A tutorial on multiple crack growth and intersections with XFEM

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Content

Problem statement

- Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

$$\mathcal{L}(\mathbf{u}, a) = \Pi(\mathbf{u}, a) + \sum_{i=1}^{n_{\text{tip}}} \int_{a_i} G_c^i \,\mathrm{d}a_i$$

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- Consider a cracked linear-elastic isotropic solid subject to an external load whose quasistatic behavior can be described by the following total Lagrangian form:

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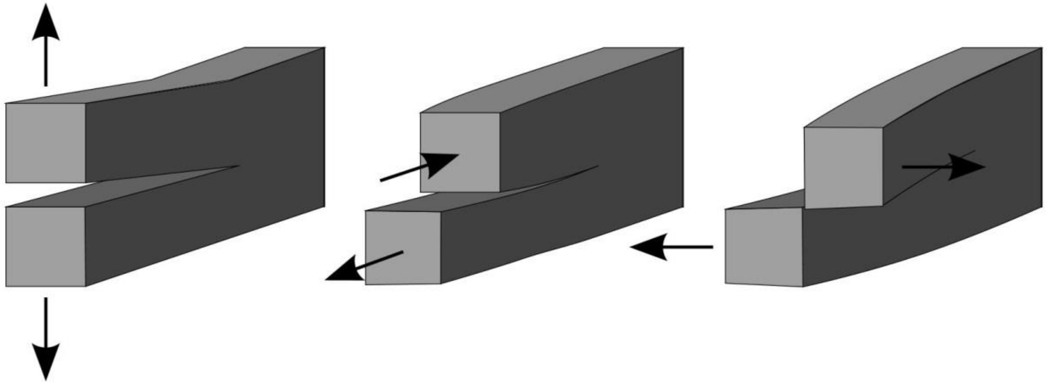
- The solution for ***u****(a)* and *a(t)* are obtained by satisfying the stationarity of ***L****(****u****,a)* during the evolution of *t*, subject to $\Delta a_i \geq 0$:

$$\delta\mathcal{L}(\mathbf{u},a) = \delta_{\mathbf{u}}\Pi(\mathbf{u},a) + \sum_{i=1}^{n_{\text{tip}}} \left[\frac{\partial \Pi(\mathbf{u},a)}{\partial a_i} + G_c^i \right] \delta a_i = 0$$

- Post processing of solution to evaluate SIF

$$I^{(1+2)} = \int_\Omega \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} \mathrm{d}\Omega = \frac{2}{E'} (K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)})$$



Mode I:
Opening

Mode II:
In-plane shear

Mode III:
Out-of-plane shear

Crack growth
maximum hoop stress

- Post processing of solution to evaluate SIF

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- Crack growth direction

$$\theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[\frac{1}{4} \left(\frac{K_I}{K_{II}} - \mathrm{sign}(K_{II}) \sqrt{ \left(\frac{K_I}{K_{II}} \right)^2 + 8 } \right) \right]$$

Crack growth
maximum hoop stress

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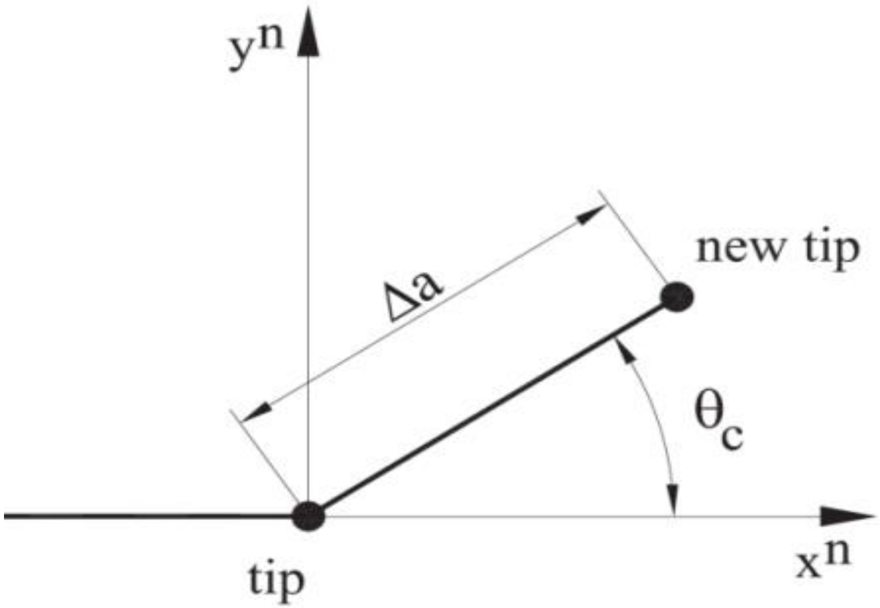
- Crack growth criterion

$$\frac{k_I(K_I, K_{II}, \theta_c)^2 + k_{II}(K_I, K_{II}, \theta_c)^2}{E'} = G_c$$

Crack growth
energy minimization

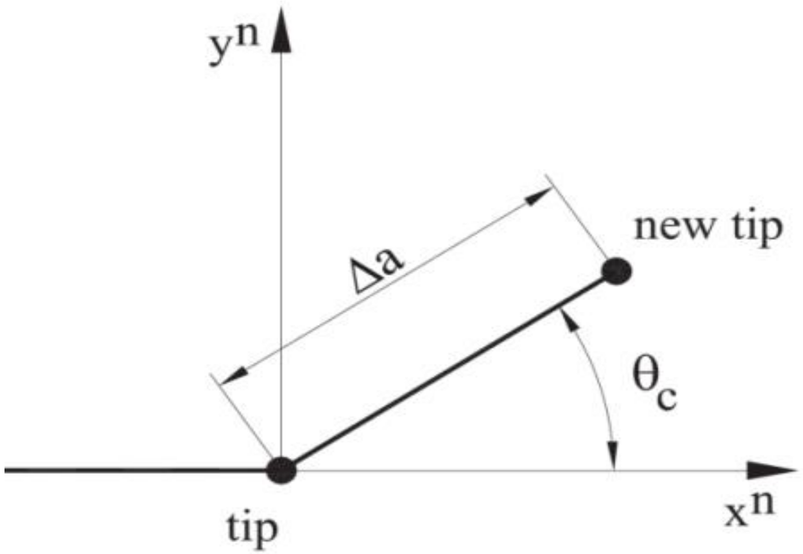
- Energy release rate w.r.t. crack increment direction, θ_i:



Crack growth
energy minimization

- Energy release rate w.r.t. crack increment direction, θ_i:

$$G_i = -\frac{\partial \Pi(\mathbf{u}, \mathbf{a} + \Delta\mathbf{a})}{\partial \theta_i}$$



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- Energy release rate w.r.t. crack increment direction, θ_i:

$$G_i = -\frac{\partial \Pi(\mathbf{u}, \mathbf{a} + \Delta\mathbf{a})}{\partial \theta_i}$$

- The rates of energy release rate:

$$H_{ij} = \frac{\partial G_i}{\partial \theta_j}$$

- Updated directions:

$$\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^{i} - \mathbf{H}^{-1}\mathbf{G}$$

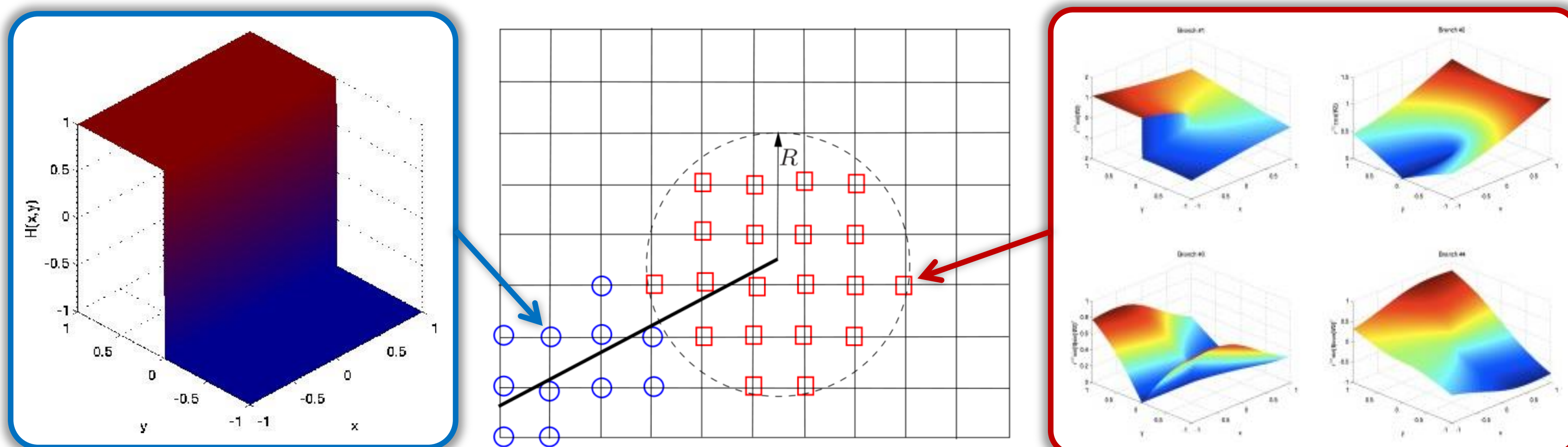
Discretization
The eXtended Finite Element Method (XFEM)

- Approximation function (single crack)

$$\mathbf{u}^h(\mathbf{x}) = \underbrace{\sum_{I \in \mathcal{N}_I} N_I(\mathbf{x})\mathbf{u}^I}_{\text{standard part}} + \underbrace{\sum_{J \in \mathcal{N}_J} N_J(\mathbf{x}) H(\mathbf{x}) \mathbf{a}^J}_{\text{discontinuous enrichment}} + \underbrace{\sum_{K \in \mathcal{N}_K} N_K(\mathbf{x}) \sum_{\alpha=1}^{4} f_\alpha(\mathbf{x}) \mathbf{b}^{K\alpha}}_{\text{singular tip enrichment}}$$

