NON-OPTIMAL LEVELS OF A REDUCIBLE MOD ℓ MODULAR REPRESENTATION

HWAJONG YOO

ABSTRACT. Let $\ell > 3$ be a prime and N be a square-free integer prime to ℓ. For each prime p dividing N, let a_p be either 1 or -1. We give a sufficient criterion for the existence of a newform f of weight 2 for $\Gamma_0(N)$ such that the mod ℓ Galois representation attached to f is reducible and $U_p f = a_p f$ for primes p dividing N. The main techniques used are level raising methods based on an exact sequence due to Ribet.

CONTENTS

1. INTRODUCTION

It has been known that newforms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ give rise to compatible systems of ℓ-adic Galois representations, and if the ℓ-adic Galois representations attached to two newforms are isomorphic for some prime ℓ, then the newforms are, in fact, equal. But the corresponding statement is not true for the semisimplifications of the mod ℓ reductions of ℓ-adic Galois representations attached to newforms, as different newforms can be congruent modulo ℓ. To study the different levels from which a given modular mod ℓ representation ρ can arise is interesting and has been discussed by several mathematicians, Carayol, Diamond, Khare, Mazur, Ribet, and Taylor in the case when ρ is (absolutely) irreducible. (e.g. see [DT94].)

For simplicity, fix a prime $\ell > 3$ and let f be a newform of weight 2 for $\Gamma_0(N)$ with a square-free integer N prime to ℓ. Assume that ρ_f, the semisimplified mod ℓ Galois representation attached to f, is reducible. Then, $\rho_f \simeq 1 \oplus \chi =: \rho$, where χ is the mod ℓ cyclotomic character (Proposition 2.1). In the sense of Serre [Se87], the optimal level of ρ is 1 because it is unramified outside ℓ. The main purpose of this paper is to find the possible non-optimal levels of ρ as in the irreducible cases due to Diamond and Taylor [DT94]. Since we consider a newform f of weight 2 and square-free level N with trivial character, an eigenvalue of the Hecke operator U_p of f is either 1 or -1 for a prime p dividing N. So, by switching prime factors of the level, we elaborate the above problem as follows.

Question 1.1. Is there a newform f of weight 2 and level $N = \prod_{i=1}^{t} p_i$ with trivial character whose mod ℓ Galois representation is reducible such that $U_{p_i} f = f$ for $1 \leq i \leq s$ and $U_{p_j} f = -f$ for $s < j \leq t$?

We call a t-tuple $(p_1, \cdots, p_t)$ for s of distinct primes *admissible* if such a newform f exists. So, our question is to find admissible t-tuples for $s \leq t$.

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In this paper, we prove the following theorems.

Theorem 1.2 (Necessary conditions). *Assume a t-tuple $(p_1, \cdots, p_t)$ for s is admissible and let $N = \prod_{i=1}^{t} p_i$. Then,*

(1) $s \geq 1$.
(2) *If* $s = t$, $\ell \mid \phi(N) := \prod_{i=1}^{t} (p_i - 1)$.
(3) $p_j \equiv -1 \pmod{\ell}$ *for* $s < j \leq t$.

Theorem 1.3 (Sufficient conditions). *Let $N = \prod_{i=1}^{t} p_i$. Then, a t-tuple $(p_1, \cdots, p_t)$ for s is admissible if one of the following holds.*

(1) $s = t$ *is odd and* $\ell \mid \phi(N)$.
(2) $s + 1 = t$, *s is odd, and* $p_t \equiv -1 \pmod{\ell}$.
(3) $s = 1$, $t > 1$, *and* $p_j \equiv -1 \pmod{\ell}$ *for* $1 < j \leq t$.
(4) $s = 2$, $t > 2$ *is even, and* $p_j \equiv -1 \pmod{\ell}$ *for* $2 < j \leq t$.
(5) *t is even,* $p_t \equiv -1 \pmod{\ell}$, *and a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for s is admissible.*
(6) $s < t$, *t is even, and a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $(s-1)$ is admissible.*

In most cases, the above necessary conditions are also sufficient for admissibility. On the other hand, there are some cases that the necessary conditions do not guarantee admissibility, for instance, $s = t = 2$.

Using above theorem we can get some results on admissible t-tuples for $t \leq 4$. When $(s, t) = (2, 2), (2, 3)$ or $(4, 4)$, we use a different argument to get some sufficient conditions on admissibility.

Theorem 1.4 (Admissible t-tuples for $t \leq 4$). *A t-tuple $(p_1, \cdots, p_t)$ for s is admissible if*

(1) $(s, t) = (1, 1)$ *if and only if* $p_1 \equiv 1 \pmod{\ell}$.
(2) $(s, t) = (1, 2)$ *if and only if* $p_2 \equiv -1 \pmod{\ell}$.
(3) $(s, t) = (2, 2)$ *if and only if some extra conditions hold (Theorem 2.4).*
(4) $(s, t) = (1, 3)$ *if and only if* $p_2 \equiv p_3 \equiv -1 \pmod{\ell}$.
(5) $(s, t) = (2, 3)$ *if some conditions hold (Theorem 6.1).*
(6) $(s, t) = (3, 3)$ *if and only if* $\ell \mid (p_1 - 1)(p_2 - 1)(p_3 - 1)$.
(7) $(s, t) = (1, 4)$ *if and only if* $p_2 \equiv p_3 \equiv p_4 \equiv -1 \pmod{\ell}$.
(8) $(s, t) = (2, 4)$ *if and only if* $p_3 \equiv p_4 \equiv -1 \pmod{\ell}$.
(9) $(s, t) = (3, 4)$ *if and only if* $p_4 \equiv -1 \pmod{\ell}$.
(10) $(s, t) = (4, 4)$ *if some conditions hold (Theorem 6.4).*

In §2, we introduce Ribet's work, which was announced in his CRM lecture [R10]. In §3, we study level raising methods which are main tools of this paper. In §4, we present a complete proof of Ribet's work on admissible tuples using results in the previous section.

In §5 and §6, we discuss generalization of Ribet's work and give some examples of admissible triples for $s = 2$ and of admissible quadruples for $s = 4$.

In Appendices, we provide some known results on arithmetic of Jacobian varieties of modular curves and Shimura curves. We include some proofs of them for reader's convenience.

1.1. **Notation.** Let B be a quaternion algebra over $\mathbb{Q}$ of discriminant D such that $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$. (Hence, D is the product of the even number of disctinct primes.) Let $\mathcal{O}$ be an Eichler order of level N of B, and set $\Gamma_0^D(N) = \mathcal{O}^{\times, 1}$, the set of (reduced) norm 1 elements in $\mathcal{O}$. Let $X_0^D(N)$ be the Shimura curve for B with $\Gamma_0^D(N)$ level structure. Let $J_0^D(N)$ be the Jacobian of $X_0^D(N)$. If $D = 1$, $X_0(N) = X_0^1(N)$ denotes the modular curve for $\Gamma_0(N)$ and $J_0(N) = J_0^1(N)$ denotes its Jacobian variety. (Note that if $D \neq 1$, $X_0^D(N)(\mathbb{C}) \simeq \mathbb{H}/\Gamma_0^D(N)$, where $\mathbb{H}$ is the complex upper half plane.) By Igusa [Ig59], Deligne-Rapaport [DR93], Cerednik [Ce76], Drinfeld [Dr76], Katz-Mazur [KM85], and Buzzard [Bu97], there is an integral model of $X_0^D(N)$. By the theory of Raynaud [Ra70], there is the Néron model of $J_0^D(N)$ over $\mathbb{Z}$, we denote it by $J_0^D(N)_{/\mathbb{Z}}$. We denote by $J_0^D(N)_{/\mathbb{F}_p}$ the special fiber of $J_0^D(N)_{/\mathbb{Z}}$ over $\mathbb{F}_p$. For a Jacobian variety J over $\mathbb{Q}$, we denote by $X_p(J)$ (resp. $\Phi_p(J)$) the character group (resp. the component group) of its special fiber $J_{/\mathbb{F}_p}$ of the Néron model $J_{/\mathbb{Z}}$.

