U$ is the usual Lebesgue integral (which converges since $g_{011}(u)$ is a smooth compactly supported function of $U$). Since $v$ is dual to $\varsigma$ and $\theta$, and the integrals with respect to these odd coordinates select the highest degree coefficients, it is natural to define the integral with respect to $v$ by

$$\int \mathrm{d}\,v\ v^k = 0, \quad \text{for all} \quad k > 0\ ,$$

and $\int \mathrm{d}\,v = 1$, or, better,

$$\int_0^1 \mathrm{d}\,v = 1\ .$$

The integration interval $[0,1]$ has of course to be properly adapted if we change coordinates. If

$$v = v(x)y + \sum_{r\geq 1} f_r^v(x) y^{2r+1} + \sum_{s \geq 0} g_s^v(x) y^{2s}\xi\eta\ ,$$

we obtain formally

$$\int \mathrm{d}\,v = \int \mathrm{d}\,y\ \left( v(x) + \sum_{r\geq 1}(2r+1)\, f_r^v(x)\, y^{2r} + \sum_{s\geq 0} 2s\, g_s^v(x)\, y^{2s-1}\xi\eta \right) = \int \mathrm{d}\,y\ v(x)\ ,$$

so that it is natural to decide that the formal integration interval $v \in [0,1]$ transforms into the formal integration interval $y \in [0, \frac{1}{v(x)}]$.

**Theorem 5.** *The integral of an integrable Berezinian section of a $\mathbb{Z}_2^2$-domain with an oriented base is well-defined.*

*Proof.* We must show that the definition is independent of the considered coordinates, i.e., that, if $(x, y, \xi, \eta)$ is another coordinate system, we have

$$\int \mathrm{d}\,x \int \mathrm{d}\,y\ \partial_\xi \partial_\eta\ f(x, y, \xi, \eta) = \int_U g_{011}(u)\,\mathrm{d}\,u\ .$$

In view of the definition of a Berezinian section, the LHS of the preceding equation is equal to

$$\int_U \mathrm{d}\,x \int_0^{\frac{1}{v(x)}} \mathrm{d}\,y\ \partial_\xi \partial_\eta\ \ g(u(x,y,\xi,\eta), v(x,y,\xi,\eta), \varsigma(x,y,\xi,\eta), \theta(x,y,\xi,\eta))\ \ \Gamma\mathrm{Ber}(\mathcal{J}(x,y,\xi,\eta))\ , \tag{25}$$

and, due to Definition 3, we can ignore in the integrated $\mathbb{Z}_2^2$-function all the terms of type $y$ (i.e., we can set $y = 0$ in the integrand). Equations (4), (22), and (23) show that, when setting $y = 0$ in

$$g(u(x,y,\xi,\eta), v(x,y,\xi,\eta), \varsigma(x,y,\xi,\eta), \theta(x,y,\xi,\eta))\ ,$$

we obtain

$$\sum g_{kab}(u(x))\,(g_0^v(x)\xi\eta)^k\,(f_0^\varsigma(x)\xi)^a\,(f_0^\theta(x)\eta)^b = \sum g_{kab}(u(x))\,(g_0^v(x)\xi\eta)^k\,(\varsigma(x)\xi)^a\,(\theta(x)\eta)^b\ . \tag{26}$$

As for the Berezinian, Theorem 3 allows to ignore in the Jacobian all the terms of type $y$. When denoting this simplified Jacobian again by $\mathfrak{J}$, we obtain (see (4))

$$\mathfrak{J} = \left(\begin{array}{cc|cc} d_x u & \xi\eta\, g_0^u(x) & 0 & 0 \\ \xi\eta\, d_x g_0^v & v(x) & \eta\, g_0^v(x) & \xi\, g_0^v(x) \\ \hline \xi\, d_x\varsigma & \eta\, g_0^\varsigma(x) & \varsigma(x) & 0 \\ \eta\, d_x\theta & \xi\, g_0^\theta(x) & 0 & \theta(x) \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\ . \tag{27}$$