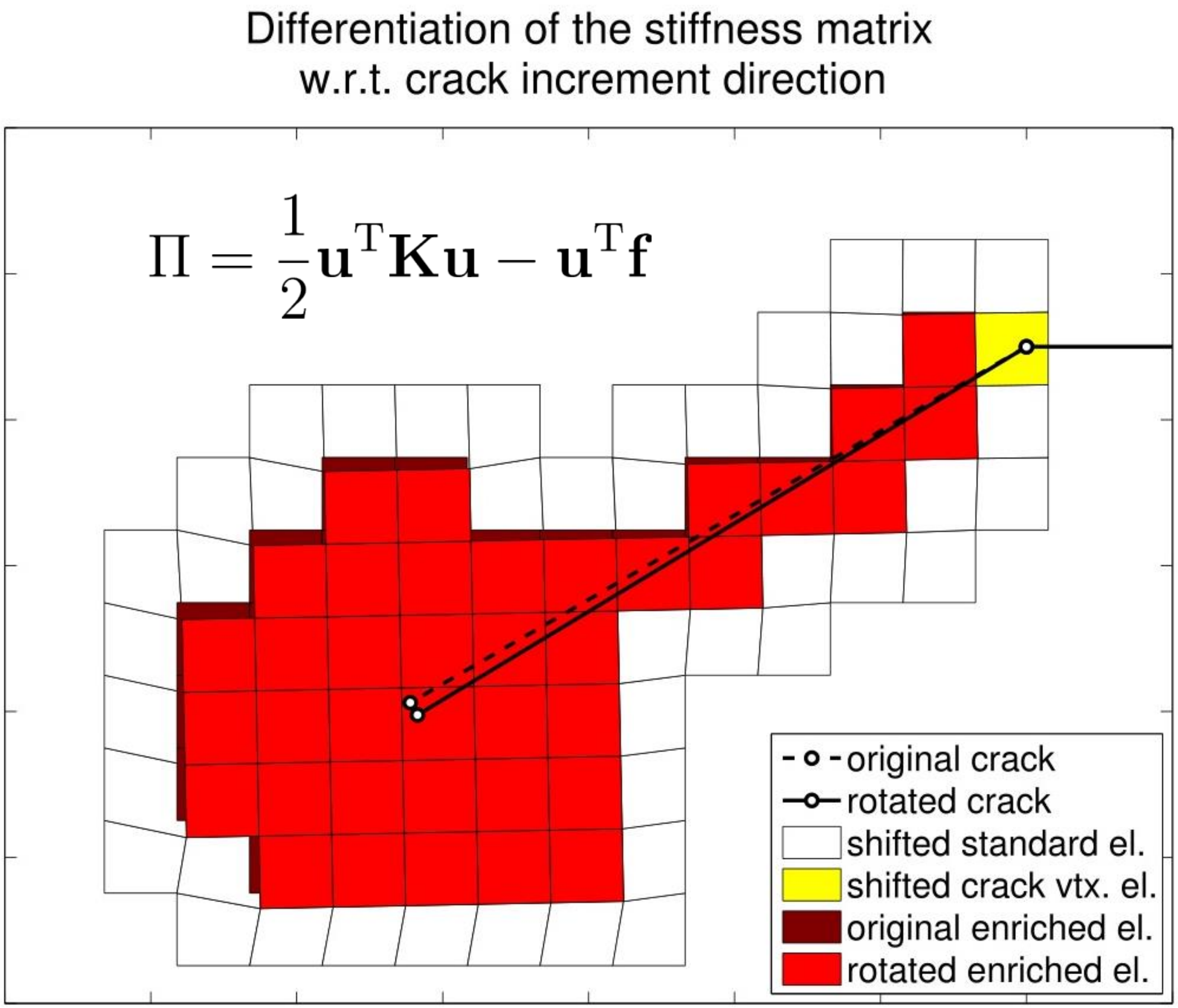
$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ above crack} \\ -1 & \text{if } \mathbf{x} \text{ below crack} \end{cases}$$

$$\{f_\alpha(r,\theta), \alpha = 1,4\} = \left\{\sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta\right\}$$



Discretization

The eXtended Finite Element Method (XFEM)



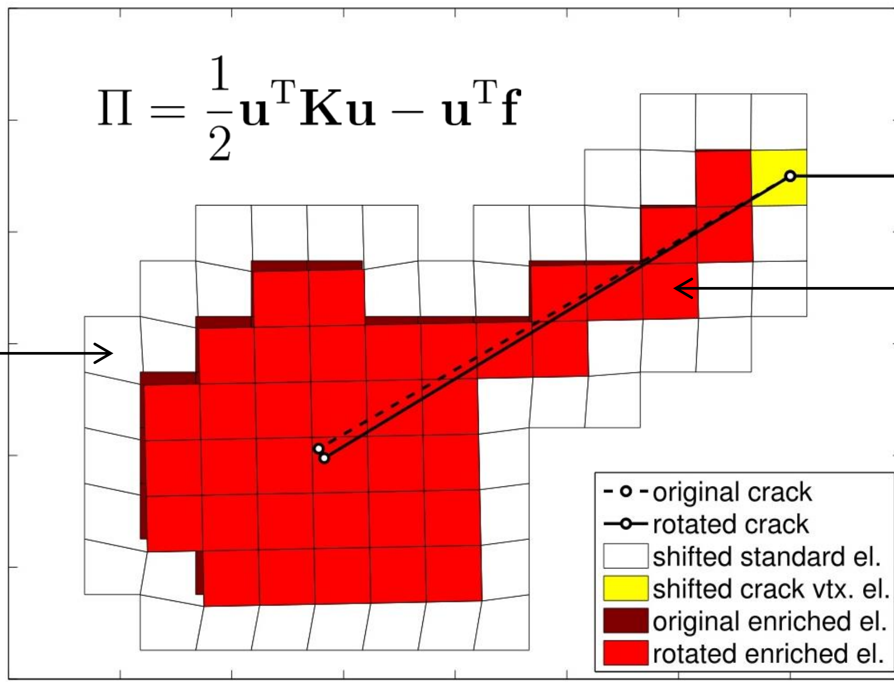
$$\delta \mathbf{K}_e = \int_{\Omega_e} \left(\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

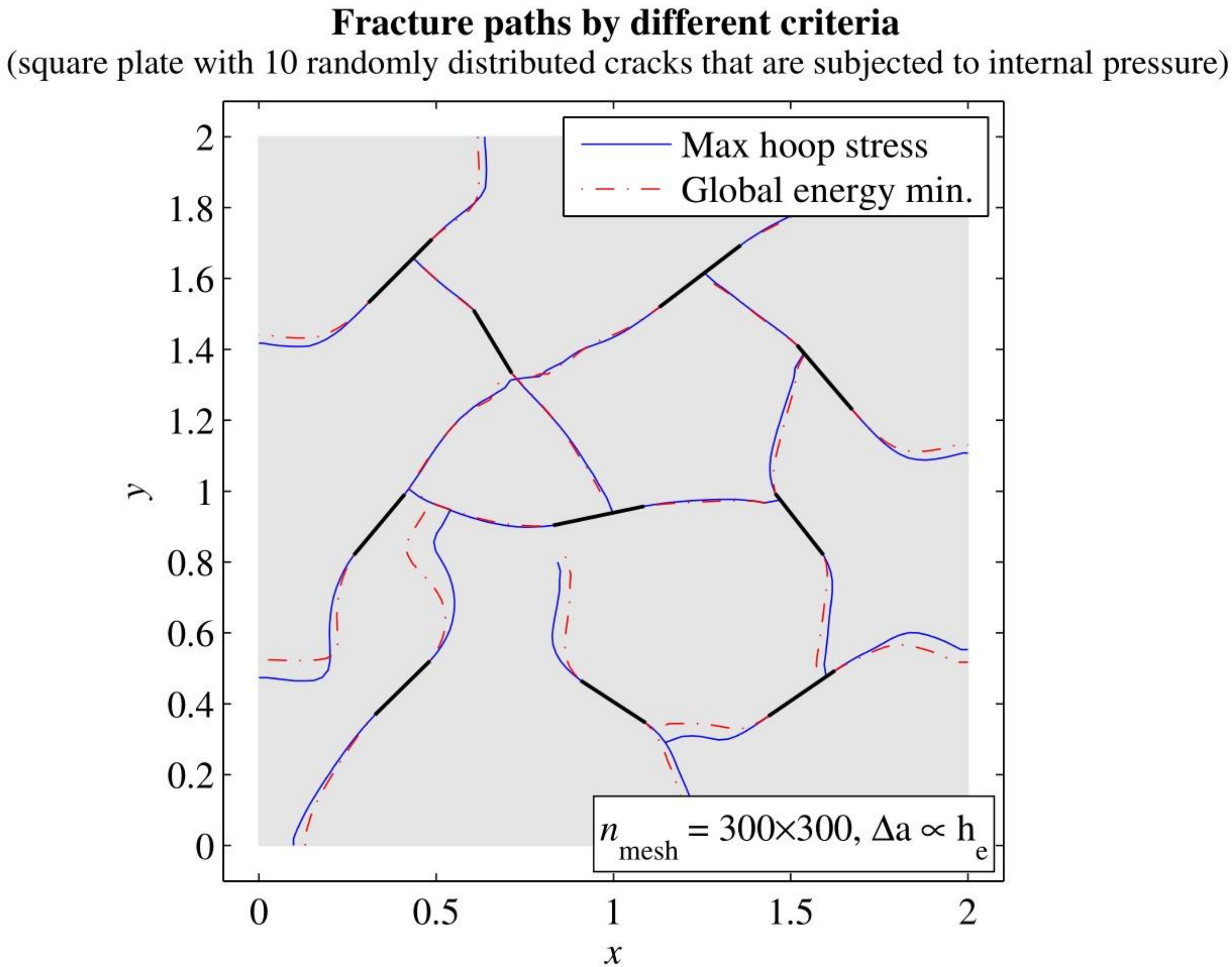
$$\delta^2 \mathbf{K}_e = \int_{\Omega_e} \left(\delta^2 \mathbf{B}^T \mathbf{D} \mathbf{B} + 2\delta \mathbf{B}^T \mathbf{D} \delta \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta^2 \mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} 2\left(\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}\right) \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta^2 \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

Differentiation of the stiffness matrix
w.r.t. crack increment direction

$$\delta \mathbf{K_e} = \mathbf{T}^T \mathbf{K_e} + \mathbf{K_e} \mathbf{T}$$

$$\delta^2 \mathbf{K_e} = 2(\mathbf{T}^T \mathbf{K_e} \mathbf{T} - \mathbf{K_e})$$

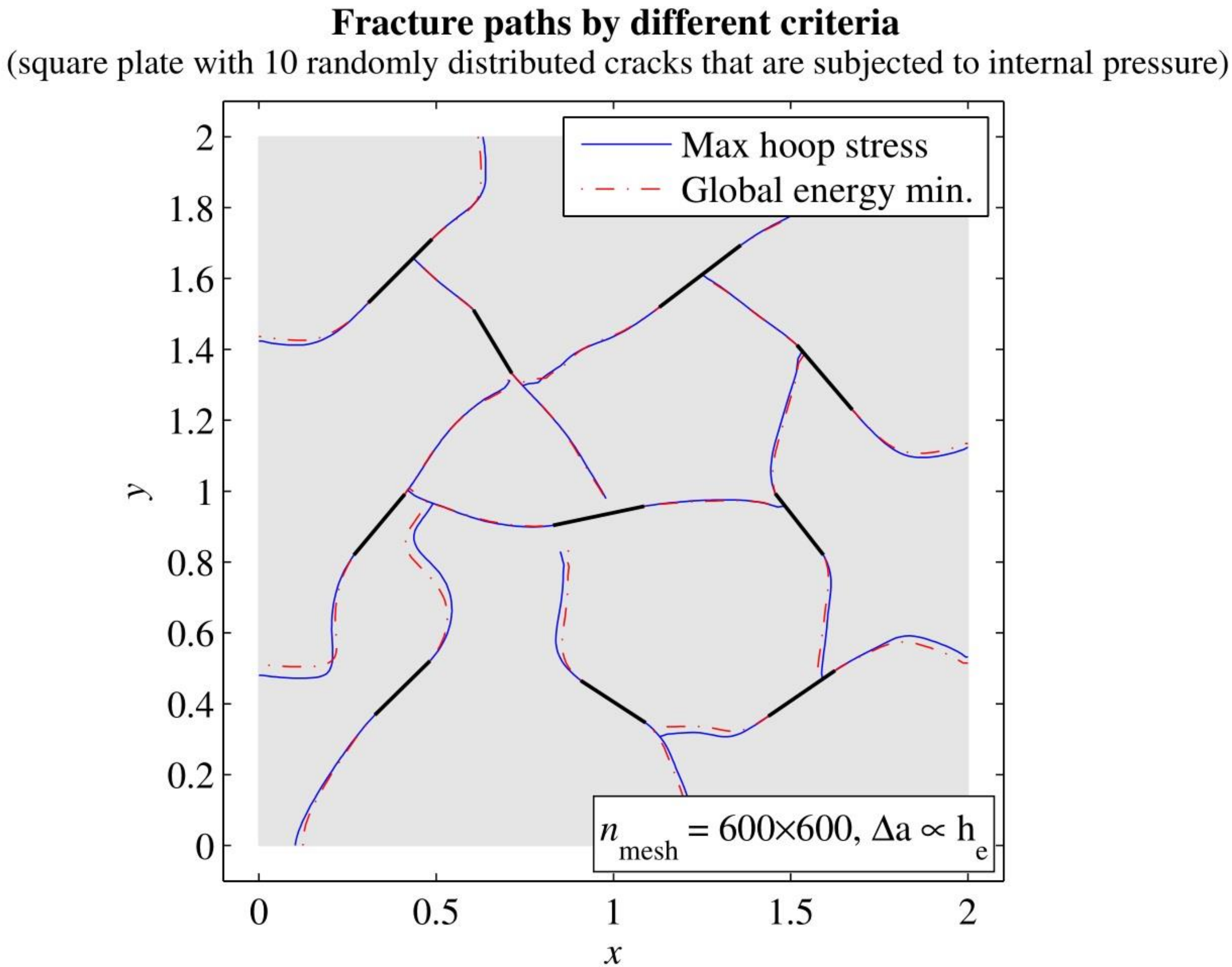




Fracture paths by different criteria
(square plate with 10 randomly distributed cracks that are subjected to internal pressure)
Max hoop stress
Global energy min.
n_{mesh} = 300×300, Δa ∝ h_e
y
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
0.5
1
1.5
2