There are Hecke operators T_p acting on $J_0^D(N)$, we denote by $\mathbb{T}_N^D$ the $\mathbb{Z}$-subalgebra of the endomorphism ring of $J_0^D(N)$ generated by all T_n. In the case that $D = 1$ (resp. $N = 1$), we denote by $\mathbb{T}_N$ (resp. $\mathbb{T}^D$) $\mathbb{T}_N^1$ (resp. $\mathbb{T}_1^D$). If p divides DN, we denote by U_p the Hecke operator T_p on $J_0^D(N)$. For a prime p dividing DN, there is also the Atkin-Lehner involution w_p on $J_0^D(N)$. For a maximal ideal $\mathfrak{m}$ of a Hecke ring $\mathbb{T}$, we denote by $\mathbb{T}_\mathfrak{m}$ the localization of $\mathbb{T}$ at $\mathfrak{m}$, i.e.,

$$\mathbb{T}_\mathfrak{m} := \varprojlim_{n} \mathbb{T}/\mathfrak{m}^n.$$

There are two degeneracy maps $\alpha_p, \beta_p : X_0^D(Np) \to X_0^D(N)$ for a prime p not dividing DN. Here, α_p (resp. β_p) is the one induced by "forgetting the level p structure" (resp. by "dividing by the level p structure"). For any divisor M of N, we denote by $J_0^D(N)_{M\text{-new}}$ the M-new subvariety of $J_0^D(N)$. We also denote by $(\mathbb{T}_N^D)^{M\text{-new}}$ the image of $\mathbb{T}_N^D$ in the endomorphism ring of $J_0^D(N)_{M\text{-new}}$. If $M = N$, we define $J_0^D(N)_{\text{new}} := J_0^D(N)_{N\text{-new}}$ and $(\mathbb{T}_N^D)^{\text{new}} := (\mathbb{T}_N^D)^{N\text{-new}}$. A maximal ideal of $\mathbb{T}_N^D$ is called *M-new* if its image in $(\mathbb{T}_N^D)^{M\text{-new}}$ is still maximal.

In this paper, we assume that $\ell > 3$ is a prime and N is a square-free integer prime to ℓ. For such an integer N, we define two arithmetic functions $\phi(N)$ and $\psi(N)$, where $\phi(N) := \prod_{p|N}(p-1)$ and $\psi(N) := \prod_{p|N}(p+1)$.

Since we focus on Eisenstein maximal ideals of residue characteristic $\ell > 3$ in this paper, we introduce the following notation for convenience.

Notation 1.5. We say that for two natural numbers a and b, a is equal to b *up to products of powers of 2 and 3* if $a = b \times 2^x 3^y$ for some integers x and y. For two finite abelian groups A and B, we denote by $A \sim B$ if $A_\ell := A \otimes \mathbb{Z}_\ell$, the ℓ-primary subgroup of A, is isomorphic to B_ℓ for all primes $\ell \nmid 6$.

For a module X, $\mathrm{End}(X)$ denotes its endomorphism ring.

We denote by χ the mod ℓ cyclotomic character, i.e.,

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{F}_\ell^\times,$$

where ζ_ℓ is a primitive ℓ-th root of unity. Note that χ is unramified outside ℓ and $\chi(\mathrm{Frob}_p) \equiv p \pmod{\ell}$ for a prime $p \neq \ell$, where Frob_p denotes an arithmetic Frobenius element for p in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For an ideal $\mathfrak{m}$ of $\mathbb{T}$ and a variety A over a field K which is a $\mathbb{T}$-module, $A[\mathfrak{m}]$ denotes the kernel of $\mathfrak{m}$ on A, i.e.

$$A[\mathfrak{m}] := \{x \in A(\overline{K}) \ : \ Tx = 0 \text{ for all } T \in \mathfrak{m}\}.$$

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2. RIBET'S WORK

In this section, we discuss Ribet's result on reducible representations arising from modular forms of weight two for $\Gamma_0(N)$ with a square-free integer N.

2.1. **Reducible mod ℓ Galois representations arising from newforms.** Let $\ell > 3$ be a prime and let N be a square-free integer prime to ℓ. Let f be a newform of weight 2 for $\Gamma_0(N)$. Assume that ρ_f, the semisimplified mod ℓ Galois representation associated to f, is reducible. Then,

Proposition 2.1 (Ribet). *ρ_f is isomorphic to $1 \oplus \chi$, where χ is the mod ℓ cyclotomic character.*

Proof. Since ρ_f is reducible, it is the direct sum of two 1-dimensional representations. Let α, β : $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}^\times$ be the corresponding characters, where $\mathbb{F}$ is some finite field of characteristic ℓ. As is well known, the hypothesis that N is square-free implies that the representation ρ_f is semistable outside ℓ in the sense that inertia subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for primes other than ℓ act unipotently in the representation ρ_f. It follows that α and β are unramified outside ℓ. Accordingly, each of these two characters is some power of χ. If $\alpha = \chi^i$ and $\beta = \chi^j$, the two exponents i and j are determined modulo $(\ell - 1)$ by the restrictions of α and β to an inertia group for ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Using the results of [Ed92], one sees easily that these exponents can only be 0 and 1 (up to permutation). □

2.2. **Admissible tuples.** Fix a prime $\ell > 3$. Fix t, the number of prime factors of N, and $s \in \{1, \cdots, t\}$, the number of plus signs. (s might be zero but in the theorem below, we will show that $s \neq 0$.)

We seek to characterize t-tuples $(p_1, \cdots, p_t)$ of distinct primes for s so that there is a newform f of level $N = \prod_{i=1}^{t} p_i$ with weight two and trivial character such that

(1) $\rho_f \simeq 1 \oplus \chi$, where ρ_f is the semisimplified mod ℓ Galois representation associated to f,
(2) $U_{p_i} f = f$ for $1 \leq i \leq s$, and
(3) $U_{p_j} f = -f$ for $s < j \leq t$.

We call these t-tuples for s *admissible*. When we discuss admissible tuples, we always fix a prime $\ell > 3$ and assume that the level N is prime to ℓ.

2.3. **Result on admissible tuples.** In this subsection, we introduce the work of Ribet on admissible tuples, which was announced in his CRM lecture [R10]. For a proof, see §4.

Theorem 2.2 (Ribet)**.** *Let a t-tuple $(p_1, \cdots, p_t)$ for s be admissible and let $N = \prod_{i=1}^{t} p_i$. Then the following hold.*

(1) $s \geq 1$.
(2) *If* $s = t$, $\ell \mid \phi(N)$.
(3) *For* $s < j \leq t$, $p_j \equiv -1 \pmod{\ell}$.

Assume that a t-tuple $(p_1, \cdots, p_t)$ for s is admissible. If $s = t$, then $\ell \mid \phi(N)$. And if $s + 1 = t$, then $p_t \equiv -1 \pmod{\ell}$. Ribet proved that this is also a sufficient condition when s is odd.

Theorem 2.3 (Ribet)**.** *Let $N = \prod_{i=1}^{t} p_i$. Then, a t-tuple $(p_1, \cdots, p_t)$ for s is admissible if one of the following holds.*

(1) *If* $s = t$ *and* s *is odd,* $\ell \mid \phi(N)$.
(2) *If* $s + 1 = t$ *and* s *is odd,* $p_t \equiv -1 \pmod{\ell}$.

By the Theorem 2.2 and 2.3, a single (p) for $s = 1$ is admissible if and only if $p \equiv 1 \pmod{\ell}$.

When $t = 2$, a pair (p, q) for $s = 1$ is admissible if and only if $q \equiv -1 \pmod{\ell}$. On the other hand, we only have a necessary condition for admissibility of a pair (p, q) for $s = 2$ that $\ell \mid \phi(pq) = (p-1)(q-1)$. Without loss of generality, we assume that $p \equiv 1 \pmod{\ell}$. Then,

Theorem 2.4 (Ribet)**.** *A pair (p, q) for $s = 2$ is admissible if and only if $q \equiv 1 \pmod{\ell}$ or q is an ℓ-th power modulo p.*

When t is even, Ribet proved a level raising theorem.

Theorem 2.5 (Ribet)**.** *Assume that a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for s is admissible and t is even. Then, a t-tuple $(p_1, \cdots, p_t)$ for s is admissible if and only if $p_t \equiv -1 \pmod{\ell}$.*

3. Level raising methods

In his paper [R84], Ribet studied the kernel of the map

$$\gamma_p : J_0(N) \times J_0(N) \to J_0(Np)$$

which is induced by the degeneracy maps. He also computed the intersection of the p-new subvariety and the p-old subvariety of $J_0(Np)$. Diamond and Taylor generalized Ribet's result [DT94], and they determined non-optimal levels of irreducible mod ℓ modular representations by level raising methods. However we cannot directly use their methods to find non-optimal levels of $1 \oplus \chi$. The reason is basically that the kernels of their level raising maps are "Eisenstein", and we do not know U_p actions on the kernels for primes p dividing the level. Instead, we introduce new level raising methods based on an exact sequence due to Ribet.

