RIEMANN SURFACES. LECTURE NOTES. WINTER SEMESTER 2014/2015.

A PRELIMINARY AND PROBABLY VERY RAW VERSION.

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1. Lecture 1

1.1. **Some prerequisites for the whole lecture course.** The following is assumed known.

1) Holomorphic functions in one variable.
2) Basics on topology: topological spaces, continuous maps.
3) basics on topological manifolds: definition.
4) Definition of a complex manifold.

1.2. **Definition of a Riemann surface.** Since this course is called “Riemann surfaces”, the first and main definition of the course is the one of a Riemann surface.

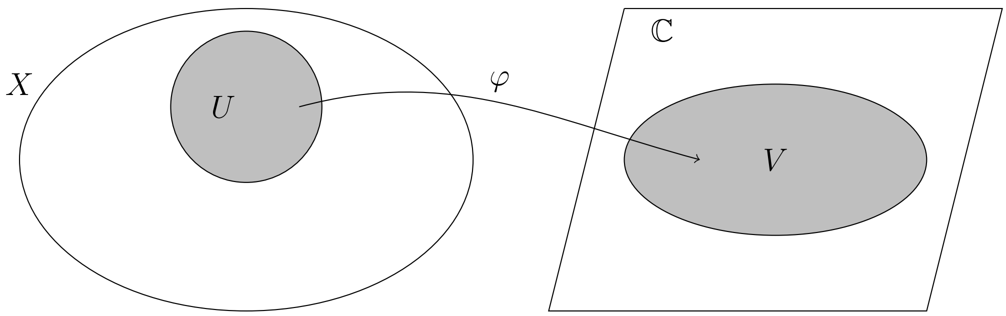
Definition 1.1. A Riemann surface is a connected 1-dimensional complex manifold.

Convention. We will usually write RS for Riemann surface.

Let us clarify the meaning of Definition 1.1.

Let X be a 2-dimensional real topological manifold.

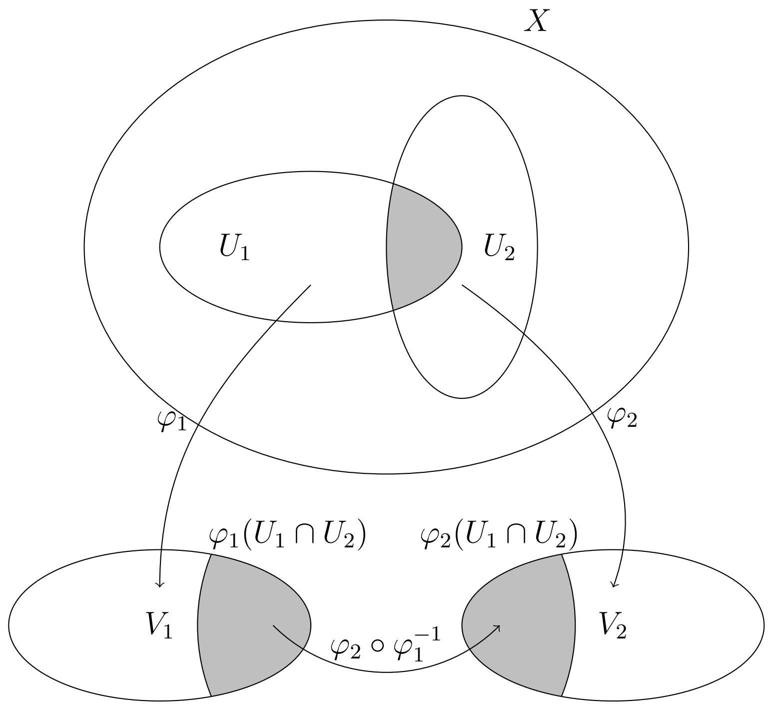
Definition 1.2. Let $U \subset X$ be an open subset. Let $V \subset \mathbb{C}$ be an open subset of the set of complex numbers (equipped with the standard Euclidean topology). Let $\varphi : U \to V$ be a homeomorphism. Then $\varphi : U \to V$ is called a complex chart on X.



Definition 1.3. Two complex charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are called holomorphically compatible if

$$\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a holomorphic map. By abuse of notation we will often denote it by $\varphi_2 \circ \varphi_1^{-1}$.



Exercise. $\varphi_2 \circ \varphi_1^{-1}$ is then automatically biholomorphic.

Definition 1.4. A system of holomorphically compatible complex charts on X

$$\mathfrak{A} = \{\varphi_i : U_i \to V_i, i \in I\}$$

such that $\bigcup_{i \in I} U_i = X$ is called a complex atlas on X.

Definition 1.5. Two atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ on X are called holomorphically compatible if every chart from $\mathfrak{A}_1$ is holomorphically compatible with every chart from $\mathfrak{A}_2$.

Exercise. Holomorphic equivalence is an equivalence relation.

Definition 1.6. A complex structure on X is an equivalence class of complex atlases.

Remark 1.7. In order to define a complex structure on X it is enough to give a complex atlas on X. Then two complex structures are equal if and only if the corresponding atlases are equivalent.

Definition 1.8. Let $\mathfrak{A}$ be a complex atlas on X. Put

$$\mathfrak{A}_{max} = \{\text{complex charts on } X \text{ holomorphically compatible with the charts from } \mathfrak{A}\}.$$

Then $\mathfrak{A}_{max}$ is the maximal atlas holomorphically compatible with $\mathfrak{A}$.

Therefore, two atlases $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent if and only if $\mathfrak{A}_{max} = \mathfrak{B}_{max}$.

Definition 1.9. A RS is a pair (X,Σ), where X is a connected 2-dimensional real topological manifold and Σ is a complex structure on X.

Equivalently: a RS is a pair $(X,\mathfrak{A})$, where X is a connected 2-dimensional real topological manifold and $\mathfrak{A}$ is a complex atlas on X.

For those who remember the definition of a complex manifold is clear now that the last definition is just the definition of a 1-dimensional complex manifold.

Convention. If (X,Σ) is a RS, then "a chart on X" means a chart in the maximal atlas on X corresponding to Σ.

Examples 1.10 (of Riemann surfaces)**.** 1) $X=\mathbb{C}$, $\mathfrak{A}=\{\mathbb{C}\xrightarrow{\text{id}}\mathbb{C}\}$.

In order to define the same complex structure one can also take the complex atlas given by $\mathfrak{A}'=\{U_n\xrightarrow{\text{id}}U_n \mid n\in\mathbb{N}\}$, where $U_n=\{z\in\mathbb{C}\mid |z|<n\}$.

2) Any domain in $U\subset\mathbb{C}$ (open connected subset of $\mathbb{C}$) , $\mathfrak{A}=\{U\xrightarrow{\text{id}}U\}$. More generally, let X be a RS and let $U\subset X$ be a domain. Then U is a RS as well. As an atlas one can take the restrictions to U of the complex charts on X.

3) Complex projective line $\mathbb{P}_1=\mathbb{P}_1(\mathbb{C})=\{(a:b)\mid (a,b)\in\mathbb{C}^2\setminus\{(0,0)\}\}$, where $(a:b)$ denotes the line in $\mathbb{C}^2$ through $(0,0)$ and (a,b). Define

$$U_0=\{(a:b)\mid a\neq 0\}=\{(1:b)\mid b\in\mathbb{C}\},\quad U_1=\{(a:b)\mid b\neq 0\}=\{(a:1)\mid a\in\mathbb{C}\}.$$

Define

$$\varphi_0: U_0\to\mathbb{C},\quad (1:b)\mapsto b,$$

and

$$\varphi_1: U_1\to\mathbb{C},\quad (a:1)\mapsto a.$$

Then $\mathfrak{A}=\{U_0\xrightarrow{\varphi_0}\mathbb{C}, U_1\xrightarrow{\varphi_1}\mathbb{C}\}$ is a complex atlas on $\mathbb{P}_1$. The transition function $\varphi_1\circ\varphi_0^{-1}|_{\varphi_0(U_0\cap U_1)}$ is

$$\varphi_0(U_0\cap U_1)=\mathbb{C}^*\to\mathbb{C}^*=\varphi_1(U_0\cap U_1),\quad a\mapsto\frac{1}{a}.$$

4) Riemann sphere $\hat{\mathbb{C}}$. As a set $\hat{\mathbb{C}}=\mathbb{C}\sqcup\{\infty\}$, where ∞ is just a symbol. The topology is defined as follows. $U\subset\hat{\mathbb{C}}$ is open if and only if **either** $\infty\notin U$ and $U\subset\mathbb{C}$ is open **or** $\infty\in U$ and $\mathbb{C}\backslash U$ is compact in $\mathbb{C}$. This defines a compact Hausdorff space homeomorphic to the two-dimensional sphere $\mathbb{S}^2$. Put $U_0=\mathbb{C}$ and $U_1=\hat{\mathbb{C}}\setminus\{0\}=\mathbb{V}^*\sqcup\{\infty\}$. Define $\varphi_0: U_0\to\mathbb{C}=\text{id}:\mathbb{C}\to\mathbb{C}$ and define $\varphi_1: U_1\to\mathbb{C}$ by

$$\varphi_1(z)=\begin{cases}\frac{1}{z}, & z\neq\infty;\\ 0, & \textit{otherwise}.\end{cases}$$

Exercise. The complex charts φ_0 and φ_1 are holomorphically compatible and constitute a complex atlas on $\hat{\mathbb{C}}$.

Indeed, it is enough to notice that the transition function $\varphi_1 \circ \varphi_0^{-1}|_{\varphi_0(U_0 \cap U_1)}$ is given by

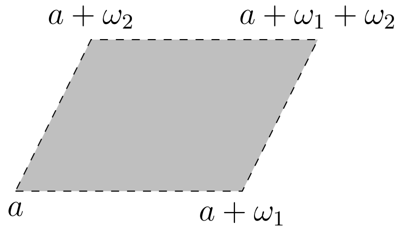
$$\varphi_0(U_0 \cap U_1) = \mathbb{C}^* \to \mathbb{C}^* = \varphi_1(U_0 \cap U_1), \quad a \mapsto \frac{1}{a}.$$

Remark. Notice that this is the same transition function as in the previous example.

5) Complex tori.

Consider $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$. Let $\{\omega_1, \omega_2\}$ be its basis over $\mathbb{R}$. Let $\Gamma = \mathbb{Z} \cdot \omega_1 + \mathbb{Z} \cdot \omega_2 = \{n\omega_1 + m\omega_2 \mid m, n \in \mathbb{Z}\}$ be the corresponding lattice. It is a subgroup in the abelian group $\mathbb{C}$. Consider the quotient homomorphism $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma$ and introduce on $\mathbb{C}/\Gamma$ the quotient topology, i. e., $U \subset \mathbb{C}/\Gamma$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}$.

For every $a \in \mathbb{C}$ put $V_a = \{a + t_1\omega_1 + t_2\omega_2 \mid t_1, t_2 \in (0,1)\}$, i. e., the interior of the parallelogram with vertices at a, $a + \omega_1$, $a + \omega_2$, $a + \omega_1 + \omega_2$.



V_a are called standard parallelograms with respect to the lattice Γ.

Put $U_a := \pi(V_a)$. Note that $\pi|_{V_a} : V_a \to U_a$ is bijective and moreover a homeomorphism. Put $\varphi_a := (\pi|_{V_a})^{-1} : U_a \to V_a$. This gives a complex atlas on $\mathbb{C}/\Gamma$.

Exercise. Check the details.

1.3. **Definition of a holomorphic function of a Riemann surface. Structure sheaf.**

Definition 1.11 (Holomorphic functions)**.** Let X be a RS. Let $Y \subset X$ be an open subset. Then a function $Y \xrightarrow{f} \mathbb{C}$ is called holomorphic on Y if for every chart $\varphi : U \to V$ on X the composition $f \circ \varphi^{-1} : \varphi(U \cap Y) \to \mathbb{C}$ is a holomorphic function.

Let $\mathcal{O}_X(Y)$ denote the set of all holomorphic functions on Y.

Exercise. $\mathcal{O}_X(Y)$ is a $\mathbb{C}$-algebra.

Remark 1.12. For every open subset $U \subset X$ we obtain a $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ of holomorphic functions on U. For every two open subsets U and W in X such that $U \subset W$, the restriction map $\mathcal{O}_X(W) \to \mathcal{O}_X(U)$, $f \mapsto f|_U$ is a homomorphism of $\mathbb{C}$-algebras. The collection of all these data is denoted $\mathcal{O}_X$ and is called the structure sheaf on X.

1.4. Exercises.

Exercise 1. 1) Check that the complex charts on $\hat{\mathbb{C}}$ introduced in the lecture are holomorphically compatible and constitute a complex atlas on $\hat{\mathbb{C}}$.

2) Prove that $\hat{\mathbb{C}}$ is homeomorphic to the complex projective line $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$.

Exercise 2. Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

1) Fill in the gaps in the definition of the complex structure on $\mathbb{C}/\Gamma$. How do the transition functions $\varphi_b \circ \varphi_a^{-1}$ look like?

2) Let S^1 denote the real 1-sphere. Show that $\mathbb{C}/\Gamma$ is homeomorphic to $S^1 \times S^1$.

Hint: *Let p_1, p_2 be the $\mathbb{R}$-basis of* $\mathrm{Hom}(\mathbb{C}, \mathbb{R})$ *dual to ω_1, ω_2. Consider the map $\mathbb{C}/\Gamma \to S^1 \times S^1$, $[z] \mapsto (\exp(2\pi i p_1(z)), \exp(2\pi i p_2(z)))$. Here $[z]$ denotes the equivalence class of a complex number z in $\mathbb{C}/\Gamma$.*

Exercise 3. In this exercise all subsets of complex manifolds are equipped with the induced topology.

1) Show that the following subspaces of $\mathbb{C}^2$ or $\mathbb{C}^3$ are complex submanifolds, hence they are Riemann surfaces. Describe the complex structures on each of them.

$$X_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid 5z_1 + 7z_2 = 0\}, \quad X_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid 3z_1 - 14z_2^2 = 0\},$$

$$X_3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 - 1 = 0\}, \quad X_4 = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \mid z_1 - z_0^2 = 0, z_2 - z_0^3 = 0\}.$$

2) Are the following subsets of $\mathbb{C}^2$ complex submanifolds?

$$X_5 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 - z_2^3 = 0\}, \quad X_6 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}.$$

Can you equip these subspaces of $\mathbb{C}^2$ with a structure of a Riemann surface?

Hint: *Have a look at the map $\mathbb{C} \to \mathbb{C}^2$, $t \mapsto (t^3, t^2)$. Study the connected components of $X_6 \setminus \{(0, 0)\}$.*

Exercise 4. 1) Describe all holomorphic functions on $\hat{\mathbb{C}}$.

Hint: *Use the compactness of $\hat{\mathbb{C}}$ and your knowledge about bounded holomorphic functions on the complex plane $\mathbb{C}$.*

2) Let Γ be a lattice in $\mathbb{C}$. Can you describe all holomorphic functions on the torus $\mathbb{C}/\Gamma$ using a similar reasoning as in part 1) of this exercise?

2. Lecture 2

2.1. Riemann removable singularities theorem for Riemann surfaces.

Theorem 2.1 (Riemann removable singularities theorem)**.** *Let X be a RS. Let $U \subset X$ be an open subset. Let $a \in U$, let $f \in \mathcal{O}_X(U \setminus \{a\})$ be bounded. Then there exists a unique $\bar{f} \in \mathcal{O}_X(U)$ such that $\bar{f}|_{U\setminus\{a\}} = f$.*

Proof. Let $\varphi : U' \to V'$ be a chart around a. Then $f \circ \varphi^{-1}$ is a holomorphic bounded function on $\varphi(U' \cap U) \setminus \{\varphi(a)\} \subset \mathbb{C}$. Therefore, there exists a unique holomorphic function F on $\varphi(U' \cap U)$ such that

$$F|_{\varphi(U'\cap U)\setminus\{\varphi(a)\}} = f \circ \varphi^{-1}.$$

Therefore, there is a unique holomorphic function g on $U \cap U'$ such that $g|_{U\cap U'\setminus\{a\}} = f|_{U\cap U'\setminus\{a\}}$. Hence $\exists!\ \bar{f} \in \mathcal{O}_X(U)$ with $\bar{f}|_{U\setminus\{a\}} = f$. □

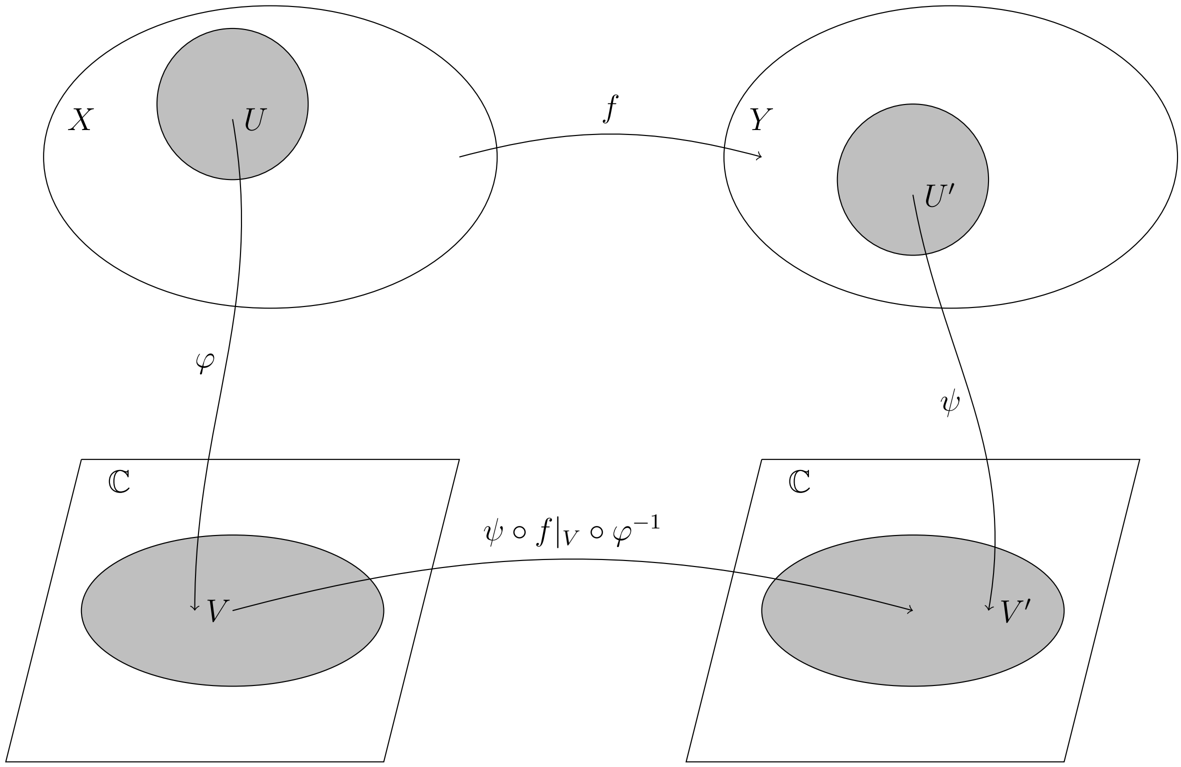
Up to now we defined

- Riemann surfaces;
- for a RS X the sheaf $\mathcal{O}_X$ of holomorphic functions on X (sheaf of $\mathbb{C}$-algebras).

In other words, we defined the objects we are going to study.

In order to be able to "compare" the objects, one usually needs morphisms (maps) between them.

Definition 2.2. 1) Let X and Y be RS. Then a map $f : X \to Y$ is called holomorphic if for every charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with $f(U) \subset U'$ the composition $\psi \circ f|_U \circ \varphi^{-1} : V \to V'$ is a holomorphic map.



2) Equivalently, the map f is holomorphic if for every open $U \subset Y$ and for every $h \in \mathcal{O}_Y(U)$ the function $f^*h := h \circ f : f^{-1}(U) \to \mathbb{C}$ belongs to $\mathcal{O}_X(f^{-1}U)$.

Exercise. Prove the equivalence of the statements of Definition 2.2.

Convention. Holomorphic maps of RS and morphisms of RS are just different names for the same notion.

Remark 2.3. It follows that the composition of morphisms is a morphism as well. Therefore, Riemann surfaces constitute a full subcategory in the category of complex manifolds.

Theorem 2.4 (Identity theorem)**.** *Let X, Y be RS, let $f_1, f_2 : X \to Y$ be two morphisms.Let $A \subset X$ be a subset such that A contains a limit point a of itself. If $f_1|_A = f_2|_A$, then $f_1 = f_2$.*

Proof. Let $S \subset X$ be the set of points $x \in X$ that have an open neighbourhood $U \ni x$ such that $f_1|_U = f_2|_U$. Then S is open by the construction. Note that $S \neq \emptyset$. Indeed, by the identity theorem for $\mathbb{C}$, $a \in S$. Our idea is to show that S is closed. Then by the connectedness of X either $S = X$ or $S = \emptyset$, hence $S = X$ and $f_1 = f_2$.

So, let b be a limit point of S. Then by the continuity of f_1 and f_2 we conclude that $f_1(b) = f_2(b)$. By the identity theorem for $\mathbb{C}$ we conclude that f_1 and f_2 equal in a neighbourhood of b, hence $b \in S$, which demonstrates that S is closed. □

Example 2.5 (Examples of morphism of RS)**.** 1) The quotient map $\mathbb{C} \to \mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$, is a holomorphic map.

2) Let Γ and Γ' be two lattices in $\mathbb{C}$. Let $\alpha \in \mathbb{C}^*$ and assume that $\alpha \cdot \Gamma \subset \Gamma'$. Then the map

$$\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma', \quad [z] \mapsto [\alpha z],$$

is a well-defined holomorphic map. Moreover, it is an isomorphism if and only if $\alpha \cdot \Gamma = \Gamma'$.

3) The map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by

$$z \mapsto \begin{cases} \frac{1}{z}, z \notin \{0, \infty\}, \\ 0, z = \infty, \\ \infty, z = 0 \end{cases}$$

is a holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

4) Consider two submanifolds X_3 and X_2 of $\mathbb{C}^2$ from Exercise 3. The map

$$X_3 \to X_2, \quad (z_1, z_2) \mapsto (z_2^2, z_2)$$

is a morphism of RS.

Definition 2.6 (Meromorphic functions)**.** 1) Let X be a RS. Let $Y \subset X$ be an open subset. A meromorphic function on Y is by definition a holomorphic function on $Y \setminus P$, where $P \subset Y$ is a subset of isolated points and and for every $p \in P$ the limit $\lim\limits_{x \to p} |f(x)|$ exists and equals ∞.

2) The points of P are called the poles of f.

3) $\mathcal{M}_X(Y)$ denotes the set of meromorphic functions on $Y \subset X$.

Exercise. Let X be a Riemann surface and let Y be an open subset in X. Check that the set $\mathcal{M}_X(Y)$ of meromorphic functions on Y has a natural structure of a $\mathbb{C}$-algebra and $\mathcal{O}_X(Y)$ is naturally included in $\mathcal{M}_X(Y)$ as a $\mathbb{C}$-subalgebra. This also defines a structure of an $\mathcal{O}_X(Y)$-module on $\mathcal{M}_X(Y)$.

Example 2.7. 1) Consider $Y = \mathbb{C} = \hat{\mathbb{C}} \setminus \{\infty\}$ as an open subset of $\hat{\mathbb{C}}$ and let f be the identity function of $\mathbb{C} \to \mathbb{C}$, $z \mapsto z$. Then f is a holomorphic function on Y. Since $\lim\limits_{z \to \infty} |f(z)| = \lim\limits_{z \to \infty} |z| = \infty$, we conclude that $\mathrm{id}_{\mathbb{C}}$ can be seen as an element of $\mathcal{M}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$.

2) Let $f \in \mathbb{C}[z]$ be a polynomial in one variable. One can consider it as a function on $\mathbb{C}$. This function is holomorphic. Using arguments similar to the previous ones, one concludes that every polynomial in one variable $f(z) \in \mathbb{C}[z]$ can be seen as an element of $\mathcal{M}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$.

Theorem 2.8. *Let X be a RS. There is a $1:1$ correspondence*

$$\mathcal{M}_X(X) \longleftrightarrow \{\textit{morphisms } X \to \hat{\mathbb{C}} \textit{ not identically } \infty\}.$$

Proof. "→". Let $f \in \mathcal{M}_X(X)$. Let P be the set of poles of f. Define $\hat{f} : X \to \hat{\mathbb{C}}$ by

$$\hat{f}(z) = \begin{cases} f(z), z \notin P \\ \infty, \text{otherwise.} \end{cases}$$

Then $\hat{f}$ is a continuous map (notice that it is enough to check it at poles). So by Riemann removable singularity theorem $\hat{f}$ is holomorphic.

"←". Consider $g : X \to \hat{\mathbb{C}}$. If the set $g^{-1}(\infty)$ contains a limit point, by identity theorem $g(z) = \infty$ for all $x \in X$, therefore $g^{-1}(\infty)$ does not contain limit points and hence it is a subset of isolated points. Denote $f = g|_{X \setminus g^{-1}(\infty)} : X \setminus g^{-1}(\infty) \to \mathbb{C}$. This is a holomorphic function on $X \setminus g^{-1}(\infty)$. For every $p \in g^{-1}(\infty)$ one checks $\lim_{z \to p} |f(z)| = \infty$. This means $f \in \mathcal{M}_X(X)$.