Results

pressure driven growth of 10 random cracks



Results
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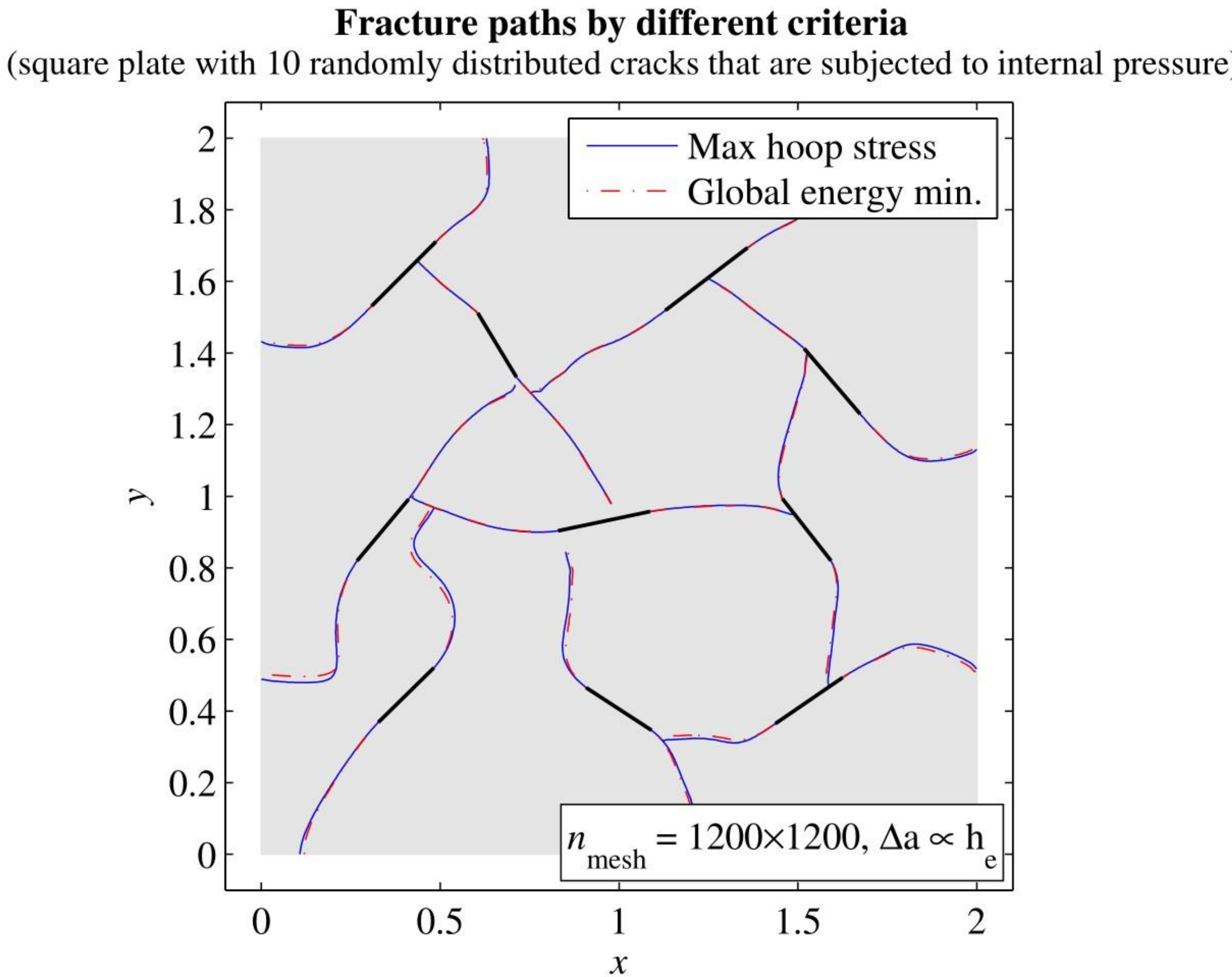
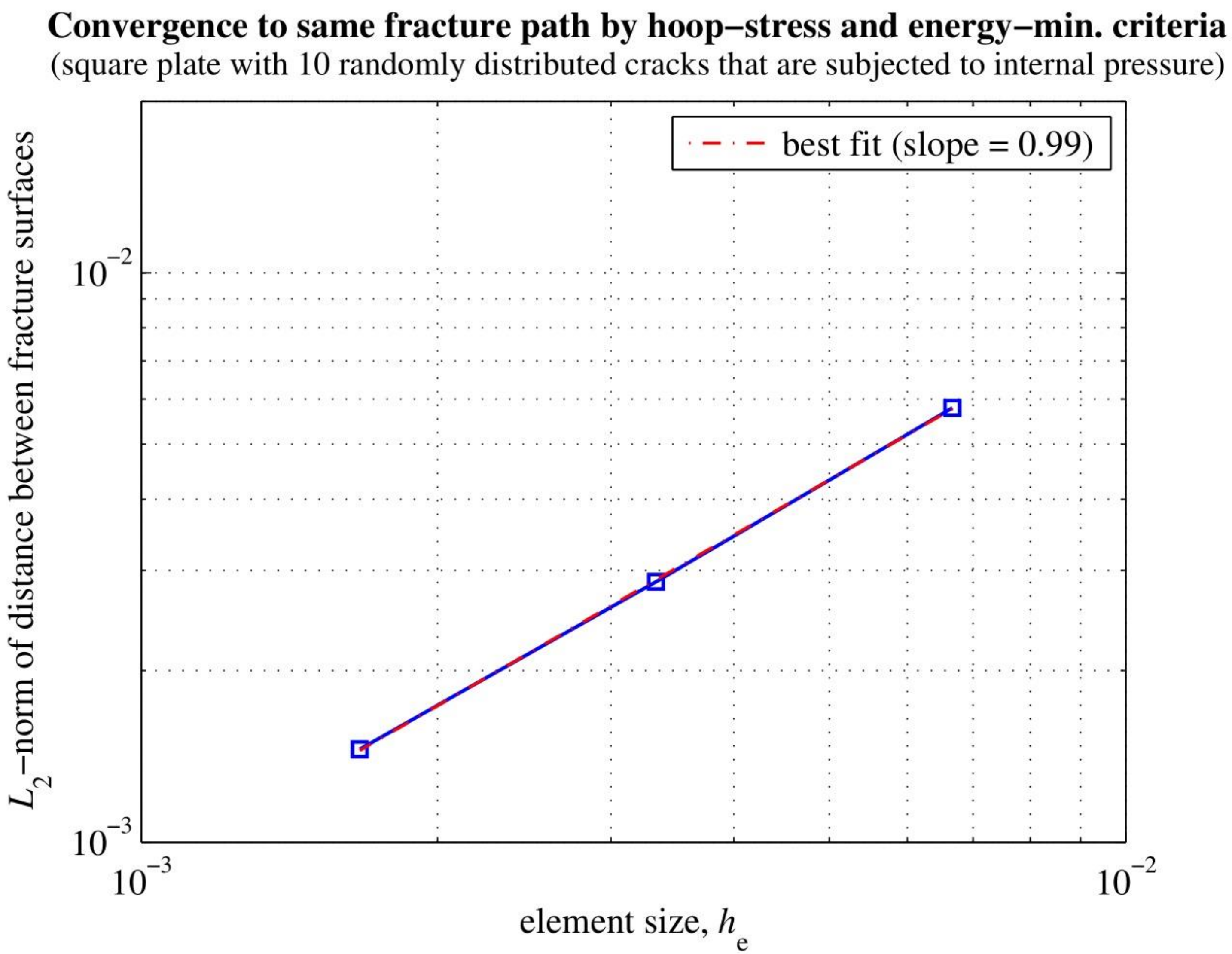