Now,

$$\Gamma\mathrm{Ber}\ \mathfrak{J} = \Gamma\mathrm{det}(A - BD^{-1}C)\ \Gamma\mathrm{det}^{-1}\,D$$

and it is straightforwardly checked that

$$\Gamma\mathrm{det}(A - BD^{-1}C) = \Gamma\mathrm{det}\left(\begin{array}{c|c} d_x u & \xi\eta\, g_0^u(x) \\ \hline \xi\eta\left(d_x g_0^v - g_0^v(x)\left(\frac{d_x\varsigma}{\varsigma(x)} + \frac{d_x\theta}{\theta(x)}\right)\right) & v(x) \end{array}\right) = v(x)\,d_x u$$

and

$$\Gamma\mathrm{det}^{-1}\,D = \Gamma\mathrm{det}^{-1}\left(\begin{array}{c|c} \varsigma(x) & 0 \\ \hline 0 & \theta(x) \end{array}\right) = \frac{1}{\varsigma(x)\theta(x)}\ . \tag{28}$$

Collecting now the relevant terms computed in (26), (27), and (28), we see that the integral (25) reads

$$\int \mathrm{d}\,x \int \mathrm{d}\,y\ \partial_\xi \partial_\eta\ f(x, y, \xi, \eta) =$$

$$\int_U \mathrm{d}\,x \int_0^{\frac{1}{v(x)}} \mathrm{d}\,y\; \partial_\xi \partial_\eta\, \frac{v(x)\, d_x u}{\varsigma(x)\theta(x)} \sum g_{kab}(u(x))\, (g_0^v(x)\xi\eta)^k\, (\varsigma(x)\xi)^a\, (\theta(x)\eta)^b\;. \tag{29}$$

There exist a priori two terms $(k, a, b)$ whose derivative $\partial_\xi \partial_\eta$ does not vanish: $(1, 0, 0)$ and $(0, 1, 1)$. As $\beta$ is integrable, the coefficient $g_{100}(u)$ of its component $g(u, v, \varsigma, \theta)$ vanishes identically. Hence, we find

$$\int_U \mathrm{d}\,x \int_0^{\frac{1}{v(x)}} \mathrm{d}\,y\; \partial_\xi \partial_\eta\, \frac{v(x)\, d_x u}{\varsigma(x)\theta(x)}\, g_{011}(u(x))\, \varsigma(x)\xi\, \theta(x)\eta =$$

$$\int_U \mathrm{d}\,x\; g_{011}(u(x))\, d_x u\, v(x) \int_0^{\frac{1}{v(x)}} \mathrm{d}\,y = \int_U g_{011}(u)\, \mathrm{d}\,u$$

($U$ is oriented: the considered coordinate transformation in $\mathcal{U}$ is assumed to induce an orientation respecting coordinate transformation in $U$). □

# 6 Further questions

In the following, we assume some familiarity with pseudodifferential forms. A short overview can be found in Section 7. Pseudodifferential forms $\widehat{\Omega}(\mathcal{M})$ of a standard $\mathbb{Z}_2$-manifold $\mathcal{M}$ are exactly the functions $C^\infty(T\mathcal{M})$ of its tangent bundle $T\mathcal{M}$, viewed as $\mathbb{Z}_2$-manifold, i.e., in coordinates $(u, \varsigma)$ of $\mathcal{M}$, they are the polynomials

$$C^\infty(u, \mathrm{d}\,\varsigma)[\varsigma, \mathrm{d}\,u] =: C^\infty(u, v)[\varsigma, \theta]$$

in the odd variables $\varsigma, \theta$ with smooth coefficients in the even $u, v$. A tangent bundle function whose dependence in $v$ is polynomial as well, i.e., a sum of terms

$$g_{kab}(u)\, v^k \varsigma^a \theta^b\;, \tag{30}$$

is an ordinary differential form of the underlying supermanifold and can of course not be integrated with respect to the canonical volume $\mathrm{d}\,u\,\mathrm{d}\,v \otimes \partial_\varsigma \partial_\theta$, because its support in $v$ is in general not compact. On the other hand, if the pseudoform has a distributional dependence in $v$, i.e., if it is a sum of terms

$$g_{ab}(u)\, \delta(v) \varsigma^a \theta^b\;, \tag{31}$$

it can be integrated. For details, see Section 7, in particular Equations (38) and (39).