3.1. **Equivalent condition.** Let $\mathfrak{m}$ be a maximal ideal of the Hecke ring $\mathbb{T}_N$ of residue characteristic ℓ. Let $\rho_{\mathfrak{m}}$ be the semisimplified mod ℓ Galois representation associated to $\mathfrak{m}$. Assume p is a prime not dividing N. We call *level raising occurs for* $\mathfrak{m}$ (from level N to level Np) if there is a maximal ideal $\mathfrak{n}$ of $\mathbb{T}_{Np}$ such that

(1) $\mathfrak{n}$ is p-new, and
(2) $\rho_{\mathfrak{n}}$, the semisimplified mod ℓ representation associated to $\mathfrak{n}$, is isomorphic to $\rho_{\mathfrak{m}}$.

Let $\mathbb{T} := \mathbb{T}_{Np}$. Since $\mathbb{T}^{p\text{-old}}$ is isomorphic to $\mathbb{T}_N[U_p]/(U_p^2 - T_pU_p + p)$, a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_N$ can be regarded as a maximal ideal of $\mathbb{T}^{p\text{-old}}$ once we choose a root γ of the polynomial $X^2 - T_pX + p \pmod{\mathfrak{m}}$ so that $U_p - \gamma \in \mathfrak{m}$. By abusing notation, let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ whose image in $\mathbb{T}^{p\text{-old}}$ is $\mathfrak{m}$. If level raising occurs for $\mathfrak{m}$, $\mathfrak{m}$ is also p-new. In other words, the image of $\mathfrak{m}$ in $\mathbb{T}^{p\text{-new}}$ is also maximal.

To detect level raising, Ribet showed that all congruences between p-new and p-old forms can be detected geometrically by the intersection between the p-old and the p-new parts of the relevant Jacobian.

Theorem 3.1 (Ribet). *Let $J := J_0(Np)$. As before, assume that Np is prime to ℓ and $p \nmid N$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{Np}$ of residue characteristic ℓ which is p-old. Then level raising occurs for $\mathfrak{m}$ if and only if*

$$J_{p\text{-old}} \bigcap J_{p\text{-new}}[\mathfrak{m}] \neq 0.$$

Proof. Let $\Omega := J_{p\text{-old}} \bigcap J_{p\text{-new}}$. If $\Omega[\mathfrak{m}] \neq 0$, $J_{p\text{-new}}[\mathfrak{m}]$ is not zero, which implies that $\mathfrak{m}$ is p-new.

Conversely, assume $\Omega[\mathfrak{m}] = 0$. Consider the following exact sequence

$$0 \longrightarrow \Omega \longrightarrow J_{p\text{-old}} \times J_{p\text{-new}} \longrightarrow J \longrightarrow 0.$$

Let $e = (1, 0) \in \mathrm{End}(J_{p\text{-old}}) \times \mathrm{End}(J_{p\text{-new}})$. If $e \notin \mathrm{End}(J)$, $J \not\simeq J_{p\text{-old}} \times J_{p\text{-new}}$. Thus, $\Omega[\mathfrak{m}] = 0$, which means that $\mathfrak{m}$ is not in the support of Ω, implies that $e \in \mathrm{End}(J) \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}}$. Moreover, $e \in \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}$ because Ω is finite. Thus,

$$e \in (\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}) \bigcap (\mathrm{End}(J) \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}}).$$

The intersection $(\mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}) \bigcap (\mathrm{End}(J) \otimes_{\mathbb{T}} \mathbb{T}_{\mathfrak{m}})$ is equal to the localization of the saturation of $\mathbb{T}$ in $\mathrm{End}(J)$ at $\mathfrak{m}$. Since $\mathbb{T}$ is saturated in $\mathrm{End}(J)$ locally at $\mathfrak{m}$ by the theorem of Agashe, Ribet, and Stein [ARS12], $e \in \mathbb{T}_{\mathfrak{m}}$.

If $\mathfrak{m}$ is also a maximal ideal after projection $\mathbb{T} \to \mathbb{T}^{p\text{-new}}$, the injection $\mathbb{T} \hookrightarrow \mathbb{T}^{p\text{-old}} \times \mathbb{T}^{p\text{-new}}$ is not an isomorphism after localizing at $\mathfrak{m}$. Thus, $e = (1, 0) \in \mathbb{T}^{p\text{-old}} \times \mathbb{T}^{p\text{-new}}$ cannot be in $\mathbb{T}_{\mathfrak{m}}$, which is a contradiction. Therefore $\mathfrak{m}$ is not p-new. ☐

Remark 3.2. The theorem of Agashe, Ribet, and Stein is as follows,

Theorem 3.3 (Agashe, Ribet, and Stein). *Let ℓ be the characteristic of $\mathbb{T}_N/\mathfrak{m}$. Then, $\mathbb{T}_N$ is saturated in $\mathrm{End}(J_0(N))$ locally at $\mathfrak{m}$ if*

(1) *$\ell \nmid N$, or*
(2) *$\ell \parallel N$ and $T_\ell \equiv \pm 1 \pmod{\mathfrak{m}}$.*

In our case, the level Np is prime to ℓ, so $\mathbb{T}_{Np}$ is saturated in $\mathrm{End}(J_0(Np))$ locally at $\mathfrak{m}$.

When we consider Jacobians of Shimura curves, we don't have the q-expansion principle, so the saturation property of the Hecke algebra is difficult to prove. However, we can prove the following.

Proposition 3.4. *Let $\mathbb{T} := \mathbb{T}_q^{pr}$, $J := J_0^{pr}(q)$, and $\mathfrak{m} := (\ell,\ U_p - 1,\ U_q - 1,\ U_r + 1,\ T_s - s - 1\ :\ \text{for primes } s \nmid pqr) \subset \mathbb{T}$. Assume that $p \not\equiv 1 \pmod{\ell}$ and $q \not\equiv 1 \pmod{\ell}$. Then, $\mathbb{T}$ is saturated in $\mathrm{End}(J)$ locally at $\mathfrak{m}$.*

Proof. It suffices to find a free $\mathbb{T}_{\mathfrak{m}}$-module of finite rank on which $\mathrm{End}(J)$ operates by functoriality (as in the paper [ARS12]).

Let Y (resp. L, X) be the character group of $J_0^{pr}(q)$ at r (resp. $J_0(pqr)$, $J_0(pq)$ at p). By Ribet [R90], there is an exact sequence

$$0 \longrightarrow Y \longrightarrow L \longrightarrow X \oplus X \longrightarrow 0.$$

Let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be the corresponding Eisenstein ideal to $\mathfrak{m}$ in $\mathbb{T}_{pqr}$ (resp. $\mathbb{T}_{pq}$). Since $p \not\equiv 1 \pmod{\ell}$ and a pair (p, q) for $s = 2$ is not admissible, $\mathfrak{b}$ is not p-new, so $X \oplus X$ does not have support at $\mathfrak{b}$. (Note that the action of $\mathbb{T}_{pq}$ on X factors through $\mathbb{T}_{pq}^{p\text{-new}}$.) Thus, $Y_{\mathfrak{m}} \simeq L_{\mathfrak{a}}$. Since $L/\mathfrak{a}L$ is of dimension 1 over $\mathbb{T}_{pqr}/\mathfrak{a}$ by Theorem B.4 and L is of rank 1 over $\mathbb{T}_{pqr}$ in the sense of Mazur [M77], $L_{\mathfrak{a}}$ is free of rank 1 over $(\mathbb{T}_{pqr})_{\mathfrak{a}}$ by Nakayama's lemma. So, $Y_{\mathfrak{m}} \simeq L_{\mathfrak{a}}$ is also a free module of rank 1 over $\mathbb{T}_{\mathfrak{m}}$. □

Remark 3.5. Ribet provides the above proof.

Using the above proposition and the proof of Theorem 3.1, we can prove the following theorem.