One sees that the constructed maps are inverse to each other. □

Corollary 2.9. *Non-trivial (non-zero) meromorphic functions may have only isolated zeroes and poles.*

Proof. Note that the poles of meromorphic function are isolated by definition.

Assume a is a non-isolated zero of $f \in \mathcal{M}_X(X)$, i. e., there exists a sequence a_i with $\lim_{i \to \infty} a_i = a$ such that $f(a_i) = 0$, $f(a) = 0$. Then by the identity theorem $\hat{f} = 0$ as a morphism $X \to \hat{\mathbb{C}}$. Therefore, $f = 0$. □

Claim. $\mathcal{M}_X(Y)$ *is a field.*

Proof. If $f \in \mathcal{M}_X(Y)$ such that $f \neq 0$, then $\dfrac{1}{f} \in \mathcal{M}_X(Y)$ as well since the zeroes of f become the poles of $\dfrac{1}{f}$. □

Example 2.10. As mentioned in Example 2.7, polynomials in one variable can be seen as meromorphic functions on $\hat{\mathbb{C}}$. By the Claim above we conclude that every rational function in one variable $\frac{f(z)}{g(z)}$, $f, g \in \mathbb{C}[z]$, $g \not\equiv 0$, can be seen as a meromorphic function on $\hat{\mathbb{C}}$ as well. So the field of the rational functions in one variable

$$\mathbb{C}(z) := \left\{ \frac{f(z)}{g(z)} \mid f, g \in \mathbb{C}[z] \text{ (polynomials in } z), g \not\equiv 0 \right\}$$

is a subfield in $\mathcal{M}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$.

Exercise. Show that every meromorphic function on $\hat{\mathbb{C}}$ is rational, i. e., $\mathcal{M}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$ coincides with $\mathbb{C}(z)$.

2.2. **Exercises.**

Exercise 5 (Examples of morphisms of Riemann surfaces)**.** Check using the definition of a holomorphic map that the following maps between Riemann surfaces are holomorphic.

1) The quotient map $\mathbb{C} \to \mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$, is a holomorphic map.

2) Let Γ and Γ' be two lattices in $\mathbb{C}$. Let $\alpha \in \mathbb{C}^*$ and assume that $\alpha \cdot \Gamma \subset \Gamma'$. Then the map

$$\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma', \quad [z] \mapsto [\alpha z],$$

is a well-defined holomorphic map. Moreover, it is an isomorphism if and only if $\alpha \cdot \Gamma = \Gamma'$.

3) The map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by

$$z \mapsto \begin{cases} \frac{1}{z}, z \notin \{0, \infty\}, \\ 0, z = \infty, \\ \infty, z = 0 \end{cases}$$

is a holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

4) Consider two submanifolds X_3 and X_2 of $\mathbb{C}^2$ from Exercise 3.

$$X_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid 3z_1 - 14z_2^2 = 0\}, \quad X_3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 - 1 = 0\},$$

The map

$$X_3 \to X_2, \quad (z_1, z_2) \mapsto \left(\frac{14}{3} z_2^2, z_2\right)$$

is a morphism of RS.

Exercise 6. Show that the set of meromorphic functions on $\hat{\mathbb{C}}$ coincide with the set of rational functions

$$\left\{\frac{f(z)}{g(z)} \mid f, g \in \mathbb{C}[z] \text{ (polynomials in } z), g \not\equiv 0\right\}.$$

Hint: One could follow the following steps. Let F, $F \not\equiv 0$, be a meromorphic function on $\hat{\mathbb{C}}$.

- Note that F has only finitely many zeros and poles.
- There are two possibilities: ∞ is either a pole of F or not.
- If ∞ is not a pole of F, consider the poles $a_1, \ldots, a_n$ of F. Consider the principal parts h_ν of F at a_ν, $\nu = 1, \ldots, n$, and observe that $F - \sum_{\nu=1}^{n} h_\nu$ is a holomorphic function on $\hat{\mathbb{C}}$. So it must be constant and hence F is a rational function.
- If ∞ is a pole of F, consider the function $\frac{1}{F}$ and show as above that it is rational.

Exercise 7. Let Γ be a lattice in $\mathbb{C}$. Then a meromorphic function $f \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ is called doubly periodic (or elliptic) with respect to Γ if $f(z) = f(z+\gamma)$ for all $z \in \mathbb{C}$ and for all $\gamma \in \Gamma$.

1) Show that there is a one-to-one correspondence between elliptic functions on $\mathbb{C}$ with respect to Γ and meromorphic functions on $\mathbb{C}/\Gamma$.

2) Show that there are only constant holomorphic doubly periodic functions.

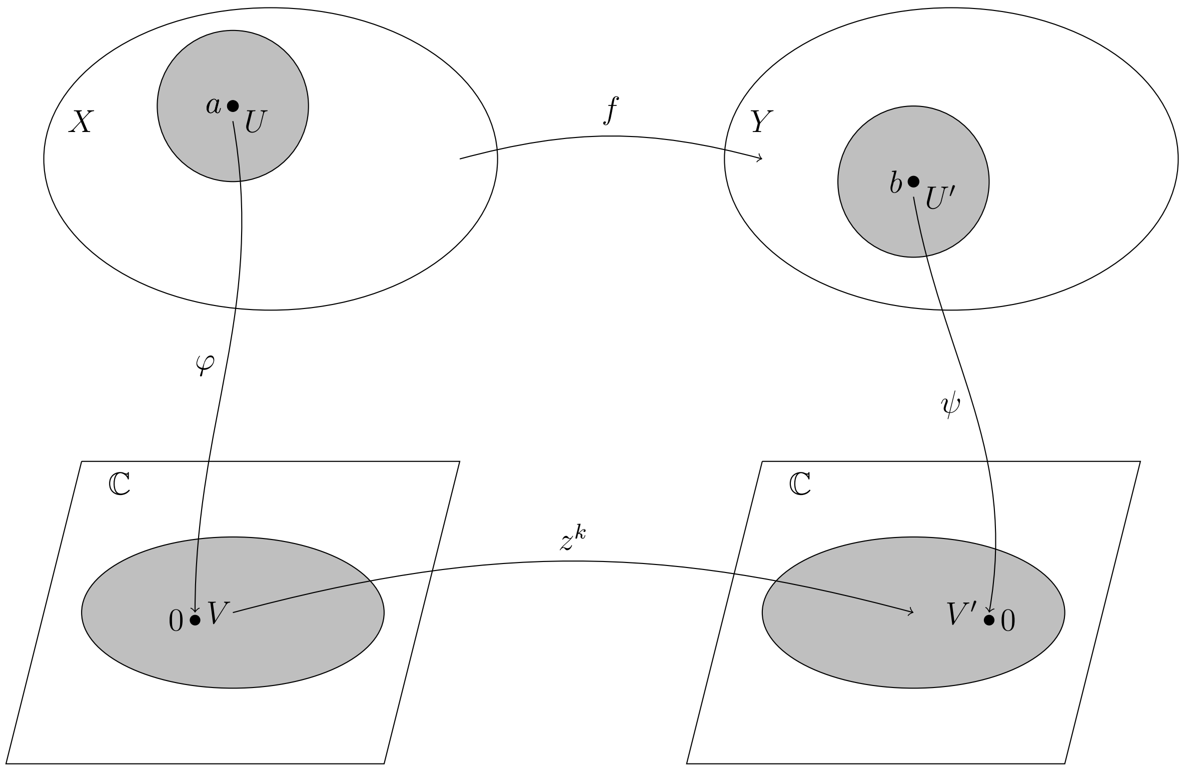
3. Lecture 3

Let us study the local behaviour of holomorphic maps of Riemann surfaces.

Theorem 3.1 (Local behaviour of holomorphic maps)**.** *Let X, Y be RS. Let $f : X \to Y$ be a non-constant holomorphic map. Let $a \in X$, $b := f(a) \in Y$. Then there exists an integer $k \geqslant 1$ such that locally around a the morphism f looks as*

$$z \mapsto z^k,$$

i. e., there exist a chart $U \xrightarrow{\varphi} V$, $a \in U$, $\varphi(a) = 0$, and a chart $U' \xrightarrow{\psi} V'$, $b \in U'$, $\psi(b) = 0$, such that $f(U) \subset U'$ and $\psi \circ f|_U \circ \varphi^{-1}(z) = z^k$.



Proof. There exists a chart $\psi : U' \to V'$ around b such that $\psi(b) = 0$. Then $f^{-1}(U')$ is open and contains a.

There exists a chart around a mapping a to 0. Intersecting with $f^{-1}(U')$ we obtain a chart $\widetilde{U} \xrightarrow{\widetilde{\varphi}} \widetilde{V}$ such that $f(\widetilde{U}) \subset U'$ and $\widetilde{\varphi}(a) = 0$.

Consider $\widetilde{F} := \psi f \widetilde{\varphi}^{-1} : \widetilde{V} \to V'$. Since $\widetilde{F}(0) = 0$, one can write $\widetilde{F}$ as $\widetilde{F}(z) = z^k \cdot \widetilde{G}(z)$, $\widetilde{G}(z) \neq 0$ in a neighbourhood W of 0. Since $\widetilde{G}(0) \neq 0$, shrinking W if necessary we may assume that there exists a holomorphic function H on W such that $H^k(z) = \widetilde{G}(z)$. Indeed, shrinking W if necessary we may assume that there exists a branch of the complex logarithmic function defined around $\widetilde{G}(W)$. Then $H(z) := \exp(\frac{1}{k} \ln \widetilde{G}(z))$ has the required property.

We obtain $\widetilde{F}(z) = z^k \cdot H^k(z) = (zH(z))^k$. Consider $\xi : W \to V'$, $z \mapsto zH(z)$. It is a biholomorphic map between W (possibly after shrinking W) and some neighbourhood of 0 in V'. Consider $\varphi : \widetilde{\varphi}^{-1}(W) \xrightarrow{\widetilde{\varphi}} W \xrightarrow{\xi} V'$. Then $\psi f \varphi^{-1}(z) = \psi f \widetilde{\varphi}^{-1}\xi^{-1}(z) = \widetilde{F}(\xi^{-1}(z)) = (\xi^{-1}(z)H(\xi^{-1}(z)))^k = (\xi(\xi^{-1}(z)))^k = z^k$. □

Definition 3.2. The number k from the previous theorem is uniquely determined for a given holomorphic map f and a given point $a \in X$. It is called the multiplicity of f at the point a and will be denoted by $\operatorname{mult}_a f$.

Exercise. Prove that $\operatorname{mult}_a f$ is well defined.

Remark 3.3 (Geometrical meaning of $\operatorname{mult}_a f$)**.** In every neighbourhood U_0 of a there exist a neighbourhood $U \ni a$ and a neighbourhood $W \ni b$ such that for every $y \in W \setminus \{b\}$

$$\# f^{-1}(y) \cap U = k,$$

i. e., U contains exactly k preimages of y.

Remark 3.4 (Computation of $\operatorname{mult}_a f$)**.** Note that in order to compute the multiplicity of a holomorphic map at a point it is enough just to go through the first part of the proof of Theorem 3.1 and to find the decomposition $\widetilde{F}(z) = z^k \widetilde{G}(z)$, $\widetilde{G}(0) \neq 0$.

Example 3.5. 1) Let f be the identity map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then $\operatorname{mult}_a f = 1$ for every $a \in \hat{\mathbb{C}}$ because f is bijective. Analogously, since $\hat{\mathbb{C}} \xrightarrow{g} \hat{\mathbb{C}}$, $g(z) = \frac{1}{z}$, is bijective, we conclude that $\operatorname{mult}_a f = 1$ for every $a \in \hat{\mathbb{C}}$.

2) Let $\hat{\mathbb{C}} \xrightarrow{f} \hat{\mathbb{C}}$ be given by $f(z) = \frac{1}{z^3}$. Then $\operatorname{mult}_0 f = 3$ and $\operatorname{mult}_i f = 1$.

Exercise. Let $f(z) \in \mathbb{C}[z]$ be a polynomial of degree k. This gives the holomorphic map

$$\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f(z) = \begin{cases} f(z), & z \in \mathbb{C} \\ \infty, & z = \infty. \end{cases}$$

Show that $\hat{f}$ has multiplicity k at ∞. What is the multiplicity of $\hat{f}$ at 0?

Corollary 3.6. *Every non-constant holomorphic map of RS $f : X \to Y$ is open.*

Proof. f is locally $z \mapsto z^k$, which is open. Since being open is a local property, f is open. □

Corollary 3.7. *Let $f : X \to Y$ be an injective morphism of RS. Then $f : X \to f(X)$ is biholomorphic.*

Proof. Injectivity implies that f is locally $z \mapsto z$. Then the inverse of f is locally $z \mapsto z$ and hence it is holomorphic. □

Corollary 3.8 (Maximum principle)**.** *Let $f \in \mathcal{O}_X(X)$ be non-constant. Then $|f|$ does not have maximum on X.*

Proof. Suppose that $|f|$ has maximum on X. Then there exists $a \in X$ such that

$$|f(a)| = \sup_{x \in X} |f(x)| =: M.$$

Consider $K := \{z \in \mathbb{C} \mid |z| \leqslant M\} \subset \mathbb{C}$. K is compact. Then $f(X) \subset K$, in particular $f(a) \in K$. Therefore, $f(a) \in \partial K$ (boundary of K). Since $f(X)$ is open, $f(a)$ must be contained in K with some neighbourhood. This is a contradiction. Hence our assumption was false and $|f|$ does not have maximum on X. □

Theorem 3.9. *Let $X \xrightarrow{f} Y$ be a non-constant morphism of RS. Let X be compact. Then f is surjective and Y is compact as well.*

Proof. Since $f(X)$ is open and compact it is open and closed. Therefore, $f(X) = Y$ since Y is connected. □

Exercise. Let Γ be a lattice in $\mathbb{C}$. Show that every non-constant elliptic function with respect to Γ attains every value $b \in \hat{\mathbb{C}}$.

Corollary 3.10. *Let X be a compact RS. Then $\mathcal{O}_X(X) = \mathbb{C}$.*

Proof. Let $f \in \mathcal{O}_X(X)$ and consider it as a holomorphic map $X \xrightarrow{f} \mathbb{C}$. If f is non-constant, then $\mathbb{C}$ must be compact, which is wrong. So f is a constant function. □

Remark 3.11. As we saw in Exercise 6 this implies that every meromorphic function on $\hat{\mathbb{C}}$ is rational.

3.1. **Exercises.**

Exercise 8. Let $X \xrightarrow{f} Y$ be a non-constant holomorphic map of Riemann surfaces and let $a \in X$. Show that the multiplicity of f at a is uniquely determined, i. e., does not depend on the choice of local charts.

Hint: *Notice that $k = \operatorname{mult}_a f$ can be thought of as the smallest k such that the k-th derivative of $F = \psi \circ f \circ \varphi^{-1}$ does not vanish at 0, where φ is a chart around a and ψ is a chart around $b = f(a)$.*

Exercise 9. Let $f(z) \in \mathbb{C}[z]$ be a polynomial of degree k. This gives a holomorphic map $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\hat{f}(\infty) = \infty$. Show that $\hat{f}$ has multiplicity k at ∞. What is the multiplicity at 0?

Exercise 10. 1) Consider the holomorphic map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^k$, where k is a positive integer. Compute $\operatorname{mult}_a f$ for an arbitrary $a \in \mathbb{C}$.

2) Consider the holomorphic map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = (z-1)^3(z-2)^7$. Compute $\operatorname{mult}_a f$ for an arbitrary $a \in \mathbb{C}$.

Exercise 11. Let $\hat{\mathbb{C}} \xrightarrow{f} \hat{\mathbb{C}}$ be a holomorphic map given by

$$f(z) = \frac{(z-3)^3}{(z+1)(z-2)^2}.$$

Compute $\operatorname{mult}_3 f$, $\operatorname{mult}_{-1} f$, $\operatorname{mult}_2 f$, $\operatorname{mult}_1 f$.

4. Lecture 4

Definition 4.1 (Elliptic functions[1])**.** Let Γ be a lattice in $\mathbb{C}$. Then a meromorphic function $f \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ is called doubly periodic (or elliptic) with respect to Γ if $f(z) = f(z+\gamma)$ for all $z \in \mathbb{C}$ and for all $\gamma \in \Gamma$.

Claim. *There is a one-to-one correspondence between elliptic functions on $\mathbb{C}$ with respect to Γ and meromorphic functions on $\mathbb{C}/\Gamma$. In particular there are only constant doubly periodic holomorphic functions on $\mathbb{C}$.*

Proof. Every elliptic function $f : \mathbb{C} \to \hat{\mathbb{C}}$ uniquely factorizes through the canonical projection $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma$ and hence defines a holomorphic map $\mathbb{C}/\Gamma \to \hat{\mathbb{C}}$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad f \quad} & \hat{\mathbb{C}} \\ & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle \hat{f}} & \\ & \mathbb{C}/\Gamma & \end{array}$$

Every holomorphic map $\hat{f} : \mathbb{C}/\Gamma \to \hat{\mathbb{C}}$ defines $f = \hat{f} \circ \pi$.

This gives the required one-to-one correspondence. ☐

Exercise. Try to invent a non-trivial elliptic function with respect to a given lattice.

Definition 4.2. Let X be a topological space. Then a path in X is a continuous map $\gamma : [0,1] \to X$. The point $\gamma(0)$ is called the initial point of γ, the point $\gamma(1)$ is called the end point of γ.

If $\gamma(0) = \gamma(1)$, then γ is called a closed path.

Definition 4.3. A topological space X is called path-connected if every two points $a, b \in X$ can be connected by a path.

Reminder 4.4. Path connectedness implies connectedness.

Exercise. Riemann surfaces are path connected.

Definition 4.5. Two paths γ, δ from a to b are called homotopic if there exists a continuous map

$$H : [0,1] \times [0,1] \to X$$

such that

$$H(t,0) = \gamma(t), \quad H(t,1) = \delta(t) \quad \text{for all } t \in [0,1]$$

[1] cf. Exercise 7

$$H(0,s)=a,\quad H(1,s)=b\quad \text{for all } s\in[0,1].$$

One writes $\gamma\sim\delta$ if γ and δ are homotopic.

Claim. *Homotopy is an equivalence relation on the set of all paths from a to b.*

Definition 4.6 (Composition)**.** Let X be a topological space. Let γ be a path from a to b. Let δ be a path from b to c. Define

$$(\gamma\cdot\delta)(t)=\begin{cases}\gamma(2t), & t\in[0,\frac{1}{2}]\\ \delta(2t-1), & t\in[\frac{1}{2},1].\end{cases}$$

Definition 4.7 (Inverse curve)**.** Let X be a topological space. Let γ be a path from a to b. Define

$$\gamma^{-1}(t)=\gamma(1-t),\quad t\in[0,1].$$

Claim. *The composition of paths and the inverse path are compatible with the homotopy equivalence, i. e., if $\gamma\sim\gamma'$, $\delta\sim\delta'$, and if $\gamma\cdot\delta$, $\gamma'\cdot\delta'$ are well-defined, then*

$$\gamma\cdot\delta\sim\gamma'\cdot\delta',\quad \textit{and}\quad \gamma^{-1}\sim\gamma'^{-1}.$$

Definition-Theorem 4.8 (Fundamental group)**.** Let $x_0\in X$. Let $\pi_1(X,x_0)$ denote the set of the homotopy classes of of closed paths from x_0 to x_0. Let $[\gamma]$ denote the homotopy class of γ. Let $[x_0]$ denote the homotopy class of the constant path

$$[0,1]\to X,\quad t\mapsto x_0.$$

Then $\pi_1(X,x_0)$ is a group with respect to the multiplication

$$[\gamma]\cdot[\delta]:=[\gamma\cdot\delta],$$

the constant class $[x_0]$ is the identity element with respect to this multiplication, for a class $[\gamma]$ its inverse is given by $[\gamma]^{-1}=[\gamma^{-1}]$.

$\pi_1(X,x_0)$ is called the fundamental group of X with respect to the base point x_0.

Proof. Exercise. □

Claim. *If $a,b\in X$ are connected by a path $\delta:[0,1]\to X$, then the map*

$$\pi_1(X,a)\to\pi_1(X,b),\quad [\gamma]\mapsto[\delta^{-1}\cdot\gamma\cdot\delta]$$

is an isomorphism of groups.

Proof. Exercise. □

Remark 4.9. Note that the isomorphism above depends on δ. It does not depend on δ if and only if $\pi_1(X, a)$ is an abelian group.

Definition 4.10. A path-connected topological space X is called simply-connected if $\pi_1(X, a)$ is trivial for some(equivalently: for every) $a \in X$. By abuse of notation we write $\pi_1(X, a) = 0$ to say that $\pi_1(X, a)$ is trivial.

Remark 4.11. 1) The fundamental group is functorial. Namely, every continuous map $f : X \to Y$, induces a homomorphism of groups

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), \quad [\gamma] \mapsto f_*([\gamma]) := [f \circ \gamma]$$

such that for two continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

it holds

$$(g \circ f)_* = g_* \circ f_*.$$

2) In particular this implies that homeomorphic path-connected topological spaces have isomorphic fundamental groups. Therefore, $\pi_1(X, a)$ (its isomorphism class to be more precise) is a topological invariant.

Claim. *Two non-homeomorphic compact RS have different fundamental groups.*

Explanation. Compact RS are orientable compact 2-dimensional real manifolds, i. e, surfaces. The latter are completely classified up to a homeomorphism.

Namely, for every non-negative integer p there is exactly one homeomorphism class.

For $p = 0$, $X \cong \hat{\mathbb{C}} \cong \mathbb{S}^2$, the corresponding fundamental group $\pi(X)$ is trivial.

For $p \geqslant 1$, the fundamental group of X can be described as

$$\pi_1(X) \cong \langle a_1, \ldots, a_p, b_1, \ldots, b_p \mid \prod_i a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle.$$

We will discuss it in more details in the next lecture. □

4.1. **Exercises.**

Exercise 12. Show that Riemann surfaces are path-connected.

Hint: *For a point x_0 of a Riemann surface X consider the set S of all points that can be connected with x_0 by a path. Show that S is non-empty, closed and open.*

Exercise 13. 1) Let a and b be two points in a topological space X. Check that the homotopy is an equivalence relation on the set of all curves from a to b.

2) Fill in the gaps and check the technical details in the definition of the fundamental group from the lecture. You may consult the *Algebraic topology* book of Allen Hatcher [8].

Exercise 14. 0) Let X be an open disc in $\mathbb{C}$ of radius 1 with centre at zero. Show that $\pi_1(X, 0) = 0$.

1) Show that the fundamental group of $\hat{\mathbb{C}}$ is trivial. Consult the *Algebraic topology* book of Allen Hatcher [8] for some technical details.

2) Compute the fundamental group of of a complex torus $\mathbb{C}/\Gamma$. Use that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ and the fact that the fundamental group of the product of two path-connected topological spaces X and Y is naturally isomorphic to the product of the corresponding fundamental groups:

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y).$$

Exercise 15. The so called uniformization theorem states that up to an isomorphism there are only 3 simply-connected Riemann surfaces, namely $\hat{\mathbb{C}}$, $\mathbb{C}$, and the open disc in $\mathbb{C}$.

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half plane. Show that $\mathbb{H}$ is simply-connected and find out to which isomorphism class it belongs.

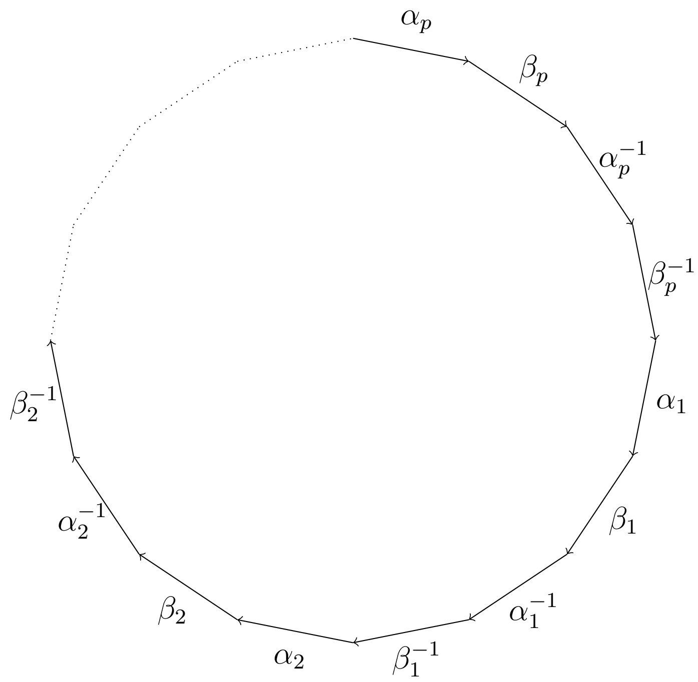
Hint: *It may help looking at the meromorphic function $\frac{z-i}{iz-1}$ on $\mathbb{C}$.*

5. Lecture 5

Last time we claimed that for every non-negative integer p there is exactly one homeomorphism class of 2-dimensional real oriented compact connected manifolds.

