Toward a Linguistic Interpretation of Deontic Paradoxes

A Beth-Reichenbach semantics approach for a new analysis of the miners scenario

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Abstract. A linguistic analysis of deontic paradoxes can be used to further develop deontic logic. In this paper we provide a Beth-Reichenbach semantics to analyze deontic paradoxes, and we illustrate it on the single agent decision problem of the miners scenario. We also introduce extensions with reactive arrows and actions, which can be used to give a linguistic interpretation of multi-agent dialogues.

1 Introduction

Consider the following discussion by Condoravdi and van der Torre [6].

> "**Example (Linguistic interpretation of Chisholm's paradox)** *The most notorious story from the deontic logic literature is known as Chisholm's paradox:*
> 1. *a certain man ought to go to the assistance of his neighbours,*
> 2. *if he goes, he ought to tell them he is coming,*
> 3. *if he does not go, he ought not to tell them he is coming,*
> 4. *he does not go.*
>
> *Analyses of the three conditional obligations have led to preference-based deontic logic, temporal deontic logic, action deontic logic, non-monotonic deontic logic, and more. A more general linguistic analysis would also question the fourth sentence: what does it mean that the man does not go? Does it mean that he cannot go, that he intends not to go, or that he did not go? Taking into account the temporal perspective of the fourth premise and, more generally, the context in which the reasoning takes place constitutes new challenges for the logical analysis of the paradox.*" [6]

The example of Condoravdi and van der Torre suggests that a linguistic analysis of deontic paradoxes can be used to further develop the logic of obligations and permissions. In this paper, we take up their challenge, and we start the development of a *semantics* for such a linguistic interpretation. We use the following fragment of the miners scenario[1] to motivate, develop and validate our approach. As in the analysis of Chisholm's

[1] Kolodny and MacFarlane [12] and Willer [16] call it a paradox, while Cariani et al. [3] call it a puzzle. In this paper, we do not consider the question whether it is a paradox, and call it "the miners scenario."

paradox above, we consider not only the obligations, but also the temporal perspective of the factual premise and, more generally, the context in which the reasoning takes place.

Example 1 (Miners scenario, single agent decision problem). The miners scenario is introduced by Kolodny and MacFarlane [12]. Ten miners are trapped either in shaft `A` or in shaft `B`, but we do not know which one. Water threatens to flood the shafts. We only have enough sandbags to block one shaft but not both. If one shaft is blocked, all of the water will go into the other shaft, killing every miner inside. If we block neither shaft, both will be partially flooded, killing one miner. The decision problem is summarised in the following table:

Action	**if miners in** `A`	**if miners in** `B`
block shaft `A`	all saved	all drowned
block shaft `B`	all drowned	all saved
block neither shaft	one drowned	one drowned

Lacking any information about the miners' exact whereabouts, and without the possibility to obtain such information, it seems acceptable to say that:

(1) a. We ought to block neither shaft.
b. If the miners are in shaft `A`, we ought to block shaft `A`.
c. If the miners are in shaft `B`, we ought to block shaft `B`.
d. Either the miners are in shaft `A` or they are in shaft `B`.

However, (1.a-d). seem to entail (2), contradicting (1.a).

(2) Either we ought to block shaft `A` or we ought to block shaft `B`.

Various consistent representations of the scenario have been given [12, 3, 4, 16, 5]. Moreover, Kolodny and MacFarlane [12] extend the single agent decision problem above to multi-agent dialogues, leading to additional logical developments. While these representations focus on the interpretation of deontic modality and conditionals in the first three sentences, leading to new developments in deontic logic like information sensitivity, decision rules, and dynamic semantics, we use in this paper ideas from intuitionistic logic [15] and reactive Kripke semantics [11] to form a new analysis focussing on the fourth sentence.

Research question Which semantics can be used to give a linguistic analysis of paradoxes and use of normative language, and thus to further develop deontic logic?

This general objective breaks down into the following three subquestions:

1. How to define a special Beth-Reichenbach semantics capable of analyzing the miners scenario?
2. How to augment the Beth-Reichenbach semantics with reactive arrows and sharpen our analysis of the miners scenario?
3. How to further extend the Beth-Reichenbach semantics to obtain a logic capable of modeling actions in the miners scenario?

Our linguistic analysis questions the disjunction in the fourth sentence. For example, is this information relevant at the current moment or in a reference time later in the narrative of the story? If we read the wording of the miners scenario and the natural flow of events involved in the situation described by the scenario, we have a story about what we know at the beginning (namely, we do not know where the miners are), we have actions we want to take (block the shafts) which intuitively we should not be taking until we know where the miners are, and when we know where the miners are we immediately have the obligation to take the proper action. Our use of the Beth-Reichenbach semantics starts by observing that, on a temporal perspective, disjunctions represent limited information. We *do not know* where the miners are, so we are only able to state a disjunction that "enumerates" the places where they could be: shaft `A` and shaft `B`. Therefore, we need a logical account where disjunctions are interpreted in that way, regardless of the actions that the agent will decide to take.

Classical logic does not have components to model the desired semantics at the object level. We need to somehow add to classical logic, at the object level, a component of knowledge, time, and actions in a natural way, where by "natural" we mean a way which mirrors our human perception of the story. Classical logic can describe the above flow of knowledge, time and actions only by acting as a meta-language, but when it is used as a meta-language, it can equally describe the cooking of an omelette. This is not what we mean by a natural logic to represent the miners scenario. We therefore do not move from classical logic to the machinery of the temporal modal action logic [2], as a sort of meta-language to describe the miners scenario [12, 3, 4]. Instead, we modify the traditional semantics for classical logic by moving to the Beth-Reichenbach semantics.

In this paper we do not introduce a full-fledged deontic logic, as there are various ways to use the Beth-Reichenbach semantics for normative reasoning. For example, we can add a modal operator for obligation to the semantics, to obtain a kind of intuitionistic standard deontic logic, or we can use the intuitionistic logic as the base logic in the input/output logic framework [8]. We leave these developments for further research, and focus in this paper on the linguistic interpretation of the miners scenario.