Theorem 3.6. *Let* $\mathbb{T} := \mathbb{T}_q^{pr}$, $J := J_0^{pr}(q)$, *and* $\mathfrak{m} := (\ell,\ U_p-1,\ U_q-1,\ U_r+1,\ T_s-s-1\ :$ *for primes* $s \nmid pqr) \subset \mathbb{T}$. *Assume that* $p \not\equiv 1 \pmod{\ell}$ *and* $q \not\equiv 1 \pmod{\ell}$. *Then, level raising occurs for* $\mathfrak{m}$ *if and only if*

$$J_{q\text{-old}} \bigcap J_{q\text{-new}}[\mathfrak{m}] \neq 0.$$

3.2. **The intersection of the p-old subvariety and the p-new subvariety.** As in the previous subsection, let p be a prime not dividing N. Let Ω be the intersection of the p-old subvariety and the p-new subvariety of $J_0(Np)$. By the degeneracy maps, we have the following maps

$$J_0(N) \times J_0(N) \xrightarrow{\gamma_p} J_0(Np) \longrightarrow J_0(N) \times J_0(N).$$

The composition of the above two maps is the matrix

$$\delta_p := \begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}.$$

Let Δ be the kernel of the above composition δ_p, i.e.

$$\Delta := J_0(N)^2[\delta_p] = \{(x, y) \in J_0(N)^2\ :\ (p+1)x = -T_p y \text{ and } T_p x = -(p+1)y\}.$$

Let Σ be the kernel of γ_p. Then Δ contains Σ and is endowed with a canonical non-degenerate alternating $\mathbb{G}_{\mathrm{m}}$-valued pairing. Let $\Sigma^{\perp}$ be the orthogonal to Σ relative to this pairing. Then, $\Sigma^{\perp}$ contains Σ and we have the formula

$$\Omega = \Sigma^{\perp}/\Sigma.$$

For more details, see [R84].

We define Δ^+ and Δ^- as follows.

$$\Delta^+ := \{(x, -x) \in J_0(N)^2\ :\ x \in J_0(N)[T_p - p - 1]\}$$

and

$$\Delta^- := \{(x,\ x) \in J_0(N)^2\ :\ x \in J_0(N)[T_p + p + 1]\}.$$

They are eigensubspaces of Δ for the Atkin-Lehner operator w_p. (w_p acts on $J_0(N)^2$ by swapping its components.) If we ignore 2-primary subgroups, $\Delta \sim \Delta^+ \oplus \Delta^-$. Furthermore we have the filtrations as follows.

$$0\ \subset\ \Sigma^+\ \subset\ (\Sigma^{\perp})^+\ \subset\ \Delta^+$$

and

$$0\ \subset\ \Sigma^-\ \subset\ (\Sigma^{\perp})^-\ \subset\ \Delta^-.$$

Since $\Delta/\Sigma^{\perp}$ is the $\mathbb{G}_{\mathrm{m}}$-dual of Σ and Σ is an antidiagonal embedding of the Shimura subgroup of $J_0(N)$ by Ribet [R84], $\Sigma^+ \sim \Sigma$ and $\Sigma^- \sim 0$. Thus,

$$(\Sigma^{\perp})^- \sim \Delta^-.$$

By the map γ_p, $(\Sigma^{\perp})^+$ maps to $(\Sigma^{\perp})^+/\Sigma$ and $(\Sigma^{\perp})^- \sim \Delta^-$ maps to Δ^- (up to 2-primary subgroups). Since $\Sigma^{\perp}/\Sigma$ lies in $J_0(Np)_{p\text{-new}}$, $U_p + w_p = 0$. Hence, $(\Sigma^{\perp})^+/\Sigma$ (resp. Δ^-) corresponds to the $+1$ (resp. -1)-eigenspace of Ω for the U_p operator (up to 2-primary subgroups).

3.3. **Ribet's exact sequence.** Let $X_p(J)$ (resp. $\Phi_p(J)$) be the character group (resp. the component group) of J at p. By the degeneracy maps, there is a Hecke equivariant map between component groups

$$\Phi_p(J_0^D(Np)) \times \Phi_p(J_0^D(Np)) \to \Phi_p(J_0^D(Npq)),$$

where q is a prime not dividing NDp. Let K (resp. C) be the kernel (resp. the cokernel) of the above map. We recall Theorem 4.3 of [R90].

Theorem 3.7 (Ribet)**.** *There is a Hecke equivariant exact sequence*

$$0 \longrightarrow K \longrightarrow X \oplus X/\mu_q(X \oplus X) \longrightarrow \Psi \longrightarrow C \longrightarrow 0,$$

where

$$X := X_p(J_0^D(Np)),\ \Psi := \Phi_q(J_0^{Dpq}(N)),\ \text{and}\ \mu_q := \begin{pmatrix} q+1 & T_q \\ T_q & q+1 \end{pmatrix}.$$

If we ignore 2-,3-primary subgroups of K (resp. C), it is isomorphic to $\Phi_p(J_0^D(Np))$ (resp. $\mathbb{Z}/(q+1)\mathbb{Z}$). For a prime $\ell > 3$, we denote by A_ℓ $A \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. We can decompose the above exact sequence into the eigenspaces by the action of the U_q operator as follows.

$$+1 \text{ eigenspaces}: \qquad 0 \longrightarrow \Phi_p(J_0^D(Np))_\ell \longrightarrow (X/(T_q - q - 1)X)_\ell \longrightarrow \Psi_\ell^+ \longrightarrow 0.$$

$$-1 \text{ eigenspaces}: \qquad 0 \longrightarrow (X/(T_q + q + 1)X)_\ell \longrightarrow \Psi_\ell^- \longrightarrow (\mathbb{Z}/(q+1)\mathbb{Z})_\ell \longrightarrow 0,$$

where Ψ^+ (resp. Ψ^-) denotes $+1$ (resp. -1) eigenspace for U_q operator on Ψ. For more details, see [R90] or Appendix A.

4. Proof of Ribet's work

Even though Ribet's work has been explained in many lectures (e.g. [R10]), a complete proof has not been published yet. In this section, we present it based on his idea.

By Mazur's approach, using ideals in the Hecke algebra $\mathbb{T}_N$, proving admissibility of a t-tuple $(p_1, \cdots, p_t)$ for s is equivalent to showing that a maximal ideal $\mathfrak{m}$ is new, where $\mathfrak{m} = (\ell,\ U_{p_i} - 1,\ U_{p_j} + 1,\ T_r - r - 1\ :\ 1 \le i \le s,\ s < j \le t,$ for all primes $r \nmid N := \prod_{i=1}^{t} p_i)$. To prove that $\mathfrak{m}$ is new, we seek a $\mathbb{T}_N^{\text{new}}$-module A such that $A[\mathfrak{m}] \neq 0$. (Since $\mathbb{T}_N^{\text{new}}$ has 1, $A[\mathbb{T}_N^{\text{new}}] = 0$.)

Proof of Theorem 2.2.

(1) When $s = 0$ and $t = 1$, Mazur proved that a single (N) is not admissible [M77]. Ribet generalized his result to the case $s = 0$ and $t = 2$ [R08]. Ribet's method also works for all $t > 2$. We present a proof of non-existence of admissible triples for $s = 0$. This method can be easily generalized to the case $s = 0$ and $t > 3$.

Assume a triple (p, q, r) for $s = 0$ is admissible and f is a newform of level pqr such that

(a) $\rho_f \simeq 1 \oplus \chi$ and

(b) $U_k f = -f$ for $k = p, q,$ and r.

This implies that $p \equiv q \equiv r \equiv -1 \pmod{\ell}$ by (3) below.

Let e be the normalized Eisenstein series of weight 2 and level 1,

$$e(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n) x^n,$$

where $\sigma(n) = \sum_{d|n, d>0} d$ and $x = e^{2\pi i \tau}$. The filtration of $e \pmod{\ell}$ is $\ell + 1$ (cf. [M77], [Se72], [Sw73]); in other words, $e \pmod{\ell}$ cannot be expressed as a sum of mod ℓ modular forms of weight 2 and level prime to ℓ. Raising the level of e, we can get Eisenstein series of weight 2 and level pqr.

Definition 4.1. For any modular form g of level N and a prime p which does not divide N,

$$[p]^+(g)(z) := g(z) - pg(pz) \quad \text{and} \quad [p]^-(g)(z) := g(z) - g(pz).$$

Recall Proposition 2.8 of [Y14].