In Subsection 5.3, we gave – in the $\mathbb{Z}_2^2$-graded context – a sense to the integral with respect to $\mathrm{d}\,u\,\mathrm{d}\,v \otimes \partial_\varsigma \partial_\theta$ of a series (in $k$) of terms (30). Observe that Equation (29) allows to see that the integrability condition on the integrated Berezinian section originates in the existence of the term $g_0^v(x)\xi\eta$ in the general $\mathbb{Z}_2^2$-coordinate-transformation (4). This term implies that the local submanifold $y = 0$ is not transformed into the local submanifold $v = 0$. In case the considered $\mathbb{Z}_2^2$-manifold admits an atlas such that for all coordinate transformations the coefficient $g_0^v(x)$ vanishes identically, the integral of a series is the integral of its pullback along a global submanifold. Moreover, the integrability condition then disappears, see (29), i.e., all Berezinian sections are integrable in that case. In the general situation of arbitrary $\mathbb{Z}_2^2$-coordinate-transformations, we compensated the problematic terms $g_0^v(x)\xi\eta$ by an *intrinsic* integrability condition.

We may also consider the $\mathbb{Z}_2^2$-integration of 'functions' of the 'type (31)'. Indeed, the above $\mathbb{Z}_2^2$-integration with respect to $v$ of a series $g(u,v,\varsigma,\theta)$ of terms (30) – which amounts to computing $g(u,0,\varsigma,\theta)$ – , reminds one the action

$$\int \mathrm{d}\, v\; g(v)\, \delta(v) = g(0)$$

of a delta 'function' on a test function $g$, see Equation (35) in Section 7, as well as the residue theorem

$$\int_{\mathcal{C}} \mathrm{d}\, v\; \frac{g(v)}{v} = (2\pi i)\, g(0)$$

of complex analysis (we do not insist on notation and conditions). Hence, the idea to integrate series $G(u,v,\varsigma,\theta)$ of the type (30), but to allow powers $v^k$, $k \in \{-N, -N+1, \ldots, -1\} \cup \mathbb{N}$, for some integer $N \geq 1$, and to develop integration theory via an algebraic multivariate residue theorem. More precisely, in the case of a $\mathbb{Z}_2^2$-manifold of dimension $1|1|1|1$, one could consider objects of the type

$$\gamma = (\mathrm{d}\, u\, \mathrm{d}\, v \otimes \partial_\varsigma \partial_\theta)\; G(u,v,\varsigma,\theta) = (\mathrm{d}\, u\, \mathrm{d}\, v \otimes \partial_\varsigma \partial_\theta) \sum_{k=-N}^{\infty} \sum_{a=0}^{1} \sum_{b=0}^{1} G_{kab}(u) v^k \varsigma^a \theta^b \ ,$$

and define their integral by

$$\int_{\mathcal{U}} \gamma := \int \mathrm{d}\, u \int \mathrm{d}\, v\; \partial_\varsigma \partial_\theta\; G(u,v,\varsigma,\theta) = \int \mathrm{d}\, u \int \mathrm{d}\, v \sum_{k=-N}^{\infty} G_{k11}(u) v^k =$$

$$\int \mathrm{d}\, u \int \mathrm{d}\, v\; \frac{1}{v} \sum_{k=-N}^{\infty} G_{k11}(u) v^{k+1} := \int_U G_{-111}(u)\; \mathrm{d}\, u \in \mathbb{R}\ . \tag{32}$$

Observe that this integral has the fundamental properties (35) and (36) of pseudodifferential forms, and that $\frac{1}{v}$ '$\sim \delta(v)$' satisfies the rescaling requirement [Wit12, Equation (3.37)]. We expect the integration rule (32) to be coordinate independent and generalizable.

# 7 Appendix: Concise review of differential superforms from the standpoint of Physics

In classical Supergeometry, one integrates compactly supported **classical Berezinian sections** over oriented supermanifolds. These Berezinian sections of a $\mathbb{Z}_2$-manifold $\mathcal{M} = (M, \mathcal{O}_M)$ are defined as the above Berezinian sections of a $\mathbb{Z}_2^2$-manifold $\mathcal{N} = (N, \mathcal{A}_N)$ – mutatis mutandis. In coordinates $(x,\xi)$, they read

$$(\mathrm{d}\, x \otimes \partial_\xi)\, f(x,\xi)\ ,$$

with a transformation rule for the components $f$ that is similar to (21). Here $x = (x^1, \ldots, x^p)$ and $\xi = (\xi^1, \ldots, \xi^q)$, where $p|q$ is the dimension of $\mathcal{M}$, and

