ANTI-DE SITTER SPACE: FROM PHYSICS TO GEOMETRY

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Many geometers have been fascinated by hyperbolic geometry, which can be defined as the geometry of Riemannian spaces of constant curvature -1. But for most physicists, Riemannian geometry is not as natural as Lorentzian metrics, which assign negative square length to tangent vectors in some directions, corresponding to "time evolution," and positive square length to others, corresponding to "space direction."

This leads to define the anti-de Sitter (AdS) n-dimensional space as the Lorentzian analog of hyperbolic space, the quadric

$$AdS_n = \{x \in \mathbb{R}^{n-1,2} \mid \langle x, x\rangle = -1\}$$

in $\mathbb{R}^{n-1,2}$, which is just $\mathbb{R}^{n+1}$ endowed with a bilinear symmetric form of signature $(2, n-1)$, just as n-dimensional hyperbolic space is defined as

$$H^n = \{x \in \mathbb{R}^{n,1} \mid \langle x, x\rangle = -1\}$$

in the Minkowski space of dimension $n+1$. AdS_n is a geodesically complete n-dimensional Lorentzian manifold of constant curvature -1.

In physical terms, it is a solution of Einstein's equation without matter but with negative cosmological constant. This space AdS_n, as well as its positive curvature cousin, the de Sitter space dS_n, is named after Willem de Sitter (1872-1934), who introduced dS_n as a cosmological model in the 1920s.

There are fundamental relations between 3-dimensional hyperbolic geometry and Teichmüller theory that we will recall briefly before considering the more recent relations between AdS geometry and surfaces. We call S a closed surface of genus at least 2, and $\mathcal{T}_S$ the Teichmüller space of S, that is, the space of complex structures on S (considered up to deformation). This finite-dimensional space is ubiquitous in mathematics, from number theory to differential geometry and mathematical physics, and it carries a rich geometric structure — including a Kähler metric of negative sectional curvature, the Weil-Petersson metric — as well as an action of a large, interesting and still somewhat mysterious entity, the mapping-class group of S.

A hyperbolic manifold is a manifold that looks locally like the hyperbolic space. Quasifuchsian hyperbolic manifolds provide the simplest non-trivial examples. They are the complete hyperbolic manifolds homeomorphic to $S \times \mathbb{R}$ that "behave well" at infinity. Let $\mathcal{QF}_S$ be the space of quasifuchsian structures on $S \times \mathbb{R}$. A quasifuchsian manifold is the quotient of H^3 by a discrete subgroup Γ of the isometry group of H^3 isomorphic to $\pi_1 S$. The *limit set* Λ_Γ of Γ is then defined as the intersection with the sphere at infinity of H^3 of the closure of the orbit $\Gamma.x$ of any point $x \in H^3$. If Γ is quasifuchsian, then Λ_Γ is a Jordan curve.

A quasifuchsian manifold M homeomorphic to $S \times \mathbb{R}$ has a boundary at infinity $\partial_\infty H^3$, which can be identified with $(\partial_\infty H^3 \setminus \Lambda_\Gamma)/\Gamma$. As such, it is endowed with a complex structure (because the action of Γ on H^3 extends as a complex action on $\partial_\infty H^3$). Since $\partial_\infty M$ is the disjoint union of two copies of S, we can associate to M two points in $\mathcal{T}_S$. According to well-known theorem of Bers [1] this correspondence between $\mathcal{QF}$ and $\mathcal{T}_S \times \mathcal{T}_S$ is one-to-one: any couple of complex structures on S can be obtained from exactly one quasifuchsian structure on $S \times R$.

This Bers double uniformization theorem is a key tool for Teichmüller theory. The following is one example (among many others) of this relation. The volume of quasifuchsian manifolds is infinite, but one can use ideas originating in mathematical physics [**?**, 4, 10] to define a finite,

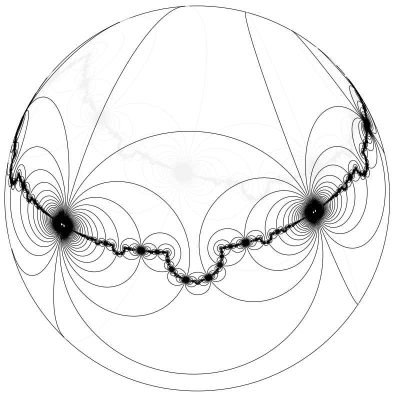

FIGURE 1. The limit set of a quasifuchsian group (picture made by C. McMullen).

"renormalized" volume. By fixing the complex structure on one boundary component of M and varying the other, one obtains a function $V_R : \mathcal{T}_S \to \mathbb{R}$, which turns out to be a Kähler potential for the Weil-Petersson metric on $\mathcal{T}_S$, and therefore a good tool to describe and understand this metric.