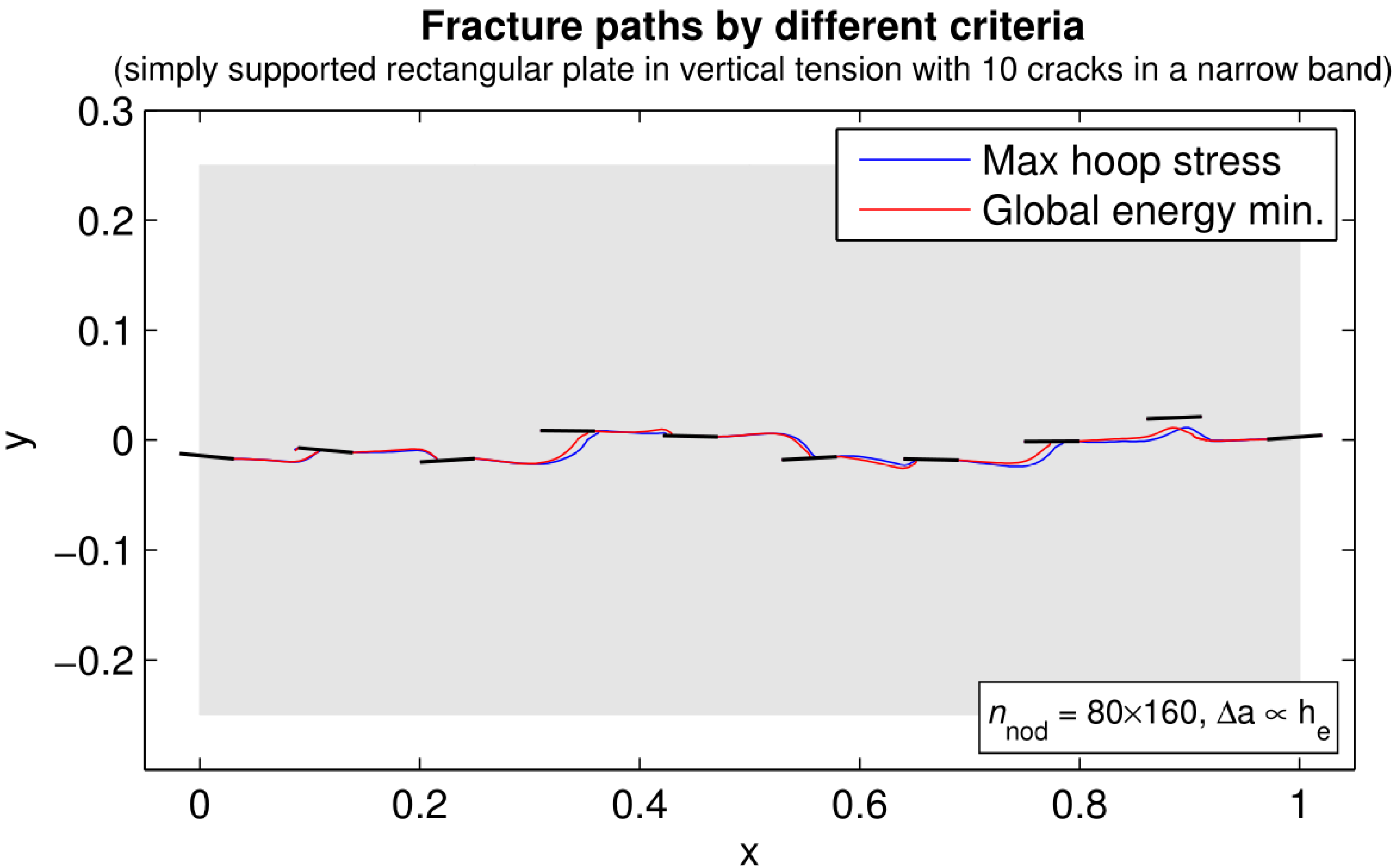
Results

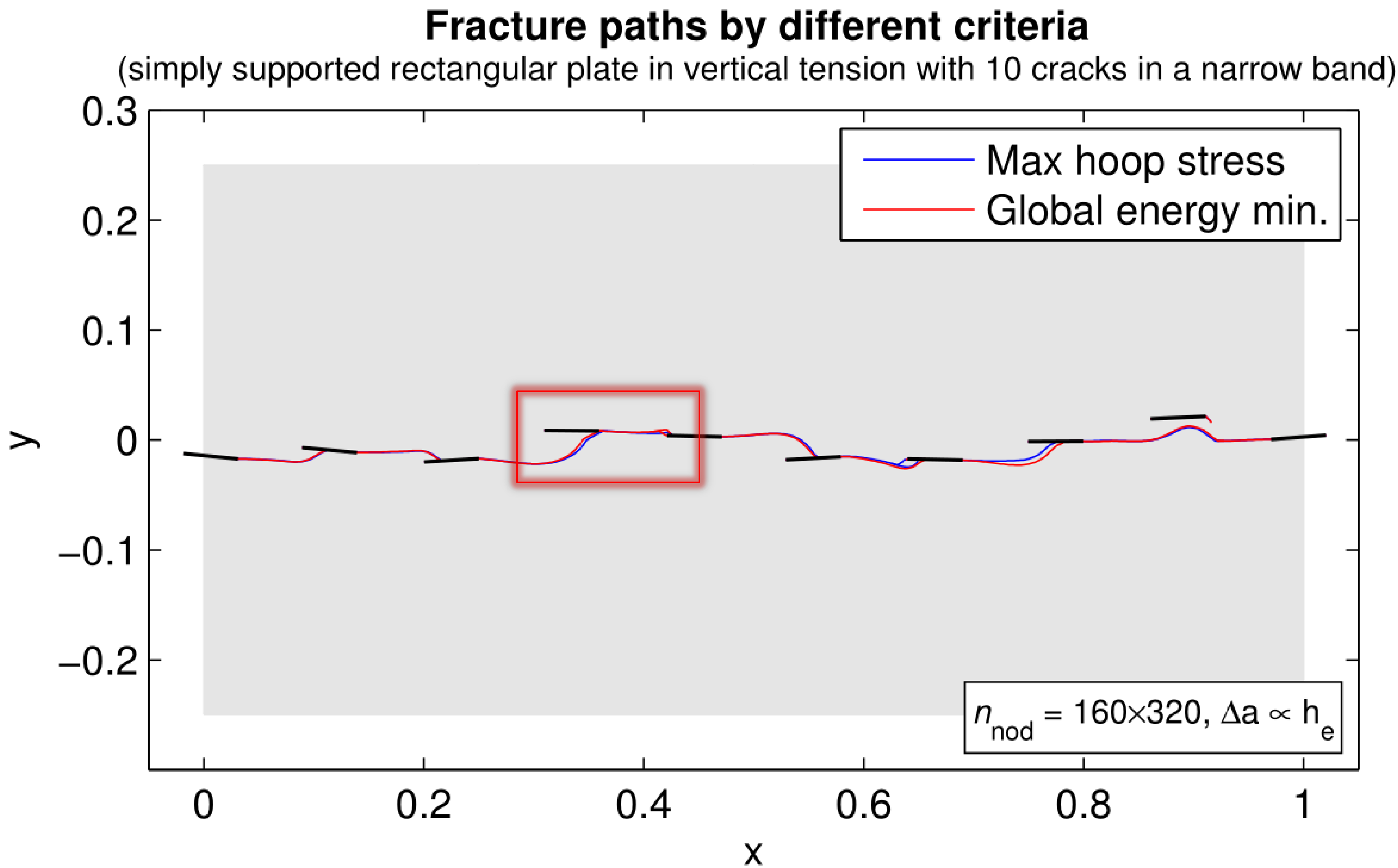
pressure driven growth of 10 random cracks



Results

tension splitting of plate containing 10 cracks

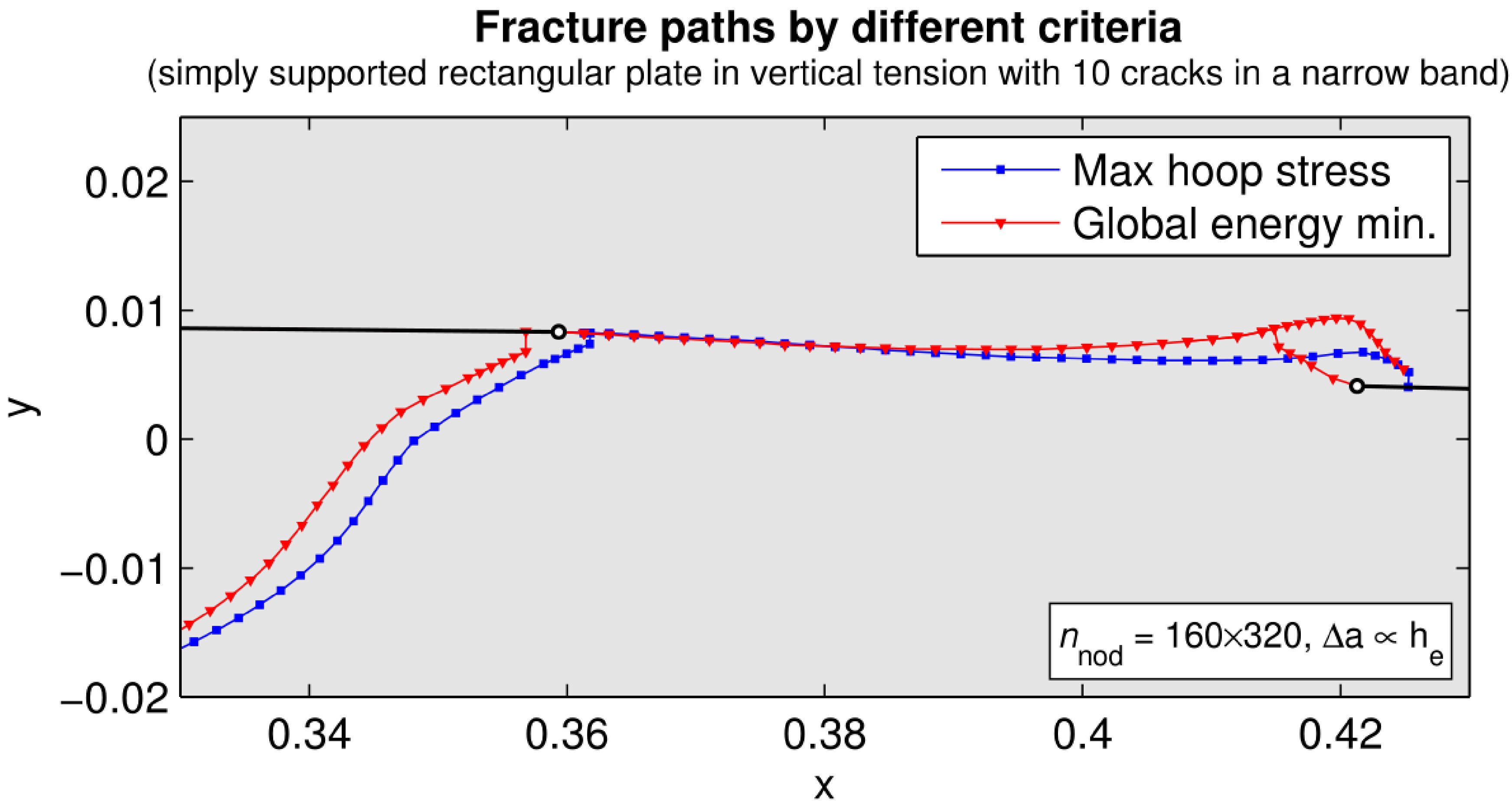




Fracture paths by different criteria
(simply supported rectangular plate in vertical tension with 10 cracks in a narrow band)
Max hoop stress
Global energy min.
n_{nod} = 160×320, Δa ∝ h_e
0.3
0.2
0.1
0
–0.1
–0.2
y
0
0.2
0.4
0.6
0.8
1
x