Proposition 4.2. *Let g be an Eisenstein series of weight 2 and level N which is an eigenform for all Hecke operators, then $[p]^+(g)$ (resp. $[p]^-(g)$) is an Eisenstein series of weight 2 and level Np such that the eigenvalue of U_p is 1 (resp. p).*

For a positive integer n, let $e_n(\tau) := e(n\tau)$. Let $P := e + e_p \equiv [p]^+(e) \pmod{\ell}$ (resp. $Q := e + e_q$, $R := e + e_r$) be the mod ℓ Eisenstein series of weight 2 and level p (resp. q, r). Raising the level, $e_p + e_{pq}$ (resp. $e_{pq} + e_{pqr}$) is a mod ℓ modular form of weight 2 and level pq (resp. pqr). Therefore,

$$e + e_{pqr} = P - (e_p + e_{pq}) + (e_{pq} + e_{pqr})$$

is also a mod ℓ modular form of weight 2 and level pqr. Let $E := \sum_{n|pqr} e_n$ and $F := \sum_{n|pqr} (-1)^{\omega(n)} e_n = [p]^- \circ [q]^- \circ [r]^-(e)$, where $\omega(n)$ is the number of distinct prime factors of n. Since $p \equiv q \equiv r \equiv -1 \pmod{\ell}$, E is congruent modulo ℓ to

$$[p]^+ \circ [q]^+ \circ [r]^+(e) = \sum_{n|pqr} (-1)^{\omega(n)} n e_n.$$

Thus E and F are mod ℓ eigenforms for all Hecke operators, and E is a mod ℓ modular form of weight 2 and level pqr. Moreover,

$$U_p(E) \equiv U_q(E) \equiv U_r(E) \equiv 1 \pmod{\ell} \quad \text{and} \quad U_p(F) \equiv U_q(F) \equiv U_r(F) \equiv -1 \pmod{\ell}.$$

Since F and f have the same Fourier expansion at $i\infty$ modulo ℓ, by the q-expansion principle, they are equal modulo ℓ. In other words, F is a mod ℓ modular form of weight 2 and level pqr. However, this is a contradiction because

$$-8e = E - F - 2P - 2Q - 2R - 2(e + e_{pqr})$$

is also a mod ℓ modular form of weight 2 and level pqr.

(2) Let $N = \prod_{i=1}^t p_i$ and let $G = [p_t]^+ \circ \cdots \circ [p_1]^+(e)$ be an Eisenstein series of level N. Then the constant term of the Fourier expansion of G at $i\infty$ is $(-1)^{t+1}\frac{\phi(N)}{24}$. (Proposition 2.13 of [Y14].)

Assume that a t-tuple $(p_1, \cdots, p_t)$ for $s = t$ is admissible. Then there is a newform f of level N such that $\rho_f \simeq 1 \oplus \chi$ and $U_{p_i} f = f$ for $1 \leq i \leq t$. Since G and f have the same eigensystem modulo a maximal ideal above ℓ, the Fourier expansion of $G - f$ at $i\infty$ is congruent to $(-1)^{t+1}\frac{\phi(N)}{24}$ modulo a maximal ideal. Hence, $(-1)^{t+1}\frac{\phi(N)}{24}$ is 0 modulo ℓ because we assume that $\ell > 3$ and there is no modular form of weight 2 and level N whose Fourier expansion at $i\infty$ is constant. (cf. Lemma 5.10 of chapter II of [M77].)

(3) Assume that a t-tuple $(p_1, \cdots, p_t)$ for s is admissible. Then, there is a newform $f = \sum a_n q^n$ of level N such that $\rho_f \simeq 1 \oplus \chi$, $U_{p_i} f = f$ for $1 \leq i \leq s$, and $U_{p_j} f = -f$ for $s < j \leq t$. The semisimplification of the local representation $\rho_f |_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ for a prime divisor p of N is $\epsilon \oplus \epsilon\chi$, where ϵ is an unramified quadratic character such that $\epsilon(\mathrm{Frob}_p) = a_p$. Thus, for $s < j \leq t$, $\epsilon(\mathrm{Frob}_{p_j}) = a_{p_j} = -1$, and

$$\rho_f(\mathrm{Frob}_{p_j}) = 1 + p_j \equiv -1 - p_j \pmod{\ell},$$

which implies that $p_j \equiv -1 \pmod{\ell}$.

□

Proof of Theorem 2.3.

(1) Let $\mathfrak{m} := (\ell,\ U_{p_i} - 1,\ T_r - r - 1\ :\ 1 \leq i \leq t,\ \text{for primes } r \nmid N)$ be an Eisenstein maximal ideal of $\mathbb{T}_N$. It is enough to show that $\mathfrak{m}$ is new.

Assume that $\ell \mid \phi(N)$. Let $p = p_1$ and $D = N/p$. Let $\Phi_p := \Phi_p(J_0^D(p))$ be the component group of $J_0^D(p)_{/\mathbb{F}_p}$. Since $\ell \mid \phi(N)$, $\Phi_p[\mathfrak{m}] \neq 0$ by Proposition A.2 and A.3. By Theorem 3.10 of [R90] and the monodromy exact sequence (A.1) in the Appendix A, the action of the Hecke ring on Φ_p factors through $(\mathbb{T}_p^D)^{p\text{-new}}$, so $\mathfrak{m}$ is p-new in $\mathbb{T}_p^D$. By the Jacquet-Langlands correspondence, $\mathfrak{m}$ is new.

(2) Let $q := p_t$ and $\mathfrak{n} := (\ell,\ U_{p_i}-1,\ U_q+1,\ T_r-r-1\ :\ 1\leq i\leq s,\ \text{for primes } r\nmid N)$ be an Eisenstein maximal ideal of $\mathbb{T}_N$. Assume that $q\equiv -1 \pmod{\ell}$. Let $p := p_1$ and $D := N/pq$ (if $s=1$, set $D=1$). Since the number of distinct prime factors of D is even, there are Shimura curves $X_0^D(p)$, $X_0^D(pq)$, and $X_0^{Dpq}(1)$. By the Ribet's exact sequence in §3.3, we have

$$\Phi_p(J_0^D(p))\times\Phi_p(J_0^D(p)) \xrightarrow{\gamma_q} \Phi_p(J_0^D(pq)) \longrightarrow C \longrightarrow 0$$

and

$$\Phi_q(J_0^{Dpq}(1)) \longrightarrow C \longrightarrow 0.$$

By Corollary A.6, C is annihilated by $\mathfrak{n}$. Therefore, $\mathfrak{n}$ is a proper maximal ideal of $\mathbb{T}^{Dpq}$. By the Jacquet-Langlands correspondence, $\mathfrak{n}$ is a new maximal ideal of $\mathbb{T}_{Dpq}$. In other words, the given t-tuple for s is admissible.

□

Proof of Theorem 2.4. Let $I := (U_p-1,\ T_r-r-1\ :\ \text{for primes } r\neq p)$ be the Eisenstein ideal of $\mathbb{T} := \mathbb{T}_p$ and $\mathfrak{m} := (\ell, I)$. By Mazur, $I_{\mathfrak{m}} := I\otimes \mathbb{T}_{\mathfrak{m}}$ is a principal ideal and it is generated by $\eta_q := T_q-q-1$ for a good prime q (Proposition 16.6 of Chap II of [M77]). Moreover, q is a good prime if and only if $q\not\equiv 1 \pmod{\ell}$ and q is not an ℓ-th power modulo p (cf. Theorem 11 of loc. cit.). In other words, the above condition on q is equivalent to the condition that η_q is not a generator of $I_{\mathfrak{m}}$.

Assume that η_q is not a generator of $I_{\mathfrak{m}}$. By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow \Phi_\ell \longrightarrow (X/\eta_q X)_\ell \longrightarrow (\Psi^+)_\ell \longrightarrow 0,$$

where $\Phi := \Phi_p(J_0(p))$, $X := X_p(J_0(p))$, and $\Psi := \Phi_q(J_0^{pq}(1))$. By Mazur, Φ is a free module of rank 1 over $\mathbb{T}/I$ and $X_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$ (§II.11 of loc. cit.). Since $(X/IX)\otimes_{\mathbb{T}}\mathbb{T}_{\mathfrak{m}} = X_{\mathfrak{m}}/I_{\mathfrak{m}}X_{\mathfrak{m}} \simeq (\mathbb{T}/I)_{\mathfrak{m}}$, if η_q is not a generator of $I_{\mathfrak{m}}$, then $\#(X/\eta_q X)_{\mathfrak{m}} > \#(X/IX)_{\mathfrak{m}} = \#(\mathbb{T}/I)_{\mathfrak{m}} = \#(\Phi)_{\mathfrak{m}}$. In other words, after localizing the above exact sequence at $\mathfrak{m}$, $\Psi^+_{\mathfrak{n}}\neq 0$, where $\mathfrak{n} = (\ell,\ U_p-1,\ U_q-1,\ T_r-r-1\ :\ \text{for primes } r\nmid pq)$ is an ideal of $\mathbb{T}^{pq}$ corresponding to $\mathfrak{m}$. Thus, $\mathfrak{n}$ is maximal. By the Jacquet-Langlands correspondence, $\mathfrak{n}$ is new.