Let's now turn to AdS, the Lorentzian cousin of hyperbolic space. Closed Lorentzian spacetimes are not too relevant from a physical point of view because they always contain closed causal curves (curves on which the metric is negative), meaning that an observer could evolve and come back to the same point in spacetime — resulting in paradoxes often used in the science-fiction literature. It is more natural to consider *globally hyperbolic* spacetimes, containing a Cauchy surface, that is, a space-like surface that any causal curves intersects exactly once.

In 1990, G. Mess [8] discovered that, in spite of superficial differences, globally hyperbolic AdS 3-manifolds have many points in common with quasifuchsian hyperbolic manifolds. There is for instance an AdS analog of the Bers double uniformization theorem: the space $\mathcal{GH}_S$ of globally hyperbolic AdS structures on $S \times \mathbb{R}$ is parameterized by $\mathcal{T}_S \times \mathcal{T}_S$. The mechanism behind this AdS version of Bers' theorem differs from the hyperbolic setting: the isometry group of AdS_3 is $O(2,2)$, which splits (up to finite index) as $O(2,1) \times O(2,1)$. So the holonomy representation of a globally hyperbolic AdS manifolds splits as two representations in the isometry group of the hyperbolic plane. Mess proved that each is the holonomy representation of a hyperbolic structures on S.

He also gave a simple and beautiful proof of Thurston's Earthquake Theorem based on globally hyperbolic AdS 3-manifolds. Thurston defined an *earthquake* to be a map sending a hyperbolic metric m and a measured lamination l (for instance, a closed curve with a positive number as "weight") to another hyperbolic metric m'. If l is a closed curve, than m' is defined by realizing l as a geodesic in (S, m), cutting S open along this geodesic, rotating the right-hand side by the weight, and then gluing back. Thurston's Earthquake Theorem asserts that given any two hyperbolic metrics m and m' on S, there is a unique measured lamination l such that an earthquake along l on m yields m'. This provides a convenient parameterization of the Teichmüller space $\mathcal{T}_S$ by the space of measured laminations, once a fixed point m has been chosen.

Thurston suggested a proof of this statement. An analytic proof was found by Kerckhoff [7]. However, the proof proposed by Mess is particularly simple. It is based on the geometric properties of the smallest non-empty subsets in globally hyperbolic AdS 3-manifolds. The boundary of this "convex core" has a hyperbolic induced metric and is "pleated" along a measured lamination. The relations between the induced metric and measured bending lamination, on one hand, and the two components of the holonomy representation, on the other hand, lead directly to the proof of Thurston's theorem.

Other valuable connections have appeared recently between closed AdS 3-manifolds and hyperbolic surfaces. In her thesis [6], Fanny Kassel gave a precise description of the holonomy representations of those closed AdS manifolds in terms of one hyperbolic surface S and a representation $\rho : \pi_1(S) \to O(2,1)$ that "shortens every curve." This led Danciger, Guéritaud and Kassel [3] to new ways of describing all length-shortening deformations of a hyperbolic surfaces, opening new developments on hyperbolic surfaces.

Recently, AdS geometry has also proved useful in a understanding basic questions on the possible combinatorics of polyhedra inscribed in quadrics. Steinitz [9] discovered that any 3-connected graph embedded in the sphere can be realized as the 1-skeleton of a polyhedron in $\mathbb{R}^3$. He also found that not all such graphs can be realized as the 1-skeleton of a polyhedron inscribed in a sphere, answering a question asked by Steiner in 1832.

Understanding the combinatorics of polyhedra inscribed in a sphere then became an fashionable question, until Hodgson, Rivin and Smith [5] gave a simple but non-explicit answer: a graph can be realized in this manner if and only if a certain system of linear equalities and inequalities has a solution.

What about realizing a polyhedron inscribed in another quadric: the one-sheeted hyperboloid or the cylinder? In some recent work with Danciger and Maloni [2], we prove that the answer is remarkably simple: *a graph embedded in the sphere can be realized as the 1-skeleton of a polyhedron inscribed in a one-sheeted hyperboloid (resp. a cylinder) if and only if it can be realized as the 1-skeleton of a polyhedron inscribed in a sphere and it admits a Hamiltonian cycle.*

Perhaps surprisingly, the proof of this statement rests on AdS geometry and more specifically on the geometry of ideal polyhedra in AdS_3 — polyhedra with all vertices on the boundary at infinity. Hodgson, Rivin and Smith describe the geometry of ideal polyhedra in hyperbolic 3-space, and the statement follows from confronting and comparing the descriptions of dihedral angles of ideal polyhedra in H^3 and AdS_3.

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