Explanation. For $p = 0$, $X \cong \hat{\mathbb{C}} \cong \mathbb{S}^2$, the corresponding fundamental group $\pi(X)$ is trivial.

For $p \geqslant 1$, X is obtained as a result of gluing of a regular $4p$-gon along its sides as shown in the following picture.



Each edge can be seen as a path on a plain. The initial and the end points are indicated by arrows. For every i one glues together inverting the orientations the edges α_i with the edges α_i^{-1} and the edges β_i with the edges β_i^{-1}.

This means that the initial point of the edge labeled by α_i or β_i is glued together with the end point of the edge labeled α_i^{-1} or β_i^{-1} respectively.

Analogously, the end point of the edge labeled by α_i or β_i is glued together with the initial point of the edge labeled α_i^{-1} or β_i^{-1} respectively.

The images of $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p$ in X are denoted by abuse of notations by the same symbols. Then the path α_i^{-1} is indeed the inverse path to α_i and the path β_i^{-1} is indeed the inverse path to β_i. Notice that each of these paths becomes a closed path at the same point (the one obtained by gluing all the vertices of the $4p$-gon).

The fundamental group of X is generated by

$$\{[\alpha_1], \ldots, [\alpha_p], [\beta_1], \ldots, [\beta_p]\}$$

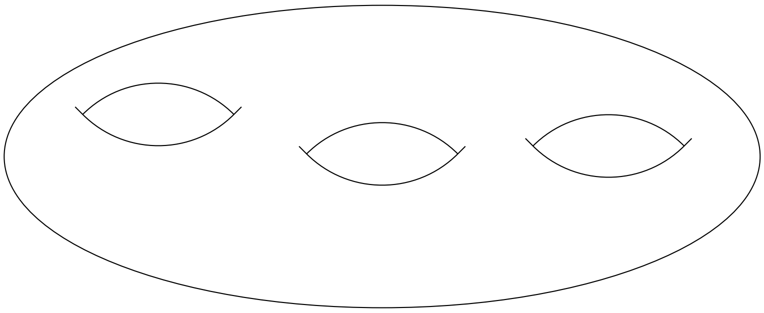
with the only relation

$$\prod_i [\alpha_i][\beta_i][\alpha_i]^{-1}[\beta_i]^{-1} = 1,$$

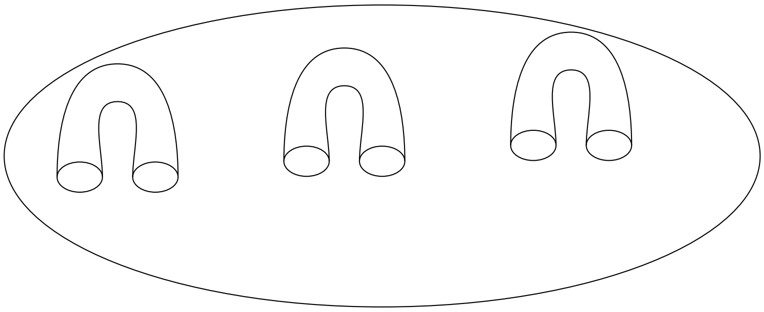
i. e.,

$$\pi_1(X) \cong \langle a_1, \dots, a_p, b_1, \dots, b_p \mid \prod_i a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle.$$

In this case X is homeomorphic to a pretzel with p holes



or equivalently to a sphere with p handles.



The relation between the generators mentioned above can be understood in the following way. Let P denote the regular $4p$-gon on a plane mentioned above. Consider the closed path

$$\gamma = \alpha_1 \cdot \beta_1 \cdot \alpha_1^{-1} \cdot \beta_1^{-1} \cdot \alpha_2 \cdot \beta_2 \cdot \alpha_2^{-1} \cdot \beta_2^{-1} \cdot \ldots \cdot \alpha_p \cdot \beta_p \cdot \alpha_p^{-1} \cdot \beta_p^{-1}$$

Then it is contractible (in P), i. e., homotopic to a constant path.

Let X be the topological space obtained as a gluing of the edges of P as explained above. Consider the corresponding quotient map $P \to X$, which is continuous by the definition of quotient topology. By composing the homotopy contracting γ to a constant path with the quotient map $P \to X$ we conclude that the image of γ in X is contractible as well, which gives $\prod_i [\alpha_i][\beta_i][\alpha_i]^{-1}[\beta_i]^{-1} = 1$. □

Exercise. Compute $\pi_1(\hat{\mathbb{C}})$, $\pi_1(\mathbb{C}/\Gamma)$, where $\Gamma \subset \mathbb{C}$ is a lattice.

Definition 5.1. Let $f : X \to Y$ be a non-constant holomorphic map. Then $x \in X$ is called a ramification point of f if there is no neighborhood U of x such that $f|_U$ is injective.

One says that f is unramified if it has no ramification points.

Remark 5.2. Ramification points are those with multiplicities $\operatorname{mult}_x f > 1$. This follows immediately from Theorem 3.1.

Corollary 5.3. *A non-constant holomorphic map of RS $f : X \to Y$ is unramified if and only if it is a local homeomorphism.*

Example 5.4. 1) $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^k$. Here 0 is the only ramification point.

2) $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ is unramified.

3) The standard projection $\mathbb{C} \to \mathbb{C}/\Gamma$ is unramified for every lattice $\Gamma \subset \mathbb{C}$.

Theorem 5.5. *Let $f : X \to Y$ be a non-constant holomorphic map of compact RS. Then for every $y \in Y$ its preimage $f^{-1}(y)$ is a finite set and the number*

$$d_y(f) := \sum_{x \in f^{-1}(y)} \operatorname{mult}_x f$$

does not depend on y.

Corollary 5.6. *If $Y = \hat{\mathbb{C}}$, then $f : X \to \hat{\mathbb{C}}$ is a meromorphic function and the number of zeroes of f is equal to the number of poles of f (counted with multiplicities).*

Definition 5.7. In the notations of Theorem 5.5 the number $d(f) := d_y(f)$ (for some/every $y \in X$) is called the degree of $f : X \to Y$.

Example 5.8. Consider the meromorphic function $f(z) = \frac{(z-2)}{(z-3)^2(z-7)^3}$ on $\hat{\mathbb{C}}$. Let us compute the number of zeroes of this function with multiplicities and thus the degree of the corresponding holomorphic map $\hat{\mathbb{C}} \xrightarrow{\hat{f}} \hat{\mathbb{C}}$.

Note that $\hat{f}^{-1}(0) = \{2, \infty\}$. Since

$$f(z) = (z-2) \cdot \frac{1}{(z-3)^2(z-7)^3}$$

and since $\dfrac{1}{(z-3)^2(z-7)^3}$ does not vanish at $z = 2$, one concludes

$$\operatorname{mult}_2 \hat{f} = 1.$$

Since

$$f(z) = \left(\frac{1}{z}\right)^4 \cdot \frac{z^4(z-2)}{(z-3)^2(z-7)^3}$$

and $\frac{z^4(z-2)}{(z-3)^2(z-7)^3}$ does not vanish at ∞, we get

$$\operatorname{mult}_\infty \hat{f} = 4.$$

Therefore, $d_0(\hat{f}) = \text{mult}_2 \hat{f} + \text{mult}_\infty \hat{f} = 1 + 4 = 5$ and hence $d(\hat{f}) = 5$.

Notice that the set of poles of f is $\{3, 7\}$. Since $\text{mult}_3(\hat{f}) = 2$ and $\text{mult}_7(\hat{f}) = 3$, we get

$$\text{mult}_3(\hat{f}) + \text{mult}_7(\hat{f}) = 2 + 3 = 5 = 1 + 4 = \text{mult}_2(\hat{f}) + \text{mult}_\infty(\hat{f}),$$

which illustrates the statement of Corollary 5.6.

Corollary 5.9. *Let $f \in \mathcal{M}(\mathbb{C}/\Gamma)$ be a non-constant meromorphic function on a torus. Then f has at least 2 poles (counted with multiplicities).*

Proof. Suppose f has less than 2 poles.

1) If f does not have poles at all, then f is a holomorphic function and hence by Corollary 3.10 f is constant, which is a contradiction.
2) If f has only one pole, then for the corresponding holomorphic map $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$ the point $\infty \in \hat{\mathbb{C}}$ has only one preimage. Therefore, for an arbitrary point $p \in \hat{\mathbb{C}}$

$$\#\hat{f}^{-1}(p) = \#\hat{f}^{-1}(\infty) = 1,$$

which means that $\hat{f} : X \to \hat{\mathbb{C}}$ is a bijection. Hence $\hat{f}$ is an isomorphism of RS (cf. Corollary 3.7 and Theorem 3.9). In particular X and $\hat{\mathbb{C}}$ must be homeomorphic as topological spaces, which is not true, since, for example, they have non-isomorphic fundamental groups.

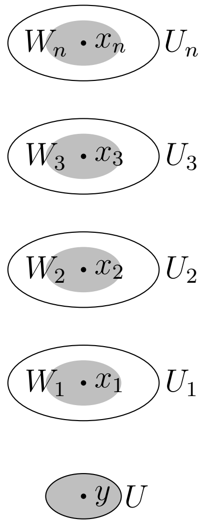
□

Remark 5.10. In fact, we showed even more. Namely, on every compact RS non-isomorphic to $\hat{\mathbb{C}}$, non-constant meromorphic functions must have at least 2 poles.

Proof of Theorem 5.5. First of all notice that $f^{-1}(y)$ must be a discrete set because of the Identity theorem (Theorem 2.4). Since X is compact, it must be finite (again by the Identity theorem). Consider now the function

$$Y \to \mathbb{Z}, \quad y \mapsto d_y(f).$$

We shall show that this function is locally constant. Since Y is connected, it would imply that $d_y(f)$ is a constant function.



Let $y \in Y$. Let $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Put $m_i = \mathrm{mult}_{x_i} f$. For every $i = 1, \ldots, n$, let U_i be an open neighbourhood of x_i such that $f|_{U_i} : U_i \to f(U_i)$ is of the form $z \mapsto z^{m_i}$ (in appropriate charts). Shrinking U_i, we can assume that $U_i \cap U_j = \emptyset$ for $i \neq j$.

Since X is compact, f is a closed map, i. e., the image of a closed set is closed. Therefore, $f(X \setminus \coprod_{i=1}^{n} U_i)$ is closed. Since y lies in its complement, which is open, there exists an open set U, $y \in U$, such that $U \subset Y \setminus f(X \setminus \coprod_{i=1}^{n} U_i)$. This implies that $f^{-1}(U) \subset \coprod_{i=1}^{n} U_i$.

Put $W_i = f^{-1}(U) \cap U_i$, then $f^{-1}(U) = \coprod_{i=1}^{n} W_i$.

For every $p \in U \setminus \{y\}$, and for every $x \in f^{-1}(p)$ the multiplicity $\mathrm{mult}_x f$ equals 1. Therefore, $d_p(f) = \sum_{x \in f^{-1}(p)} \mathrm{mult}_x f = \sum_{i=1}^{n} \#(f^{-1}(p) \cap W_i) = \sum_{i=1}^{n} m_i$.

On the other hand $d_y(f) = \sum_{i=1}^{n} m_i$ as well.

This shows that $d_p(f)$ is constant on U, so it is locally constant and hence constant, which concludes the proof. $\square$

5.1. **Exercises.**

Exercise 16. Compute the degrees $d(\hat{f})$, $d(\hat{g})$ of the holomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ corresponding to the following meromorphic functions on $\hat{\mathbb{C}}$:

$$f(z) = \frac{(z-17)^2}{z^{13}+2}, \quad g(z) = \frac{(z-1)^3}{z^2+11}.$$

Exercise 17. As we already know every meromorphic function f on $\hat{\mathbb{C}}$ is rational, i. e.,

$$f(z) = \frac{P(z)}{Q(z)}, \quad P(z), Q(z) \in \mathbb{C}[z], \quad Q(z) \neq 0.$$

Show that the degree of the corresponding holomorphic map $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ equals

$$\max\{\deg P, \deg Q\}.$$

Exercise 18. Find all ramification points of the morphism $\hat{g}$ from Exercise 16.

Exercise 19. 1) Let a be a complex number. Let f be a meromorphic function on $\hat{\mathbb{C}}$ with the only pole of multiplicity 1 at a. Show that

$$f(z) = \mu + \frac{\lambda}{z-a}$$

for some non-zero complex number λ and some $\mu \in \mathbb{C}$.

2) Consider the meromorphic function $f(z) = \frac{\cos(z)}{z}$ on $\mathbb{C}$. Find all zeroes and poles of f and the corresponding multiplicities. Compare your results with the statements from the last lecture.

6. Lecture 6

6.1. **Divisors.** Let X be a compact RS.

Definition 6.1. Let $\mathrm{Div}(X)$ be the free abelian group generated by the points of X. It is called the divisor group of X.

Elements of $\mathrm{Div}(X)$ are linear combinations

$$\sum_{x\in X} n_x \cdot x, \quad n_x \in \mathbb{Z}, \quad \text{finitely many } n_x \neq 0.$$

For a divisor

$$D = \sum_{x\in X} n_x \cdot x$$

let $D(x) := n_x$. This way, one can identify divisors with the functions $X \to \mathbb{Z}$ with finite support.

Let $\deg D = \sum_{x\in X} n_x$ be the degree of D.

Notice that

$$\deg : \mathrm{Div}\, X \to \mathbb{Z}, \quad D \mapsto \deg D$$

is a group homomorphism. Its kernel consists of all divisors of degree zero and is denoted by $\mathrm{Div}^0(X)$.

Let $f \in \mathcal{M}_X(X)$ be a non-zero meromorphic function. Identify f with the corresponding holomorphic map $X \to \hat{\mathbb{C}}$ and for $p \in X$ define

$$\mathrm{ord}_p f := \begin{cases} \mathrm{mult}_p f, & \text{if } f(p) = 0 \\ -\mathrm{mult}_p f, & \text{if } f(p) = \infty \\ 0, & \text{otherwise.} \end{cases}$$

Notice that this definition implies $\mathrm{mult}_p \lambda = 0$ for a non-zero constant function $\lambda \in \mathbb{C}^*$. It is useful to put $\mathrm{ord}_p 0 = \infty$.

The number $\mathrm{ord}_p f$ is called the order of p with respect to f. So the points with positive order are zeros of f, the points with negative order are poles of f, and the points with zero order are neither zeroes nor poles of f.

Definition 6.2. For a meromorphic non-zero function $f \in \mathcal{M}_X(X)$ put

$$(f) := \sum_{x\in X} (\mathrm{ord}_x f) \cdot x \in \mathrm{Div}\, X.$$

Divisors of this form are called principal divisors.

Remark 6.3. Notice that (f) keeps all the information about the zeroes and the poles of f.

Observation. $(f \cdot g) = (f) + (g)$, $(1/f) = -(f)$.

Therefore, the set of the principal divisors is a subgroup in $\operatorname{Div} X$, it is denoted by $\operatorname{PDiv} X$. Since by Theorem 5.5 $d_0(f) = d_\infty(f)$, we conclude that $\deg(f) = 0$ for every meromorphic function f on X. Therefore, $\operatorname{PDiv} X$ is a subgroup of $\operatorname{Div}^0(X)$ and we have an inclusion of groups

$$\operatorname{PDiv} X \subset \operatorname{Div}^0 X \subset \operatorname{Div} X.$$

The quotient group

$$\operatorname{Pic}(X) := \operatorname{Div} X / \operatorname{PDiv} X$$

is called the Picard group of X. Its elements are called divisor classes.

The group

$$\operatorname{Pic}^0(X) := \operatorname{Div}^0 X / \operatorname{PDiv} X,$$

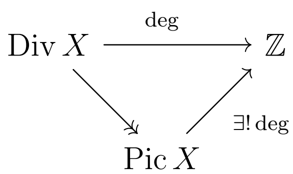
which is a subgroup of $\operatorname{Pic} X$, is called the restricted Picard group.

We say that two divisors D and D' are linearly equivalent and write $D \sim D'$ if D and D' represent the same element in $\operatorname{Pic} X$, i. e., if $D - D' = (f)$ for some meromorphic function f.

Since $\operatorname{PDiv} X$ lies in the kernel of the degree homomorphism, we get a factorization homomorphism

$$\operatorname{Pic} X \to \mathbb{Z}, \quad [D] \mapsto \deg D,$$

which is denoted (by abuse of notation) by deg as well.



Let $D, D' \in \operatorname{Div} X$. Then we say $D \geqslant D'$ or $D' \leqslant D$ if

$$D(x) \geqslant D'(x) \text{ for all } x \in X.$$

Let $D \in \operatorname{Div} X$, let $U \subset X$ be open. Put

$$\mathcal{O}_D(U) := \mathcal{O}_X(D)(U) := \{f \in \mathcal{M}_X(U) \mid \operatorname{ord}_x f \geqslant -D(x) \text{ for all } x \in U\}.$$

This defines a sheaf on X, denoted by $\mathcal{O}_X(D)$. This is a sheaf of $\mathcal{O}_X$-modules, in particular this means that $\mathcal{O}_X(D)(U)$ is an $\mathcal{O}_X(U)$ module for every open $U \subset X$.

Indeed, for $f \in \mathcal{O}_X(D)(U)$ and $u \in \mathcal{O}_X(U)$, it holds $\operatorname{ord}_x(uf) = \operatorname{ord}_x u + \operatorname{ord}_x f$. Since $\operatorname{ord}_x u \geqslant 0$, one concludes that $\operatorname{ord}_x(uf) \geqslant \operatorname{ord}_x f \geqslant -D(x)$, i. e., $u \cdot f \in \mathcal{O}_X(D)(U)$.

If $V \subset U$ are two open sets, then there is a restriction homomorphism

$$\mathcal{O}_X(D)(U) \to \mathcal{O}_X(D)(V), \quad f \mapsto f|_V$$

compatible with the module structure, i. e.,

$$(u \cdot f)|_V = u|_V \cdot f|_V, \quad u \in \mathcal{O}_X(U), f \in \mathcal{O}_X(D)(U).$$

Remark 6.4. $\mathcal{O}_X(0) = \mathcal{O}_X$, i. e., $\mathcal{O}_X(0)(U) = \mathcal{O}_X(U)$ for all open subsets $U \subset X$.

Proposition 6.5. *Let $D, D' \in \operatorname{Div} X$. Assume $D \sim D'$, then the sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are isomorphic.*

Remark 6.6. $\mathcal{O}_X(D) \cong \mathcal{O}_X(D)$ means that for every open $U \subset X$ there exists an isomorphism of $\mathcal{O}_X(U)$-modules

$$\mathcal{O}_X(D)(U) \xrightarrow{\eta(U)} \mathcal{O}_X(D')(U)$$

compatible with the restriction maps, i. e., for an inclusion of open sets $W \subset U \subset X$

$$\eta(U)(s)|_W = \eta(W)(s|_W) \quad \text{for every } s \in \mathcal{O}_X(D)(U),$$

or, equivalently, there is the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(D)(U) & \xrightarrow{\eta(U)} & \mathcal{O}_X(D')(U) \\ \downarrow \rho_{UW} & & \downarrow \rho_{UW} \\ \mathcal{O}_X(D)(W) & \xrightarrow{\eta(W)} & \mathcal{O}_X(D')(W), \end{array}$$

where ρ_{UW} denotes the restriction map $s \mapsto s|_W$.

Proof of Proposition 6.5. $D \sim D'$ means $D - D' = (s)$ for some $s \in \mathcal{M}_X(X)$. Then for every open $U \subset X$ and $f \in \mathcal{O}_X(D)(U)$ (i. e. $\operatorname{ord}_x f \geqslant -D(x)$ for all $x \in X$) we conclude that

$$\operatorname{ord}_x(s|_U \cdot f) = \operatorname{ord}_x(s) + \operatorname{ord}_x f \geqslant \operatorname{ord}_x s - D(x) = \operatorname{ord}_x s - (D' + (s))(x) = -D'(x)$$

and hence the map

$$\mathcal{O}_X(D)(U) \xrightarrow{\eta(U)} \mathcal{O}_X(D')(U), \quad f \mapsto s|_U \cdot f$$

is well defined. One sees that it is an homomorphism of $\mathcal{O}_X(U)$-modules and it possesses the inverse map given by $g \mapsto s^{-1}|_U \cdot g$. Therefore, $\eta(U)$ is an isomorphism. The compatibility with the restrictions follows as well. □

Remark 6.7. Even more is true. Let $D, D' \in \operatorname{Div} X$. Then the sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ are isomorphic if and only if $D \sim D'$.

Exercise. Try to prove this. You could follow the following steps.

1) Notice that for small enough $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(D)(U)$ is isomorphic to $\mathcal{O}_X(U)$.
2) Let R be an arbitrary $\mathbb{C}$-algebra. Notice that defining a homomorphism of R-modules $R \to R$ is equivalent to choosing $r \in R$ (the image of $1 \in R$).
3) Using the previous observations show that every isomorphism $\eta(U) : \mathcal{O}_X(D)(U) \to \mathcal{O}_X(D')(U)$ is of the form $f \mapsto s \cdot f$, $s \in \mathcal{M}_X(U)$, for small enough U.
4) Analyze the situation and obtain the required statement.

Definition 6.8. Let $D \in \operatorname{Div} X$. Then

$$\mathcal{L}(D) := \mathcal{O}_X(D)(X) = \{f \in \mathcal{M}_X(X) \mid \operatorname{ord}_x f \geqslant -D(x)\} = \{f \in \mathcal{M}_X(X) \mid (f) \geqslant -D\} \cup \{0\}$$

is called the Riemann-Roch space of D. It is a vector space over $\mathbb{C}$.

Example 6.9. 1) Let $D = a$ for some $a \in X$. Then

$$\mathcal{L}(D) = \{f \in \mathcal{M}_X(X) \mid (f) \geqslant -a\} = \left\{ f \in \mathcal{M}_X(X) \,\middle|\, \begin{array}{l} f \text{ has at most 1 pole of multiplicity 1} \\ \text{and this pole can only be at } a \end{array} \right\}.$$

2) Let $D = n \cdot a$ for some $a \in X$ and a positive integer n. Then

$$\mathcal{L}(D) = \{f \in \mathcal{M}_X(X) \mid (f) \geqslant -n \cdot a\} = \left\{ f \in \mathcal{M}_X(X) \,\middle|\, \begin{array}{l} f \text{ has at most 1 pole of multiplicity at} \\ \text{most } n \text{ and this pole can only be at } a \end{array} \right\}.$$

3) Let $D = -n \cdot a$ for some $a \in X$ and a positive integer n. Then

$$\mathcal{L}(D) = \{f \in \mathcal{M}_X(X) \mid (f) \geqslant n \cdot a\} = \left\{ f \in \mathcal{M}_X(X) \,\middle|\, \begin{array}{l} f \text{ does not have any poles and must} \\ \text{have a zero of multiplicity at least } n \text{ at} \\ a \end{array} \right\}.$$

6.2. **Exercises.**

Exercise 20. Compute the principal divisors (f), (g) of the following meromorphic functions on $\hat{\mathbb{C}}$ (cf. Exercise 16):

$$f(z) = \frac{(z-17)^2}{z^{13}+2}, \quad g(z) = \frac{(z-1)^3}{z^2+11}.$$

Exercise 21. Show that $\operatorname{Pic}^0 \hat{\mathbb{C}} = 0$, i. e., $\operatorname{PDiv} X = \operatorname{Div}^0 X$. Conclude that $\operatorname{Pic} \hat{\mathbb{C}} \cong \mathbb{Z}$.

Exercise 22. Let $X = \hat{\mathbb{C}}$.

1) Compute the Riemann-Roch space $\mathcal{L}_{\hat{\mathbb{C}}}(D)$ for

$$D = n \cdot p, \quad p = 0 \in X, \quad n \in \mathbb{Z}.$$

2) Notice that Exercise 21 says that two divisors on $\hat{\mathbb{C}}$ are linearly equivalent if and only if they have the same degree, in particular for every divisor D on $\hat{\mathbb{C}}$ and every $p \in \hat{\mathbb{C}}$

$$D \sim \deg D \cdot p.$$

In the lecture we mentioned that two linearly equivalent divisors have isomorphic Riemann-Roch spaces. If $D - D' = (s)$ for some $s \in \mathcal{M}_X(X)$, then the isomorphism is given by

$$\mathcal{L}(D) \to \mathcal{L}(D'), \quad f \mapsto s \cdot f.$$

Using this and your computations from part 1) of this exercise compute the Riemann-Roch spaces $\mathcal{L}(D)$ for the following divisors.

$$D = p, \quad p = 5 + 2i;$$

$$D = p - q, \quad p = 3, q = 4 - i;$$

$$D = 2p + 3q - 18r, \quad p = 6 - 2i, q = 47i, r = 356 - 3i;$$

$$D = 2 \cdot x_1 + 8 \cdot x_2 - 6 \cdot x_3 - 3 \cdot x_4, \quad x_1 = 11i, \quad x_2 = (2 - i), \quad x_3 = 44, \quad x_4 = \infty.$$

3) Check which of the following divisors on $\hat{\mathbb{C}}$ are linearly equivalent and describe the isomorphisms of the corresponding Riemann-Roch spaces for the pairs of linearly equivalent divisors.

$$D_1 = 3 \cdot (5 + 8i) + 27 \cdot (1 - i) - 6 \cdot (8i), \quad D_2 = 5 \cdot (i), \quad D_3 = 7 \cdot (28 + 3i) - 1 \cdot (i) - 1 \cdot (48),$$

$$D_4 = 4 \cdot (18) + 20 \cdot (33i), \quad D_5 = 3 \cdot (16 + 11i).$$

Exercise 23. Consider the complex torus $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} + \mathbb{Z} \cdot 3i$. Compute $\mathcal{L}(D)$ for

$$D = p, \quad p = [4 + 5i] \in X;$$

$$D = p - q, \quad p = [8], q = [2i].$$

7. Lecture 7

It turns out that the Riemann-Roch spaces are finite dimensional.