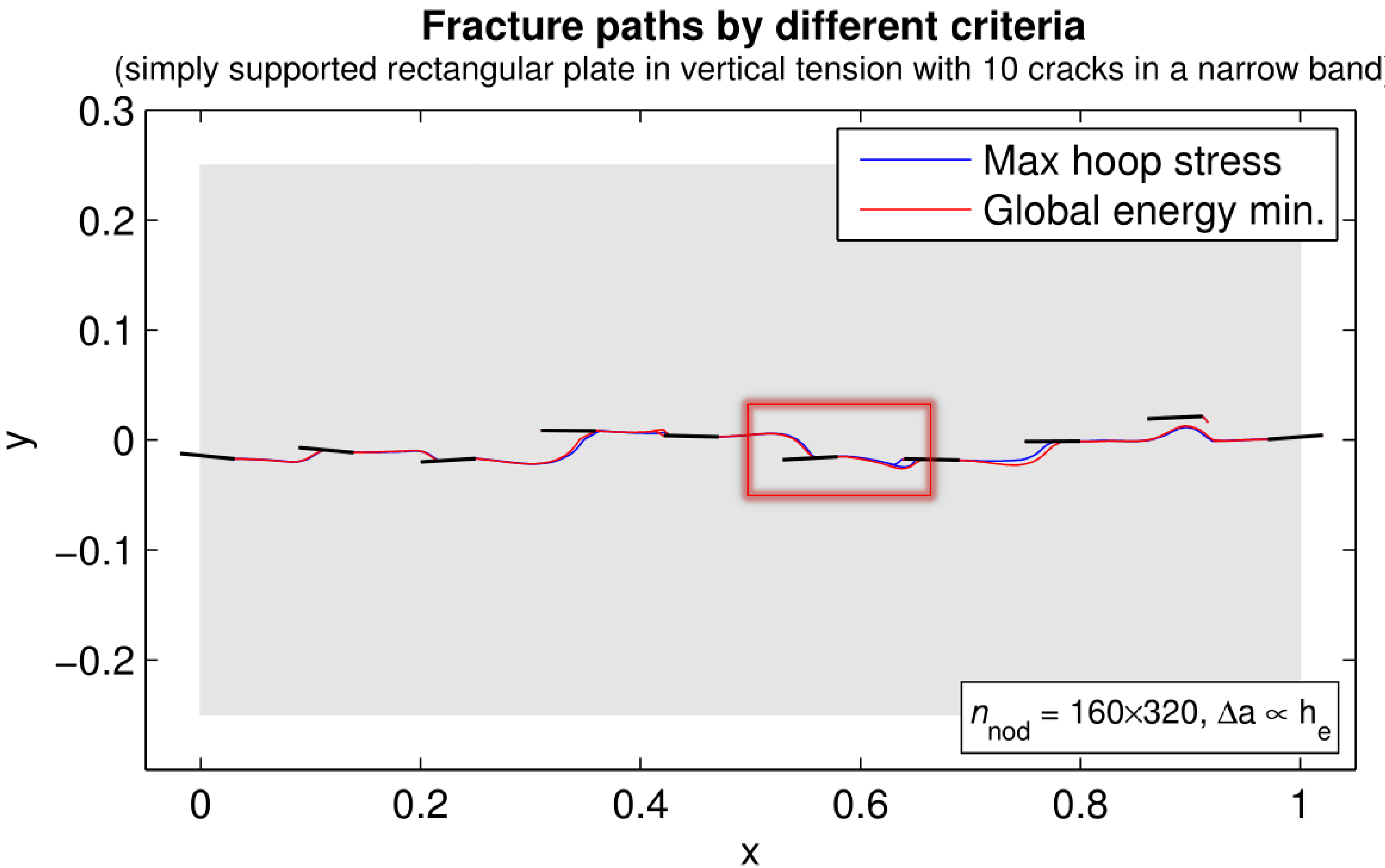
Results

tension splitting of plate containing 10 cracks

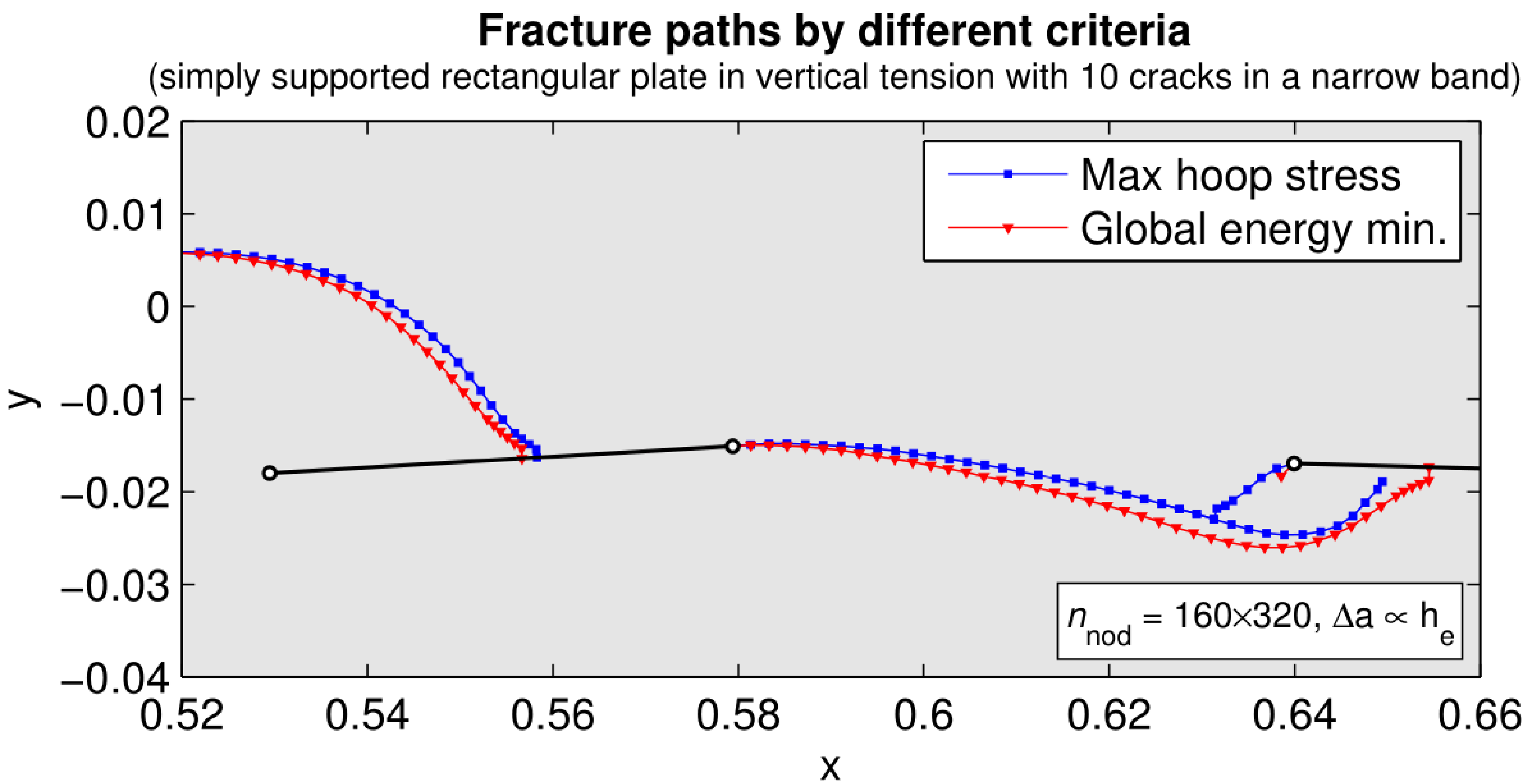


Results

10 crack problem

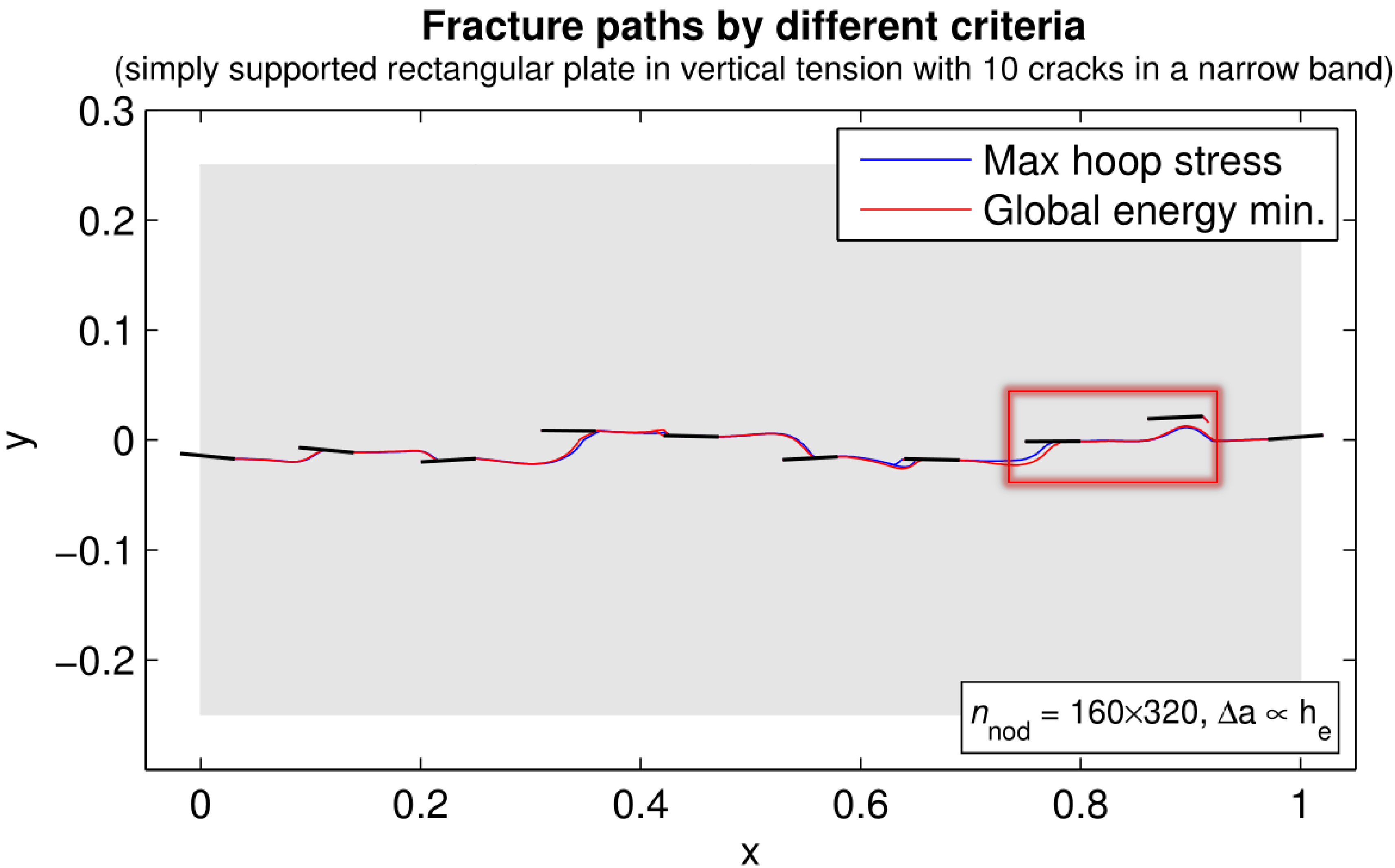


Results
tension splitting of plate containing 10 cracks



Results

tension splitting of plate containing 10 cracks



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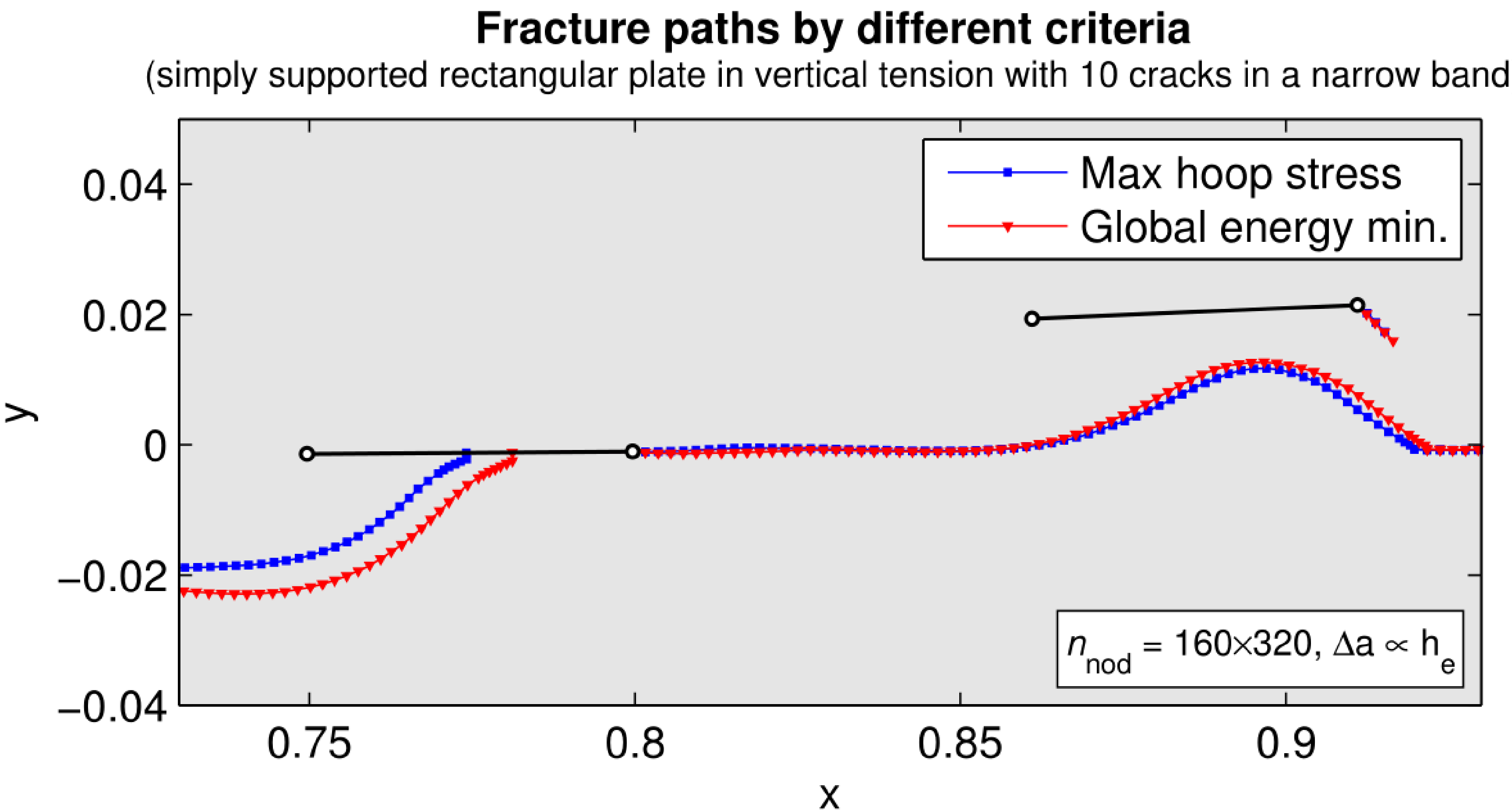
tension splitting of plate containing 10 cracks