$$\mathrm{d}\,x \otimes \partial_\xi = \mathrm{d}\,x^1 \ldots \mathrm{d}\,x^p \otimes \partial_{\xi^1} \ldots \partial_{\xi^q} = \mathrm{d}\,x^1 \wedge \ldots \wedge \mathrm{d}\,x^p \otimes \partial_{\xi^1} \circ \ldots \circ \partial_{\xi^q} ,$$

which is often also denoted by $\mathcal{D}(x, \xi)$ or $[\mathrm{d}\,x|\,\mathrm{d}\,\xi]$. In view of what has been said, compactly supported functions $C^\infty(\mathcal{M}) = \mathcal{O}_M(M)$ can be integrated over oriented manifolds $\mathcal{M}$ with respect to a fixed Berezinian section.

In particular, the tangent bundle $T\mathcal{M}$ may be viewed as a supermanifold with coordinates $(x, y, \xi, \eta) := (x, \mathrm{d}\,\xi, \xi, \mathrm{d}\,x)$ of parities $(0, 0, 1, 1)$. An orientation on $\mathcal{M}$ induces an orientation on $T\mathcal{M}$. In fact, we consider here a supermanifold $\mathcal{M}$ with a so-called $(-,-)$ orientation. This means that $\mathcal{M}$ is endowed with an atlas, such that the coordinate transformations satisfy $\det J_{00} \det J_{11} > 0$, i.e., such that the determinants of the even-even and odd-odd parts of the Jacobian matrix have the same sign. In this case, the coordinate Berezinian volume $\mathcal{D}(x, y, \xi, \eta)$ is invariant under coordinate transformation – essentially because coordinates appear pairwise – , so that $T\mathcal{M}$ has a **canonical Berezinian section**. Hence, compactly supported functions $C^\infty(T\mathcal{M})$ are integrable over the oriented $T\mathcal{M}$.

The functions $C^\infty(T\mathcal{M})$, i.e., locally, the polynomials $C^\infty(x, y)[\xi, \eta]$ in $(\xi, \eta)$ with coefficients that are smooth with respect to $(x, y)$, are called **pseudodifferential forms** $\widehat{\Omega}(\mathcal{M})$ of $\mathcal{M}$ [BL77b]. Pseudodifferential forms with **polynomial** dependence in $y$, in one coordinate system, have this property in all systems. They are ordinary differential forms $\Omega(\mathcal{M})$ of $\mathcal{M}$ and are clearly not integrable. However, pseudodifferential forms with compact support in $(x, y)$, or sufficiently quick decrease at infinity in $(x, y)$, or, still, whose support in $x$ is compact and which with respect to $y$ are a **distribution supported at the origin** – also a property that is coordinate independent – can be integrated [BL77b], [Wit12].

Pseudodifferential forms correspond in some sense to the inhomogeneous differential forms in classical Differential Geometry. Just as the latter can be projected onto $k$-forms, with $0 \le k \le p$ – where $p$ is the dimension of the underlying smooth manifold – , the Baranov-Schwarz transformation allows to project the former onto $r|s$**-forms**, for all possible $r, s$. The full complex of all $r|s$-forms of a supermanifold of dimension $p|q$ can be found in [GM07]. Note that $0 \le s \le q$ and that, for fixed $s$, the complex is unbounded in both directions – so that $r \in \mathbb{Z}$ – , except for $s = 0$ (resp., $s = q$). In this case, the $r|s$-forms coincide with **ordinary differential superforms** (resp., with **integral forms** [BL77a], at least for $r \ge 0$), and the considered complex is bounded to the left (resp., to the right). Further, $r|s$-forms can be integrated over sub- or singular manifolds of dimension $r|s$, although the integrals do not necessarily converge.

There is no good description of $r|s$-forms with $0 < s < q$. In the following, we provide insight from the standpoint of Physics [CDGM10], [Wit12].

Consider a supermanifold $\mathcal{M} = (M, \mathcal{O}_M)$ of dimension $p|q$. In coordinates $(x, \xi)$, its space of '$r|s$-forms' is obtained from the usual basic forms $\mathrm{d}\,x, \mathrm{d}\,\xi$ and additional basic forms $\delta^{(\alpha)}(\mathrm{d}\,\xi)$.