Theorem 7.1. $\dim \mathcal{L}(D) < \infty$ *for all* $D \in \operatorname{Div} X$.

Notation. $l(D) := \dim_{\mathbb{C}} \mathcal{L}(D)$.

Proof. **Idea**. We are going to follow the following steps.

1) $l(D) = 0$ for D with $\deg D < 0$, $l(0) = 1$.
2) For $D' = D + a$ for some $a \in X$ there is an inclusion of vector spaces $\mathcal{L}(D) \subset \mathcal{L}(D')$ and $\dim \mathcal{L}(D')/\mathcal{L}(D) \leqslant 1$.
3) Hence, by induction, $\dim \mathcal{L}(D) < \infty$ for every divisor D.

Details.

1) Let $\deg D < 0$. Assume $l(D) \neq 0$, then $\mathcal{L}(D) \neq 0$. Take some non-zero $f \in \mathcal{L}(D) \subset \mathcal{M}_X(X)$. Then $(f) \geqslant -D$ and in particular $\deg f \geqslant \deg(-D) = -\deg D > 0$. This is a contradiction.

   Since $\mathcal{L}(0) = \mathcal{O}_X(X) = \mathbb{C}$, one gets $l(0) = 1$.

   This gives a basis of the induction.
2) Let $D \in \operatorname{Div} X$, let $a \in X$, let $D' = D + a$. Then $D' \geqslant D$ and hence $-D(x) \geqslant -D'(x)$ and $\mathcal{L}(D) \subset \mathcal{L}(D')$.

   Choose a chart $\varphi : U \to V$ around a such that $\varphi(a) = 0$. For every $f \in \mathcal{L}(D')$ put $f_\varphi := f|_U \circ \varphi^{-1}$. Then f_φ is a meromorphic function on V. Consider its Laurent expansion at 0. Since $f \in \mathcal{L}(D')$, f_φ may have at 0 a pole of order at most $D'(a) = 1 + D(a) = 1 + d$, where $d = D(a)$.

   So
   $$f_\varphi(z) = a_{-d-1}(f) \cdot z^{-d-1} + a_{-d} \cdot z^{-d} + \cdots = \sum_{i=-d-1}^{\infty} a_i(f) \cdot z^i, \quad a_i(f) \in \mathbb{C}$$
   around 0.

   Now consider the map $\mathcal{L}(D') \xrightarrow{\xi} \mathbb{C}, \quad f \mapsto a_{-d-1}(f)$. It is a linear map. Its kernel coincides with $\mathcal{L}(D)$. So $\mathcal{L}(D')/\mathcal{L}(D) = \mathcal{L}(D')/\ker \xi \cong \operatorname{Im} \xi \subset \mathbb{C}$ and hence $\dim_{\mathbb{C}} \mathcal{L}(D')/\mathcal{L}(D) \geqslant 1$.
3) Notice that every divisor D' can be written as $D' = D + a$ for some $a \in X$ and $D \in DivX$. Moreover $\deg D < \deg D'$. This provides the step of the induction.

This concludes the proof. □

Example 7.2. 1) Let $p, q \in X$, $p \neq q$.

(a) If $D = p$, then $l(D) \leqslant 2$ because $D = 0 + p$ and $l(0) = 1$.

(b) If $D = -p$, then $l(D) = 0$.

(c) If $D = p - q$, then $l(D) \leqslant 1$ because $D = (-q) + p$ and $l(-q) = 0$.

2) Let $X = \mathbb{C}/\Gamma$ be a complex torus. Then $l(p) = 1$ for every $p \in X$.

3) Let $X = \hat{\mathbb{C}}$. Then $l(p) = 2$ for every $p \in X$.

Stalks of the structure sheaf. Let $a \in X$. Consider the set of pairs

$$\{(U, f) \mid U \subset X \text{ open}, a \in U, f \in \mathcal{O}_X(U)\}.$$

One defines the relation

$$(U, f) \sim (V, g) \overset{df}{\Longleftrightarrow} \exists \text{ open } W \subset U \cap V,\ a \in W \text{ such that } f|_W = g|_W.$$

Claim. *"$\sim$" is an equivalence relation.*

Proof. Exercise. □

Definition 7.3. The set of the equivalence classes is denoted by $\mathcal{O}_{X,a}$ and is called the **stalk** of the structure sheaf $\mathcal{O}_X$ at the point a.

We write $[(U, f)]$ of $[U, f]$ for the equivalence class of (U, f). By abuse of notation one also writes f_a, which means the equivalence class of a holomorphic function f defined in some neighbourhood of a. This equivalence class is called the **germ** of (U, f) (or simply the germ of f) at a.

Claim. *$\mathcal{O}_{X,a}$ is a $\mathbb{C}$-algebra with operations defined by*

$$f_a + g_a = (f + g)_a, \quad f_a \cdot g_a = (fg)_a, \quad \lambda \cdot f_a = (\lambda f)_a.$$

Proof. Exercise. □

Claim (Model example)**.** *$\mathcal{O}_{\mathbb{C},a} \cong \mathbb{C}\{z - a\} \cong \mathbb{C}\{z\}$ (convergent power series).*

Proof. Define

$$\mathcal{O}_{\mathbb{C},a} \mapsto \mathbb{C}\{z - a\}, \quad [U, f] \mapsto \text{Taylor expansion of } f \text{ at } a\text{: } f(z) = \sum_{i \geqslant 0} c_i (z - a)^i.$$

This gives the required isomorphism. □

Since every RS is locally isomorphic to $\mathbb{C}$, we conclude that $\mathcal{O}_{X,a} \cong \mathbb{C}\{z\}$ for every $a \in X$.

Indeed, fix a chart $\varphi : U \to V$ around $a \in X$. Then

$$\mathcal{O}_{X,a} \to \mathcal{O}_{\mathbb{C},\varphi(a)}, \quad f_a \mapsto (f \circ \varphi^{-1})_{\varphi(a)}$$

gives an isomorphism of $\mathbb{C}$-algebras $\mathcal{O}_{X,a} \cong \mathcal{O}_{\mathbb{C},\varphi(a)} \cong \mathbb{C}\{z\}$.

Exercises.

Exercise 24. Let D be a divisor on a compact Riemann surface. Let

$$\mathcal{L}(D) = \{f \in \mathcal{M}_X(X) \mid (f) \geqslant -D\} \cup \{0\}$$

be its Riemann-Roch space. In the lecture we proved that $\mathcal{L}(D)$ is a finite dimensional vector space over $\mathbb{C}$. Assume that $\deg D \geqslant 0$ and using our proof obtain the following estimation for the dimension $l(D)$ of $\mathcal{L}(D)$:

$$l(D) \leqslant \deg D + 1.$$

Exercise 25. Let $X = \hat{\mathbb{C}}$ and let $D \in \operatorname{Div} \hat{\mathbb{C}}$ be a divisor with non-negative degree. Show that the inequality from the previous exercise becomes an equality, i. e.,

$$l(D) = \deg D + 1.$$

Hint: *It is enough to find* $\deg D + 1$ *linear independent meromorphic functions from* $\mathcal{L}(D)$*. Have a look at Exercise 22.*

Exercise 26. Define $X = \{\langle x_0, x_1, x_2 \rangle \in \mathbb{P}_2 \mid x_1^2 - x_0x_2 = 0\}$. Then X is a 1-dimensional complex submanifold of $\mathbb{P}_2$. Let $p = \langle 0, 0, 1 \rangle \in X$, let $D = p$. Compute $l(D) = \dim_{\mathbb{C}} \mathcal{L}(D)$.
Hint: *Study the map* $\mathbb{P}_1 \to X$, $\langle s, t \rangle \mapsto \langle s^2, st, t^2 \rangle$.

Exercise 27. (0) Let a be a point of a Riemann surface X. Show that the stalk $\mathcal{O}_{X,a}$ is a $\mathbb{C}$-algebra with the operations defined in the lecture:

$$f_a + g_a := (f + g)_a, \quad f_a \cdot g_a := (fg)_a, \quad \lambda \cdot f_a := (\lambda f)_a, \quad f_a, g_a \in \mathcal{O}_{X,a}, \lambda \in \mathbb{C}.$$

In particular check that the definitions given in the lecture are well-defined, i. e., do not depend on the choice of representatives.

(1) Consider the evaluation homomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}_{X,a} \xrightarrow{\text{ev}} \mathbb{C}, \quad [U, f] \mapsto f(a).$$

Show that its kernel is the only maximal ideal of $\mathcal{O}_{X,a}$.

8. Lecture 8

Consider the evaluation homomorphism

$$\mathrm{ev} : \mathcal{O}_{X,a} \to \mathbb{C}, \quad f_a \mapsto f(a).$$

Its kernel is an ideal $\mathfrak{m}_{X,a} \subset \mathcal{O}_{X,a}$ given by

$$\mathfrak{m}_{X,a} = \{[U, f] \in \mathcal{O}_{X,a} \mid f(a) = 0\}.$$

Since $\mathcal{O}_{X,a}/\mathfrak{m}_{X,a} \cong \mathbb{C}$ and $\mathbb{C}$ is a field we conclude that $\mathfrak{m}_{X,a}$ is a maximal ideal of $\mathcal{O}_{X,a}$.

Claim. *$\mathfrak{m}_{X,a}$ is the only maximal ideal of $\mathcal{O}_{X,a}$. One says that $\mathcal{O}_{X,a}$ is the local algebra (or the local ring) of X at a.*

Remark 8.1. Recall that a ring with only one maximal ideal is called local.

Under the isomorphism $\mathcal{O}_{X,a} \cong \mathbb{C}\{z\}$ the ideal $\mathfrak{m}_{X,a}$ corresponds to the ideal in $\mathbb{C}\{z\}$ consisting of all convergent power series with trivial free term, i. e., the principal ideal $\langle z \rangle$ generated by z.

Remark 8.2. Notice that $\mathbb{C}\{z\}$ is a principal domain,i. e., all ideals are principal, i. e., generated by a single element. Moreover, every ideal of $\mathbb{C}\{z\}$ is of the form $\langle z^m \rangle$ for some $m \geqslant 0$.

Proof. Exercise. □

Let $\mathfrak{m}^2_{X,a}$ be the ideal generated by the products $s_1 \cdot s_2$, $s_1, s_2 \in \mathfrak{m}_{X,a}$. It corresponds to the principal ideal $\langle z^2 \rangle$. Clearly $\mathfrak{m}^2_{X,a} \subset \mathfrak{m}_{X,a}$. Consider the quotient $\mathcal{O}_{X,a}$-module and the corresponding quotient $\mathbb{C}\{z\}$-module $\langle z \rangle / \langle z^2 \rangle$. Then

$$\mathfrak{m}_{X,a}/\mathfrak{m}^2_{X,a} \cong \langle z \rangle / \langle z^2 \rangle \cong \mathbb{C} \cdot [z],$$

where $[z]$ denotes the class of z in $\langle z \rangle / \langle z^2 \rangle$.

We see that though $\mathfrak{m}_{X,a}$ and $\mathfrak{m}^2_{X,a}$ are infinite dimensional vector spaces over $\mathbb{C}$, their quotient $\mathfrak{m}_{X,a}/\mathfrak{m}^2_{X,a}$ is a 1-one dimensional vector space over $\mathbb{C}$.

Definition 8.3. The vector space $\mathfrak{m}_{X,a}/\mathfrak{m}^2_{X,a}$ is called the cotangent space of X at a and will be denoted in this lecture by $\mathrm{CT}_a X$.

Its dual space

$$(\mathfrak{m}_{X,a}/\mathfrak{m}^2_{X,a})^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,a}/\mathfrak{m}^2_{X,a}, \mathbb{C})$$

is called the tangent space of X at a and is denoted by $\mathrm{T}_a X$.

Definition 8.4. Let $[U, f] \in \mathcal{O}_{X,a}$. Put $d_a f := [f - f(a)] \in \operatorname{CT}_a X$.

For every open $U \subset X$ this defines the map

$$df : U \to \bigsqcup_{a \in U} \operatorname{CT}_a X, \quad a \mapsto d_a f.$$

Definition 8.5. Let $\varphi : U \to V$ be a chart of a Riemann surface X. Let $a \in U$. We call φ a local coordinate at a if $\varphi(a) = 0$.

We will often denote local coordinates by Latin letters, say $z : U \to V \subset \mathbb{C}$.

Let $z : U \to V \subset \mathbb{C}$ be a local coordinate at $a \in U$. Then $d_a z$ is a non-zero element in $\operatorname{CT}_a X$. Therefore, it can be taken as a basis of $\operatorname{CT}_a X$.

In particular one should be able to write $df(x) = g(x) \cdot dz(x)$ for some function $g : U \to \mathbb{C}$. Let us study this in more details.

Consider the composition $F = f \circ z^{-1}$. It is a holomorphic function in a neighbourhood V of $0 \in \mathbb{C}$. For $b \in U$, take the Taylor expansion of F at $z(b) \in V$.

$$F(t) = \sum_{i \geqslant 0} c_i (t - z(b))^i.$$

Then

$$f(x) = f \circ z^{-1} \circ z(x) = F(z(x)) = \sum_{i \geqslant 0} (z(x) - z(b))^i$$

and hence

$$d_b f = [f - f(b)] = [\sum_{i \geqslant 1} c_i (z - z(b))^i] = [c_1(z - z(b)) + (z - z(b))^2 \sum_{i \geqslant 2} c_i (z - z(b))^{i-2}] =$$

$$[c_1(z - z(b))] = c_1[z - z(b)] = F'(z(b)) \cdot d_b z.$$

Definition 8.6. Let $z : U \to V$ be a local coordinate at $a \in U$. Let $f \in \mathcal{O}_X(U)$. Put as above $F = f \circ z^{-1}$ and define

$$\frac{\partial f}{\partial z}(b) := F'(z(b)) = \frac{\partial F}{\partial t}(z(b)).$$

In these notations $d_b f = \frac{\partial f}{\partial z}(b) \cdot d_b z$ and finally

$$df = \frac{\partial f}{\partial z} \cdot dz, \tag{1}$$

a formula which looks familiar.

Sheaf of differential forms. Let $U \subset X$ be an open subset of a RS X. We have just seen that every $f \in \mathcal{O}_X(U)$ gives us a map

$$df : U \to \bigsqcup_{a \in U} \mathrm{CT}_a\, X, \quad a \mapsto d_a f.$$

Moreover, we computed that for a local coordinate $z : W \to \mathbb{C}$, $W \subset U$ it holds $df|_W = \frac{\partial f}{\partial z} \cdot dz$.

Let now $\omega : U \to \bigsqcup_{a \in U} \mathrm{CT}_a\, X$ be an arbitrary map such that $\omega(a) \in \mathrm{CT}_a\, X$. Then, as above, for a local coordinate $z : W \to \mathbb{C}$, $W \subset U$, we conclude that

$$\omega|_W = g \cdot dz$$

for some function $g : W \to \mathbb{C}$.

Definition 8.7. Let ω be as above. If g is a holomorphic function for every local coordinate $z : W \to \mathbb{C}$, then ω is called a holomorphic differential form on U.

Equivalently, ω is a holomorphic differential form if U can be covered by open sets U_i with local coordinates $z_i : U_i \to \mathbb{C}$ such that after representing the restrictions of ω as $\omega|_{U_i} = f_i \cdot dz_i$, the functions $f_i : U_i \to \mathbb{C}$ are holomorphic.

The set of all holomorphic differential forms on U is denoted by $\Omega_X(U)$. It is naturally an $\mathcal{O}_X(U)$-module. This defines a sheaf of $\mathcal{O}_X$-modules. The sheaf Ω_X is called the sheaf of differential forms on X.

Example 8.8. As we saw above, df is a holomorphic differential form on U for every $f \in \mathcal{O}_X(U)$.

Remark 8.9. For every open set $U \subset X$ the map

$$\mathcal{O}_X(U) \to \Omega_X(U), \quad f \mapsto df$$

is a linear map of $\mathbb{C}$-vector spaces, which gives a morphism of sheaves of $\mathbb{C}$-vector spaces $\mathcal{O}_X \to \Omega_X$.

Example 8.10. Let us compute $\Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$. Let $\omega \in \Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$. Let $z_0 : U_0 \to \mathbb{C}$ and $z_1 : U_1 \to \mathbb{C}$ be the standard charts of $\hat{\mathbb{C}}$. Then $\omega|_{U_0} = f_0 dz_0$ and $\omega|_{U_1} = f_1 dz_1$ for some holomorphic functions f_0 and f_1 on U_0 and U_1 respectively. It should also hold $f_0 dz_0|_{U_0 \cap U_1} = f_1 dz_1|_{U_0 \cap U_1}$. Since $z_0 = 1/z_1$ on $U_0 \cap U_1 = \mathbb{C}^*$, using (1) one gets $dz_0 = (-1/z_1^2) dz_1$, hence $f_0(1/z_1) \cdot (-1/z_1^2) dz_1 = f_1(z_1) dz_1$, and therefore $f_0(1/z_1) = -z_1^2 f_1(z_1)$. Comparing the Laurent expansions of these two holomorphic functions on $\mathbb{C}^*$, one immediately concludes that $f_0 = 0$, $f_1 = 0$, which means $\Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}}) = 0$.

Definition 8.11. Let U be an open subset of a Riemann surface X. A meromorphic differential form on U is an element $\omega \in \Omega_X(U \setminus S)$ for some discrete set S such that for every chart $U' \xrightarrow{z} V'$ with $U' \subset U$ the local expressions $\omega|_{U' \setminus S} = f dz$ are given by meromorphic functions $f \in \mathcal{M}_X(U')$.

Let $\mathcal{K}_X(U)$ denote the set of all meromorphic differential forms on U.

Remark 8.12. $\mathcal{K}_X(U)$ is naturally an $\mathcal{M}_X(U)$-module: for $f \in \mathcal{M}_X(U)$ and for $\omega \in \mathcal{K}_X(U)$

$$(f \cdot \omega)(x) = f(x) \cdot \omega(x).$$

Moreover, $\mathcal{K}_X$ is a sheaf of $\mathcal{M}_X$-modules. In particular, $\mathcal{K}_X$ is a sheaf of $\mathcal{O}_X$-modules.

Analogously to the case of holomorphic differential forms, there is the homomorphism of sheaves of vector spaces over $\mathbb{C}$ (note that it is not a homomorphism of $\mathcal{O}_X$-modules!)

$$\mathcal{M}_X \xrightarrow{d} \mathcal{K}_X.$$

Namely, for every open $U \subset X$ there is the linear map of vector spaces

$$\mathcal{M}_X(U) \to \mathcal{K}_X(U), \quad f \mapsto df$$

and the commutative diagram

$$\begin{array}{ccccccc} \mathcal{O}_X(U) & \hookrightarrow & \mathcal{M}_X(U) & \quad & f & \longmapsto & f \\ \downarrow d & & \downarrow d & & \downarrow & & \downarrow \\ \Omega_X(U) & \hookrightarrow & \mathcal{K}_X(U), & & df & \longmapsto & df. \end{array}$$

Definition 8.13. Let $\omega \in \mathcal{K}_X(U)$ for some open $U \subset X$. Let $a \in U$, let $z : U' \to V'$ be a local coordinate at a. Write $\omega|_{U'} = f dz$ for some $f \in \mathcal{M}_X(U')$. Define now the order of ω at a by

$$\operatorname{ord}_a \omega := \operatorname{ord}_a f.$$

Claim. $\operatorname{ord}_a \omega$ *does not depend on the choice of* z.

Proof. Exercise. □

Definition 8.14. Let X be a compact RS. Let $\omega \in \mathcal{K}_X(X)$. Define the divisor associated to ω by

$$(\omega) := \sum_{x \in X} \operatorname{ord}_x \omega \cdot x \in \operatorname{Div} X.$$

Example 8.15. Let $X = \hat{\mathbb{C}}$. We know already (cf. Example 8.10) that there are no non-trivial holomorphic differential forms on $\hat{\mathbb{C}}$.

Let us mimic the reasoning from Example 8.10 in order to find a non-trivial meromorphic differential form on $\hat{\mathbb{C}}$.

Let $\omega \in \mathcal{K}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$. Let $z_0 : U_0 \to \mathbb{C}$ and $z_1 : U_1 \to \mathbb{C}$ be the standard charts of $\hat{\mathbb{C}}$. Then $\omega|_{U_0} = f_0 dz_0$ and $\omega|_{U_1} = f_1 dz_1$ for some meromorphic functions f_0 and f_1 on U_0 and U_1 respectively. It should also hold $f_0 dz_0|_{U_0 \cap U_1} = f_1 dz_1|_{U_0 \cap U_1}$. Since $z_0 = 1/z_1$ on $U_0 \cap U_1 = \mathbb{C}^*$, using (1) one gets $dz_0 = (-1/z_1^2) dz_1$, hence $f_0(1/z_1) \cdot (-1/z_1^2) dz_1 = f_1(z_1) dz_1$, and therefore $f_0(1/z_1) = -z_1^2 f_1(z_1)$. Take $f_0(z_0) = 1$. Then $1 = -z_1^2 f_1(z_1)$, i. e., $f_1(z_1) = -1/z_1^2$. Thus we have just found a non-trivial meromorphic differential form ω on $\hat{\mathbb{C}}$. This form coincides with dz_0 on U_0 and equals $-\frac{1}{z_1^2} dz_1$ on U_1.

Let us compute the divisor corresponding to ω. Since $\operatorname{ord}_a \omega = \operatorname{ord}_a 1 = 0$ for $a \in \mathbb{C}$ and $\operatorname{ord}_\infty \omega = \operatorname{ord}_\infty(-\frac{1}{z_1^2}) = -2$, we conclude that

$$(\omega) = -2 \cdot \infty.$$

In particular $\deg(\omega) = -2$.

Exercise. Find a non-trivial meromorphic differential form ω' on $\hat{\mathbb{C}}$ different from the one presented in Example 8.15. Compute the corresponding divisor $(\omega') \in \operatorname{Div} \hat{\mathbb{C}}$ and its degree $\deg(\omega')$.

Exercises.

Exercise 28. Consider the following holomorphic functions on $\mathbb{C}$.

$$f_1(z) = (z-3)(z+5i)^6 + 11, \quad f_2(z) = \exp(z), \quad f_3(z) = \sin(z^2).$$

For $a = 0, 3, -5i$, find a generator of the cotangent space $\mathrm{CT}_a\,\mathbb{C}$ and express $d_a f_i$, $i = 1, 2, 3$, in terms of this generator.

Exercise 29. Consider the Riemann sphere $\hat{\mathbb{C}}$ and let $z_0 = \varphi_0$ and $z_1 = \varphi_1$ be its standard charts. Consider the meromorphic function

$$f(z) = \frac{z(z+1)}{(z-1)(z-2)^3} \in \mathcal{M}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$$

as a holomorphic function on $\hat{\mathbb{C}} \setminus \{1, 2\}$.

Compute

$$\frac{\partial f}{\partial z_0}(0), \quad \frac{\partial f}{\partial z_1}(\infty), \quad \frac{\partial f}{\partial z_0}(-1), \quad \frac{\partial f}{\partial z_1}(-1), \quad \frac{\partial f}{\partial z_0}(3), \quad \frac{\partial f}{\partial z_1}(3).$$

For $a = 0, \infty, -1, 3$ express if possible $d_a f$ in terms of $d_a z_0$ and $d_a z_1$.

Exercise 30. Let $X = \mathbb{C}/\Gamma$ be a complex torus. Find a non-trivial holomorphic differential form ω_0 on X. Compute the corresponding divisor (ω_0).

Exercise 31. Find two linear independent non-trivial meromorphic differential forms ω_1 and ω_2 on $\hat{\mathbb{C}}$. Compute the corresponding divisors $(\omega_1), (\omega_2) \in \operatorname{Div} \hat{\mathbb{C}}$ and their degrees $\deg(\omega_1)$ and $\deg(\omega_2)$.