Results
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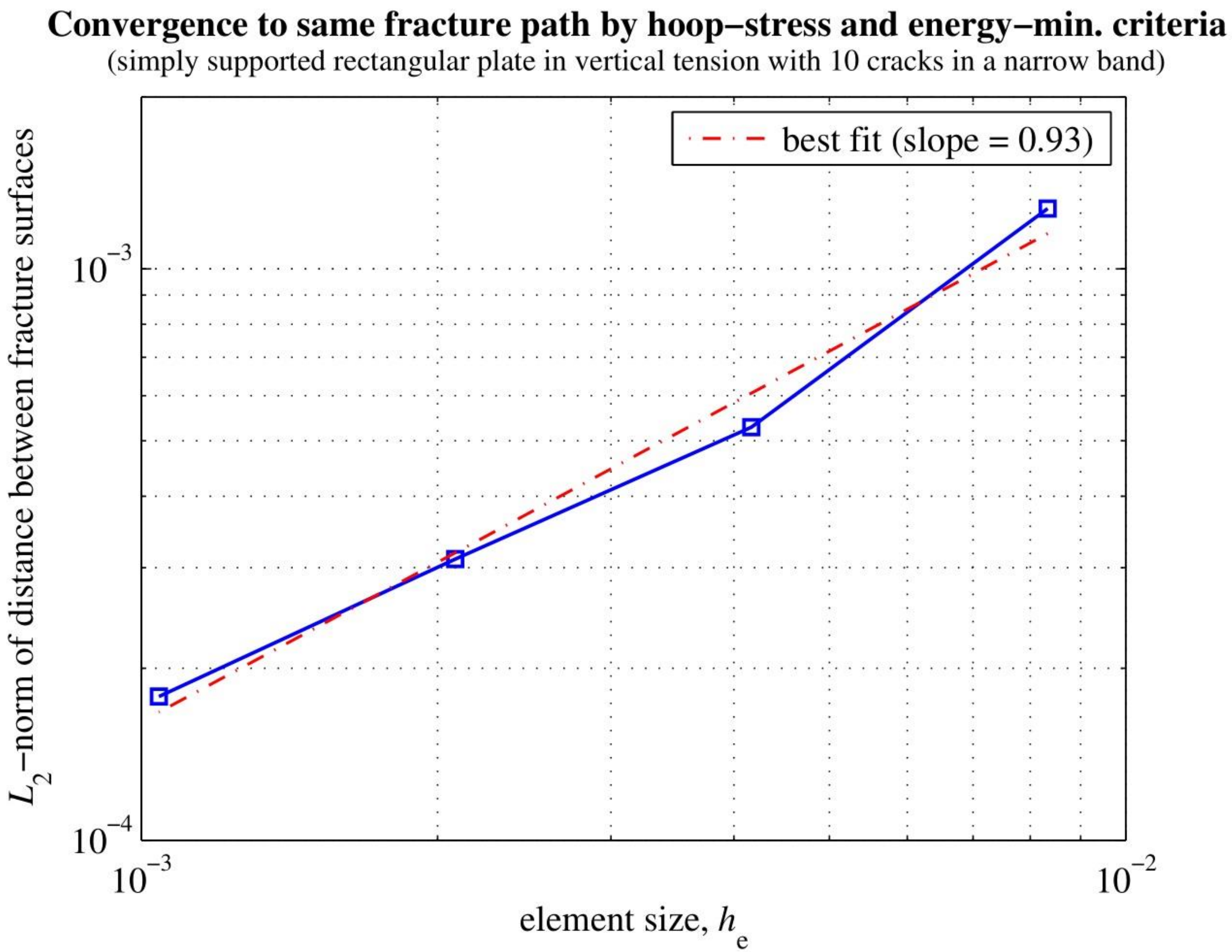


Summary

1. Robust approach to determining multiple crack growth directions based on the principle of minimum global energy

2. The criteria converge to fracture paths that are, in the global sense, in close agreement. Consequence of local-symmetry.

3. The criteria can be used to estimate the upper/lower bound of the true fracture path for smooth crack growth problems

Managing crack intersections

overview

- Elements that feature crack intersections need multiple jump enrichments to capture the kinematics of crack opening correctly (Daux *et al.* 2000, Budyn *et al*. 2004)
- Multiple jump enrichments are superposed on an element by extending the element's approximation space for each crack that the element contains whilst averting linear dependence between approximations.
- A dedicated book-keeping practice is required to efficiently manage multi-layer enrichments (e.g. as pertains to intersections) in a topologically consistent manner with regard to each crack occurrence.

Managing crack intersections
merging cracks in 2D

- Minimum distance criterion:
  - intersection of crack **A** onto crack **B** is forced once the distance between **A**’s tip and **B**’s surface becomes less than a prescribed tolerance (e.g. typically, the size of **A**’s tip enrichment radius)
  - crack **A** is extended normal to **B**’s surface and deflected along it
- As intersection happens:
  - intersection between crack **A** and crack **B** is registered once it is detected that **A**’s tip increment crosses **B** thereby forming an ‘X’-type intersection
  - crack **A** is pulled back until it’s tip lies on **B**; **A** is then extended along **B**

Managing crack intersections

blending crack intersections

- The deflected fracture extent of **A** is called the blending region of the fracture junction. The size of it needs to be large enough to smoothly merge **A** onto **B**
- Crack merging is accommodated by blending elements that lie on the blending region a short distance from the junction.
- Blending elements are jump-enriched elements that are cut by the blending region and that have at least one node whose support is cut by the part of **A** that can be thought as the relative complement of **A** with respect to **B**.
- In a 3D context, where cracks are polygonal surfaces, the approach to crack intersections is analogous to 2D in that the surface of crack **A** needs to be deflected along surface **B** followed by an effective blending procedure.

Managing crack intersections

multi-jump enrichment

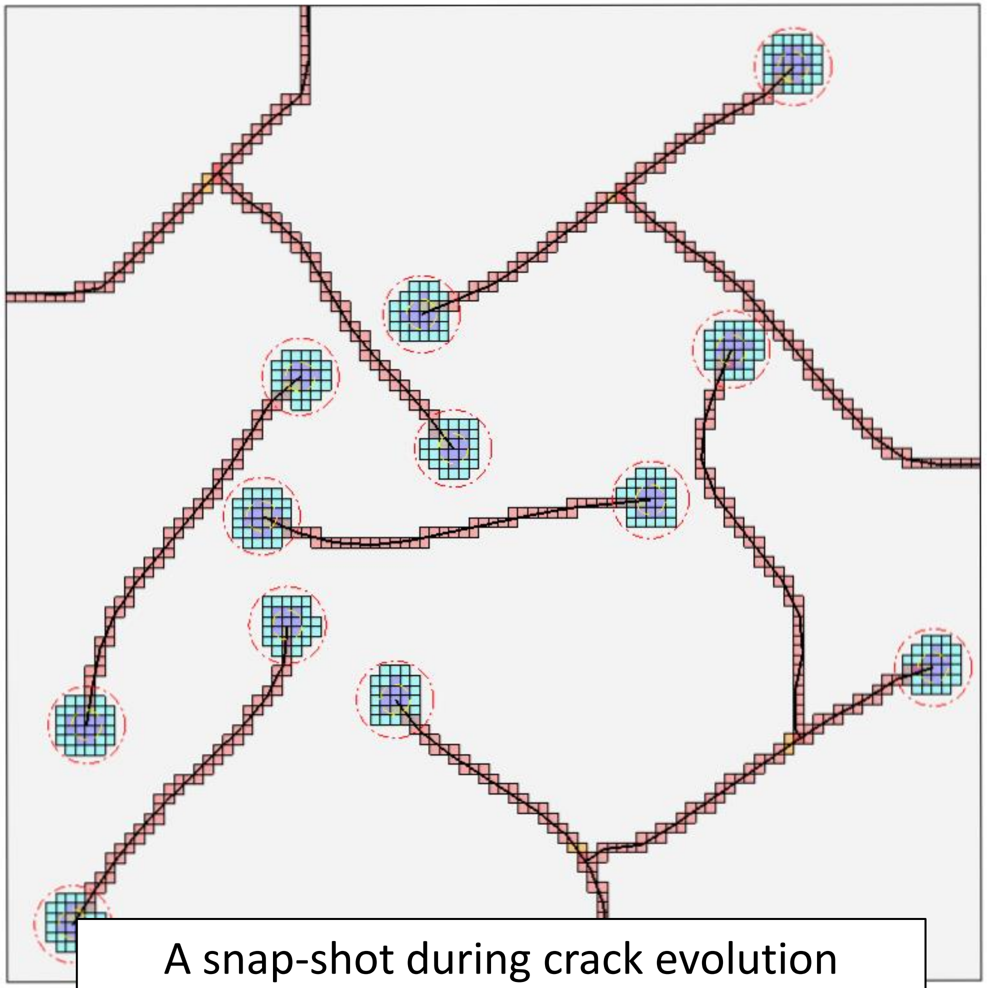
- The discontinuous part of the displacement approximation over an element cut by several cracks can be expressed (using shifted enrichment) as follows:

$$\mathbf{u}^h_{\text{disc}}(\mathbf{x}) = \sum_{i=1}^{n_{\text{crk}}} \sum_{I \in \mathcal{N}_H^i} N_I(\mathbf{x}) \Big(H_i(\mathbf{x}) - H_i(\mathbf{x}_I) \Big) \mathbf{a}_{iI}$$

- If the element serves to blend a particular crack (e.g. **A**) onto another crack (e.g. **B**), some of the enriched DOFs pertaining to **A** will have to be set to zero.
- This is to prevent linear dependence between the enriched shapes relating to **A** with those relating to **B**, because they are identical by virtue of **A** perfectly overlying **B** in the blending region.
- The blending zero-degrees of freedom are determined as those DOFs whose corresponding nodal support is not cut by the main branch of crack **A**, yet the DOF belongs to at least one element that has at least one of its nodal supports cut by the main branch of **A**