The paper is structured as follows. In section 2, we present the Beth-Reichenbach semantics for classical logic. In section 3, we present our case study of the miners scenario. In section 4 we introduce reactive arrows and in section 5, we introduce actions. Section 6 formalizes the miners scenario and section 7 compares our representation of the miners scenario with the literature. We conclude in section 8.

2 The Beth-Reichenbach semantics for classical logic

In this paper our starting point is propositional logic. Therefore for brevity we only introduce the propositional fragment of Beth-Reichenbach semantics. Drawing from Reichenbach [14], Beth [1] introduced his semantics in 1956 as a candidate semantics for intuitionistic logic, and it is combined with Kripke semantics to form the Beth-Kripke semantics [9]. It became popular as the semantics of intuitionistic logic. Finite Beth-Reichenbach models comprize semantics for classical logic. The basic idea of finite Beth-Reichenbach models can be described by the following example.

Example 2 (Police scenario). Imagine a police officer collecting evidence on a murder case and preparing a file for the prosecution of a certain suspect for the murder. At any given moment of time the police officer can go in different directions collecting evidence and according to what he finds, different statements can be verified to be true. There are three options for a statement A:

(a) There is enough evidence now to prove A.
(b) It is clear now that no matter how our investigations will proceed, there will not be enough evidence to prove that A is true; therefore for the purpose of prosecution it is acceptable that $\neg A$ is true.
(c) Although there is not enough evidence to establish that A is true, it may be possible in the future that some new evidence will be uncovered that will establish the truth of A. Therefore neither A nor $\neg A$ are established as true now.

The police has a deadline by which time the investigation and the prosecution file has to be prepared, and therefore the model is finite.

A Beth-Reichenbach model is an overview of the different states of evidence in the investigation. It is a finite ordered set with the relation '$\leq$' such that '$\mathtt{t}\leq\mathtt{s}$' means that $\mathtt{s}$ has more established evidence than $\mathtt{t}$. Thus if at $\mathtt{t}$ statement A can be proven true, it could be also be proven true at $\mathtt{s}$. Such a finite Beth-Reichenbach model provides semantics for classical logic, because if we look at the endpoints of the process, namely all the possible files where no further investigation and collection of evidence is performed, we get a classical model. What is true in that final node is what can be proven and what is false in that final node is what cannot be proven (which may be seen as a kind of close world assumption). There are some immediate properties of this mental picture:

(a) The nodes together with $\leq$ relation forms an acyclic order (for example a tree-structure) with finite depth.
(b) If a statement is proven at moment $\mathtt{t}$, then it will remain proven later than $\mathtt{t}$.
(c) To prove a statement of the form "not φ" at moment $\mathtt{t}$, the policeman must be certain that no matter what further investigation is done: φ will *never* be proven. In other words, "$\neg\varphi$" is proven at moment $\mathtt{t}$ iff φ is not proven later than $\mathtt{t}$.
(d) A statement of the form "φ or ψ" is proven by the policeman at moment $\mathtt{t}$ iff no matter how he stops his investigation, at least one of φ and ψ will have been proven when he stops.

The Beth-Reichenbach semantics for classical logic contains a component of progressive knowledge which is compatible with the progression of knowledge aspects we find in the miners scenario. Thus we use the Beth-Reichenbach semantics to model the miners scenario. But first we have to give formal definitions.

The language we use contains atomic formulas q, negations of atoms $\neg q$, conjunctions and disjunctions. Every well-formed formula is a disjunction of conjunctions of atoms or their negations, and we do not have implication in our language.

Definition 1 (Beth-Reichenbach semantics for classical logic). *Consider the language* $\mathtt{L}_C$ *of classical propositional logic with atoms* Q=$\{p, q, \ldots\}$ *and the connectives* $\neg$, $\wedge$, $\vee$. *Well-formed formulae (wff) be of the following form* $\bigvee_i \bigwedge_j \pm q_{i,j}$ *where* $q_{i,j}$ *is atomic, +q is q and -q is* $\neg q$. *A Beth-Reichenbach model for* $\mathtt{L}_C$ *has the form* (T, R, h), *where* T *is a set of reference points (worlds, information states),* $R \subseteq T \times T$, (T, R) *is a finite tree.* h *is an assignment giving a subset* $h(q) \subseteq T$ *to each atomic* q *. Furthermore, let* $<$ *be the transitive closure of* R*, and* $\mathtt{x} \leq \mathtt{y}$ *be* $\mathtt{x} = \mathtt{y}$ *or* $\mathtt{x} < \mathtt{y}$. *We require the following to hold:*

(a) For each $\mathtt{t} \in T$*, say that* $\mathtt{t}$ *is an endpoint iff* $\neg\exists_x(\mathtt{t} < x)$. *Let* $\mathtt{E}_t$=$\{\mathtt{x} \in T \mid \mathtt{t} \leq \mathtt{x}$ *and* $\neg\exists_{\mathtt{y} \in T}(\mathtt{x} < \mathtt{y})\}$. *Intuitively* $\mathtt{E}_t$ *is the set of all endpoints of* $\mathtt{t}$. *We require that for each* $\mathtt{t} \in \mathtt{T}$, $\mathtt{E_t} \neq \emptyset$. *This means that:* $\forall_{\mathtt{t}}\exists_{\mathtt{s}}(\mathtt{t} \leq \mathtt{s}$ *and* $\mathtt{s}$ *is an endpoint).*
(b) $\mathtt{E_t} \subseteq \mathtt{h}(q)$ *iff* $\mathtt{t} \in \mathtt{h}(q)$.

We now define satisfaction of a formula A in a model. We write $\mathtt{t} \models_{index} A$, where the $index$ gives the type of satisfaction we are defining. In our formalization, there will be two possible values for $index$: *bs* and *br*. In BS-semantics, we only need satisfaction on bs, while satisfaction on *br* will be used in our extended BR-semantics. Satisfaction on bs in $\mathtt{t}$ depends on the truth values at the endpoints and the ones assigned by $\mathtt{h}$ in $\mathtt{t}$.