9. Lecture 9

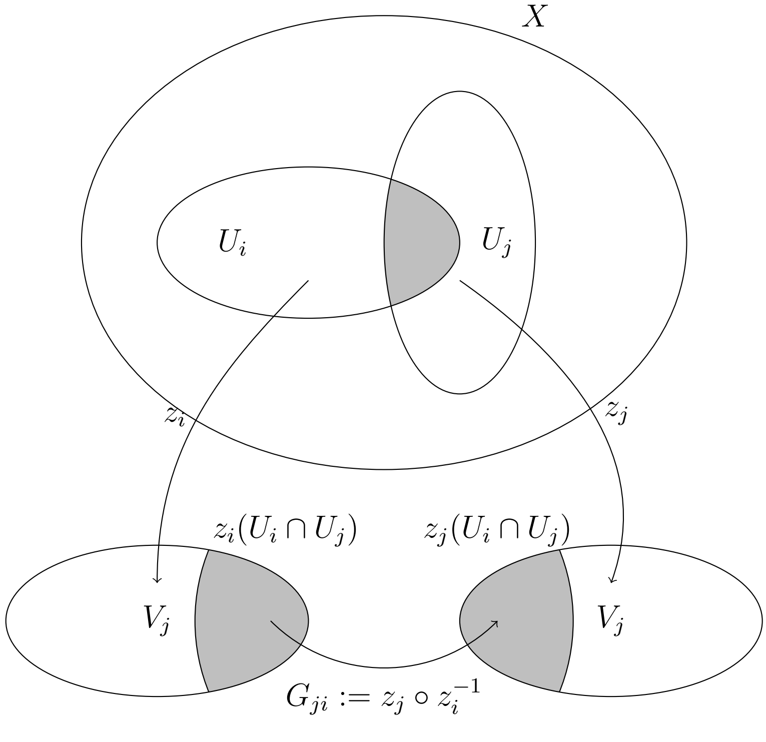
Proposition 9.1. *Let* $\omega_0 \in \mathcal{K}_X(X)$, $\omega_0 \not\equiv 0$. *Then* $\mathcal{K}_X(X) = \{f \cdot \omega_0 \mid f \in \mathcal{M}_X(X)\}$, *i. e.,*

$$\mathcal{M}_X(X) \to \mathcal{K}_X(X), \quad f \mapsto f \cdot \omega_0$$

is an isomorphism of $\mathbb{C}$*-vector spaces.*

Proof. Let $\omega \in \mathcal{K}_X(X)$ be an arbitrary meromorphic differential form on X. Let $\bigcup U_i = X$ be a covering of X by charts $z_i : U_i \to V_i$ such that $\omega_0|_{U_i}$ is given by $f_i dz_i$ and $\omega|_{U_i}$ is given by $g_i dz_i$ for some meromorphic functions f_i and g_i on U_i.

Note that $f_i \not\equiv 0$ for every i. Otherwise, by an argument similar to the one from the proof of Theorem 2.4 (identity theorem), $\omega_0 \equiv 0$. Consider $h_i = g_i/f_i \in \mathcal{M}_X(U_i)$.



Using (1) we get

$$dz_j = \frac{\partial z_j}{\partial z_i} dz_i.$$

So on $U_i \cap U_j$ we obtain

$$\omega_0|_{U_i \cap U_j} = f_j dz_j = f_j \cdot \frac{\partial z_j}{\partial z_i} dz_i = f_i dz_i, \quad \omega|_{U_i \cap U_j} = g_j dz_j = g_j \cdot \frac{\partial z_j}{\partial z_i} dz_i = g_i dz_i.$$

Therefore,

$$f_i = f_j \cdot \frac{\partial z_j}{\partial z_i}, \quad g_i = g_j \cdot \frac{\partial z_j}{\partial z_i},$$

and finally

$$h_i|_{U_i \cap U_j} = g_i/f_i = \frac{g_j \cdot \frac{\partial z_j}{\partial z_i}}{f_j \cdot \frac{\partial z_j}{\partial z_i}} = f_j/g_j = h_j|_{U_i \cap U_j}.$$

This means that there exists $h \in \mathcal{M}_X(X)$ such that $h|_{U_i} = h_i$.

We conclude that $g_i = h_i f_i = h f_i$ for every i. This means $\omega = h \cdot \omega_0$.

This concludes the proof. □

Definition 9.2. Let $D \in \operatorname{Div} X$. Let $U \subset X$ be an open subset. Define

$$\Omega_X(D)(U) := \{\omega \in \mathcal{K}_X(U) \mid \operatorname{ord}_a \omega \geqslant -D(a) \text{ for all } a \in U\}.$$

Then $\Omega_X(D)(U)$ is an $\mathcal{O}_X(U)$-module, in particular $\Omega_X(D)(X) = \{\omega \in \mathcal{K}_X(X) \mid (\omega) \geqslant -D\}$ is a $\mathbb{C}$-vector space.

Moreover, $\Omega_X(D)$ is a sheaf of $\mathcal{O}_X$-modules.

Definition 9.3. Let $\omega_0 \in \mathcal{K}_X(X)$, $\omega_0 \neq 0$. Then the divisor $K = (\omega_0)$ is called the canonical divisor on X.

Remark 9.4. On a compact Riemann surface there always exists a non-zero meromorphic differential form.

Note however that this fact is not at all trivial!

Remark 9.5. Note that K is not uniquely determined, it depends on ω_0. However, its divisor class

$$[K] \in \operatorname{Pic} X = \operatorname{Div} X / \operatorname{PDiv} X$$

does not depend on the choice of ω_0.

Proposition 9.6. *Let X be a compact Riemann surface. Let $K = (\omega_0)$. For every divisor $D \in \operatorname{Div}(X)$ there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(D) \to \Omega_X(D - K)$ defined for every open $U \subset X$ by*

$$\mathcal{O}_X(D)(U) \to \Omega_X(D - K)(U), \quad f \mapsto f \cdot \omega_0.$$

Equivalently: $\mathcal{O}_X(K + D) \cong \Omega_X(D)$,

$$\mathcal{O}_X(K + D)(U) \to \Omega_X(D)(U), \quad f \mapsto f \cdot \omega_0.$$

Corollary 9.7. $\Omega_X(D)(X) \cong \mathcal{O}_X(K + D)(X) = \mathcal{L}(K + D)$, *in particular*

$$\dim_{\mathbb{C}} \Omega_X(D)(X) < \infty$$

for every divisor $D \in \operatorname{Div} X$.

Definition 9.8. The dimension of $\mathcal{L}(K) \cong \Omega_X(0)(X) = \Omega_X(X)$ is called the genus of X and is denoted by

$$g = g_X := \dim_{\mathbb{C}} \Omega_X(X).$$

Example 9.9. 1) Since by Example 8.10 $\Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}}) = 0$, one concludes that $g_{\hat{\mathbb{C}}} = 0$.

2) By Exercise 32 $g_{\mathbb{C}/\Gamma} = 1$ for every complex torus $\mathbb{C}/\Gamma$.

Theorem 9.10 (Riemann-Roch)**.**

$$l(D) - l(K - D) = \deg D + 1 - g.$$

Equivalently,

$$l(D) - \dim \Omega_X(-D)(X) = \deg D + 1 - g.$$

Proof. No proof. □

Example 9.11. 1) Let $D = 0$. Then Theorem 9.10 reads as $l(0) - l(K) = \deg 0 + 1 - g$, hence $g = l(K)$, i. e., we get back the definition of the genus.

2) Let $D = K$. Then $l(K) - l(0) = \deg K + 1 - g$ and therefore

$$\deg K = 2g - 2.$$

3) If $\deg D \geqslant 2g-1$, then $\deg(K-D) = \deg K - \deg D = 2g-2-\deg D < 0$, thus $l(K-D) = 0$ and finally

$$l(D) = \deg D + 1 - g.$$

One can summarize this as follows.

$$\begin{cases} l(D) = 0, & \text{if } \deg D < 0; \\ l(D) \geqslant \deg D + 1 - g, & \text{if } 0 \leqslant \deg D < 2g - 1; \\ l(D) = \deg D + 1 - g, & \text{if } \deg D \geqslant 2g - 1. \end{cases}$$

Theorem 9.12 (Riemann-Hurwitz formula)**.** *Let $f : X \to Y$ be a non-constant holomorphic map of compact RS. Then*

$$2g_X - 2 = d(f)(2g_Y - 2) + \sum_{x \in X} (\operatorname{mult}_x f - 1)$$

Equivalently $\deg K_X = d(f) \deg K_Y + \deg R_f$, *where K_X and K_Y are canonical divisors on X and Y respectively and $R_f = \sum_{x\in X}(\operatorname{mult}_x f - 1) \cdot x$ is the so called ramification divisor of f.*

Remark 9.13. Note that $\operatorname{mult}_x f > 1$ only for finitely many points of X (ramification points, cf. Definition 5.1).

Exercises.

Exercise 32. Let $X = \mathbb{C}/\Gamma$ be a complex torus.

1) Find a non-trivial holomorphic differential form ω_0 on X. Compute the corresponding divisor (ω_0).

2) Let ω be an arbitrary holomorphic differential form on X. Then $\omega = f\omega_0$ for some meromorphic function f. Conclude that f must be holomorphic.

3) Conclude that $\Omega_X(X) = \mathbb{C} \cdot \omega_0$, i. e., vector space generated by ω_0.

Exercise 33. 1) Let X be a compact Riemann surface of genus g. Let $p \in X$ and let $D = (g+1)p$. Apply the Riemann-Roch formula to D and conclude that $l(D) \geqslant 2$. The latter means that there exists a non-constant meromorphic function $f \in \mathcal{L}(D)$.

2) Estimate the degree of the corresponding holomorphic map $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$?

3) Conclude that every compact Riemann surface of genus 0 is isomorphic to $\hat{\mathbb{C}}$.

Exercise 34. Using your computations from Exercise 32 compute the genus of a complex torus $X = \mathbb{C}/\Gamma$ using two different methods.

(1) Compute the degree of the canonical divisor and use the the Riemann-Roch formula.

(2) Compute explicitly $\Omega_X(X)$ and its dimension.

Exercise 35. 1) Let $X \subset \mathbb{P}_2$ be the subspace

$$X_2 = \{\langle z_0, z_1, z_2\rangle \in \mathbb{P}_2 \mid z_0^2 + z_1^2 + z_2^2 = 0\}.$$

Show that X_2 is a submanifold of $\mathbb{P}_2$, i. e., a Riemann surface. Consider the map

$$X_2 \xrightarrow{f} \hat{\mathbb{C}}, \quad \langle z_0, z_1, z_2\rangle \mapsto \frac{z_1}{z_2},$$

where $\frac{a}{0}$ is assumed to be ∞. Show that this is a holomorphic map of RS. Apply the Riemann-Hurwitz formula and compute the genus of X_2. Conclude that X_2 is isomorphic to the Riemann sphere.

Hint: *Compute the number of preimages of $f^{-1}(p)$ for every $p \in \hat{\mathbb{C}}$. Using that there can be only finitely many ramification points, find the ramification points and obtain the value of $d(f)$.*

2) Generalize the computations to the case of

$$X_d = \{\langle z_0, z_1, z_2\rangle \in \mathbb{P}_2 \mid z_0^d + z_1^d + z_2^d = 0\}, \quad d \in \mathbb{N}.$$

What is the genus of X_d?

10. Lecture 10

Let us consider some corollaries from the Riemann-Roch theorem.

Corollary 10.1. *On every compact RS X there exists a non-constant meromorphic function $f \in \mathcal{M}_X(X)$.*

Proof. Let $p \in X$ be an arbitrary point, take $D = (g+1) \cdot p$. Then $l(D) \geqslant g+1+1-g = 2$. This means that the dimension of the Riemann-Roch space $\mathcal{L}(D)$ is at least 2. Therefore, this space must contain a non-constant meromorphic function. □

Observation. *Take $f \in \mathcal{L}(D)$ as above. The only point that could be a pole of this meromorphic function is p. Its multiplicity is at most $g+1$, therefore the degree of the corresponding holomorphic non-constant map $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$ is at most $g+1$.*

Corollary 10.2. *Every compact RS of genus 0 is isomorphic to $\hat{\mathbb{C}}$*

Proof. As above one gets a holomorphic map $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$ of degree 1, which must be an isomorphism (cf. Theorem 3.9 and Corollary 3.7). □

Some facts about coverings.

Definition 10.3. A continuous map of topological spaces $X \xrightarrow{f} Y$ is called a covering if for every $y \in Y$ there exists an open neighbourhood U of y such that $f^{-1}(U) = \bigsqcup_i V_i$ and $f|_{V_i} : V_i \to U$ is a homeomorphism.

Observation. *If Y is a RS and $X \xrightarrow{f} Y$ is a covering, then there is a unique complex structure on X such that f is a holomorphic map.*

Proof. Exercise. □

So every covering of a RS is then a locally biholomorphic map.

Remark 10.4. Not every local biholomorphism is a covering. For example, take $X = B(0,1) = \{z \in \mathbb{C} \mid |z| < 1\}$, $Y = \mathbb{C}$. Then the natural inclusion $X \subset Y$ is locally biholomorphic but not a covering.

Claim. *Every locally biholomorphjic map of compact RS is a covering.*

Proof. Use an argument similar to the one from the proof of Theorem 5.5. □

Definition 10.5. Let $\widetilde{X} \xrightarrow{f} X$ be a covering of RS. Then it is called a universal covering if $\widetilde{X}$ is simply connected, i. e., if $\pi_1(\widetilde{X}) = 0$.

Proposition 10.6. *1) A universal covering exists for every RS.*

2)(Universal propery): $\widetilde{X} \xrightarrow{f} X$ *is a universal covering if and only if for every covering* $Y \xrightarrow{g} X$ *and every choice of points* $x_0 \in X$, $y_0 \in g^{-1}(x_0)$, $\widetilde{x}_0 \in f^{-1}(x_0)$ *there exists a unique holomorphic map* $\widetilde{X} \xrightarrow{h} Y$ *with* $h(\widetilde{x}_0) = y_0$ *such that* $g \circ h = f$.

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{h}{\swarrow} & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}, \quad \begin{array}{ccc} & & \widetilde{x}_0 \\ & \overset{h}{\swarrow} & \downarrow f \\ y_0 & \overset{g}{\longmapsto} & x_0 \end{array}$$

Proof. Topology. □

Morphisms of complex tori. Let $X = \mathbb{C}/\Gamma$ and $Y = \mathbb{C}/\Gamma'$ be two complex tori. Our aim is to describe all holomorphic maps $X \to Y$.

Reminder 10.7. Remind (cf. Example 2.5) that for $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$ one obtains a holomorphic map

$$X \to Y, \quad [z] \mapsto [\alpha \cdot z].$$

Let $X \xrightarrow{f} Y$ be an arbitrary non-constant holomorphic map. Then by Riemann-Hurwitz formula (Theorem 9.12), one concludes that f has no ramification points. So it must be a covering.

Note that the canonical maps $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma$, $z \mapsto [z]$, and $\mathbb{C} \xrightarrow{\pi'} \mathbb{C}/\Gamma'$, $z \mapsto [z]$ are coverings and even universal coverings. Then by the universal property of universal coverings there exists a holomorphic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi' \circ F = f \circ \pi$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{f} & Y. \end{array} \tag{2}$$

Consider now for a fixed $\gamma \in \Gamma$ the function $\Phi_\gamma(z) = F(z+\gamma) - F(z)$. From the commutativity of diagram (2) we get that $\Phi_\gamma(z) \in \Gamma'$ for every $z \in \mathbb{C}$. Since Φ_γ is continuous, there exists $\gamma' \in \Gamma'$ such that $\Phi_\gamma(z) = \gamma'$ for all $z \in \mathbb{C}$. Hence $\Phi'_\gamma(z) = 0$ and thus $F'(z+\gamma) - F'(z) = 0$. This means that F' is a doubly periodic (elliptic) holomorphic function on $\mathbb{C}$, therefore it must be constant, i. e., there exists $a \in \mathbb{C}$ such that $F'(z) = a$ for all $z \in \mathbb{C}$. This implies $F(z) = az + b$ for some $a, b \in \mathbb{C}$. Therefore, $f([z]) = [az] + [b]$. This can only be well-defined if for every $\gamma \in \Gamma$ it holds $f([z+\gamma]) = f([z])$, which implies $a\Gamma \subset \Gamma'$.

On the other hand one sees that for every choice of $a, b \in \mathbb{C}$ such that $a\Gamma \subset \Gamma'$ the map

$$X \to Y, \quad [z] \mapsto [az] + [b]$$

is holomorphic. It can be represented as a composition of

$$X \to Y, \quad [z] \mapsto [az]$$

with the automorphism of $Y = \mathbb{C}/\Gamma'$

$$Y \to Y, \quad [z] \mapsto [z] + [b].$$

We obtained the following.

Proposition 10.8. *Every holomorphic map of complex tori $\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ can be represented as a composition of a holomorphic map*

$$\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma', \quad [z] \mapsto [az], \quad a \in \mathbb{C}, \quad a\Gamma \subset \Gamma',$$

and an isomorphism

$$\mathbb{C}/\Gamma' \to \mathbb{C}/\Gamma', \quad [z] \mapsto [z] + [b], \quad b \in \mathbb{C}.$$

Isomorphism classes of complex tori. Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Let $\Gamma' = \mathbb{Z} + \mathbb{Z} \cdot \frac{\omega_2}{\omega_1}$. Then $\omega_1\Gamma' = \Gamma$ and

$$\mathbb{C}/\Gamma' \to \mathbb{C}/\Gamma, \quad [z] \mapsto [\omega_1 z]$$

is an isomorphisms of complex tori.

So, while studying the isomorphism classes of complex tori, it is enough to consider only the lattices

$$\mathbb{Z} + \mathbb{Z} \cdot \tau, \quad \operatorname{Im} \tau \neq 0.$$

Moreover, if $\operatorname{Im} \tau < 0$, then $\operatorname{Im} \tau^{-1} > 0$ and $\tau(\mathbb{Z} + \mathbb{Z}\tau^{-1}) = (\mathbb{Z} + \mathbb{Z}\tau)$, i. e., the lattices $\mathbb{Z} + \mathbb{Z}\tau^{-1}$ and $\mathbb{Z} + \mathbb{Z}\tau$ define isomorphic tori. Therefore, it is enough to consider only lattices

$$\mathbb{Z} + \mathbb{Z} \cdot \tau, \quad \operatorname{Im} \tau > 0.$$

Notation. Let $\mathbb{H}$ denote the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$.

For $\tau \in \mathbb{H}$ denote $\Gamma(\tau) := \mathbb{Z} + \mathbb{Z} \cdot \tau$.

Let now $\Gamma_1 = \Gamma(\tau_1) = \mathbb{Z} + \mathbb{Z} \cdot \tau_1$, $\Gamma_2 = \Gamma(\tau_2) = \mathbb{Z} + \mathbb{Z} \cdot \tau_2$. Assume they define isomorphic tori $\mathbb{C}/\Gamma_1 \cong \mathbb{C}/\Gamma_2$. Then the isomorphism is given by $[z] \mapsto [az] + [b]$. Since the translation $[z] \mapsto [z] + [b]$ is an isomorphism, the map $[z] \mapsto [az]$ must be an isomorphism as well. So it must hold $a\Gamma_1 = \Gamma_2$ (cf. Example 2.5).

In particular it means that $a \cdot \tau_1$ and $a \cdot 1$ belong to Γ_2. Write

$$a\tau_1 = \alpha\tau_2 + \beta, \quad a = \gamma\tau_2 + \delta, \quad \alpha, \beta, \delta, \gamma \in \mathbb{Z}.$$

In other words
$$a\cdot\begin{pmatrix}\tau_1\\ 1\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\begin{pmatrix}\tau_2\\ 1\end{pmatrix}.$$
Analogously, since the equality $a\Gamma_1=\Gamma_2$ is equivalent to $a^{-1}\Gamma_2=\Gamma_1$, one concludes that
$$a^{-1}\cdot\begin{pmatrix}\tau_2\\ 1\end{pmatrix}=\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\cdot\begin{pmatrix}\tau_1\\ 1\end{pmatrix}$$
for some integer matrix $\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$.

One has
$$\begin{pmatrix}\tau_2\\ 1\end{pmatrix}=aa^{-1}\begin{pmatrix}\tau_2\\ 1\end{pmatrix}=\left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot[a^{-1}\cdot\begin{pmatrix}\tau_2\\ 1\end{pmatrix}]\right]=$$
$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\cdot\begin{pmatrix}\tau_2\\ 1\end{pmatrix}=\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix}\tau_2\\ 1\end{pmatrix}=\begin{pmatrix}c_{11}\tau_2+c_{12}\\ c_{21}\tau_2+c_{22}\end{pmatrix},$$
where
$$\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}.$$
Therefore, from the equalities $\tau_2=c_{11}\tau_2+c_{12}$ and $1=c_{21}\tau_2+c_{22}$ we get
$$\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
which means that $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}$ are invertible to each other integer matrices. Therefore, their determinants equal either 1 or -1.

Since $\tau_1=\dfrac{a\tau_1}{a}=\dfrac{\alpha\tau_2+\beta}{\gamma\tau_2+\delta}$, we obtain
$$\tau_1=\frac{\alpha\tau_2+\beta}{\gamma\tau_2+\delta}=\frac{(\alpha\tau_2+\beta)(\gamma\bar{\tau}_2+\delta)}{|\gamma\tau_2+\delta|^2}=\frac{\alpha\gamma|\tau_2|^2+\beta\delta+\alpha\delta\tau_2+\beta\gamma\bar{\tau}_2}{|\gamma\tau_2+\delta|^2}.$$
Hence
$$\operatorname{Im}\tau_1=\frac{1}{|\gamma\tau_2+\delta|^2}\cdot(\alpha\delta-\beta\gamma)\operatorname{Im}\tau_2 \tag{3}$$
Since $\operatorname{Im}\tau_1>0$ and $\operatorname{Im}\tau_2>0$, one concludes that $\alpha\delta-\beta\gamma=\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}>0$ and hence $\det\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)=1$. We have shown that $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$.

So, if Γ_1 and Γ_2 define isomorphic tori, then $\tau_1 = \dfrac{\alpha\tau_2 + \beta}{\gamma\tau_2 + \delta}$ for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

On the other hand, if $\tau_1 = \dfrac{\alpha\tau_2 + \beta}{\gamma\tau_2 + \delta}$ for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then $a\Gamma_1 = \Gamma_2$ for $a = \gamma\tau_2 + \delta$. We obtained the following result.

Theorem 10.9. *Two lattices $\Gamma(\tau_1)$ and $\Gamma(\tau_2)$, $\tau_1, \tau_2 \in \mathbb{H}$, define isomorphic complex tori if and only if*

$$\tau_1 = \frac{\alpha\tau_2 + \beta}{\gamma\tau_2 + \delta}$$

for $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

In other words, if one defines an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by

$$\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \cdot \tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta},$$

the set of its orbits $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ can be seen as the set of all isomorphism classes of complex tori.

Exercises.

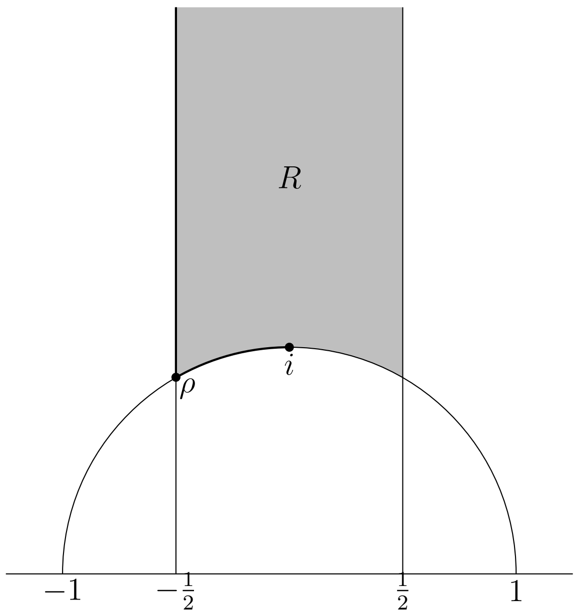
Exercise 36. Let $\Gamma = \mathbb{Z} + \mathbb{Z} \cdot \tau$, $\tau \in \mathbb{C}$, be a lattice in $\mathbb{C}$. Let n be a natural number and let $\Gamma' = \mathbb{Z} + \mathbb{Z} \cdot (n\tau)$. Put $X = \mathbb{C}/\Gamma$ and $X' = \mathbb{C}/\Gamma'$ and consider the map

$$X \to X', \quad [z] \mapsto [nz].$$

By Exercise 5 it is a holomorphic map of Riemann surfaces. Prove that it is a covering. What is the number of points in the fibres?

Exercise 37. Let $R = \{z \in \mathbb{C} \mid |z| > 1, |\operatorname{Re} z| < \frac{1}{2}\}$ and let

$$F = R \cup \{z \in \mathbb{C} \mid \operatorname{Re}(z) = -\frac{1}{2}, |z| \geqslant 1\} \cup \{z \in \mathbb{C} \mid |z| = 1, -\frac{1}{2} \leqslant \operatorname{Re}(z) \leqslant 0\}.$$



Let $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ denote the space of the orbits of the action of $\operatorname{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Prove that the restriction of the projection map

$$\pi : \mathbb{H} \to \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}), \quad \tau \mapsto \text{orbit of } \tau = \{\tau' \in \mathbb{H} \mid \tau' = \frac{a\tau + b}{c\tau + d}, \ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{SL}_2(\mathbb{Z})\}$$

to F is a bijection. This means that the points of F are in one-to-one correspondence with the isomorphism classes of complex tori.

Exercise 38. In the lecture we realized the group $\operatorname{SL}_2(\mathbb{Z})$ as the group of transformations of the upper half-plane $\mathbb{H}$ of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \operatorname{SL}_2(\mathbb{Z}).$$

(1) Show that this group is generated by the transformations

$$\tau \mapsto \tau + 1 \quad \text{and} \quad \tau \mapsto -\frac{1}{\tau}.$$

(2) What is the image of the region R from the previous exercise under the generators of $\mathrm{SL}_2(\mathbb{Z})$ from the first part of this exercise?