Here

$$\delta^{(\alpha)}(\mathrm{d}\,\xi) = \delta^{(\alpha_1)}(\mathrm{d}\,\xi^1) \dots \delta^{(\alpha_q)}(\mathrm{d}\,\xi^q) ,$$

where $\alpha_j \in \mathbb{N}$, where $\delta^{(\alpha_j)}(\mathrm{d}\,\xi^j)$ is interpreted as the $\alpha_j$-th derivative of $\delta(\mathrm{d}\,\xi^j)$, whereas $\delta(\mathrm{d}\,\xi^j)$ is viewed as the (standard) $\delta$-function in the variable $\mathrm{d}\,\xi^j$.

Recall that the delta 'function' (at the origin) in a real variable $x$ has value 0 except that $\delta(0) = +\infty$. Further, the integral

$$\int_{\mathbb{R}} \mathrm{d}\,x \; \delta(x) = 1 . \tag{33}$$

In fact, the delta 'function' is a generalized function or distribution, also denoted by $\delta$, which assigns to each locally integrable function $\phi \in L^1_{\mathrm{loc}}(\mathbb{R})$, the value

$$\delta(\phi) := \text{“} \int_{\mathbb{R}} \mathrm{d}\,x \; \phi(x)\delta(x) \text{”} = \phi(0) . \tag{34}$$

Just as $\delta(x)$ is thus an operator that acts on test functions $\phi(x)$, we think about the $\delta(\mathrm{d}\,\xi^j)$ as operators that act on ordinary differential superforms $\omega(x, \xi, \mathrm{d}\,x, \mathrm{d}\,\xi)$ (in the following, we only write the relevant variable – and suppress all the others) according to the rule

$$\int \mathrm{d}(\mathrm{d}\,\xi^j) \; \omega(\mathrm{d}\,\xi^j) \; \delta(\mathrm{d}\,\xi^j) = \omega(0) . \tag{35}$$

Moreover, it is natural to set

$$\int \mathrm{d}(\mathrm{d}\,\xi^j) \; \omega(\mathrm{d}\,\xi^j) \; \delta^{(1)}(\mathrm{d}\,\xi^j) = -\int \mathrm{d}(\mathrm{d}\,\xi^j) \; \partial_{\mathrm{d}\,\xi^j}\omega \;\; \delta(\mathrm{d}\,\xi^j) . \tag{36}$$

The results (35) and (36) show that the **products** of the basic forms $\mathrm{d}\,\xi^j$ and the corresponding new basic forms $\delta^{(\alpha)}(\mathrm{d}\,\xi^j)$ satisfy, as operators on superforms, the equalities

$$\mathrm{d}\,\xi^j \; \delta(\mathrm{d}\,\xi^j) = 0 \quad \text{and} \quad \mathrm{d}\,\xi^j \; \delta^{(1)}(\mathrm{d}\,\xi^j) = -\delta(\mathrm{d}\,\xi^j) . \tag{37}$$

As for the commutation rules between the generators, note that the sign convention met in the literature is usually the so-called Bernstein-Leites convention, i.e., the generators $\mathrm{d}\,x^i$ (resp., $\mathrm{d}\,\xi^j$) are odd (resp., even), and the commutation rule is given by the parities, e.g.,

$$\mathrm{d}\,x^{i_1} \, \mathrm{d}\,x^{i_2} = -\,\mathrm{d}\,x^{i_2} \, \mathrm{d}\,x^{i_1} .$$

Further, the additional generators $\delta(\mathrm{d}\,\xi^j)$ are **odd** (parity shift) and thus anticommute among themselves, for instance.

Moreover, the **cohomological degree** of a derivative $\delta^{(\alpha_j)}(\mathrm{d}\,\xi^j)$ is, logically, $-\alpha_j$, so that forms of negative degree do exist. Besides the **parity** and the (cohomological) degree, one introduces the **picture number**, which counts the number of delta functions and of their derivatives. For instance, the form

$$\mathrm{d}\,x^{i_1} \dots \mathrm{d}\,x^{i_\ell} \;\; \mathrm{d}\,\xi^{j_1} \dots \mathrm{d}\,\xi^{j_m} \; \delta^{(\alpha_1)}(\mathrm{d}\,\xi^{k_1}) \dots \delta^{(\alpha_s)}(\mathrm{d}\,\xi^{k_s}) \; f(x, \xi)$$