Conversely, η_q is a local generator of $I_{\mathfrak{m}}$. Let Ω be the intersection of the q-old subvariety and the q-new subvariety of $J_0(pq)$. Let $\Delta := J_0(p)^2[\delta_q]$ and let Σ be the kernel of γ_q as in §3.2. We have a filtration of Δ^+

$$0\ \subset\ \Sigma\ \subset\ (\Sigma^{\perp})^+\ \subset\ \Delta^+$$

and Δ^+ is isomorphic to $J_0(p)[\eta_q]$. Since η_q is a generator of $I_{\mathfrak{m}}$, $(\Delta^+)_{\mathfrak{m}}$ is isomorphic to $J_0(p)[I]_{\mathfrak{m}}$. By Mazur (loc. cit.), $J_0(p)[I]$ is free of rank 2 and Σ is free of rank 1 over $\mathbb{T}/I$. Thus, the $\mathfrak{m}$-primary subgroup of $(\Sigma^{\perp})^+/\Sigma$ is 0 because $\Delta^+/(\Sigma^{\perp})^+$ is the $\mathbb{G}_m$-dual of Σ. In other words, the $\mathfrak{m}$-primary subgroup of Ω is 0, i.e., $\mathfrak{m}$ is not in the support of Ω. By Theorem 3.1, $\mathfrak{m}$ is not q-new. In other words, a pair (p,q) for $s=2$ is not admissible. □

Proof of Theorem 2.5. Assume that a $(t-1)$-tuple $(p_1,\cdots,p_{t-1})$ for s is admissible and t is even.

Let $p=p_1$, $D=\prod_{j=2}^{t-1}p_j$, and $q=p_t$. (If $t=2$, let $D=1$.) Since the number of prime factors of D is even, there are Shimura curves $X_0^D(p)$, $X_0^D(pq)$, and $X_0^{Dpq}(1)$. If a t-tuple $(p,\dots,p_{t-1},q)$ for s is admissible, then $q\equiv -1 \pmod{\ell}$ by Theorem 2.2.

Conversely, assume $q\equiv -1 \pmod{\ell}$. Since a $(t-1)$-tuple $(p,\cdots,p_{t-1})$ for s is admissible, there is a new Eisenstein maximal ideal $\mathfrak{m} := (\ell,\ U_{p_i}-1,\ U_{p_j}+1,\ T_r-r-1\ :\ 1\leq i\leq s,\ s<j\leq t-1,\ \text{for primes } r\nmid pD)$ in $\mathbb{T}_{pD}$. Let $X := X_p(J_0^D(p))$ be the character group of $J_0^D(p)_{/\mathbb{F}_p}$. Then, by the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow (X/(T_q+q+1)X)_\ell \longrightarrow (\Psi^-)_\ell,$$

where $\Psi := \Phi_q(J_0^{Dpq}(1))$. Because $q\equiv -1 \pmod{\ell}$, $\ell\in\mathfrak{m}$, and $T_q-q-1\in\mathfrak{m}$, $T_q+q+1\in\mathfrak{m}$. By the Jacquet-Langlands correspondence and the fact that $(\mathbb{T}_p^D)^{p\text{-new}}$ acts faithfully on X, $X/\mathfrak{m}X\neq 0$. Therefore $(X/(T_q+q+1)X)_\ell$ has support at $\mathfrak{m}$, so $\Psi^-[\mathfrak{n}]\neq 0$, where $\mathfrak{n} := (\ell,\ U_{p_i}-1,\ U_{p_j}+1,\ T_r-r-1\ :\ 1\leq i\leq s,\ s<j\leq t,\ \text{for primes } r\nmid Dpq)\subset\mathbb{T}^{Dpq}$. In other words, $\mathfrak{n}$ is maximal. By the Jacquet-Langlands correspondence, $\mathfrak{n}$ is new. □

5. ADMISSIBLE TUPLES FOR $s = 1$ OR EVEN t

In this section we present new results on admissible tuples for $s = 1$ or even t.

Theorem 5.1. *Assume $t > s$. A t-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible if and only if $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.*

Proof. Assume a t-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible. Then, by Theorem 2.2, $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.

Conversely, assume $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.

(1) Case 1 : Assume that t is odd and a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for $s = 1$ is admissible. Let $p := p_1$, $q := p_t$, $C = \prod_{i=3}^{t-1} p_i$, and $D = \prod_{i=2}^{t-1} p_i$. (If $t = 3$, set $C = 1$.) Let $\mathfrak{m} := (\ell,\ U_p - 1,\ U_{p_i} + 1,\ T_r - r - 1\ :\ 2 \leq i \leq t-1,\ \text{for primes } r \nmid pD)$ be a new Eisenstein maximal ideal of $\mathbb{T}_{pD}$. Let $\mathfrak{n} := (\ell,\ U_p - 1,\ U_q + 1,\ U_{p_i} + 1,\ T_r - r - 1\ :\ 2 \leq i \leq t-1,\ \text{for primes } r \nmid pDq)$ be an Eisenstein maximal ideal of $\mathbb{T}_{pDq}$. It suffices to show that $\mathfrak{n}$ is new.

Since t is odd, there are Shimura curves $X_0^C(pp_2)$ and $X_0^{Dq}(p)$. By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow (X/(T_q + q + 1)X)_\ell \longrightarrow \Psi_\ell^- \longrightarrow (\mathbb{Z}/(q+1)\mathbb{Z})_\ell \longrightarrow 0,$$

where $X := X_{p_2}(J_0^C(pp_2))$ and $\Psi := \Phi_q(J_0^{Dq}(p))$. Since $(\mathbb{T}_{pp_2}^C)^{p_2\text{-new}}$ acts faithfully on X and $\mathfrak{m}$ is new, $X/\mathfrak{m}X \neq 0$. Moreover, $T_q + q + 1 \in \mathfrak{m}$ because $q \equiv -1 \pmod{\ell}$, $\ell \in \mathfrak{m}$, and $T_q - q - 1 \in \mathfrak{m}$. Thus, $\mathfrak{m}$ is in the support of $(X/(T_q + q + 1)X)_\ell$, so $\mathfrak{n}$ is also in the support of Ψ_ℓ^-. Therefore $\mathfrak{n}$ is a proper maximal ideal of $\mathbb{T}_p^{Dq}$. If $\mathfrak{n}$ is p-old, then there is a new maximal ideal $(\ell,\ U_{p_i} + 1,\ T_r - r - 1\ :\ 2 \leq i \leq t,\ \text{for primes } r \nmid Dq)$ of $\mathbb{T}_{Dq}$ by the Janquet-Langlands correspondence. In other words, a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $s = 0$ is admissible, which contradicts Theorem 2.2. Thus, $\mathfrak{n}$ is p-new, so by the Janquet-Langlands correspondence $\mathfrak{n}$ is a new maximal of $\mathbb{T}_{pDq}$.

(2) Case 2 : Assume that t is even and a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for $s = 1$ is admissible. Then, since $p_t \equiv -1 \pmod{\ell}$, by Theorem 2.5, a t-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible.

When $t = 2$, by Theorem 2.3, a pair (p_1, p_2) for $s = 1$ is admissible. Thus, by induction on t, a t-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible for all $t \geq 2$. □

Using the same method as above, we can prove the following level raising theorem, which is almost complement of the case in Theorem 2.5 when t is even. (This excludes the case $s = t$ only.)

Theorem 5.2. *Assume t is even and $t > s$. And assume that a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $(s-1)$ is admissible. Then, a t-tuple $(p_1, \cdots, p_t)$ is admissible for s.*

In contrast to Theorem 2.5, there is no condition on p_1 for raising the level.

Proof. Let $\mathfrak{m} := (\ell,\ U_{p_i} - 1,\ U_{p_j} + 1,\ T_r - r - 1\ :\ 2 \leq i \leq s,\ s < j \leq t,\ \text{for primes } r \nmid pD)$ be an Eisenstein maximal ideal of $\mathbb{T}_{pD}$, where $D := \prod_{k=2}^{t-1} p_k$ and $p := p_t$. By our assumption, $\mathfrak{m}$ is new. Let $\mathfrak{n} := (\ell,\ U_{p_i} - 1,\ U_{p_j} + 1,\ T_r - r - 1\ :\ 1 \leq i \leq s,\ s < j \leq t,\ \text{for primes } r \nmid Dpq)$ be an Eisenstein maximal ideal of $\mathbb{T}_{Dpq}$. It suffices to show that $\mathfrak{n}$ is new.