Definition 2 (Satisfaction in bs **in Beth-Reichenbach model (**T**,** R**,** $\mathtt{h}$**)).**

1. $\mathtt{t} \models_{bs} q$ *iff* $\mathtt{t} \in \mathtt{h}(q)$*, where* $\mathtt{t} \in T$ *and q atomic*
2. $\mathtt{t} \models_{bs} \neg A$ *iff for all* $\mathtt{t'}$, $\mathtt{t} \leq \mathtt{t'}$ *implies* $\mathtt{t'} \not\models_{bs} A$
3. $\mathtt{t} \models_{bs} A \wedge B$ *iff* $\mathtt{t} \models_{bs} A$ *and* $\mathtt{t} \models_{bs} B$.
4. $\mathtt{t} \models_{bs} A \vee B$ *iff* $\mathtt{t'} \models_{bs} A$ *or* $\mathtt{t'} \models_{bs} B$*, for all* $\mathtt{t'} \in \mathtt{E}_t$.

Remark 1. The interpretation of the atoms, $\neg$ and $\vee$ can be understood as modal S4 interpretation. The atoms q are understood as $\Box q$, and $\neg$ and $\vee$ are understood as $\Box\neg$ and $\Box\vee$.

We analyze the miners scenario by dropping condition (b) of Definition 1 to obtain our BR-semantics. A BR model is defined as an BS model as in Definition 1, but without condition (b), i.e. there is no connection between $\mathtt{E}_t \subseteq \mathtt{h}(q)$ and $\mathtt{t} \in \mathtt{h}(q)$.

Definition 3 (BR-model). *A BR-model has the form of a model of Definition 1 without requirement (b).*

Satisfaction on *br* is the same as satisfaction on *bs*, except the condition on negation. To highlight the difference we use a different symbol for negation, namely '$\sim$'. The second interpretation rule in definition 4 implements close world assumption, and evaluate a negative atomic formula without looking at the endpoints. In case $\mathtt{h}$ does not assign a positive value to an atom q, then q is asserted as false.

Definition 4 (Satisfaction $\models_{br}$ **in Beth-Reichenbach model (**$\mathtt{T}$**,** R**,** $\mathtt{h}$**)).**

1. $\mathtt{t} \models_{br} q$ *iff* $\mathtt{t} \in \mathtt{h}(q)$*, where* $\mathtt{t} \in \mathtt{T}$ *and q atomic*
2. $\mathtt{t} \models_{br} \sim A$ *iff* $\mathtt{t} \not\models_{br} A$.
3. $\mathtt{t} \models_{br} A \wedge B$ *iff* $\mathtt{t} \models_{br} A$ *and* $\mathtt{t} \models_{br} B$.

4. $\texttt{t} \models_{br} A \vee B$ *iff* $\texttt{t'} \models_{br} A$ *or* $\texttt{t'} \models_{br} B$*, for all* $\texttt{t'} \in \texttt{E}_t$.

Remark 2. When we evaluate atoms we do not look at the endpoints. On the other hand, for evaluating disjunction, we look at the endpoints. For example, let $\texttt{T}=\{\texttt{r},\texttt{t}, \texttt{s}\}$, $\texttt{r} < \texttt{t}$, $\texttt{r} < \texttt{s}$, and $\texttt{h(Z)}=\{\texttt{t}, \texttt{s}\}$, $\texttt{h(X)}=\{\texttt{t}\}$, $\texttt{h(Y)}=\{\texttt{s}\}$. In this example, the reference point "r" does not belong to h(Z). This is to show that even if Z may be true at all endpoints, in *br* it is false r'. On the other hand, in *br* (Z $\vee$ Z) holds at now, according to the interpretation rule of disjunction. This represents a big difference with respect to our logic and classical logic. In our logic, it is not true that (Z $\vee$ Z) $\models_{br}$ Z.

Disjunction constitutes a "connection" between the two kinds of satisfactions we are going to use ($\models_{br}$ and $\models_{bs}$). The way formulae are satisfied is different for both atomic formulae and boolean operators, except the disjunction where we look at the endpoints in both kinds of satisfaction.

In BR semantics we read disjunction as modal, namely we read $A \vee B$ as $\Box(A \vee B)$. So our semantics is modal logic in disguise. The advantage is that we are adding to classical logic just enough modal properties to address the miners scenario in a natural way, without having to bring in and commit to a lot of unnecessary modal machinery.

3 A case study: the miners scenario

Several authors provide consistent representations of the miners scenario. Kolodny and MacFarlane [12] give a detailed discussion of various consistent representations, but they conclude that the only satisfactory representation of the scenario is to invalidate the argument from (1.b-d) to (2) by rejecting modus ponens. Willer [16] argues that there are good reasons to preserve modus ponens and develops another consistent representation by falsifying monotonicity. Charlow [5] proposes a comprehensive representation which requires rethinking the relationship between relevant information (what we know) and practical rankings of possibilities and actions (what to do). Cariani et al. [3] argue that the traditional Kratzer's semantics [13] of deontic conditionals is not capable of representing the scenario satisfactorily. They propose to extend Kratzer's standard account by adding a parameter representing a "decision problem" to solve the scenario. Finally, Carr [4] argues that the proposal of Cariani et al. is still problematic, in that it packs decision theory in the semantics of modals.

We choose an approach different from the above mentioned treatment. In a nutshell, instead of invalidating the argument from (1.b-d) to (2), we address the scenario by making (1.a-d) and (2) compatible. According to our BR-semantics, the problem with the miners scenario is that we have three reference points. See Figure 1. Each reference point represents an information state. At now we do not have information where the miners are. Later, at point $\texttt{l}_1$, we know in which shaft the miners are, and later still (point $\texttt{l}_2$) we block the correct shaft. The meaning of "(Miners in A $\vee$ Miners in B)" is that no matter how information evolves, either we have that the assertion "Miners in A" holds or that "Miners in B" holds.