Exercise 39. Let Γ be a lattice in $\mathbb{C}$ and let $\mathbb{C}/\Gamma$ be the corresponding complex torus. In the lecture we showed that the automorphisms of X must be of the form

$$[z] \mapsto [az] + [b], \quad a, b \in \mathbb{C}, \quad a \cdot \Gamma = \Gamma.$$

Let $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$ denote the subgroup in the group of all automorphisms of $\mathbb{C}/\Gamma$ consisting of the automorphisms $\mathbb{C}/\Gamma \xrightarrow{f} \mathbb{C}/\Gamma$ such that $f([0]) = [0]$, i. e.,

$$\mathrm{Aut}_0(\mathbb{C}/\Gamma) = \{\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma, [z] \mapsto [az] \mid a \in \mathbb{C}, a \cdot \Gamma = \Gamma\}.$$

(0) Show that $a \cdot \Gamma = \Gamma$ implies $|a| = 1$.

(1) Compute $\mathrm{Aut}_0(\mathbb{C}/\Gamma(i)) \cong \mathbb{Z}/4\mathbb{Z}$, where $\Gamma(i) = \mathbb{Z} + \mathbb{Z} \cdot i$.

(2) Compute $\mathrm{Aut}_0(\mathbb{C}/\Gamma(\rho)) \cong \mathbb{Z}/6\mathbb{Z}$, where $\Gamma(\rho) = \mathbb{Z} + \mathbb{Z} \cdot \rho$, $\rho = e^{\frac{2}{3}\pi i} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

(3) Compute $\mathrm{Aut}_0(\mathbb{C}/\Gamma(\tau)) \cong \mathbb{Z}/2\mathbb{Z}$, where $\Gamma(\tau) = \mathbb{Z} + \mathbb{Z} \cdot \tau$, for $\tau = 2i$ and $\tau = \frac{1}{2} + i$.

(4) Try to compute $\mathrm{Aut}_0(\mathbb{C}/\Gamma(\tau))$, for an arbitrary $\tau \in F$.

11. Lecture 11

In the last lecture we obtained a description of the isomorphism classes of complex tori.

Consider now the quotient map

$$\mathbb{H} \xrightarrow{\pi} \mathbb{H}/\operatorname{SL}_2(\mathbb{Z}), \quad \tau \mapsto \text{orbit of } \tau.$$

Introduce on $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ the quotient topology, i. e., call the set $U \subset \mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ open if and only if $\pi^{-1}U \subset \mathbb{H}$ is open.

Exercise. π is a local homeomorphism outside of the orbits of the points $i, \rho \in \mathbb{H}$, $\rho = \exp(\frac{2\pi}{3} \cdot i) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. This allows us to introduce a structure of a Riemann surface on

$$(\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})) \setminus \{\pi(i), \pi(\rho)\},$$

i. e., on the quotient space without the two points $\pi(i)$ and $\pi(\rho)$.

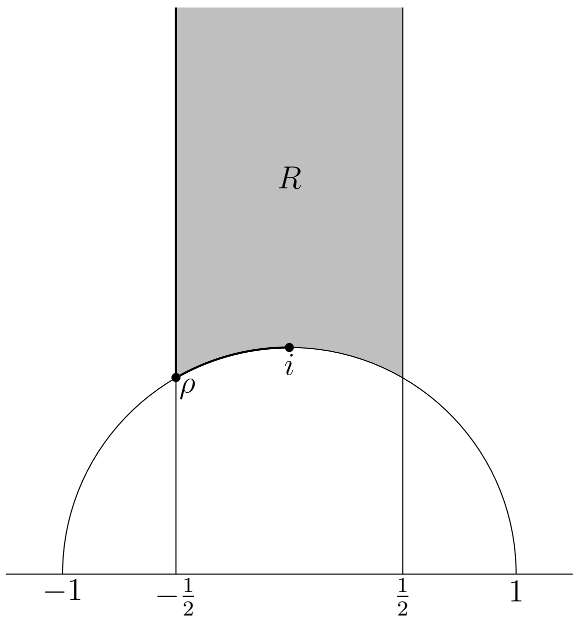
Remark 11.1. Notice that the restriction of π to every neighbourhood of i or ρ is never injective. This shows that π can not be a local homeomorphism around these points.

Let us visualize the space $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$. Let

$$R = \{z \in \mathbb{C} \mid |z| > 1, |\operatorname{Re} z| < \frac{1}{2}\}$$

and take

$$F = R \cup \{z \mid \operatorname{Re} z = -\frac{1}{2}, |z| \geqslant 1\} \cup \{z \mid |z| = 1, -\frac{1}{2} \leqslant \operatorname{Re} z \leqslant 0\}.$$



Exercise. Then the restriction of π to F is a bijection, i. e., F can be seen as the set of all isomorphism classes of complex tori.

Proof of the surjectivity. Let $\tau \in \mathbb{H}$. Let us show that there exists $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $A \cdot \tau \in F$. More details can be found in [5].

First of all notice that

$$\operatorname{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{1}{|c\tau + d|^2} \cdot \operatorname{Im}\tau, \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

This assures for a fixed τ the existence of

$$\max_{A \in \mathrm{SL}_2(\mathbb{Z})} \{\operatorname{Im}(A \cdot \tau)\}.$$

Therefore, there exists $A_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that for $\tau_0 = A_0 \cdot \tau$

$$\operatorname{Im}\tau_0 \geqslant A \cdot \tau, \quad \text{for every } A \in \mathrm{SL}_2(\mathbb{Z}).$$

Since $\operatorname{Im}(\tau_0 + n) = \operatorname{Im}\tau_0$ for every $n \in \mathbb{Z}$, we may assume, possibly taking $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \cdot A_0$ instead of A_0, that $|\operatorname{Re}\tau_0| \leqslant \frac{1}{2}$.

Since $\operatorname{Im}\tau_0 \geqslant \operatorname{Im} A\tau$ for every $A \in \mathrm{SL}_2(\mathbb{A})$, let us apply this to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot A_0$. We get

$$\operatorname{Im}\tau_0 \geqslant \operatorname{Im}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) A_0 \cdot \tau) = \operatorname{Im}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot \tau_0) = \operatorname{Im}(-1/\tau_0) = \frac{\operatorname{Im}\tau_0}{|\tau_0|^2},$$

which implies $|\tau_0| \geqslant 1$.

If τ_0 does not belong to F, then either $\operatorname{Re}\tau_0 = \frac{1}{2}$ or $|\tau_0| = 1$ and $0 < \operatorname{Re}\tau_0 \leqslant \frac{1}{2}$. One can easily correct this. Namely, if $\operatorname{Re}\tau_0 = \frac{1}{2}$, then $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) A_0 \cdot \tau = \tau_0 - 1 \in F$; if $|\tau_0| = 1$ and $0 < \operatorname{Re}\tau_0 \leqslant \frac{1}{2}$, then $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) A_0 \cdot \tau = -1/\tau_0 \in F$. $\square$

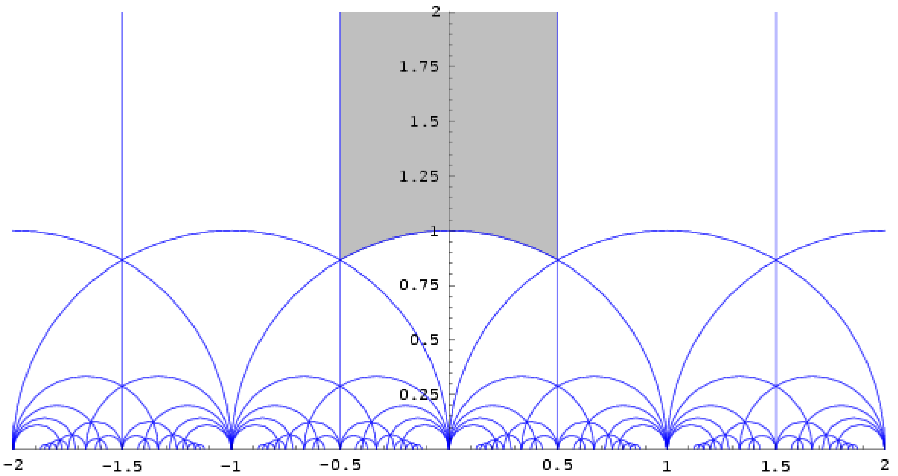


FIGURE 1. The interior of every triangular region (with one of the vertices lying possibly "at infinity") is the image of R under the action of some element from $\mathrm{SL}_2(\mathbb{Z})$.

Automorphism of complex tori. Let us study the automorphism of complex tori. By Proposition 10.8 it is enough to study the automorphisms $\mathbb{C}/\Gamma \xrightarrow{f} \mathbb{C}/\Gamma$ such that $f([0]) = 0$. So let $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$ denote the subgroup in the group of all automorphisms of $\mathbb{C}/\Gamma$ consisting of the automorphisms $\mathbb{C}/\Gamma \xrightarrow{f} \mathbb{C}/\Gamma$ such that $f([0]) = [0]$. Then, as already mentioned,

$$\mathrm{Aut}_0(\mathbb{C}/\Gamma) = \{\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma, [z] \mapsto [az] \mid a \in \mathbb{C}, a \cdot \Gamma = \Gamma\}.$$

An automorphism from $\mathrm{Aut}_0(\mathbb{C}/\Gamma(\tau))$, $\tau \in \mathbb{H}$, is given by a matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. Namely, the automorphism is given by the rule

$$[z] \mapsto [az], \quad a = \gamma\tau + \delta.$$

Notice that (3) implies in this case $|a| = 1$.

If $\gamma = 0$, then this provides two different automorphisms of $\mathbb{C}/\Gamma(\tau)$: the identity $[z] \mapsto [z]$ and $[z] \mapsto -[z]$.

Analyzing the case of $\gamma \neq 0$ one can obtain the following statement.

Claim. *Let $\tau \in F$. If $\tau \neq i$ and $\tau \neq \rho$, then*

$$\mathrm{Aut}_0(\mathbb{C}/\Gamma(\tau)) = \{\pm \mathrm{id}_{\mathbb{C}/\Gamma(\tau)}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

It holds also

$$\mathrm{Aut}_0(\mathbb{C}/\Gamma(i)) \cong \mathbb{Z}/4\mathbb{Z}, \quad \mathrm{Aut}_0(\mathbb{C}/\Gamma(\rho)) \cong \mathbb{Z}/6\mathbb{Z}.$$

Proof. Exercise. □

Remark 11.2. 1) Notice that the automorphism group of the Riemann sphere $\mathrm{Aut}(\hat{\mathbb{C}})$ coincides with the group of the transformations

$$\hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad x \mapsto \frac{ax+b}{cx+d}, \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}).$$

which is isomorphic to the quotient of the general linear group $\mathrm{GL}_2(\mathbb{C})$ by the subgroup of the matrices $\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) \mid \lambda \in \mathbb{C}^*\}$. This quotient is denoted by $\mathrm{PGL}_2(\mathbb{C})$. Notice that $\mathrm{PGL}_2(\mathbb{C})$ is an infinite group. The subgroup $\mathrm{Aut}_0(\hat{\mathbb{C}})$ of the automorphisms preserving $0 \in \hat{\mathbb{C}}$ consists of the transformations

$$\hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad x \mapsto \frac{ax}{cx+d}, \quad \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}).$$

This group is infinite as well.

2) Notice that though $\mathrm{Aut}_0(\mathbb{C}/\Gamma)$ is finite for every lattice Γ, the whole automorphism group $\mathrm{Aut}(\mathbb{C}/\Gamma)$ is infinite.

3) The Hurwitz's automorphisms theorem says that for a compact Riemann surface X of genus $g \geqslant 2$ the automorphism group $\operatorname{Aut}(X)$ is finite and

$$|\operatorname{Aut}(X)| \leqslant 84(g-1).$$

Meromorphic functions on complex tori. Consider the Riemann-Roch formula from Theorem 9.10 for a complex torus $X = \mathbb{C}/\Gamma$. We know that $g = g_X = 1$, hence $2g - 1 = 1$ and thus for every divisor D on X with $\deg D > 0$ it holds $\deg D \geqslant 2g - 1$ and we obtain

$$l(D) = \deg D + 1 - g = \deg D.$$

In particular for $D = n \cdot [0]$ we obtain

$$l(D) = \begin{cases} 1, & \text{if } n = 0; \\ n, & \text{if } n \geqslant 1. \end{cases} \tag{4}$$

This gives $l(2 \cdot [0]) = 2$, i. e., there exists a non-constant meromorphic function on X with the only pole at $[0]$ or multiplicity 2.

Reminder 11.3. Recall that meromorphic functions on $\mathbb{C}/\Gamma$ are in one-to-one correspondence with doubly periodic (elliptic) meromorphic functions on $\mathbb{C}$ with respect to Γ (Theorem 2.8).

So there must exist an elliptic function on $\mathbb{C}$ with respect to Γ with poles of order 2 at the points of Γ.

A naïve attempt to construct such a function could be to consider the sum

$$\sum_{\gamma \in \Gamma} \frac{1}{(z-\gamma)^2},$$

but this sum is infinite and is not convergent in any reasonable sense. However one can slightly modify this idea in order to get the required function. Put

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \gamma \in \Gamma} \left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right).$$

This infinite sum is summable (one can read about this (in German) in [11]) and defines an elliptic function on $\mathbb{C}$ with respect to Γ with poles of order 2 at the points of Γ. Of course, this function depends on a given $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ or $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, so to indicate this dependence one uses the notations

$$\wp(z) = \wp(z; \Gamma) = \wp(z; \omega_1, \omega_2) = \wp(z; \tau).$$

Definition 11.4. $\wp$ is called the Weierstraß $\wp$-function.

The derivative of the Weierstraß $\wp$-function

$$\wp'(z) = -\sum_{\gamma\in\Gamma} \frac{2}{(z-\gamma)^3}$$

has clearly poles of order 3 at the points of Γ, so it defines a meromorphic function on $\mathbb{C}/\Gamma$ with the only pole of multiplicity 3 at $[0]$. Note that $\wp(z)$ and $\wp'(z)$ are linearly independent. Therefore, (4) implies

$$\mathcal{L}([0]) = \mathbb{C}\cdot 1, \quad \mathcal{L}(2\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z), \quad \mathcal{L}(3\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z),$$

where we use the same notations for elliptic functions and the corresponding meromorphic functions on $\mathbb{C}/\Gamma$.

Combining $\wp(z)$ and $\wp'(z)$ with each other one easily produces examples of meromorphic functions from $\mathcal{L}(n\cdot[0])$ for every $n\in\mathbb{N}$. For example $\wp^2(z)\in\mathcal{L}(4\cdot[0])$, $\wp(z)\wp'(z)\in\mathcal{L}(5\cdot[0])$. Of course, one can also take higher derivatives, then $\wp''(z)\in\mathcal{L}(4\cdot[0])$, etc.

Combining $\wp(z)$ and $\wp'(z)$ and using (4) one easily computes $\mathcal{L}(4\cdot[0])$ and $\mathcal{L}(5\cdot[0])$.

Exercise. $\mathcal{L}(4\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z)$, $\mathcal{L}(5\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z) + \mathbb{C}\cdot\wp(z)\wp'(z)$.

Let now $n=6$. Then $l(6\cdot[0])=6$. However the functions

$$1,\quad \wp,\quad \wp',\quad \wp^2,\quad \wp\wp',\quad \wp^3,\quad (\wp')^2$$

all belong to $\mathcal{L}(6\cdot[0])$. Therefore they must be linearly dependent. This means that there must exist a polynomial in two variables $f(x,y)\in\mathbb{C}[x,y]$, with monomials $1, x, y, x^2, xy, x^3, y^2$ such that

$$f(\wp,\wp') = 0.$$

Let us find this polynomial.

Algebraic relation between $\wp$ and $\wp'$.

Claim. *The Weierstraß $\wp$-function can be given as*

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2(n+1)}\cdot z^{2n},$$

where the coefficients

$$G_m = \sum_{0\neq\gamma\in\Gamma}\gamma^{-m}, \quad m\geqslant 3.$$

are called the Eisenstein series.

Proof. Exercise. □

One computes

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \dots,$$
$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \dots,$$
$$(\wp'(z))^2 = \frac{4}{z^6} - 24G_4 \frac{1}{z^2} - 80G_6 + \dots,$$
$$\wp^3(z) = \frac{1}{z^6} + 9G_4 \frac{1}{z^2} + 15G_6 + \dots.$$

Therefore,

$$(\wp'(z))^2 - 4\wp^3(z) = -60G_4 \frac{1}{z^2} - 140G_6 + \dots,$$
$$(\wp'(z))^2 - 4\wp^3(z) + 60G_4\wp(z) = -140G_6 + \dots,$$

which means that $(\wp'(z))^2 - 4\wp^3(z) + 60G_4\wp(z)$ is holomorphic, thus it must be constant, i. e.,

$$(\wp'(z))^2 - 4\wp^3(z) + 60G_4\wp(z) = -140G_6.$$

We obtained the following statement.

Proposition 11.5. *Let $g_2 = 60G_4$, $g_3 = 140G_6$. Put*

$$f(x, y) = y^2 - 4x^3 + g_2 x + g_3.$$

Then $f(\wp, \wp') = 0$.

Exercises.

Exercise 40. Let $\pi : \mathbb{H} \to \mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ be the projection map and let the set of orbits $\mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$ be equipped with the quotient topology.

(1) Let $\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2} = e^{\frac{2\pi}{3}i}$. Show that π is a local homeomorphism outside of the orbits of i and ρ.

(2) Show that for every $\tau \in \mathbb{H}$ from the orbit of i or ρ every open neigbourhood of τ contains different points with the same image under π.

Hint: *For small ϵ consider in the case $\tau = i$ the pair of numbers $e^{i(\frac{\pi}{2}+\epsilon)}$ and $e^{i(\frac{\pi}{2}-\epsilon)}$; for $\tau = \rho$ consider the pair $e^{i(\frac{2\pi}{3}+\epsilon)}$ and $-1 + e^{i(\frac{\pi}{3}-\epsilon)}$.*

Exercise 41. Let Γ be a lattice in $\mathbb{C}$, let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus, and let $\wp(z)$ be the corresponding Weierstraß function.

Notice that by the Riemann-Roch theorem $l(4 \cdot [0]) = 4$. On the other hand the functions $1, \wp, \wp', \wp^2, \wp''$ belong to $\mathcal{L}(4 \cdot [0])$. Conclude that they are linear dependant and find a linear relation between them. You could do it directly or using the relation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

from the lecture.

Exercise 42. Consider for a lattice $\Gamma \subset \mathbb{C}$ the Eisenstein series $G_4 = G_4(\Gamma) = \sum_{0\neq\gamma\in\Gamma} \gamma^{-4}$, $G_6 = G_6(\Gamma) = \sum_{0\neq\gamma\in\Gamma} \gamma^{-6}$. Let $\Gamma(\tau) = \mathbb{Z} + \mathbb{Z}\cdot\tau$. As in the lecture, denote $\rho = e^{i\frac{2\pi}{3}}$. Compute

$$G_4(\Gamma(\rho)) = 0, \quad G_6(\Gamma(i)) = 0.$$

Hint: *Notice that one can exchange the order of the summands in the Eisenstein series.*
For $\Gamma = \Gamma(\rho)$ define the subset $\Gamma' \subset \Gamma$ by $\Gamma' = \{\gamma \in \Gamma \mid \gamma = r \cdot e^{i\varphi} \text{ with } 0 \leqslant \varphi < \frac{\pi}{3}\}$. Observe that Γ can be seen as the disjoint union of the rotations of Γ', namely of the sets $e^{i\frac{\pi k}{3}} \cdot \Gamma'$, $k = 0, 1, \ldots, 5$. Notice that $\sum_{k=0}^{5} e^{-4i\frac{\pi k}{3}} = 0$.
For $\Gamma = \Gamma(i)$ define $\Gamma' = \{\gamma \in \Gamma \mid \gamma = r \cdot e^{i\varphi} \text{ with } 0 \leqslant \varphi < \frac{\pi}{2}\}$. Observe that Γ is the disjoint union of Γ', $i\Gamma'$, $-\Gamma'$, and $-i\Gamma'$. Use that $\sum_{k=0}^{3} e^{-6i\frac{\pi k}{2}} = 0$.

Exercise 43. Let Γ be a lattice in $\mathbb{C}$ and let $\wp$ be the corresponding Weierstraß function.

(1) Notice that $\wp'(z)$ considered as a meromorphic function on $\mathbb{C}/\Gamma$ has its only pole at $[0]$ of multiplicity 3. How many zeroes could $\wp'(z)$ have? Using that $\wp'$ is elliptic and odd, show that the points $[\frac{\omega_1}{2}]$, $[\frac{\omega_2}{2}]$, $[\frac{\omega_1+\omega_2}{2}]$ are zeroes of $\wp'(z)$. Are there any other zeroes of $\wp'(z)$?

(2) Show that $\wp(z) = \wp(w)$ if and only if either $z = w \mod \Gamma$ or $z = -w \mod \Gamma$.

Hint: *For a fixed w consider $h(z) = \wp(z) - \wp(w)$ and study its set of zeroes using that $\wp(z)$ is an even function. How many zeroes can $h(z)$ have? When can $h(z)$ have a multiple zero?*

12. Lecture 12

Our next aim is to determine the field $\mathcal{M}_X(X)$ of meromorphic functions on a complex torus X.

Identify $\mathcal{M}_X(X)$ with the field of elliptic functions on $\mathbb{C}$ with respect to Γ.

Let $f(z)$ be an elliptic function, then

$$f(z) = \frac{1}{2}(f(z) + f(-z)) + \frac{1}{2}(f(z) - f(-z)).$$

Put $g(z) = \frac{1}{2}(f(z)+f(-z))$ and $h(z) = \frac{1}{2}(f(z)-f(-z))$, then $f(z) = g(z)+h(z)$, $g(-z) = g(z)$ and $h(-z) = -h(z)$, i. e., g is even and h is odd. This proves the following.

Claim. *Every elliptic function on $\mathbb{C}$ can be represented as a sum of an even elliptic function f with an odd elliptic function h.*

Even elliptic functions. Our first observation is that $\wp(z)$ is even.

Theorem 12.1. *Let $f(z)$ be an even elliptic function. Then there exists a rational function in one variable $\Phi(t) \in \mathbb{C}(t)$ such that $f = \Phi(\wp)$. Moreover, if the poles of f are contained in Γ, then Φ can be taken polynomial.*

Proof. Assume that the poles of f are contained in Γ. Consider the Laurent expansion of f at 0. Since f is even, we get

$$f = \sum_{i>-n} a_{2i} z^{2i}.$$

Hence the poles of f must have an even order. Consider the principal part of f at 0:

$$a_{-2n} z^{-2n} + \cdots + a_{-1} z^{-2}.$$

Note that the Laurent expansion of $\wp(z)$ at zero is

$$\frac{1}{z^2} + b_2 z^2 + b_4 z^4 + \ldots.$$

Its principal part is $\frac{1}{z_2}$. One concludes that the principal part of $\wp^l(z)$ is of the form

$$\frac{1}{z^{2l}} + \text{linear combination of } \frac{1}{z^{2\nu}} \text{ with } \nu < l.$$

Then $f - a_{-2n}\wp^n(z)$ has poles of smaller multiplicity that f. So, by induction one gets that for some coefficients $\lambda_i \in \mathbb{C}$ the function $f - \sum_{i\geqslant 1}^{n} \lambda_i \wp^i$ is holomorphic, hence constant, say λ_0. Then

$$f = \sum_{i\geqslant 0}^{n} \lambda_i \wp^i = \Phi(\wp), \quad \Phi(t) = \sum_{i\geqslant 0}^{n} \lambda_i t^i.$$

Let now f be an arbitrary even elliptic function. Modulo Γ it can have only finitely many poles outside Γ. Let $p_1, \ldots, p_r$ be the corresponding representatives of all poles not belonging to Γ. Then $\wp(z) - \wp(p_i)$ has a zero at p_1. Let ν_i be the multiplicity of the pole p_i of f. Then

$$h(z) = f \cdot \prod_{i=1}^{r} (\wp(z) - \wp(p_i))^{\nu_i}$$

does not have any poles outside of Γ and therefore there exists a polynomial $\Psi(t) \in \mathbb{C}[t]$ such that $\Psi(\wp) = h(z)$. Then

$$f = \frac{h(z)}{\prod_{i=1}^{r} (\wp(z) - \wp(p_i))^{\nu_i}} = \frac{\Psi(\wp)}{\prod_{i=1}^{r} (\wp(z) - \wp(p_i))^{\nu_i}},$$

i. e., $f = \Phi(\wp)$ for

$$\Phi(t) = \frac{\Psi(t)}{\prod_{i=1}^{r} (t - \wp(p_i))^{\nu_i}} \in \mathbb{C}(t).$$

This concludes the proof. □

Odd elliptic functions. Notice that $\wp'(z)$ is odd. Let f be an arbitrary odd elliptic function. Then $\frac{f}{\wp'}$ is an even elliptic function, hence there exists $\Phi(t) \in \mathbb{C}(t)$ such that $f = \wp' \cdot \Phi(\wp)$. Finally we get

Theorem 12.2. *Let $X = \mathbb{C}/\Gamma$ be a complex torus. Let $\wp(z) = \wp(z; \Gamma)$ be the corresponding Weierstraß $\wp$-function. Then $\mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma) = \mathbb{C}(\wp) + \wp'(z)\mathbb{C}(\wp)$*

Remark 12.3. Notice that the proof of Theorem 12.2 is constructive

Corollary 12.4. $\mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma) \cong \mathbb{C}(x)[y]/(y^2 - 4x^3 + g_2 x + g_3)$*, where* $g_2 = 60 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^4}$, $g_3 = 140 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^6}$

Proof. Define a surjective homomorphism

$$\mathbb{C}(x)[y] \to \mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma), \quad x \mapsto \wp(z), \quad \mapsto \wp'(z).$$

Then by Proposition 11.5 $y^2 - 4x^3 + g_2 x + g_3$ lies in the kernel and we obtain a surjection

$$\mathbb{C}(x)[y]/(y^2 - 4x^3 + g_2 x + g_3) \to \mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma).$$

Since f is irreducible polynomial over $\mathbb{C}(x)$, we conclude that $\mathbb{C}(x)[y](y^2 - 4x^3 + g_2 x + g_3)$ is a field. Since non-zero field homomorphisms are injective, we conclude that $\mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma) \cong \mathbb{C}(x)[y]/(y^2 - 4x^3 + g_2 x + g_3)$. This concludes the proof. □

Complex tori as smooth projective algebraic plane curves. Recall that the projective plane

$$\mathbb{P}_2 = \{\langle x_0, x_1, x_2\rangle \mid (x_0, x_1, x_2) \in \mathbb{C}^3 \setminus \{0\}\},$$

has a natural structure of a complex manifold.