Since t is even, there are Shimura curves $X_0^D(p)$ and $X_0^{Dpq}(1)$. By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow \Phi_\ell \longrightarrow (X/(T_q - q - 1)X)_\ell \longrightarrow \Psi_\ell^+ \longrightarrow 0,$$

where $\Phi := \Phi_p(J_0^D(p))$, $X := X_p(J_0^D(p))$, and $\Psi := \Phi_q(J_0^{Dpq}(1))$. Since $\mathfrak{m}$ is new and $(\mathbb{T}_p^D)^{p\text{-new}}$ acts faithfully on X, $X/\mathfrak{m}X \neq 0$. Because $T_q - q - 1 \in \mathfrak{m}$, $\mathfrak{m}$ lies in the support of $(X/(T_q - q - 1)X)_\ell$. Since $\mathfrak{m}$ is not in the support of Φ_ℓ by Proposition A.2, $\mathfrak{n}$ is in the support of Ψ_ℓ^+. Thus, $\mathfrak{n}$ is a maximal ideal of $\mathbb{T}^{Dpq}$. By the Jacquet-Langlands correspondence, $\mathfrak{n}$ is new. □

Corollary 5.3. *Assume $t > 2$ is even. Then, a t-tuple $(p_1, \cdots, p_t)$ for $s = 2$ is admissible if and only if $p_i \equiv -1 \pmod{\ell}$ for $3 \leq i \leq t$.*

Proof. If a t-tuple is admissible, then by Theorem 2.2, $p_i \equiv -1 \pmod{\ell}$ for $3 \leq i \leq t$.

Conversely, assume that $p_i \equiv -1 \pmod{\ell}$ for $3 \leq i \leq t$. By Theorem 5.1, a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $s = 1$ is admissible. Thus, a t-tuple $(p_1, \cdots, p_t)$ for $s = 2$ is admissible. □

6. Admissible triples and quadruples

In this section, we classify admissible triples and quadruples.

6.1. **Admissible triples.** By Theorem 2.2 and 2.3, a triple (p, q, r) for $s = 3$ is admissible if and only if $\ell \mid \phi(pqr)$. And by Theorem 5.1, a triple (p, q, r) for $s = 1$ is admissible if and only if $q \equiv r \equiv -1 \pmod{\ell}$.

However if $s = 2$, we cannot directly use above theorems to get admissible triples. By Theorem 2.2, if (p, q, r) for $s = 2$ is admissible, then $r \equiv -1 \pmod{\ell}$. Assume that $r \equiv -1 \pmod{\ell}$. Let $I := (U_p - 1,\ U_r + 1,\ T_s - s - 1\ :\ \text{for primes } s \nmid pr)$ be an Eisenstein ideal of $\mathbb{T} := \mathbb{T}_{pr}$ and $\mathfrak{m} := (\ell, I)$. Since $r \equiv -1 \pmod{\ell}$, $\mathfrak{m}$ is new maximal at level pr. We want to understand admissibility of a triple (p, q, r) for $s = 2$ by a level raising method.

Theorem 6.1. *Assume $p \not\equiv 1 \pmod{\ell}$ and if $q \equiv 1 \pmod{\ell}$, assume further that p is not an ℓ-th power modulo q. Let $\eta_q := T_q - q - 1$. Then, a triple (p, q, r) for $s = 2$ is admissible if η_q is not a generator of $I_{\mathfrak{m}}$.*

Assume further that $q \not\equiv 1 \pmod{\ell}$ and $r \not\equiv -1 \pmod{\ell^2}$. Then, a triple (p, q, r) for $s = 2$ is not admissible if η_q is a generator of $I_{\mathfrak{m}}$.

Proof. By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow \Phi_\ell \longrightarrow (X/\eta_q X)_\ell \longrightarrow \Psi_\ell^+ \longrightarrow 0,$$

where $\Phi := \Phi_p(J_0(pr))$, $X := X_p(J_0(pr))$, and $\Psi := \Phi_q(J_0^{pq}(r))$. By the Appendix A, $\#\Phi_{\mathfrak{m}} = \ell^n$, where ℓ^n is the power of ℓ exactly dividing $r + 1$. By the result in [Y14], $X_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$ and $(\mathbb{T}/I)_{\mathfrak{m}} \simeq \mathbb{Z}/\ell^n\mathbb{Z}$. Assume η_q is not a generator of $I_{\mathfrak{m}}$. Then,

$$\#(X/\eta_q X)_{\mathfrak{m}} > \#(X/IX)_{\mathfrak{m}} = \#(\mathbb{T}/I)_{\mathfrak{m}} = \ell^n.$$

Thus, $\Psi_{\mathfrak{n}}^+ \neq 0$, where $\mathfrak{n} := (\ell,\ U_p - 1,\ U_q - 1,\ U_r + 1,\ T_w - w - 1\ :\ \text{for primes } w \nmid pqr) \subset \mathbb{T}_r^{pq}$. In other words, $\mathfrak{n}$ is maximal. If it is r-old, then a pair (p, q) for $s = 2$ is admissible, which contradicts our assumption. Therefore, $\mathfrak{n}$ is r-new, and by the Jacquet-Langlands correspondence, $\mathfrak{n}$ is new.

Assume further that $q \not\equiv 1 \pmod{\ell}$, $r \not\equiv -1 \pmod{\ell^2}$, and η_q is a local generator of I, i.e., $I_{\mathfrak{m}} = (\eta_q) = \mathfrak{m}_{\mathfrak{m}}$.

Let K be the kernel of the map $J_0^{pr}(1) \times J_0^{pr}(1) \to J_0^{pr}(q)$ by degeneracy maps. Then, as in §3.2, we have

$$0 \subset K_{\mathfrak{m}}^+ \subset (K^\perp)_{\mathfrak{m}}^+ \subset J_0^{pr}(1)[\eta_q]_{\mathfrak{m}}.$$

Since by Proposition C.4, K^+ contains the Skorobogatov subgroup of $J_0^{pr}(1)$ at r, which is of order $r+1$ up to products of powers of 2 and 3, $\#K_{\mathfrak{m}}^+ \geq \ell$. Since $(\eta_q) = I_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}$, $J_0^{pr}(1)[\eta_q]_{\mathfrak{m}} = J_0^{pr}(1)[\mathfrak{m}]$ and it is of dimension 2 over $\mathbb{T}^{pr}/\mathfrak{m} \simeq \mathbb{F}_\ell$ by Theorem B.1. Because $J_0^{pr}(1)[\eta_q]/(K^\perp)^+$ is the $\mathbb{G}_{\mathrm{m}}$-dual of K^+, $(K^\perp/K)_{\mathfrak{m}}^+ = 0$. Since $K^\perp/K$ is isomorphic to the intersection of the q-old subvariety and the q-new subvariety of $J_0^{pr}(q)$ and $\mathfrak{m}$ is not in the support of $K^\perp/K$, by Theorem 3.6, level raising does not occur. Thus, $\mathfrak{n}$ is not q-new. □

6.1.1. *Examples.* When N is prime, η_q is a local generator of an Eisenstein ideal $I = (T_r - r - 1\ :\ \text{for primes } r \neq N) \subset \mathbb{T}_N$ at $\mathfrak{m} := (\ell, I)$ if and only if $q \not\equiv 1 \pmod{\ell}$ and q is not an ℓ-th power modulo N. However, when N is composite, we don't know what congruence implies local generation.

Consider the easiest case. As in Theorem 6.1, we assume that $r \equiv -1 \pmod{\ell}$ and $p \not\equiv 1 \pmod{\ell}$. Assume further that $r \not\equiv -1 \pmod{\ell^2}$. In this case, $I_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}$.

Let $f(\tau) = \sum a_n x^n$ be a newform of weight 2 for $\Gamma_0(pr)$ whose mod ℓ Galois representation is reducible such that $a_p = 1$, $a_r = -1$, where $x = e^{2\pi i \tau}$. If $a_q \equiv q + 1 \pmod{\mathfrak{m}^2}$, $\eta_q := T_q - q - 1 \in \mathfrak{m}^2$, so η_q is not a local generator of I at $\mathfrak{m}$.