In a sense, we are operating in modal logic without bringing an explicit modality into the language, and we read "Either the miners are in shaft A or they are in shaft B" as $\Box$ (Either the miners are in shaft A or they are in shaft B.) and we read "we ought

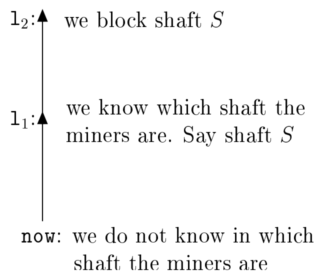

Fig. 1. Reference points in the miners scenario

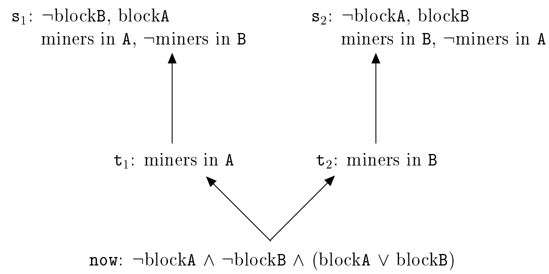

Fig. 2. Items in our solution to the miners scenario

to block neither shaft" as $\neg\Box$ block $A \wedge \neg\Box$ block B. If we regard the worlds in the Beth-Reichenbach model as possible worlds with a reflexive and transitive accessibility relation, then the semantic condition we gave to formulas of the form $x \wedge y$ is as if we have a necessity operator $\Box$ in front of it. We can thus have that $(X \vee Y) \wedge \neg X \wedge \neg Y$ can be consistent:

(3.a) $\texttt{now} \models_{br} (X \vee Y) \wedge \neg X \wedge \neg Y$

By instantiating (3.a), we can get (3). We thus have that at reference point now, "(block A ∨ block B)" is true but "block A" and "block B" are both false.

(3) $\texttt{now} \models_{br}$ (block A ∨ block B) ∧ ¬block A ∧ ¬block B

Moreover, if we add the deontic modal operator '$\bigcirc$' to the language, we can instantiate (3.a) differently as in (4) (we will formally define deontic modality below in Section 5.1).

(4) $\texttt{now} \models_{br}$ ($\bigcirc$ block A ∨ $\bigcirc$ block B) ∧ ¬ $\bigcirc$ block A ∧ ¬ $\bigcirc$ block B

This means that at now "we ought to block A or we ought to block B" is true but "we ought to block A " is false and "we ought to block B " are false. Therefore (1.a-d) and (2) are compatible. (4) moreover gives the right prediction to the miners scenario: the prediction given by (4) is "not block A" and "not block B" at now, although given more information "we will eventually either block A or block B".

Some readers may think that we can represent the miners scenario in standard deontic logic (SDL), augmented with a K (Knowledge) modality:

1. $\bigcirc(\neg A \wedge \neg B)$
2. $KA \rightarrow \bigcirc A$
3. $KB \rightarrow \bigcirc B$
4. $K(A \vee B)$

Now $\bigcirc A \vee \bigcirc B$, which would conflict with 1, is not derivable. This could be a simple and effective solution based on recognizing the epistemic context. However, SDL has its problems and of course we need to give problems free axioms for the $\bigcirc, K$ logic.

The perceptive reader might think that the Beth-Reichenbach semantics approach seems to provide a solution by coincidence. This is not the case. Indeed that there is an epistemic reading of the Beth-Reichenbach semantics itself enabling it to give a solution. Given that SDL has its problems we believe that it is better to offer a simple well known semantics , namely the Beth-Reichenbach semantics. The question to ask now is whether we can use the Beth-Reichenbach semantics to solve the difficulties of SDL. This could be a future research, where SDL is based on intuitionistic logic.

4 Reactive semantics

We now describe the ReBR-semantics. This is an intermediate semantics where we augment BR models with reactive double arrows, as defined in Gabbay [11], but no actions. It is only a temporary step, for reasons of exposition, to lead the reader towards the final ReBRA semantics with actions. The reactive arrows are not needed to solve the miners scenario but it is compatible with it and can expand and solve other problems related to multi-agent and their respective progression of knowledge in time.

So we explain the idea with a diagram. Consider Figure 3. As the agents traverse an arc, if there is a double arrow emanating from the arc to another arc, the double arrow will disconnect the target arc.

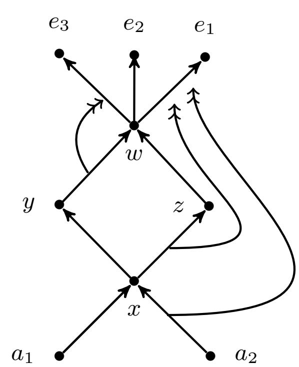

Fig. 3. Double arrow model

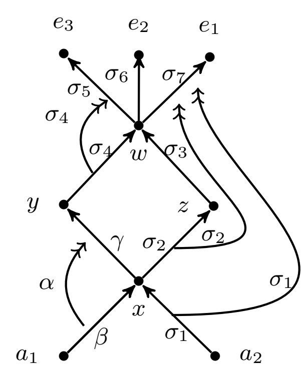

Fig. 4. Model for Figure 3 enriched with actions and reactivity

So if an agent passes through the path $\mathtt{a}_2 \rightarrow \mathtt{x} \rightarrow \mathtt{z} \rightarrow \mathtt{w}$, passing through the arc $\mathtt{x} \rightarrow \mathtt{z}$, the double arrow $(\mathtt{x} \rightarrow \mathtt{z}) \twoheadrightarrow (\mathtt{w} \rightarrow \mathtt{e}_3)$ gets active and disconnects (blocks) the arc $(\mathtt{w} \rightarrow \mathtt{e}_3)$. Similarly, if the agent moves to node $\mathtt{w}$ along the path $\mathtt{a}_1 \rightarrow \mathtt{x} \rightarrow \mathtt{y} \rightarrow \mathtt{w}$, then he cannot go to $\mathtt{e}_1$ because by passing through $\mathtt{y} \rightarrow \mathtt{w}$ there is a double arrow disconnects the arow from $\mathtt{w}$ to $\mathtt{e}_1$. We identify an agent with the path of the following form:

$$\Pi = (\mathtt{x}_1, \mathtt{x}_2, \ldots, \mathtt{x}_n), \text{ such that:}$$
$$\mathtt{x}_1 \rightarrow \mathtt{x}_2 \rightarrow \mathtt{x}_3 \rightarrow \ldots \rightarrow \mathtt{x}_n$$

x_1 is the starting point of the agent and x_n is the current point of the agent. Mathematically, if $((x_i, x_{i+1}), (u, v))$ is a double arrow, then $u \to v$ is blocked, in case $x_i \to x_{i+1}$ is in Π. We say in this case that $u \to v$ is blocked by Π. In particular, we may have that $(u \to v) \in \Pi$.

Now, we continue to explain the diagram in Figure 3. e_1, e_2, and e_3 are endpoints, while x, y, and z are reference points without being endpoints. In the BS-semantics for this figure, *atomic* formulae may be true or false at the reference points. Their value is *given* in terms of an assignment h. On the other hand, complex formulae (negation, disjunctions, and conjunctions) are *evaluated* in terms of the truth values of the atoms at the endpoints above them according with the interpretation rules given in Definition 2 for the BS-semantics. In other words, atomic values are always given, while for evaluating complex formulae we must look at the endpoints.

The role of reactivity is in the notion of what it means to be above a point. For an endpoint e to be above a point t we need to have a path from t leading to e such that none of its arcs are blocked by the path itself. For example, the path $y \to w \to e_1$ blocks its own arc $w \to e_1$. If the agent goes to w through y, he can reach the endpoints $\{e_2, e_3\}$, while if he goes from x to w through z he can reach $\{e_1, e_2\}$. This means that passing the path from x to z does not let the agent reach the endpoint e_3.
We define now a legitimate path.

Definition 5 (Legitimate path). *Given a set of point S and a relation $R \subseteq S \times S$ and a relation $R^* \subseteq R \times R$, a path Π is legitimate if it does not block itself, i.e. if there is not an arc in Π that activates a double arrow in R^* that blocks another arc in Π.*

Two paths can be concatenated if the last point of one is the starting point of the other. We refer to the concatenation of two paths Π and Π' via the notation $\Pi * \Pi'$

Definition 6. *A reactive Beth-Reichenbach model for L_C has the form (T, R, R^*, h), where T is a set of worlds, $R \subseteq T \times T$, $R^* \subseteq R \times R$, and h is an assignment giving a subset h(q)$\subseteq$T for each atomic q.*

We require the following to hold: Given a path Π with last point t, let E_Π be the set of all endpoints x such that there exists a path Π' beginning with t and ending with x such that the concatenation of Π with Π' is legitimate. We require that for each legitimate Π, $E_\Pi \neq \emptyset$. Note that this means that every legitimate path has an endpoint above it that can be legitemetely reached.

Definition 7. *Satisfaction $\models_{br}$ in reactive Beth-Reichenbach model (T, R, R^*, h), with respect to a legitimate path Π.*

1. *($\Pi \models_{br} q$) iff t is in h(q), where q is atomic and t is the last point of Π.*
2. *($\Pi \models_{br} \neg A$) iff ($\Pi \not\models_{br} A$).*
3. *($\Pi \models_{br} A \wedge B$) iff ($\Pi \models_{br} A$) $\wedge$ ($\Pi \models_{br} B$)*
4. *($\Pi \models_{br} X \vee Y$) iff in any endpoint $t \in E_\Pi$ we have: either ($t \models_{br} X$) or ($t \models_{br} Y$)*

5 Action involved model

In this section we add action to our semantics. We begin with an explanation and then give a formal definition. Consider Figure 5. In the state $\mathtt{a}_1$ Mary has the laptop. John wants the laptop and in state $\mathtt{x}$ John has the laptop. On our previous model there is nothing else to say. We do not know *in what way* John has become the owner of the laptop. In our new model we want to add actions and specify that the move from $\mathtt{a}_1$ to $\mathtt{x}$ was the result of an action. Let us list (all) the actions available to John.

- Action α: steal the laptop
- Action β: buy the laptop
- Action γ: buy insurance for the laptop

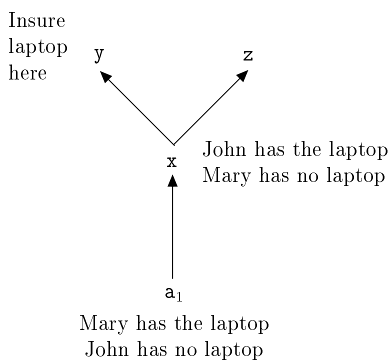

Fig. 5. A model explaining the need of actions

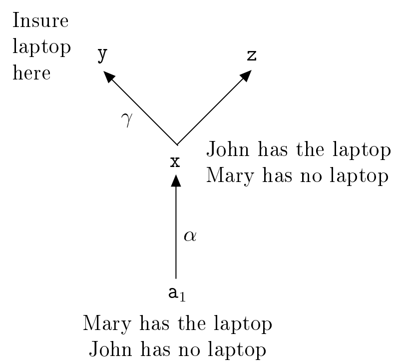

Fig. 6. Model of Figure 5 enriched with actions

In our new model we need to write annotation as to which action was used in the transitions. Therefore if John stole the laptop then he cannot insure it. So we have Figure 7. But if he bought the laptop then he can insure it. So we have Figure 8.

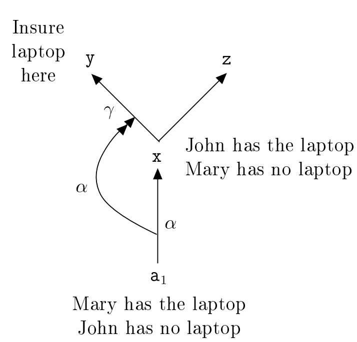

Fig. 7.