Managing crack intersections
system updating

- Evaluate the fracture growth criterion to determine which crack grow

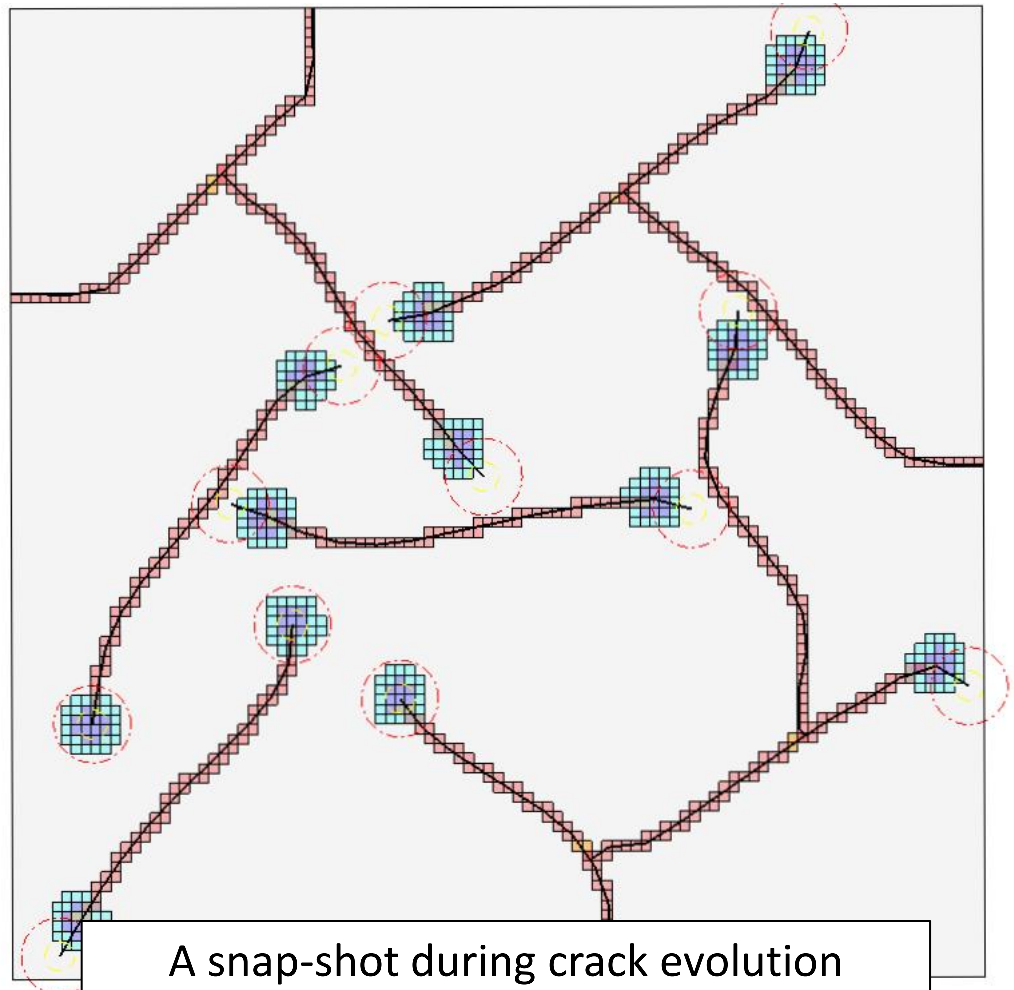


A snap-shot during crack evolution
(fig. shows enriched elements)

Managing crack intersections

system updating

- Evaluate the fracture growth criterion to determine which crack grow
- Advance cracks



A snap-shot during crack evolution
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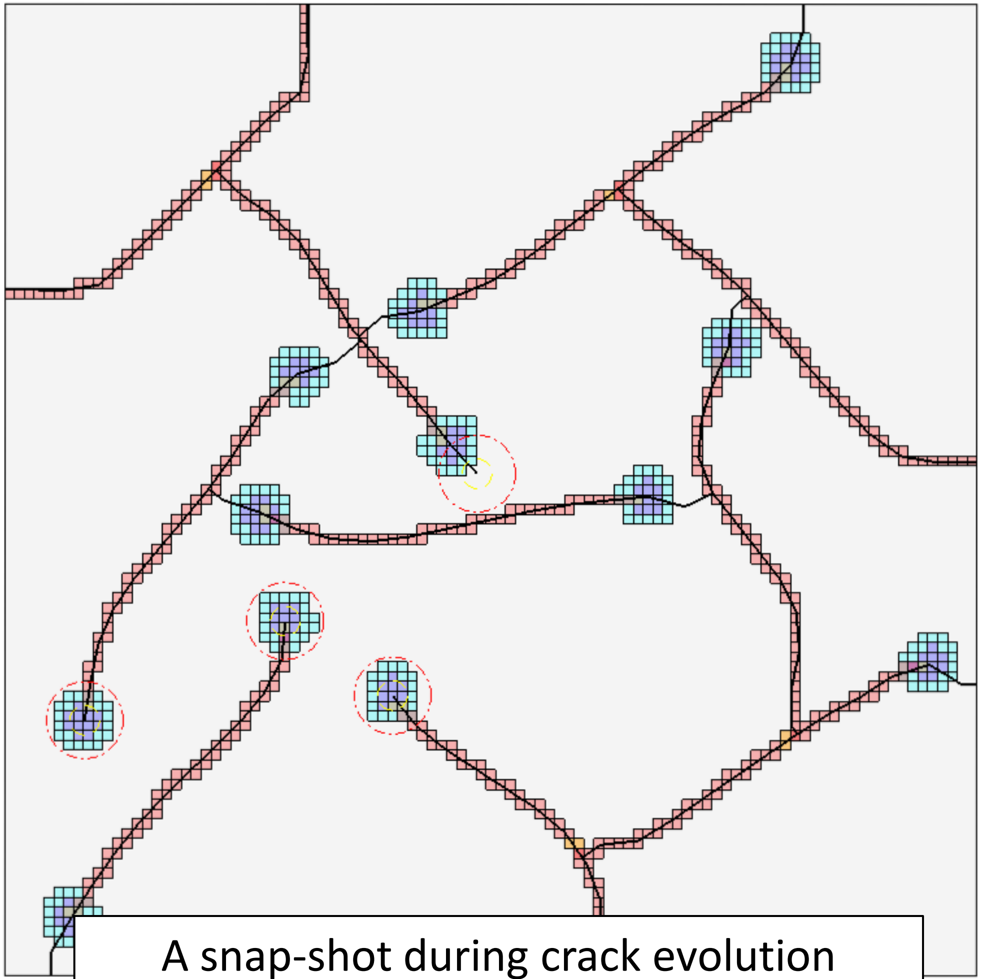
Managing crack intersections
system updating

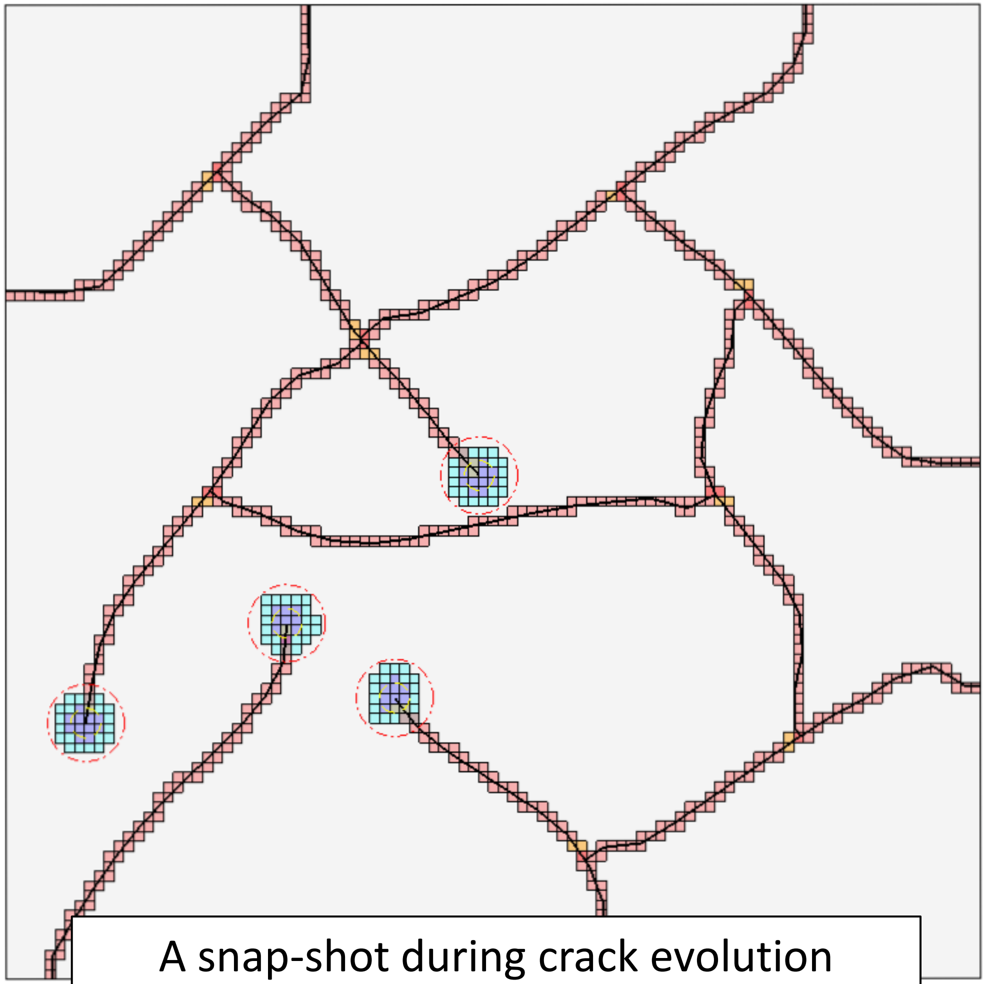
- Evaluate the fracture growth criterion to determine which crack grow
- Advance cracks
- Use an intersection criterion to merge cracks (e.g. min. dist.)



A snap-shot during crack evolution (fig. shows enriched elements)

Managing crack intersections
system updating

- Evaluate the fracture growth criterion to determine which crack grow
- Advance cracks
- Use an intersection criterion to merge cracks (e.g. min. dist.)
- Update enrichment topology



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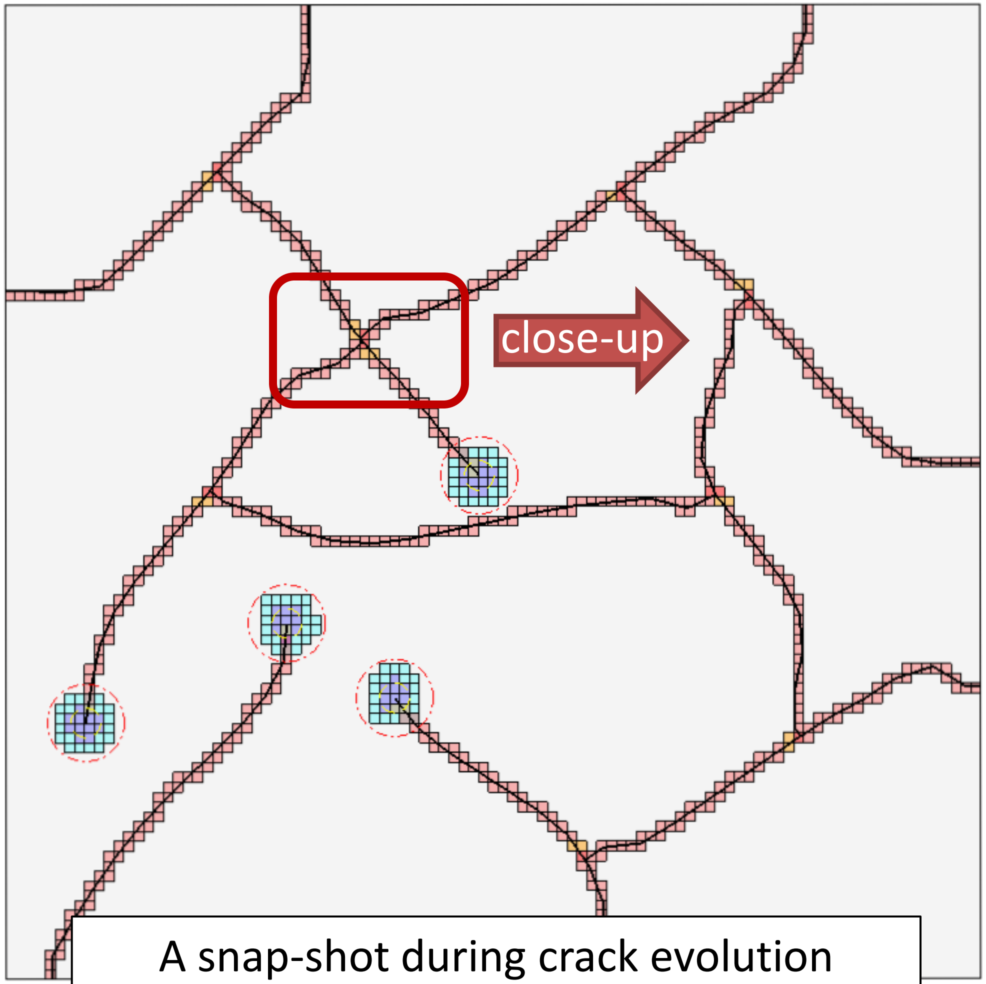