Definition 12.5. A plane projective curve C is the set of zeroes of a homogeneous polynomial $f \in \mathbb{C}[z_0, z_1, z_2]$

$$C = Z(f) = \{\langle x_0, x_1, x_2 \in \mathbb{P}_2 \mid f(x_0, x_1, x_2) = 0\rangle\}.$$

C is called smooth is it is a complex submanifold of $\mathbb{P}_2$ (in this case it is a Riemann surface).

Claim. *$C = Z(f) \subset \mathbb{P}_2$ is smooth if and only if*

$$Z(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}) = \{\langle x_0, x_1, x_2\rangle \in \mathbb{P}_2 \mid \frac{\partial f}{\partial z_i}(x_0, x_1, x_2) = 0, i = 0, 1, 2\}$$

is empty, i. e., the partial derivatives of f do not have common zeroes in $\mathbb{P}_2$.

Proof. Exercise. □

Theorem 12.6. *Every complex torus $\mathbb{C}/\Gamma$ is isomorphic to a smooth projective plane cubic curve. More precisely, $\mathbb{C}/\Gamma \cong Z(f)$, where*

$$f = z_0 z_2^2 - 4z_1^3 + g_2 z_0^2 z_1 + g_3 z_0^3, \quad g_2 = 60 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^4}, \quad g_3 = 140 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^6}.$$

The isomorphism is given by the map

$$\mathbb{C}/\Gamma \xrightarrow{\varphi} \mathbb{P}_2, \quad [z] \mapsto \begin{cases} \langle 1, \wp(z), \wp'(z)\rangle, [z] \neq [0]; \\ \langle 0, 0, 1\rangle, [z] = [0]. \end{cases}$$

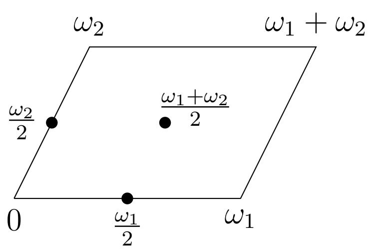
Proof(Sketch). Let $C = Z(f)$. From the discussion above it is clear that $\varphi(\mathbb{C}/\Gamma) \subset C$.

I. **Bijectivity of $\varphi : \mathbb{C}/\Gamma \to C$.**

I.1. **Injectivity.**

Lemma 12.7. *1) $\wp(z) = \wp(w)$ if and only if $z = w \mod \Gamma$ or $z = -w \mod \Gamma$.*

2) $\wp'(z) = 0$ if and only if $2z \in \Gamma$, i. e., there are three different $\mod \Gamma$ zeroes $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_1+\omega_2}{2}$.



So if $z, w \notin \Gamma$ such that $\varphi(z) = \varphi(w)$, then $\wp(z) = \wp(w)$, $\wp'(z) = \wp'(w)$. So either $w = z \mod \Gamma$ (and hence $[z] = [w]$) or $z = -w \mod \Gamma$ and $\wp'(z) = \wp'(-w) = -\wp'(w) = -\wp'(z)$. In the second case $2\wp'(z) = 0$, thus $\wp'(z) = 0$. Then by Lemma 12.7 $2z \in \Gamma$ and finally $z = w \mod \Gamma$. Since $\varphi([z]) \neq \langle 0, 0, 1\rangle$ for all $[z] \neq [0]$, we conclude that φ is injective.

Remark 12.8. In particular $\wp$ takes different values at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ (i. e., at zeroes of $\wp'$). Put $h(x) = 4x^3 - g_2x - g_3$. Then since $\wp'(z)^2 = h(\wp(z))$, we conclude that $\wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$ are 3 different zeroes of h, thus

$$h(x) = 4\left(x - \wp\left(\frac{\omega_1}{2}\right)\right) \cdot \left(x - \wp\left(\frac{\omega_2}{2}\right)\right) \cdot \left(x - \wp\left(\frac{\omega_1 + \omega_2}{2}\right)\right).$$

I.2 **Surjectivity.** It is clear that $\langle 0, 0, 1\rangle \in \varphi(\mathbb{C}/\Gamma)$.

Take an arbitrary $\langle 1, a, b\rangle \in C$. Since $\wp$ takes all values,there exists $z \in \mathbb{C}$ with $\wp(z) = a$. Since $b^2 = \wp'(z)^2 = h(\wp(z)) = h(a)$ we conclude $\wp'(z) = \pm b$. If $\wp'(z) = b$, then $\varphi([z]) = \langle 1, a, b\rangle$. If $\wp'(z) = -b$, then $\varphi([-z]) = \langle 1, \wp(-z), \wp'(-z)\rangle = \langle 1, \wp(z), -\wp'(z)\rangle = \langle 1, a, b\rangle$.

II. C **is a smooth curve in $\mathbb{P}_2$ (i. e., submanifold).** Indeed. Suppose the contrary. Then there exists $s = \langle s_0, s_1, s_2\rangle \in \mathbb{P}_2$ such that

$$\frac{\partial f}{\partial z_0}(s) = \frac{\partial f}{\partial z_1}(s) = \frac{\partial f}{\partial z_2}(s) = 0.$$

One computes that this implies that

$$\Delta = g_2^3 - 27g_3^2 = 0.$$

On the other hand one notes that Δ is the discriminant of $h(x) = 4x^3 - g_2x - g_3$. Since the latter has 3 zeroes, we get $\Delta \neq 0$ and thus a contradiction. Therefore C is smooth.

III. From the definition of φ it follows that it is continuous. Clearly φ is holomorphic on $\mathbb{C}/\Gamma \setminus \{[0]\}$. By Theorem 2.1 φ is a holomorphic map to $\mathbb{P}_2$. Its image C is a submanifold, so $\varphi : \mathbb{C}/\Gamma \to C$ is a holomorphic map of Riemann sirfaces. Since it is bijective, we conclude that φ is an isomorphism, which concludes the proof. □

Definition 12.9. Smooth projective plane cubic curves are called elliptic curves. So complex tori are elliptic curves.

j-invariant. We defined for $\tau \in \mathbb{H}$ $g_2 = g_2(\tau)$, $g_3 = g_3(\tau)$. Thus one can consider g_2 and g_3 as functions on $\mathbb{H}$. These functions are holomorphic on $\mathbb{H}$. One can show that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$

$$g_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^4 \cdot g_2(\tau), \quad g_3\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^6 \cdot g_3(\tau).$$

One says in this situation that g_2 is a modular form of weight 4 and g_3 is a modular form of weight 6.

Then $\Delta = g_2^3 - 27g_3^2$ has the property

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12} \cdot \Delta(\tau)$$

and one says that Δ is a modular form of weight 12. We showed above that $\Delta = g_2^3 - 27g_3^2 \neq 0$, so one obtains the following holomorphic function on $\mathbb{H}$:

$$j(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)}.$$

Then

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau),$$

so j is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Definition 12.10. The holomorphic function $j : \mathbb{H} \to \mathbb{C}$ is called j-invariant.

Therefore, there exists a unique factorization through $\mathbb{H} \xrightarrow{\pi} \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$, which by abuse of notation is denoted by j as well.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad j \quad} & \mathbb{C} \\ & {\scriptstyle\pi}\searrow \quad \exists! \quad \swarrow & \\ & \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) & \end{array}$$

Theorem 12.11. *The map*

$$\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{j} \mathbb{C}, \quad [\tau] \mapsto j(\tau)$$

is a bijection, i. e., two complex tori $\mathbb{C}/\Gamma(\tau)$ *and* $\mathbb{C}/\Gamma(\tau')$ *are isomorphic if and only if* $j(\tau) = j(\tau')$.

Proof. No proof. A proof can be found for example in [5]. □

Exercises.

Exercise 44. Let Γ be a lattice in $\mathbb{C}$ and let $\wp$ be the corresponding Weierstraß function. Notice that the elliptic functions $\wp''''(z)$ and $\wp'(z)\cdot\wp'''(z)$ are even with poles in Γ. Represent them as polynomials in $\wp$.

Exercise 45. Let Γ be a lattice in $\mathbb{C}$ and let $\wp$ be the corresponding Weierstraß function. Notice that the elliptic functions $\wp'''(z)$ and $\wp^{(5)}(z)$ are odd. Represent them as $\wp'\cdot\Psi(\wp)$ for some $\Psi(t)\in\mathbb{C}(t)$.

Exercise 46. In the lecture we showed that

$$\mathcal{M}_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma)\cong\mathbb{C}(x)[y]/(y^2-4x^3+g_2x+g_3).$$

Find the inverse of y^3 in $\mathbb{C}(x)[y]/(y^2-4x^3+g_2x+g_3)$. Use it to express $(1/\wp'(z))^3$ as a polynomial in $\wp'$ with coefficients in $\mathbb{C}(\wp)$.

Exercise 47. In the lecture we defined j-invariant

$$j(\tau)=\frac{g_2^3(\tau)}{\Delta(\tau)},\quad \Delta(\tau)=g_2^3(\tau)-27g_3^2(\tau).$$

Compute the following values of j-invariant:

$$j\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=0,\quad j(i)=1.$$

In other words show that

$$g_2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=0,\quad g_3(i)=0.$$

13. Lecture 13

Integration of differential forms. Let $U \subset X$ be an open subset of a Riemann surface X. Let $\omega \in \Omega_X(U)$.

Let $\gamma : [a, b] \to U$ be a smooth (i. e., piece-wise differentiable) path. This means that for every chart $\varphi_i : U_i \to V_i$, $U_i \subset U$, the functions $\varphi_i \circ \gamma : \gamma^{-1}(U_i) \to V_i$ are piece-wise differentiable.

I. Assume there exists a chart $\varphi : W \to V$, $W \subset U$ such that $\gamma([a, b]) \subset W$. Write $\omega|_W = f \cdot d\varphi$ for $f \in \mathcal{O}_X(W)$ and define

$$\int_\gamma \omega := \int_a^b f(\gamma(t)) \cdot (\varphi(\gamma(t)))' dt$$

Claim. *This definition does not depend on the choice of φ.*

Proof. Exercise. □

II. One can always choose a partition of the interval $[a, b]$, i. e.,

$$a = a_0 < a_1 < \cdots < a_m = b$$

such that for $\gamma_i := \gamma|_{[a_{i-1}, a_i]} : [a_{i-1}, a_i] \to X$ there exists a chart $\varphi_i : U_i \to V_i$ of X with $\gamma_i([a_{i-1}, a_i]) \subset U_i$. Define now

$$\int_\gamma \omega := \sum_{i=1}^m \int_{\gamma_i} \omega.$$

Claim. *This definition does not depend on the choice of the partition.*

Proof. Exercise. □

So, for every open subset $U \subset X$, for every $\omega \in \Omega_X(U)$, and for every smooth path $\gamma : [a, b] \to U$, we get

$$\int_\gamma \omega \in \mathbb{C}.$$

Remark 13.1. Analogously, for an open set $U \subset X$, for $\omega \in \mathcal{K}_X(U)$, and and for a smooth path $\gamma : [a, b] \to U$ such that $\gamma([a, b])$ does not contain poles of ω, one gets $\int_\gamma \omega$ as well. Indeed, just replace U by $U' = U \setminus \{\text{poles of } \omega\}$. Then $\omega \in \Omega_X(U')$ and $\gamma([a, b]) \subset U'$.

Properties. I. Reparameterisation invariance. Let $[a', b'] \xrightarrow{\alpha} [a, b]$ be a smooth map such that $\alpha(a') = a$, $\alpha(b') = b$. Let $\gamma : [a, b] \to X$ be a smooth path. Then $\gamma \circ \alpha : [a', b']$ is a smooth path as well and

$$\int_\gamma \omega = \int_{\gamma\circ\alpha} \omega.$$

II. Linearity. $\int\limits_{\gamma}(\lambda\omega_1 + \mu\omega_2) = \lambda\int\limits_{\gamma}\omega_1 + \mu\int\limits_{\gamma}\omega_2$ for differential forms ω_1, ω_2 around γ and for $\lambda, \mu \in \mathbb{C}$.

III. Let $\gamma : [a, b] \to X$ be a smooth path, let U be a neighbourhood of $\gamma([a, b])$, let $f \in \mathcal{O}_X(U)$. Then

$$\int\limits_{\gamma} df = f(\gamma(b)) - f(\gamma(a)).$$

IV. Let $\{\gamma_i\}_1^n$ be a partition of a smooth path γ, i. e, $\gamma = \gamma_1\gamma_2\dots\gamma_n$. Then

$$\int\limits_{\gamma} \omega = \sum_{i=1}^{n} \int\limits_{\gamma_i} \omega.$$

V. Let γ^{-1} be the inverse path to a smooth path γ. Then

$$\int\limits_{\gamma^{-1}} \omega = -\int\limits_{\gamma} \omega.$$

Remark 13.2. Every continuous path can be approximated by smooth paths. This allows to define integrals of differential forms over arbitrary continuous paths.

Theorem 13.3. *Let X be a Riemann surface. Let $\omega \in \Omega_X(X)$. Let $\gamma \sim \delta$ be two homotopic paths. Then*

$$\int\limits_{\gamma} \omega = \int\limits_{\delta} \omega.$$

Proof (hint). This is a consequence of the Stokes' theorem. □

Corollary 13.4. *Let X be a RS, let $x_0 \in X$. Consider the fundamental group $\pi_1(X, x_0)$. Let $\omega \in \Omega_X(X)$, then*

$$\pi_1(X, x_0) \to \mathbb{C}, \quad [\gamma] \mapsto \int\limits_{\gamma} \omega$$

is a well-defined group homomorphism.

Proof. The map is well-defined by the previous theorem. Let γ, δ be two closed paths at x_0. By property (IV) of integrals it holds

$$\int\limits_{\gamma\cdot\delta} \omega = \int\limits_{\gamma} \omega + \int\limits_{\delta} \omega.$$

Thus the map $[\gamma] \mapsto \int\limits_{\gamma} \omega$ is a group homomorphism for every $\omega \in \Omega_X(X)$. □

Definition 13.5. The number $\int_\gamma \omega$ is called period of γ with respect to ω. The homomorphism

$$\int_{-} \omega : \pi_1(X, x_0) \to \mathbb{C}, \quad [\gamma] \mapsto \int_\gamma \omega,$$

is called the period homomorphism.

Exercise. Compute the periods of the generators of $\pi_1(\mathbb{C}/\Gamma)$ with respect to some generator ω of $\Omega_{\mathbb{C}/\Gamma}(\mathbb{C}/\Gamma)$.

Definition 13.6. Let $\omega \in \mathcal{K}_X(U)$ for some open subset U of a RS X. Let $a \in U$. Let $z : U' \to V$ be a local coordinate at a. Let $\omega|_{U'} = f dz$ for some $f \in \mathcal{M}_X(U')$. Define

$$\operatorname{res}_a \omega := \operatorname{res}_{z(a)}(f \circ z^{-1}),$$

this number is called the residue of ω at a.

Reminder 13.7. Let $U \subset \mathbb{C}$ be open, let $b \in U$, $f \in \mathcal{O}_X(U \setminus \{b\})$, and let

$$f(z) = \sum_i c_i (z - b)^i$$

be its Laurent power series at b. Then

$$\operatorname{res}_b f = c_{-1}.$$

Equivalently

$$\operatorname{res}_b f = \frac{1}{2\pi i} \oint_b f dz.$$

Remark 13.8. It makes no sense to define residues of meromorphic functions on RS because it would depend on the choice of local coordinates.

Claim. $\operatorname{res}_a \omega$ *defined as in Definition 13.6 does not depend on the choice of a local coordinate.*

Theorem 13.9 (Residue theorem)**.** *Let X be a compact RS, let $\omega \in \mathcal{K}_X(X)$. Then*

$$\sum_{x \in X} \operatorname{res}_x \omega = 0.$$

Proof (hint). Follows from the Stokes' theorem. □

Example 13.10. Let $f \in \mathcal{M}_X(X)$. Put $\omega = \frac{df}{f}$. The residue theorem reads then as

$$\sum_{p \in X} \operatorname{res}_p \frac{df}{f} = 0.$$

For every $p \in X$ choose a local coordinate z at p and write f locally around p as $f = z^k \widetilde{f}$, where $\widetilde{f}$ is a holomorphic function around p such that $\widetilde{f}(p) \neq 0$ and $k = \operatorname{ord}_p f$. Then

$$df = (kz^{k-1}\widetilde{f} + z^k \frac{\partial \widetilde{f}}{\partial z})dz$$

and therefore

$$\frac{df}{f} = \left(\frac{k}{z} + \frac{\frac{\partial \widetilde{f}}{\partial z}}{\widetilde{f}} \right) dz.$$

This means $\operatorname{res}_p \frac{df}{f} = k = \operatorname{ord}_p f$, so the residue theorem reads as

$$\sum_{p \in X} \operatorname{ord}_p f = 0,$$

which we already know.

Theorem 13.11. *Let $S \subset X$ be a finite set. For $a \in S$ let U_a be an open neighbourhood such that $U_a \cap U_b = \emptyset$ for $a \neq b$. Let $\omega_a \in \mathcal{K}_X(U_a)$ such that $\omega_a \in \Omega_X(U_x \setminus \{a\})$. Let $\sum\limits_{a \in S} \operatorname{res}_a \omega_a = 0$. Then there exists $\omega \in \mathcal{K}_X(X)$ such that S is its set of poles and $\omega|_{U_a} - \omega_a \in \Omega_X(U_a)$.*

Proof. Without. □

Remark 13.12. This means that the the condition $\sum_{x \in X} \operatorname{res}_x \omega = 0$ from the residue theorem is the only restriction for the existence of meromorphic differential forms.

Corollary 13.13. *On every compact Riemann surface X there exists a non-constant meromorphic function $f \in \mathcal{M}_X(X)$.*

Proof. For every two different points $p_1, p_2 \in X$ there exist differential forms $\omega_1, \omega_2 \in \mathcal{K}_X(X)$ such that p_1 is the only pole of ω_1 with $\operatorname{ord}_{p_1} \omega_1 = -2$, p_2 is the only pole of ω_2, $\operatorname{ord}_{p_2} \omega_2 = -2$. Then $\omega_1 = f \cdot \omega_2$ for some $f \in \mathcal{M}_X(X)$. One sees that f should be non-constant. □

Exercises.

Exercise 48. Consider the lattice $\Gamma = \mathbb{Z} \cdot 5 + \mathbb{Z} \cdot (2+3i)$. Let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus. Consider the path $\gamma : [0,1] \to X, \quad \gamma(t) = [(12+9i) \cdot t]$. Let ω be the standard generator of $\Omega_X(X)$, i. e., for every chart $\varphi : U \to V$ it holds $\omega|_U = d\varphi$. Compute

$$\int_\gamma \omega.$$

Exercise 49. Let $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ be a lattice in $\mathbb{C}$. Let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus.

Define $\delta_1 : [0,1] \to X$ by $\delta_1(t) = [t \cdot \gamma_1]$ and $\delta_2 : [0,1] \to X$ by $\delta_2(t) = [t \cdot \gamma_2]$. Notice that δ_1 and δ_2 are smooth closed paths at the point $[0] \in X$. Moreover, they generate the fundamental group of X.

Let ω be the standard generator of $\Omega_X(X)$, i. e., for every chart $\varphi : U \to V$ it holds $\omega|_U = d\varphi$. Compute the integrals

$$\int_{\delta_1} \omega \quad \text{and} \quad \int_{\delta_2} \omega.$$

Exercise 50. Consider the Riemann sphere $\hat{\mathbb{C}}$. Let $z = \varphi_0 : U_0 \to \mathbb{C}$ and $w = \varphi_1 : U_1 \to \mathbb{C}$ be the standard charts. Consider the meromorphic function $f = \frac{z^3}{z^2-1}$ on $\hat{\mathbb{C}}$ and define $\omega \in \mathcal{K}_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$ by the condition $\omega|_{U_0} = f dz$. Compute $\operatorname{res}_1 \omega$ and $\operatorname{res}_{-1} \omega$. Use the Residue theorem to obtain the value of $\operatorname{res}_\infty \omega$.

Exercise 51. Let $D = \sum\limits_{i=1}^{r} a_i \cdot x_i$ be a principal divisor on a complex torus $X = \mathbb{C}/\Gamma$, i. e., $D = (f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$. Show that

$$\sum_{i=1}^{r} a_i \cdot x_i = 0$$

as an element of $X = \mathbb{C}/\Gamma$.

Hint: Let $\pi : \mathbb{C} \to X$ be the canonical projection. Consider $F(z) = f \circ \pi(z)$. Choose a fundamental parallelogram V in $\mathbb{C}$ such that there are no poles or zeros of F on its boundary ∂V. Consider the integral

$$\int_{\partial V} z \cdot \frac{F'(z)}{F(z)}\, dz$$

and apply the standard residue theorem.

Theorem. *For a meromorphic function g on an open set $V \subset \mathbb{C}$ which possesses a continuous extension to the closure of V one has*

$$\frac{1}{2\pi i}\int\limits_{\partial V} g(z)dz = \sum_{a\in V} \operatorname{res}_a g.$$

14. Lecture 14

Definition 14.1. Let X be a compact RS, let

$$\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p$$

be some representatives of generators of the fundamental group $\pi_1(X)$ of X (cf. Lecture 4). Let $\omega \in \Omega_X(X)$, define $A_i(\omega) = \int\limits_{\alpha_i} \omega$, $B_i(\omega) = \int\limits_{\beta_i} \omega$. We obtain the linear maps

$$\Omega_X(X) \xrightarrow{A} \mathbb{C}^p, \quad \omega \mapsto (A_1(\omega), A_2(\omega), \dots, A_p(\omega)),$$

$$\Omega_X(X) \xrightarrow{B} \mathbb{C}^p, \quad \omega \mapsto (B_1(\omega), B_2(\omega), \dots, B_p(\omega)).$$

Theorem 14.2. *A and B are isomorphisms of vector spaces.*

Proof. No proof. A proof can be deduced from the theory of harmonic functions. □

Corollary 14.3. *Let $\omega \in \Omega_X(X)$. Then*

$$\omega = 0 \quad \Leftrightarrow A_i(\omega) = 0 \ \forall i \quad \Leftrightarrow \ B_i(\omega) = 0 \ \forall i.$$

Definition 14.4. Fix a basis of $\Omega_X(X)$, say $\{\omega_1, \dots, \omega_g\}$ (assume $g \geqslant 1$). Then for every closed curve α in X at $x_0 \in X$ the vector

$$(\int\limits_{\alpha} \omega_1, \dots, \int\limits_{\alpha} \omega_g) \in \mathbb{C}^g$$

is called a period of X with respect to $\{\omega_1, \dots, \omega_g\}$.

Denote by $L = L(\omega_1, \dots, \omega_g) \subset \mathbb{C}^g$ the set of all periods of X with respect to $\{\omega_1, \dots, \omega_g\}$. Since

$$\int\limits_{\alpha} \omega + \int\limits_{\beta} \omega = \int\limits_{\alpha \cdot \beta} \omega,$$

we see that L is subgroup of $\mathbb{C}^g$.

Consider an arbitrary period $(\int\limits_{\alpha} \omega_1, \dots, \int\limits_{\alpha} \omega_g)$. Since $[\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g]$ generate the fundamental group, $[\alpha]$ can be expressed as a product of their powers. Then

$$(\int\limits_{\alpha} \omega_1, \dots, \int\limits_{\alpha} \omega_g)$$

is a linear combination of

$$(\int\limits_{\alpha_i} \omega_1, \dots, \int\limits_{\alpha_i} \omega_g), \quad i = 1, \dots, g, \quad \text{and} \quad (\int\limits_{\beta_j} \omega_1, \dots, \int\limits_{\beta_j} \omega_g), \quad j = 1, \dots, g,$$

with integer coefficients. In other words,

$$(\int_\alpha \omega_1, \dots, \int_\alpha \omega_g)$$

is a linear combination with integer coefficients of the rows of the **period matrix**

$$\begin{pmatrix} A_1(\omega_1) & \dots & A_1(\omega_g) \\ \vdots & \ddots & \vdots \\ A_g(\omega_1) & \dots & A_g(\omega_g) \\ B_1(\omega_1) & \dots & B_1(\omega_g) \\ \vdots & \ddots & \vdots \\ B_g(\omega_1) & \dots & B_g(\omega_g) \end{pmatrix}.$$

So the rows of the period matrix generate L as an abelian group.