is of (cohomological) degree $r := \ell + m - \alpha_1 - \ldots - \alpha_s$ and has picture number $s$. We denote the space of **$r|s$-forms**, i.e., the space of $r$-forms with picture number $s$, by $\Omega^{r|s}(\mathcal{M})$. If $s = 0$ (resp., $0 < s < q$), we have $r \in \mathbb{N}$ (resp., $r \in \mathbb{Z}$). As mentioned above, $r|s$-forms with picture $s = 0$ are just **ordinary differential superforms**. In view of (37), an $r|s$-form with picture $s = q$ is of the type

$$\mathrm{d}\, x^{i_1} \ldots \mathrm{d}\, x^{i_\ell}\ \delta^{(\alpha_1)}(\mathrm{d}\,\xi^1) \ldots \delta^{(\alpha_q)}(\mathrm{d}\,\xi^q)\ f(x, \xi)\ ,$$

with $r = \ell - \alpha_1 - \ldots - \alpha_q$. Hence, $r|q$-forms are **distributional pseudodifferential forms** in the sense defined above. Let us mention that it can be shown that the subcomplexes

$$\Omega^{\bullet|s}(\mathcal{M}) = \bigoplus_{r=0}^{p} \Omega^{r|s}(\mathcal{M})\ ,$$

obtained for $s \in \{0, \ldots, q\}$, are isomorphic – the idea of 'change of picture' originates and is of importance in superstring theory. Further, it is now clear that, whereas ordinary superforms define a complex which is bounded below (by 0-forms), but not above (there are no top differential forms), the $r|q$-superforms provide a complex which is bounded above (since $r \in \{\ldots, -1, 0, \ldots, p\}$), but not below (of course).

A top $r|q$-form, i.e., a form in $\Omega^{p|q}(\mathcal{M})$, is of the type

$$\omega(x, y, \xi, \eta) = \omega(x, \mathrm{d}\,\xi, \xi, \mathrm{d}\,x) :=$$

$$\mathrm{d}\,x^1 \ldots \mathrm{d}\,x^p\ \delta(\mathrm{d}\,\xi^1) \ldots \delta(\mathrm{d}\,\xi^q)\ f(x, \xi) = \eta^1 \ldots \eta^p\ \delta(y^1) \ldots \delta(y^q)\ f(x, \xi)\ .$$

Since, as indicated above, $r|s$-forms can be integrated over $r|s$-dimensional submanifolds, **a form $\omega \in \Omega^{p|q}(\mathcal{M})$ can be integrated over the $p|q$-dimensional smooth supermanifold $\mathcal{M}$ itself**. Indeed, such a form is a function on $T\mathcal{M}$ that is a distribution supported at 0 with respect to $\mathrm{d}\,\xi = (\mathrm{d}\,\xi^1, \ldots, \mathrm{d}\,\xi^q)$, so it can be integrated with respect to the natural Berezinian section $\mathcal{D}(x, \mathrm{d}\,\xi, \xi, \mathrm{d}\,x)$, provided it is compactly supported in $x = (x^1, \ldots, x^p)$ and $\mathcal{M}$ is endowed with a $(-,-)$ orientation:

$$\int_{\mathcal{M}} \omega = \int_{T\mathcal{M}} \mathcal{D}(x, \mathrm{d}\,\xi, \xi, \mathrm{d}\,x)\ \ \omega(x, \mathrm{d}\,\xi, \xi, \mathrm{d}\,x) = \int_{T\mathcal{M}} \mathrm{d}\,x\,\mathrm{d}\,y \otimes \partial_\xi \partial_\eta\ \ \omega(x, y, \xi, \eta)\ , \tag{38}$$

where the integral over $T\mathcal{M}$ is the Berezinian integral and where the integral over $y$ is given by (33). If $f_{1\ldots q}(x)$ is the compactly supported coefficient of $\xi^1 \ldots \xi^q$, we find, up to sign,

$$\int_{\mathcal{M}} \omega = \int \mathrm{d}\,x\ f_{1\ldots q}(x)\ . \tag{39}$$

Of course, the above intuitive approach is mainly presented as a coordinate viewpoint. However, pseudodifferential forms, distributional pseudoforms, $r|s$-forms, ordinary superforms, and integral forms, as well as the just considered integral, can be shown to be global concepts. For a more detailed and rigorous exposition, see [Vor14].

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