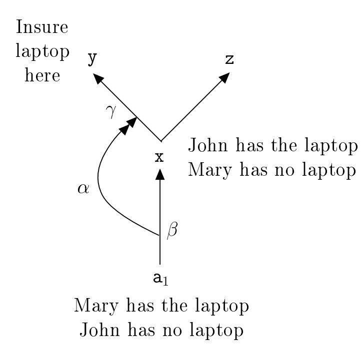

Fig. 8.

Therefore, in our modelization, reactive arrows define a *dependency* among actions. Action α can block the execution of action γ. This is like a deontic rule on action: $\alpha \rightarrow$O($\neg$ γ). Indeed, Gabbay [10] uses reactive arrows to give semantics to contrary-to-duty obligations.

In other words, we have an opportunity to use the reactive arrows, in the following sense: if the laptop is stolen it *ought* not to be insured. Therefore we model this deontic/legal rule in Figure 8 using annotated arrows with actions to indicate which action was taken to move across the arrows and we also annotate double arrows by actions to indicate what actions activate the double arrow.

The double arrow $(\mathtt{a}_1 \rightarrow_\alpha \mathtt{x}) \twoheadrightarrow_\alpha (\mathtt{x} \rightarrow_\gamma \mathtt{y})$ cancels $(\mathtt{x} \rightarrow_\gamma \mathtt{y})$, but the double arrow $(\mathtt{a}_1 \rightarrow_\beta \mathtt{x}) \twoheadrightarrow_\alpha (\mathtt{x} \rightarrow_\gamma \mathtt{y})$ does not cancel $(\mathtt{x} \rightarrow_\gamma \mathtt{y})$, because α is different from β (the laptop was bought, not stolen). So for a reactive cancellation to work $(\mathtt{t}_1 \rightarrow_\gamma \mathtt{t}_2) \twoheadrightarrow_\varepsilon (\mathtt{s}_1 \rightarrow_\eta \mathtt{s}_2)$, we must have γ=ε.

Note that the above extra action structure does not affect our solution of the miners scenario. It just gives extra information. Consider Figure 4, which is an extension of Figure 3 with action annotations. An action symbol annotating an arc in the figure represents the action which triggered the transition indicated by this arc. Again, note that the arrow $(\mathtt{x} \rightarrow \mathtt{y})$ is not cancelled, because $\beta \neq \alpha$,

5.1 The nature of action

In our model, we need two pure types of actions:

Knowledge information actions. For example, let us introduce the action δ = "get info about the location of the miners". δ is non-deterministic we can find out the miners are in shaft $\mathtt{A}$ or in shaft $\mathtt{B}$.

Facts actions For example, let us introduce the action ε = "kill all the miners". ε also yields information, e.g. that all miners are dead.

In order to model actions, it is necessary to add a definition of how actions are to be executed, and what is the form of actions. We adopt a traditional AI view:

Action α: (precondition, postcondition)

Both precondition and postcondition are represented by wff of our language $\mathtt{L}_C$. In the miners scenario we have:

Action block-A: (miners in A, A is blocked)
Action block-B: (miners in B, B is blocked)

The two actions are blocked when their preconditions do not hold at the current reference point. For example, most likely the action "kill all the miners" cannot be executed because there could be severe preconditions to execute the action, e.g. finding out that the miners are terrorist hiding in the shaft.

5.2 ReBRA-semantics

Having introduced all ingredients, namely basic BR-Semantics, reactivity and actions, we are ready now to define our final semantics, which we call ReBRA-Semantics.

Definition 8. *ReBRA-Semantics model.*
Let ($\mathtt{S}$, $\mathtt{R}$), $\mathtt{R} \subseteq \mathtt{S} \times \mathtt{S}$ *be a finite network. This is our basic set of states/worlds/reference points and the transition relation* $\mathtt{R}$*. Let* $\mathtt{E}$ *be the set of endpoints of* ($\mathtt{S}$, $\mathtt{R}$)*, i.e.* $e \in \mathtt{E}$ *iff* $\neg\exists_y(e\mathtt{R}y)$*. Let* $<$ *be the transitive closure of* $\mathtt{R}$ *and let* $\mathtt{x} \leq \mathtt{y}$ *be* $\mathtt{x} = \mathtt{y}$ *or* $\mathtt{x} < \mathtt{y}$*. Let* $\mathtt{E}_t$ = {*set of all endpoints* $\mathtt{s}$ *such that* $\mathtt{t} \leq \mathtt{s}$ }*. We also have a stock of actions* $\mathtt{A}$ *of the form* α=(Pre_α, $Post_\alpha$)*, where* Pre_α *is a statement being the precondition of* α *and* $Post_\alpha$ *is the postcondition. Then:*

A model is a system with ($\mathtt{S}$, $\mathtt{R}$, $\mathtt{R}^*$, $\mathtt{A}$, $\mathtt{h}$)*, where* $\mathtt{S}$ *is the set of states,* $\mathtt{A}$ *is the set of actions,* $\mathtt{R} \subseteq \mathtt{S} \times \mathtt{S}$*, and* $\mathtt{R}^* \subseteq \mathtt{A} \times \mathtt{R} \times \mathtt{R}$*, and for each atomic q,* $\mathtt{h}(q) \subseteq \mathtt{S}$*.*

Satisfaction of a model is given with respect to an annotated path:

Definition 9. *Annotated paths.*
Taken $\mathtt{S}$, $\mathtt{R}$*, and* $\mathtt{A}$ *of definition 8, let* Π=($\mathtt{s}_0$, $\mathtt{s}_1$, ..., $\mathtt{s}_k$) *to denote a path. We have* $\mathtt{s}_i\mathtt{R}\mathtt{s}_{i+1}$, $0 \leq i \leq k$*. We also write* $\mathtt{s}_0 \to \mathtt{s}_1 \to \ldots \to \mathtt{s}_k$*. A path is the history of the agent as he moves around the network from one state to the next.* $\mathtt{s}_0$ *is the beginning state of the path and* $\mathtt{s}_k$ *is the last state of the path.*