One sees that the rank (over $\mathbb{C}$) of the period matrix is g. Moreover, one can show that its rows are linearly independent over $\mathbb{R}$. This means that L is a free abelian subgroup of $\mathbb{C}^g$ of rank $2g$, i. e., a lattice in $\mathbb{C}^g$.

Definition 14.5. Define the Jacobian of X by

$$\operatorname{Jac}(X) := \mathbb{C}^g/L.$$

One introduces a complex structure on $\operatorname{Jac}(X)$ as for one-dimensional complex tori (page 5). Then $\operatorname{Jac}(X)$ is a complex manifold of dimension g.

Exercise. $\operatorname{Jac}(\mathbb{C}/\Gamma) \cong \mathbb{C}/\Gamma$.

Fix a point $q \in X$ of a compact Riemann surface X. For a point $x \in X$ take some path γ_x from q from x and consider

$$\left(\int_q^x \omega_1, \int_q^x \omega_2, \dots, \int_q^x \omega_g\right) := \left(\int_{\gamma_x} \omega_1, \dots, \int_{\gamma_x} \omega_g\right).$$

It is an element in $\mathbb{C}^g$. Of course it depends on the choice of γ_x. However if δ_x is another path connecting q and x, for every $\omega \in \Omega_X(X)$

$$\int_{\gamma_x} \omega - \int_{\delta_x} \omega = \int_{\gamma_x} \omega + \int_{\delta_x^{-1}} \omega = \int_{\gamma_x \cdot \delta_x^{-1}} \omega,$$

where $\alpha_x = \gamma_x \cdot \delta_x^{-1}$ is a closed path at q. Therefore,

$$\left(\int_{\gamma_x} \omega_1, \dots, \int_{\gamma_x} \omega_g\right) - \left(\int_{\delta_x} \omega_1, \dots, \int_{\delta_x} \omega_g\right) = \left(\int_{\alpha_x} \omega_1, \dots, \int_{\alpha_x} \omega_g\right) \in L.$$

Thus the map

$$\lambda_q : X \to \operatorname{Jac}(X) = \mathbb{C}^g/L, \quad x \mapsto [\left(\int\limits_q^x \omega_1, \dots, \int\limits_q^x \omega_g\right)]$$

is well-defined.

Moreover, it is holomorphic.

Exercise. Show that λ_q is holomorphic.

Since $\operatorname{Jac}(X)$ has a natural structure of an abelian group, one can extend λ_q by linearity to a homomorphism

$$\Lambda_q : \operatorname{Div} X \to \operatorname{Jac} X, \quad \sum_{x \in X} a_x \cdot x \mapsto \sum_{x \in X} a_x \cdot \lambda_q(x).$$

Remark 14.6. Λ_q depends on the choice of $q \in X$.

Consider its restriction to the subgroup $\operatorname{Div}^0 X \subset \operatorname{Div} X$.

Claim.

$$\Lambda_q|_{\operatorname{Div}^0 X} : \operatorname{Div}^0 X \to \operatorname{Jac} X$$

does not depend on the choice of q.

Proof. Since every $D \in \operatorname{Div}^0 X$ is a sum of divisors of the form $a - b$, $a, b \in X$, $a \neq b$, it is enough to check the statement for $D = a - b$, $a \neq b$. Then

$$\Lambda_q(D) = [(\int_q^a \omega_1, \dots, \int_q^a \omega_g)] - [(\int_q^b \omega_1, \dots, \int_q^b \omega_g)] =$$

$$[(\int_q^a \omega_1 - \int_q^b \omega_1, \dots, \int_q^a \omega_g - \int_q^b \omega_g)] = [(\int_b^a \omega_1, \dots, \int_b^a \omega_g)],$$

i. e., does not depend on q. □

Definition 14.7. Define $\Lambda := \Lambda_q|_{\operatorname{Div}^0 X}$ for some (every) $q \in X$.

We obtained a homomorphism $\Lambda : \operatorname{Div}^0 X \to \operatorname{Jac} X$. Recall that for $f \in \mathcal{M}_X(X)$, $(f) \in \operatorname{Div}^0 X$. Notice that $(f) = (g)$ for $f, g \in \mathcal{M}_X(X)$ implies that $\frac{f}{g} \in \mathcal{O}_X(X) = \mathbb{C}$. Hence, to know the divisor of $f \in \mathcal{M}_X(X)$ is the same as to know f up to a multiplication by a scalar. So, to describe $\mathcal{M}_X(X)$ is the same as to describe $\operatorname{PDiv} X \subset \operatorname{Div}^0 X$.

Theorem 14.8. *I. (Abel)* $\operatorname{PDiv} X = \operatorname{Ker} \Lambda$, *i. e., a divisor* $D \in \operatorname{Div}^0 X$ *is a divisor of some meromorphic function* $f \in \mathcal{M}_X(X)$ $(D = (f))$ *if and only if* $\Lambda(X) = 0$. *In particular* $\operatorname{Pic}^0 X = \operatorname{Div}^0 X / \operatorname{PDiv} X$ *can be seen as a subgroup of* $\operatorname{Jac} X$ *by means of the induced embedding*

$$\operatorname{Pic}^0 X \to \operatorname{Jac} X, \quad [D] \mapsto \Lambda(D).$$

II. (Jacobi) Λ *is surjective, in particular*

$$\operatorname{Pic}^0 X \to \operatorname{Jac} X, \quad [D] \mapsto \Lambda(D).$$

is an isomorphism of abelian groups.

Proof. No proof. □

Corollary 14.9. $\lambda_q : X \to \operatorname{Jac} X$ *is injective for every* $q \in X$

Proof. Suppose that λ_q is not injective. Then there exist $a, b \in X$, $a \neq b$, with $\lambda_q(a) = \lambda_q(b)$. Then for $D = a - b$, $\Lambda(D) = \lambda_q(a) - \lambda_q(b) = 0$, hence there exists $f \in \mathcal{M}_X(X)$ such that $D = (f)$. Then f has degree 1 as a map of Riemann surfaces $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$. Therefore $X \cong \hat{\mathbb{C}}$, which is a contradiction because we assumed $g_X \geqslant 1$. □

Corollary 14.10. *If* $g_X = 1$*, then* $\lambda_q : X \to \operatorname{Jac} X = \mathbb{C}/L$ *is an isomorphism, i. e., complex tori are the only compact Riemann surfaces of genus* 1*.*

Proof. λ_q is a holomorphic injective map of Riemann surfaces $X \to \mathbb{C}/L$, hence surjective, and hence an isomorphism. □

Corollary 14.11 (Abel-Jacobi theorem for complex tori)**.** *Let* $X = \mathbb{C}/\Gamma$ *be a complex torus.*

(0) Then $\operatorname{Jac} X$ *can be identified with* X *itself.*

(1) Let $D = \sum_i a_i \cdot [x_i] \in \operatorname{Div} X$ *be a divisor on* X*,* $a_i \in \mathbb{Z}$*,* $x_i \in \mathbb{C}$*. Let* $D_{\mathbb{C}} = \sum_i a_i x_i \in \mathbb{C}$*. Then under the identification* $\operatorname{Jac} X = X$*, the map* $\Lambda : \operatorname{Div}^0 X \to \operatorname{Jac} X = X$ *is given by*

$$D \mapsto [D_{\mathbb{C}}] = D_{\mathbb{C}} + \Gamma \in X = \mathbb{C}/\Gamma.$$

Hence

$$\operatorname{Pic}^0 X \to X, \quad [D] \mapsto [D_{\mathbb{C}}],$$

is an isomorphism of abelian groups.

(2) In other words, for $D \in \operatorname{Div}^0 X$ *there exists* $f \in \mathcal{M}_X(X)$ *with* $D = (f)$ *if and only if* $D_{\mathbb{C}} \in \Gamma$*.*

Proof. Exercise. □

Some final remarks. Let X be a compact Riemann surface of genus $g_X \geqslant 1$. Then $\operatorname{Jac} X$ can be embedded into $\mathbb{P}_n$ for some n. Then the chain of the embeddings

$$X \subset \operatorname{Jac} X \subset \mathbb{P}_n$$

gives an embedding of X into $\mathbb{P}_n$ as a submanifold.

Remark 14.12. Note that not every higher dimensional torus can be embedded into $\mathbb{P}_n$. However this is the case for the tori defined by period lattices.

Definition 14.13. A projective variety is a zero set of homogeneous polynomials $f_1, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$

$$Z(f_1, \dots, f_m) = \{\langle x_0, \dots, x_n \rangle \in \mathbb{P}_n \mid f_i(x_0, \dots, x_n) = 0 \quad \forall i = 1, \dots, m\}.$$

Theorem 14.14 (Chow)**.** *Compact complex submanifolds of $\mathbb{P}_n$ are projective varieties.*

Corollary 14.15. *Every compact Riemann surface can be realized as a projective variety, i. e., a projective algebraic curve.*

Remark 14.16. Let $C = Z(f) \subset \mathbb{P}_2$ be a smooth plane algebraic curve, $\deg f = d$. Then its genus is

$$g_C = \frac{(d-1)(d-2)}{2}.$$

In particular, $g_C = 0$ for $d = 1$ and $d = 2$, $g_C = 1$ for $d = 3$, $g_C = 3$ for $d = 4$, $g_C = 6$ for $d = 5$, so one sees that not all compact Riemann surfaces can be realized as plane algebraic curves (for example Riemann surfaces of genus 2).

Dimension of the moduli space. In our course we showed that the space of isomorphism classes (so called **moduli space**) of compact Riemann surfaces of genus

- $g = 0$ consists of one point;
- $g = 1$ has dimension 1 and can be identified with $\mathbb{C}$ (using j-invariant).

One can show that for $g \geqslant 2$, the space $\mathcal{M}_g$ of the isomorphism classes of compact Riemann surfaces of genus g has dimension $3g - 3$.

Exercises.

Exercise 52. (1) Let Γ be a lattice in $\mathbb{C}$ and let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus. Fix some generators α_1 and β_1 of the fundamental group of X, fix a basis of $\Omega_X(X)$, and compute the corresponding period matrix. You could use some of your results from Exercise 49.

(2) Let Γ be a lattice in $\mathbb{C}$ and let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus. Show that $\mathrm{Jac}(X) \cong X$.

Exercise 53. Let X be a compact Riemann surface of genus $g \geqslant 1$. Let $\{\omega_1, \dots, \omega_g\}$ be a basis of $\Omega_X(X)$. Let $L \subset \mathbb{C}^g$ be the corresponding lattice of periods. For a fixed point $q \in X$ we constructed the map

$$\lambda_q : X \to \mathrm{Jac}(X) = \mathbb{C}^g/L, \quad x \mapsto [(\int_q^x \omega_1, \dots, \int_q^x \omega_g)].$$

Prove that λ_q is a holomorphic map.

Hint: Notice that it is enough to understand the following.
(1) Let w be a point in $\mathbb{C}$. Let f be a holomorphic function in some open neighbourhood W of w. Then in every open ball U around w, $U \subset W$, for every point $x \in U$, and for every path γ_x that connects w and x, the integral

$$\int_{\gamma_x} f dz$$

depends only on x and not on the choice of γ_x, hence the notation $\int_w^x f dz := \int_{\gamma_x} f dz$ makes sense.
(2) Moreover, there exists an open ball U around w where f has a primitive function, i. e., a holomorphic function F such that $F'(z) = f(z)$. Then $\int_w^x f dz = \int_w^x F'(z) dz = F(x) - F(w)$ and hence the function

$$U \ni x \mapsto \int_w^x f dz$$

is holomorphic.

Exercise 54. Let $X = \mathbb{C}/\Gamma$ be a complex torus, $\Gamma = \mathbb{Z} \cdot \omega_1 + \mathbb{Z} \cdot \omega_2$. Let $D_1 = \left[\frac{\omega_1}{2}\right] + \left[\frac{\omega_2}{2}\right] - \left[\frac{\omega_1+\omega_2}{2}\right]$, $D_2 = \left[\frac{\omega_1}{2}\right] + \left[\frac{\omega_2}{2}\right] - 2 \cdot \left[\frac{\omega_1+\omega_2}{2}\right]$, $D_3 = \left[\frac{\omega_1}{2}\right] + \left[\frac{\omega_2}{2}\right] - 2 \cdot \left[\frac{\omega_1+\omega_2}{4}\right]$.

Check whether D_1, D_2, D_3 are principal divisors.

Appendix A. Examples of compact Riemann surfaces with different genera.

For an arbitrary genus $g \in \mathbb{Z}_{\geqslant 0}$ we are going to present an example of a compact Riemann surface of genus g.

A.1. **Genus 0.** We know that up to an isomorphism there is only one Riemann compact surface of genus 0. This is the Riemann sphere $\hat{\mathbb{C}}$ or the projective line $\mathbb{P}_1$.

A.2. **Genus 1.** We know that the only compact Riemann surfaces of genus 1 are complex tori. These can be seen as plane projective cubic curves given by the equation

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3.$$

In other words, complex tori are just closures in $\mathbb{P}_2$ of the affine curves $C \subset \mathbb{C}^2$,

$$C = \{(x, y) \mid y^2 = 4x^3 - g_2x - g_3\},$$

where $\mathbb{C}^2$ is embedded into $\mathbb{P}_2$ by

$$(x, y) \mapsto \langle x, y, 1\rangle.$$

So, we can see elliptic curves as the closures in $\mathbb{P}_2$ of the affine curves of the form

$$C = \{(x, y) \mid y^2 = h(x)\},$$

where h is a cubic polynomial with 3 different roots.

Reminder A.1. Notice that for a polynomial $f \in \mathbb{C}[x, y]$ of degree d the closure of the affine zero set

$$Z(f) = \{(x, y) \mid f(x, y) = 0\} \subset \mathbb{C}^2$$

is a zero set of the homogenized polynomial $F \in \mathbb{C}[x, y, z]$ defined by $F(x, y, z) = z^d \cdot f(\frac{x}{z}, \frac{y}{z})$. Namely,

$$\overline{Z(f)} = Z(F) = \{\langle x, y, z\rangle \mid F(x, y, z) = 0\}.$$

A.3. **Generalizing elliptic curves.** One could try to generalize the construction of elliptic curves in order to get examples of Riemann surfaces of higher genera.

A.3.1. *Trying a straightforward approach.* One easily notes that for a polynomial $h \in \mathbb{C}[x]$ the curve

$$C = \{(x, y) \mid y^2 = h(x)\} \subset \mathbb{C}^2$$

is smooth (is a submanifold of $\mathbb{C}^2$) if and only if all roots of h are different. Let $h = c \cdot \prod_1^d (x - a_i)$, $d \geqslant 3$, with $a_i \neq a_j$ for $i \neq j$.

Embed $\mathbb{C}^2$ into $\mathbb{P}_2$ as above by the map $(x, y) \mapsto \langle x, y, 1\rangle$ and consider the closure $\bar{C}$ of C in $\mathbb{P}_2$. Then $\bar{C}$ is defined by the equation

$$y^2 z^{d-2} = c \cdot \prod_1^d (x - a_i z).$$

One sees that $\langle 0, 1, 0\rangle$ is a singular point of $\bar{C}$ if $d > 3$, so taking the closure in $\mathbb{P}_2$ of a smooth curve in $\mathbb{C}^2 \subset \mathbb{P}_2$ does not always produce a submanifold of $\mathbb{P}_2$, i. e., $\bar{C}$ is not always a Riemann surface.

A.3.2. *Another approach.* Let us look at $\mathbb{C}^2$ as at the product $\mathbb{C} \times \mathbb{C}$ keeping in mind that $\mathbb{C}$ can be seen as an open subset of $\hat{\mathbb{C}} \cong \mathbb{P}_1$. This suggests to realize $\mathbb{C}^2$ as an open subset of a line bundle over $\hat{\mathbb{C}} \cong \mathbb{P}_1$.

Reminder A.2. A line bundle over $\hat{\mathbb{C}}$ is a 2-dimensional complex manifold E and a holomorphic map $E \xrightarrow{\pi} \hat{\mathbb{C}}$ such that over the standard open charts U_0 and U_1 of $\hat{\mathbb{C}}$ the restrictions $E|_{U_0} = \pi^{-1}(U_0)$ and $E|_{U_1} = \pi^{-1}(U_1)$ are isomorphic to $U_0 \times \mathbb{C}$ and $U_0 \times \mathbb{C}$ via isomorphisms ϕ_0 and ϕ_1 respectively such that $\pi|_{\pi^{-1}(U_0)} = pr_1 \circ \phi_0$ and $\pi|_{\pi^{-1}(U_1)} = pr_1 \circ \phi_1$ and the transition map

$$(U_0 \cap U_1) \times \mathbb{C} \xrightarrow{\phi_1 \phi_0^{-1}} (U_0 \cap U_1) \times \mathbb{C}, \quad (x, v) \mapsto (x, g_{10}(x)(v))$$

is given in the fibre over $x \in U_0 \cap U_1$ by a linear map $g_{10}(x) : \mathbb{C} \to \mathbb{C}$, i. e., g_{10} can be seen as a holomorphic map $g_{10} : U_0 \cap U_1 \to \mathbb{C}^*$.

Notice that it is enough to know g_{10} in order to reconstruct E up to an isomorphism. It is known that up to an isomorphism E is defined by a gluing map g_{10} of the form $g_{10}(t) = t^n$ for some $n \in \mathbb{Z}$.

Let E be given by the cocycle $g_{10}(t) = t^n$. Then E can be glued together from two pieces $U_0 \times \mathbb{C}$ and $U_1 \times \mathbb{C}$, each of which is identified with $\mathbb{C}^2$, the gluing is given by the map

$$\mathbb{C}^* \times \mathbb{C} \cong (U_0 \cap U_1) \times \mathbb{C} \xrightarrow{\phi_1 \phi_0^{-1}} (U_0 \cap U_1) \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}, \quad (x, y) \mapsto (1/x, yx^n).$$

Then the point (x, y) is mapped to $(\xi, \eta) = (1/x, yx^n)$. Since $y^2 = h(x)$ and $x = 1/\xi$, one obtains $y = \eta/x^n = \eta\xi^n$ and therefore

$$\eta^2\xi^{2n} = h(1/\xi) = \frac{1}{\xi^d} \cdot c \cdot \prod_1^d (1 - a_i\xi).$$

Notice that the polynomial $g(\xi) = c \cdot \prod_1^d (1 - a_i\xi)$ does not vanish at 0 and has different roots.

If $\delta = 2n + d > 0$, then

$$\eta^2\xi^\delta = g(\xi).$$

The curve in $\mathbb{C}^2$ given by

$$C_1 = \{(\xi, \eta) \mid \eta^2\xi^\delta - g(\xi) = 0\}$$

is smooth. So the union of $C_0 = C$ and C_1 is a Riemann surface in E. However, since C_1 does not contain any points of the form $(0, \eta)$, C_1 is contained in C_0. So this construction does not add any points to C_0 and hence does not provide a compact Riemann surface.

If $\delta = 2n + d \leqslant 0$, then for $\epsilon = -\delta$

$$\eta^2 = \xi^\epsilon g(\xi).$$

The curve in $\mathbb{C}^2$ given by

$$C_1 = \{(\xi, \eta) \mid \eta^2 - \xi^\epsilon g(\xi) = 0\}$$

is smooth if only if the polynomial $\xi^\epsilon g(\xi)$ does not have multiple roots, i. e., since g has only simple roots different from zero, if and only if $\epsilon = 0$ or $\epsilon = 1$. Let X be the union of $C_0 = C$ and C_1. Then X is a Riemann surface in E. Moreover, X is compact as a union of two compact sets

$$\{(x, y) \mid y^2 = h(x), |x| \leqslant 1\} \cup \{(\xi, \eta) \mid \eta^2 = \xi^\epsilon g(\xi), |\xi| \leqslant 1\}.$$

The Riemann surfaces of this type are called hyperelliptic curves.

A.3.3. *Genus of X.* Since X is constructed as a submanifold of a line bundle E over $\hat{\mathbb{C}}$, one obtains a natural holomorphic map

$$X \xrightarrow{\pi} \hat{\mathbb{C}}$$

which is given over U_0 and U_1 by $(x, y) \mapsto x$ and $(\xi, \eta) \mapsto \xi$ respectively.

First of all, let us compute the degree of $X \xrightarrow{\pi} \hat{\mathbb{C}}$. Notice that for every $x \in U_0 \subset \hat{\mathbb{C}}$ such that $h(x) \neq 0$, there are exactly 2 points in the preimage $\pi^{-1}(x)$. Since there can be only finitely many ramification points, one concludes that $d(\pi) = 2$.

The set of the ramification points coincides with the preimages of the points $x \in \hat{\mathbb{C}}$ such that either $h(x) = 0$ if $x \in U_0$ or $\xi^\epsilon g(\xi) = 0$ if $x = 1/\xi \in U_1$. There are d such points lying over

U_0 and 1 more point over $\infty \in \hat{\mathbb{C}}$ in the case $\epsilon = 1$, i. e., if d is odd. The multiplicity of each ramification point is 2, therefore

$$\sum_{x \in X} (\operatorname{mult}_x \pi - 1) = d + \epsilon.$$

Let g denote the genus of X. Let us apply the Riemann-Hurwitz formula to this map. It reads

$$2g - 2 = 2(-2) + d + \epsilon.$$

Therefore, $g = \frac{d+\epsilon}{2} - 1$, so one can obtain this way a compact Riemann surface of an arbitrary genus $g \in \mathbb{N}$.

Remark A.3. We have shown that a hyperelliptic curve X of genus g comes together with a holomorphic map $\pi : X \to \hat{\mathbb{C}}$ of degree 2.

One can also show that the converse is true: every compact Riemann surface of genus g with a holomorphic map $\pi : X \to \hat{\mathbb{C}}$ of degree 2 is isomorphic to a hyperelliptic curve.

Remark A.4. A hyperelliptic curve of genus g and the corresponding holomorphic map $X \to \hat{\mathbb{C}}$ define $2(g+1)$ points on $\hat{\mathbb{C}}$ (images of the ramification points). Acting by an automorphism of $\hat{\mathbb{C}}$, i. e., by the transformations $x \mapsto \frac{ax+b}{cx+d}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, we can always assume that 3 of the points are, for example, $0, 1, \infty$. Then the remaining $2g - 1$ points parameterize the isomorphism classes of hyperelliptic curves of genus g. Moreover, different $(2g-1)$-tuples of points in $\hat{\mathbb{C}}$ provide different isomorphism classes of hyperelliptic curves.

The latter means that the subspace of the hyperelliptic curves in the moduli space $\mathcal{M}_g$ (cf. page 77) has dimension $2g - 1$. Since $\dim \mathcal{M}_g = 3g - 3$ for $g \geqslant 2$, one concludes that the codimension of the hyperelliptic locus in $\mathcal{M}_g$ equals $g - 2$.

So, for $g \geqslant 3$ there are compact Riemann surfaces that are not hyperelliptic.

A.4. **Genus** 2. In order to obtain a hyperelliptic Riemann surface of genus 2, it should hold $d + \epsilon = 6$, so one can take $d = 5$ or $d = 6$.

Remark A.5. It can be shown that every compact Riemann surface of genus 2 is a hyperelliptic curve. By Remark A.3 it is enough to show the existence of a holomorphic map $X \to \hat{\mathbb{C}}$ of degree 2, or, equivalently, it is enough to find a meromorphic function on X with two poles.

A.5. **Higher genera.** As mentioned above, there must exist a non-hyperelliptic Riemann surface of genus $g \geqslant 3$.

Example A.6. Let C be a plane projective curve smooth curve of degree 4, for example

$$C = \{\langle x, y, z\rangle \in \mathbb{P}_2 \mid x^4 + y^4 + z^4 = 0\}.$$

As we know, the genus of C is 3. However, C is not hyperelliptic.

More generally, a hyperelliptic curve can not be realized as a submanifold of $\mathbb{P}_2$.

Remark A.7. Notice that X is obtained from $C = C_0$ by adding one point if d is odd. In this case our construction is just a one-point compactification and therefore there is a natural homeomorphism of X and $\bar{C}$.

If d is even, X is obtained from $C = C_0$ by adding two points.

Remark A.8. Notice that the closure of $C = C_0$ in $\mathbb{P}_2$ is also a one-point compactification. However, as we noticed above, $\bar{C}$ is a submanifold of $\mathbb{P}_2$ only for $d = 3$. In the case $d = 3$ the genus of X is 1 and our one-point compactification construction of X is isomorphic to $\bar{C}$.

For $d > 3$, $\bar{C}$ is singular. So, though X and $\bar{C}$ are homeomorphic as topological spaces, the complex structure on X is not induced by the complex structure of $\mathbb{P}_2$.

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