A snap-shot during crack evolution
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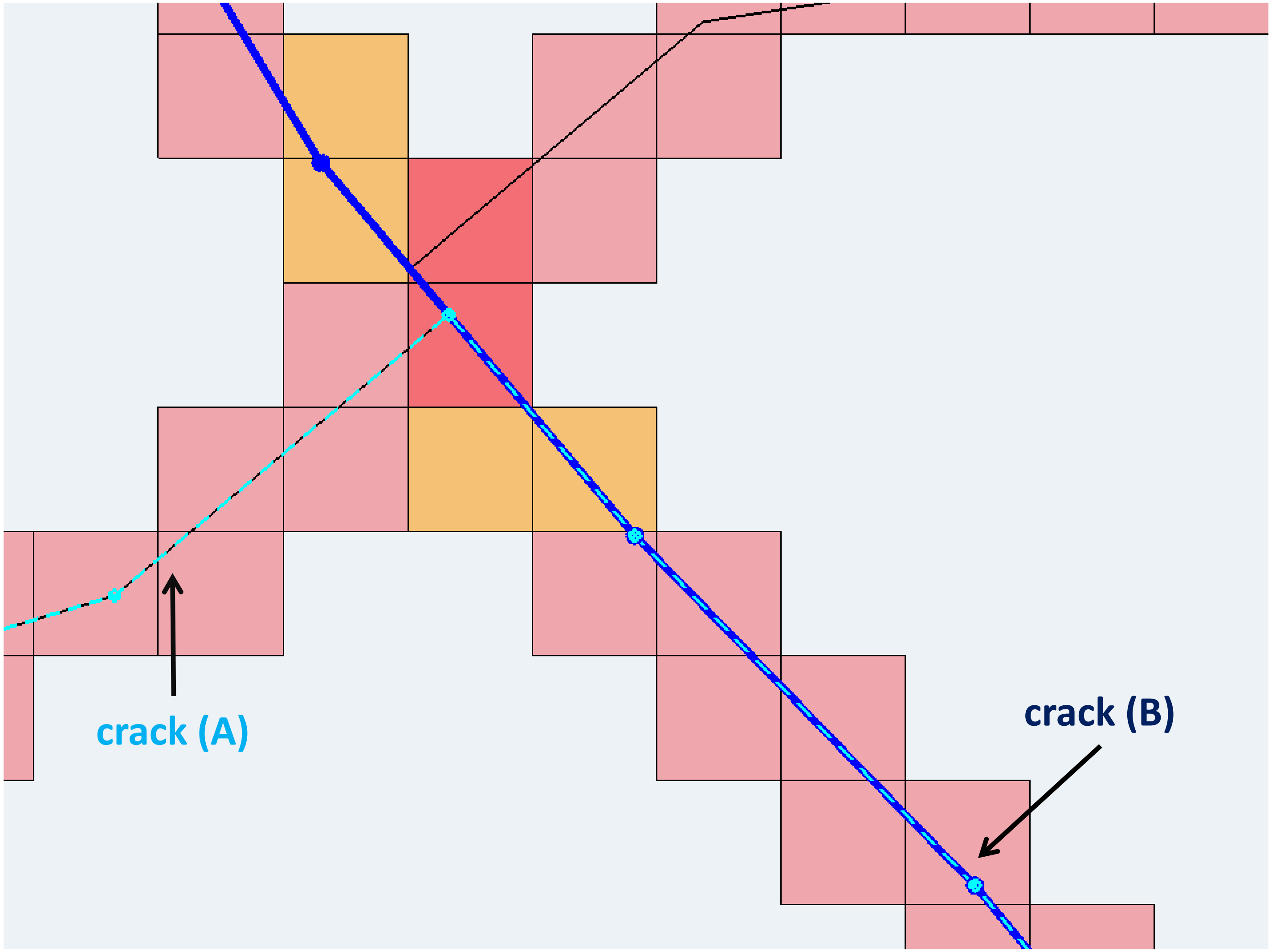
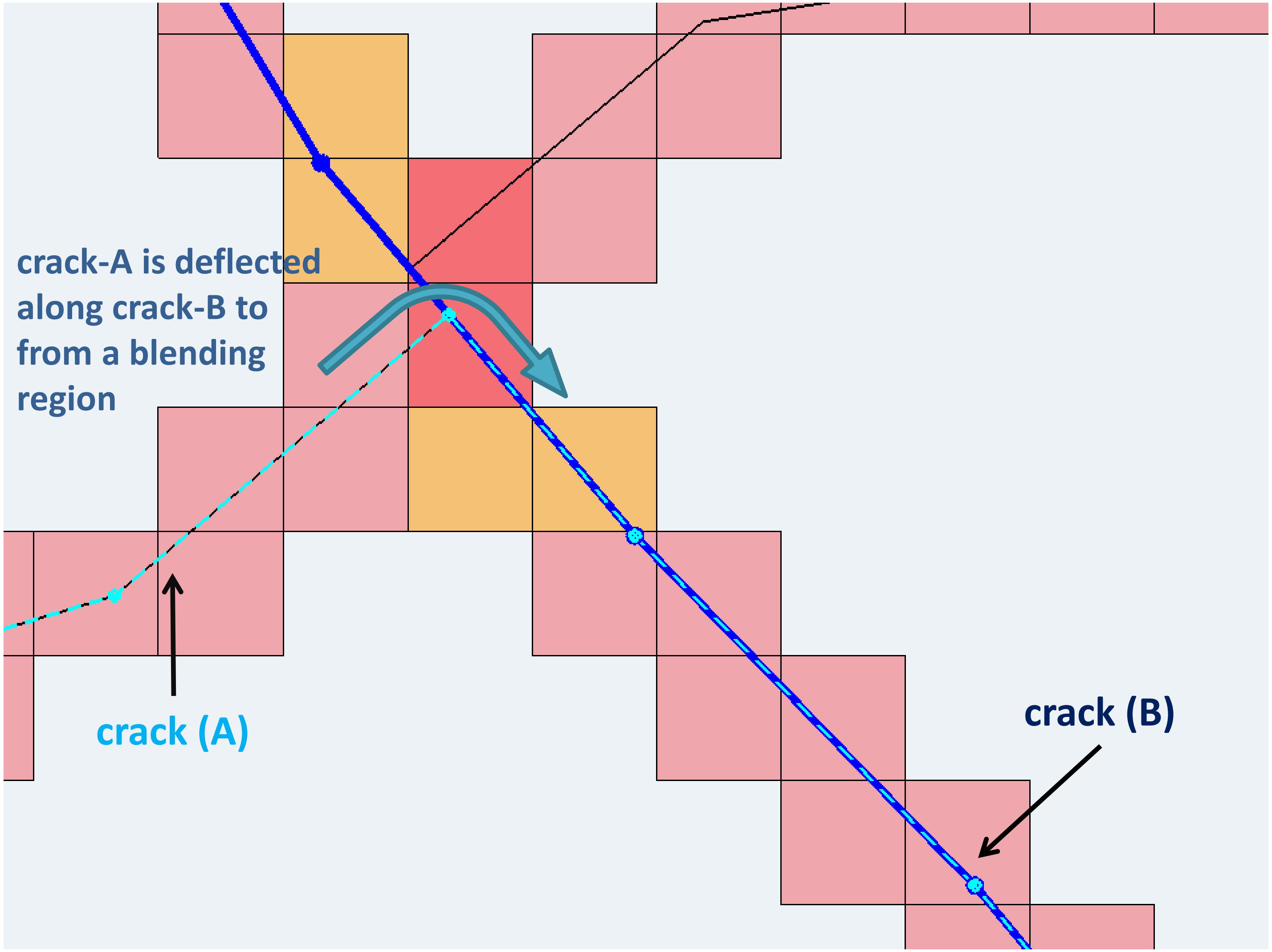
Managing crack intersections

system updating (close-up view)



Managing crack intersections

system updating (close-up view)



Managing crack intersections

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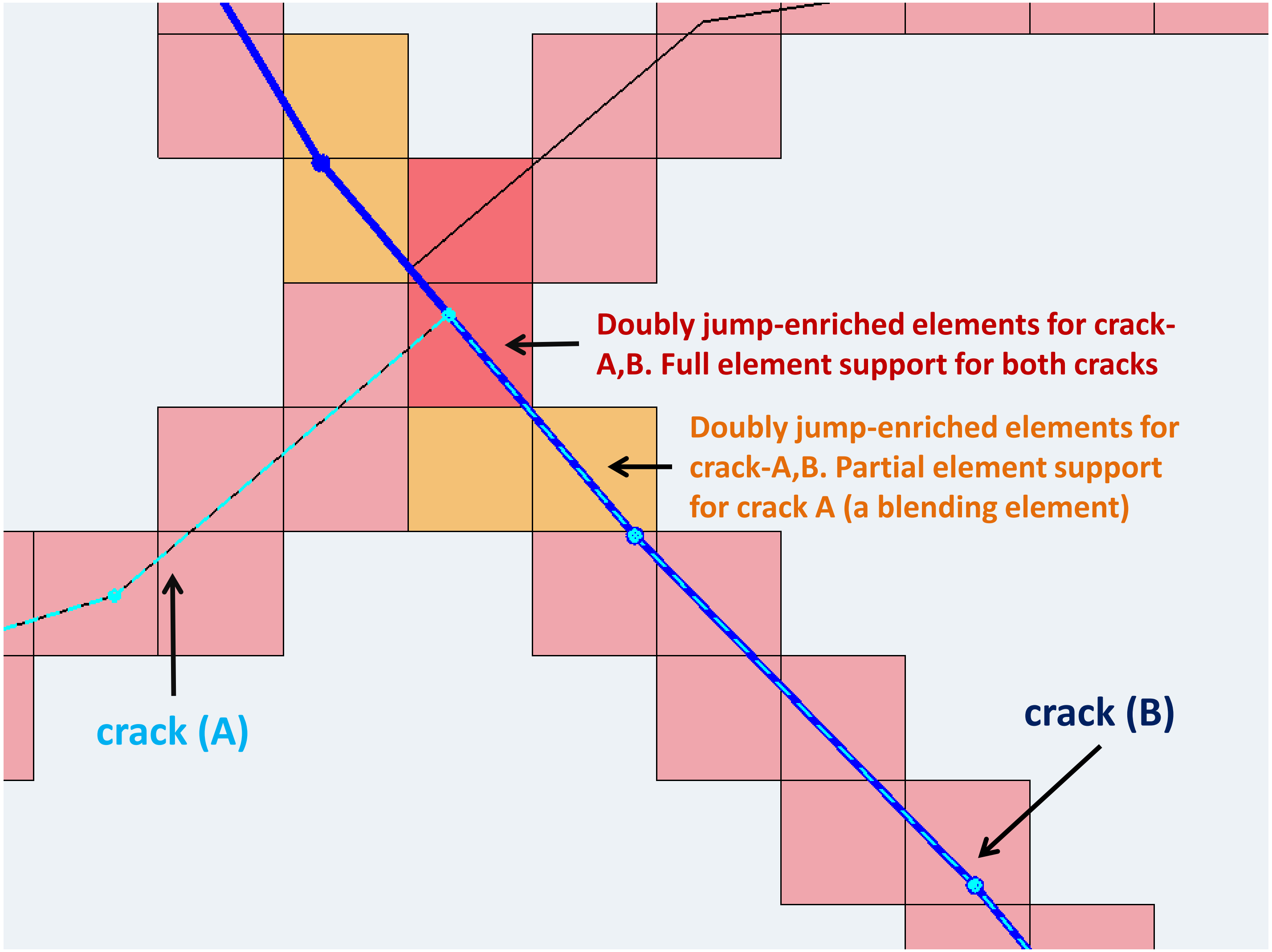

Managing crack intersections

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