An annotated path Π *has the form* ($\mathtt{s}_0 \to_{\alpha_0} \mathtt{s}_1 \to_{\alpha_1} \ldots \to_{\alpha_{k-1}} \mathtt{s}_\mathtt{k}$)*. This is a path where each transition* $\mathtt{s}_i\mathtt{R}\mathtt{s}_{i+1}$ *is labelled by actions. We imagine an agent moving along the path* ($\mathtt{s}_0 \to \mathtt{s}_1 \to \ldots \to \mathtt{s}_k$)*. Each move from state* $\mathtt{s}_i$ *to state* $\mathtt{s}_{i+1}$ *is done by action* α_i*.* $\mathtt{s}_0$ *is where the agent started and* $\mathtt{s}_k$ *is where the agent is currently situated. Arrows annotated by actions are elements of* $\mathtt{A} \times \mathtt{S} \times \mathtt{S}$*.*

We also have annotated reactive double arrows:

Annotated reactive double arrows have the form ($\mathtt{t} \to \mathtt{s}$)$\twoheadrightarrow_\alpha$($\mathtt{x} \to \mathtt{y}$)*, where* $\mathtt{t}$, $\mathtt{s} \in \mathtt{S}$, $\mathtt{t}\mathtt{R}\mathtt{s}$, $\mathtt{x}, \mathtt{y} \in \mathtt{S}$, $\mathtt{x}\mathtt{R}\mathtt{y}$*. The annotated double arrows are elements in* $\mathtt{A} \times \mathtt{R} \times \mathtt{R}$*.*

We need now to define satisfaction in a model.

Definition 10. *Satisfaction, Legitimacy, and Coherence with respect to actions. Let* Π *be an annotated path.* Π *is a legitimate annotated path iff both (a) and (b) hold.*

- *(a) There does not exist two arcs of the form* $\mathtt{x} \to_\alpha \mathtt{y}$ *and* $\mathtt{u} \to_\beta \mathtt{v}$ *in* Π *such that* ($\mathtt{x} \to_\alpha \mathtt{y}$)$\twoheadrightarrow_\alpha$($\mathtt{u} \to_\beta \mathtt{v}$) *is in* $\mathtt{R}^*$*. I.e.* (α, (α, $\mathtt{x}$, $\mathtt{y}$), (β, $\mathtt{u}$, $\mathtt{v}$)) $\in \mathtt{R}^*$*.*
- *(b) Whenever* $\mathtt{x} \to_\alpha \mathtt{y}$ *is in the path then* $\Pi_\mathtt{x} \models_{br} A$*, where* $\Pi_\mathtt{x}$ *is the initial path of* Π *up to node* $\mathtt{x}$ *and* A *is the precondition of* α *and* $\models_{br}$ *is the br-satisfaction defined in the next item 3.*

Two paths can be concatenated if the last point of one is the starting point of the other. We refer to the concatenation of two paths Π *and* Π' *via the notation* $\Pi * \Pi'$*.*

Let Π *be a legitimate path. We define* $\mathtt{E}_\Pi$*, the set of legitimate endpoints of* Π*, is defined as follows: a point* $\mathtt{t}$ *is in* $\mathtt{E}_\Pi$ *iff* $\mathtt{t}$ *is an endpoint and for some* Π' *such that the* *first element of Π' is equal to the last element of Π and the last element of Π' is* $\mathtt{t}$ *and* $\Pi * \Pi'$ *is legitimate.*

Finally, let Π be a legitimate path. We define satisfaction $\models_{br}$ as follows:

(a) $(\Pi \models_{br} q)$ iff $\mathtt{t}$ *is in* $\mathtt{h}(q)$*, where q is atomic and* $\mathtt{t}$ *is the last point of Π.*
(b) $(\Pi \models_{br} \neg \mathtt{A})$ iff $(\Pi \not\models_{br} \mathtt{A})$.
(c) $(\Pi \models_{br} \mathtt{A} \wedge \mathtt{B})$ iff $(\Pi \models_{br} \mathtt{A}) \wedge (\Pi \models_{br} \mathtt{B})$
(d) $(\Pi \models_{br} \mathtt{X} \vee \mathtt{Y})$ iff in any endpoint $\mathtt{t} \in \mathtt{E}_{\Pi}$ we have: either $(\mathtt{t} \models_{br} \mathtt{X})$ or $(\mathtt{t} \models_{br} \mathtt{Y})$

The last component we need to complete our formal framework is a mechanism that allows agents to choose the actions they will execute, among those available.
In the present paper, given an action "α: (prec(α), post(α))" and a path Π whose last point is $\mathtt{t}$, we say that α can be executed given Π iff $\Pi \models_{br}$ prec(α) holds. In other words, we understand every action as a conditional norm. There we define an action to be obligatory iff the precondition of the action is satisfied. Formally,

Definition 11. *Obligatory actions.*

$$\Pi \models_{br} \bigcirc\alpha \text{ iff } \Pi \models_{br} prec(\alpha)$$

Agents must execute all actions whose precondition holds.

Of course, actions could be selected in other different ways. According to [4], actions should be ranked according to probability, expected values, goals of the agents, etc. The present account only uses agents in deontic mode, i.e. agents are always obligated to perform an action when the preconditions hold, while an extension to ordinary action is left as future work.
Given the definition of obligation in def.11, we deduce the following:

(5) if $\Pi \not\models_{br}$ prec(α), then $\Pi \models_{br} \neg \bigcirc \alpha$.

In the miners scenario, (5) can be used to derive $\neg \bigcirc block\ A$ and $\neg \bigcirc block\ B$ as long as the precondition of $block\ A$ and $block\ B$ are not satisfied in the reference point $\mathtt{now}$.

6 Using ReBRA-semantics for the miners scenario

We now illustrate the use of actions in the context of the miners scenario. We are not going to use reactive arrows for the moment. Figure 2 becomes now Figure 9. Let us summarize figure 9. At $\mathtt{now}$ we have true that: "A is not blocked", "B is not blocked", "A is blocked $\vee$ B is blocked". We do not know where the miners are. We take action "get-info" and get two non-deterministic results: "$\mathtt{t}_1$: miners in A" and "$\mathtt{t}_2$: miners in B".

The preconditions of the action "get-info" is $\top$. Therefore, we can get-info at any point. The postcondition of "get-info" is that we know where the miners are and so we reduce the number of endpoints. After the execution of get-info, for instance, we could move to either $\mathtt{t}_1$ or $\mathtt{t}_2$ so that we will see a single endpoint.

We take action "block-A" at $\mathtt{t}_1$. We take action "block-B" at $\mathtt{t}_2$. We get $\mathtt{s}_1$ and $\mathtt{s}_2$ respectively. However, at $\mathtt{now}$, actions "block-A" cannot be executed because we

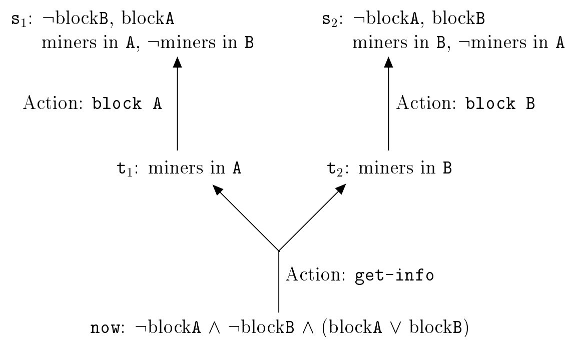

Fig. 9. Items in the miners scenario with actions

require the precondition of "block-A" to be "miners in A", and similarly for the action "block-B". We therefore accept "not ought to block A" at reference point now by referring the Kantian law "ought implies can". Moreover, we take for granted that at t_1 "ought to block A" is true and at t_2 "ought to block B" is true. Then by the semantics of disjunction we have "ought to block A or ought to block B" is true at now. That is to say, we have (4) in the miners scenario. As we state at the end of Section 3, (4) is logically consistent in our semantics and gives the right prediction.

7 Related work

We have proposed a new approach based on reactive semantics that appears to be promising for handling normative multi-agents systems [7] and their respective progression of knowledge in time. We used our new logic in this paper to specifically solve a well-known puzzle that recently gained popularity in the scientific community: the miners scenario. This section highlights the differences between our approach and some recent solutions to that scenario in the literature.

Several authors observe that the paradox arises by applying deduction rules that are commonly assumed to be valid in deontic logic, and they propose a revision of such rules. [12] examine various options, i.e. rejecting either one of the premises or one of the three deduction rules used in the derivation (disjunction introduction, disjunction elimination, or modus ponens). They come to the conclusion that we must reject modus ponens for indicative conditionals. The validity of modus ponens - they argue - must be "information-sensitive", i.e. it must be defined with respect to the knowledge that is at the disposal of the agent at the time of the inference.

Other authors do not accept the solution by [12], focussed on modus ponens, arguing that contextual (pragmatic) preferences ought to be included in the semantics of modal operators. [3] and [5] belong to this school of thought. They modify the semantics of modal operators by including some kinds of decision rules that allow prefererence for one of the available options, so that inconsistency does not arise.

Some have questioned this solution, e.g. Carr and Willer [4], [16]. Carr, in particular, shows that encapsulating pragmatic decision rules within the meaning of modals

- a solution that seems at odds with standard literature on modals - makes the formal framework too rigid. It is no longer possible, for instance, to rank the available actions that an agent can perform according to the probability of getting the expected outcome. With respect to the miners scenario, for instance, it is not possible to model a scenario where the agent knows that there is a 99% probability that the miners are in shaft A, and so he could decide to take the risk and block the shaft.

We acknowledge that Carr is on the right track. However, in our view, she has not fully achieved the goal of keeping pragmatic constraints distinct from the semantics. Carr's work started from a question by Krazter and von Fintel: "why pack information about rational decision making into the meaning of modals?". She developed a formal theory where preference rules are asserted in terms of separate (context-dependent) functions that affect the truth conditions of modals and conditionals, but they are not part of the formal representations of their meaning. She states that the decision rules regarding actions only "determine the meanings of modals".

Although this allows for a more expressive and flexible management of pragmatic constraints, modals still need decision rules to be interpreted in a model. In other words, we do not achieve neat independence between semantics and pragmatics if the choice of a certain action is needed to determine the truth values of modals and conditionals.

Our basic BR-semantics is already capable of solving the miners scenario as it adopts a different account of disjunction. Disjunctions, used to express limited knowledge, are interpreted with respect to the endpoints, not the current reference point. In this respect, our approach is more similar to that of [12]. However, rather than rejecting modus ponens, we reject disjunction introduction since in our logic "$A \rightarrow (A \vee B)$" does not hold. We have also shown that our basic BR-semantics may be extended with actions into a new semantics that we call ReBR-semantics, to enable the implementation of all pragmatic preferences and constraints that affect the selection of the proper actions to be taken. A complete and exhaustive formalization of such constraints, however, deserves much further work.

8 Conclusion

This paper presents the following contributions. First, we show how a linguistic interpretation of deontic paradoxes can be used to further develop deontic logic, by introducing a special Beth-Reichenbach semantics and using it to represent the single agent decision problem of the miner's scenario. In further research we will complete this argument, for example by extending the semantics with a deontic modal operator, or by using the language as a base logic in the input/output logic framework.

Second, we give a new analysis of the single agent decision problem of the miners scenario. We bring modal meaning to disjunction which mirror our intuitions and eliminates the cause of the scenario, without bringing in the full machinery of modal or non-classical logic.

Third, we augment the BR-semantics with reactive arrows and actions, obtaining a new semantic that we call ReBR- and ReBRA-semantics. In future work we will illustrate how this extended semantics can be used to represent the multi-agent dialogues of the miners scenario.

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