Moreover, in our examples below, all newforms are defined over $\mathbb{Q}$, i.e., $\mathbb{T}_{pr} = \mathbb{Z}$ and $\mathfrak{m} = \ell\mathbb{Z}$. Thus, η_q is not a local generator if and only if $\eta_q \equiv 0 \pmod{\ell^2}$. In the examples below, we follow the notation in Stein's table [St].

(1) Admissibility of $(2, q, 19)$ for $s = 2$ when $\ell = 5$.
A newform f of level $pr = 38$ (as above) is $E[38, 2]$. Let a_n be the eigenvalue of T_n for $E[38, 2]$. Then $a_q \equiv 1 + q \pmod{25}$ when q = 23, 41, 97, 101, 109, 113, 149, 151, 193, 199, 239, 241, 251, 257, 277, 347, 359, 431, and 479 for primes $q < 500$. Since only $(2, 151)$, $(2, 241)$, $(2, 251)$, and $(2, 431)$ for $s = 2$ are admissible, a triple $(2, q, 19)$ for $s = 2$ is admissible if
$$q = 23, 41, 97, 101, 109, 113, 149, 193, 199, 239, 257, 277, 347, 359, \text{ or } 479$$
for $q < 500$.

Remark 6.2. A newform of level $2 \times 23 \times 19$ with $a_2 = a_{23} = 1$ and $a_{19} = -1$ is $E[874, 8]$.

(2) Admissibility of $(3, q, 19)$ for $s = 2$ when $\ell = 5$.
A newform f of level $pr = 57$ (as above) is $E[57, 3]$. Let b_n be the eigenvalue of T_n for $E[57, 3]$. Then $b_q \equiv 1 + q \pmod{25}$ when q = 41, 97, 101, 167, 197, 251, 257, 269, 313, 349, 409, 419, 431, and 491 for primes $q < 500$. Since only $(3, 41)$, $(3, 431)$, and $(3, 491)$ for $s = 2$ are admissible, a triple $(3, q, 19)$ for $s = 2$ is admissible if
$$q = 97, 101, 167, 197, 251, 257, 269, 313, 349, 409, \text{ or } 419$$
for $q < 500$.

(3) Admissibility of $(2, q, 29)$ for $s = 2$ when $\ell = 5$.
A newform f of level $pr = 58$ (as above) is $E[58, 1]$. Let c_n be the eigenvalue of T_n for $E[58, 1]$. Then $c_q \equiv 1 + q \pmod{25}$ when q = 89, 97, 137, 151, 181, 191, 223, 241, 251, 347, 367, 401, 431, 433, and 491 for primes $q < 500$. Since only $(2, 151)$, $(2, 241)$, $(2, 251)$, and $(2, 431)$ for $s = 2$ are admissible, a triple $(2, q, 29)$ for $s = 2$ is admissible if
$$q = 89, 97, 137, 181, 191, 223, 347, 367, 401, 433, \text{ or } 491$$
for $q < 500$.

(4) Admissibility of $(2, q, 13)$ for $s = 2$ when $\ell = 7$.
A newform f of level $pr = 26$ (as above) is $E[26, 2]$. Let d_n be the eigenvalue of T_n for $E[26, 2]$. Then $d_q \equiv 1 + q \pmod{49}$ when q = 43, 101, 223, 229, 233, 269, 307, 311, and 349 for primes $q < 500$. Since a pair $(2, q)$ for $s = 2$ is not admissible for $q < 500$, a triple $(2, q, 13)$ for $s = 2$ is admissible if
$$q = 43, 101, 223, 229, 233, 269, 307, 311, \text{ or } 349$$
for $q < 500$.

Remark 6.3. In the last case, a pair $(2, q)$ for $s = 2$ is admissible when $q = 631$ and $q = 673$. As before,
$$d_{631} \equiv 1 + 631 \pmod{49} \text{ and } d_{691} \equiv 1 + 691 \pmod{49}.$$
In other words, computations tells that if a pair (p, q) for $s = 2$ is admissible then $\eta_q := T_q - q - 1$ is not a local generator of I at $\mathfrak{m}$.

6.2. **Admissible quadruples.** By Theorem 5.1, a quadruple (p, q, r, w) for $s = 1$ is admissible if and only if $q \equiv r \equiv w \equiv -1 \pmod{\ell}$. And by Corollary 5.3, a quadruple (p, q, r, w) for $s = 2$ is admissible if and only if $r \equiv w \equiv -1 \pmod{\ell}$. Moreover, by Theorem 2.2 and 2.3, a quadruple (p, q, r, w) for $s = 3$ is admissible if and only if $w \equiv -1 \pmod{\ell}$.

Even though we don't know what are the necessary and sufficient conditions for admissibility of quadruples for $s = 4$, we can mimic the strategy for the case $(s, t) = (2, 3)$.

By Theorem 2.2, a quadruple (p, q, r, w) for $s = 4$ is admissible only if $\ell \mid \phi(pqrw)$. So, without loss of generality, assume that $p \equiv 1 \pmod{\ell}$. Assume further that $\ell^2 \nmid \phi(pqr)$. Let $I := (U_p - 1,\ U_q - 1,\ U_r - 1,\ T_k - k - 1\ :\ \text{for primes } k \nmid pqr)$ be an Eisenstein ideal of $\mathbb{T} := \mathbb{T}_{pqr}$ and $\mathfrak{m} := (\ell, I)$. By Theorem 2.3, $\mathfrak{m}$ is new.

Theorem 6.4. *A quadruple (p, q, r, w) for $s = 4$ is admissible if $\eta_w := T_w - w - 1$ is not a local generator of I at $\mathfrak{m}$.*

Before we prove the above theorem, we need some lemmas about the character group and the component group of $J_0^{qr}(p)$ over $\mathbb{F}_p$.

Lemma 6.5. *Let $\Phi := \Phi_p(J_0^{qr}(p))$. Then, the order of Φ is $\phi(pqr)$ up to products of powers of 2 and 3 and $\Phi \otimes \mathbb{T}_{\mathfrak{m}} \simeq \mathbb{Z}/\ell\mathbb{Z}$.*

Proof. This follows from Proposition A.2 and A.3. □

Lemma 6.6. *Let $X := X_p(J_0^{qr}(p))$. Then, $\#(X/IX) \otimes \mathbb{T}_{\mathfrak{m}} \geq \#(\mathbb{T}/I) \otimes \mathbb{T}_{\mathfrak{m}} = \ell$.*

Proof. Since X is a $(\mathbb{T}_p^{qr})^{p\text{-new}}$-module of rank 1 in the sense of Mazur (§II.6 in [M77]), $\#(X/IX) \otimes (\mathbb{T}_p^{qr})_{\mathfrak{m}}^{p\text{-new}} \geq \#(\mathbb{T}_p^{qr}/I) \otimes (\mathbb{T}_p^{qr})_{\mathfrak{m}}^{p\text{-new}}$. By the Janquent-Langlands correspondence, $(\mathbb{T}_p^{qr})^{p\text{-new}} \simeq \mathbb{T}^{\text{new}}$ and $(\mathbb{T}_p^{qr})_{\mathfrak{m}}^{p\text{-new}} \simeq \mathbb{T}_{\mathfrak{m}}^{\text{new}}$. Since $I_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}$ from the assumption $\ell^2 \nmid \phi(pqr)$, $(\mathbb{T}_p^{qr}/I) \otimes (\mathbb{T}_p^{qr})_{\mathfrak{m}}^{p\text{-new}} \simeq (\mathbb{T}^{\text{new}}/\mathfrak{m}) \otimes \mathbb{T}_{\mathfrak{m}}^{\text{new}} \simeq \mathbb{F}_\ell$. Thus, the result follows. □

Proof of Theorem 6.4. It suffices to show that an ideal $\mathfrak{n} := (\ell,\ U_p - 1,\ U_q - 1,\ U_r - 1,\ U_w - 1,\ T_k - k - 1\ :\ \text{for primes } k \nmid pqrw)$ of $\mathbb{T}_{pqrs}$ is new. By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow \Phi_\ell \longrightarrow (X/\eta_w X)_\ell \longrightarrow \Psi_\ell^+ \longrightarrow 0,$$

where $\Psi := \Phi_w(J_0^{pqrw}(1))$. Assume that η_w is not a local generator of I at $\mathfrak{m}$. Since η_w is not a local generator of I at $\mathfrak{m}$, We have

$$\#(X/\eta_w X) \otimes \mathbb{T}_{\mathfrak{m}} > \#(X/IX) \otimes \mathbb{T}_{\mathfrak{m}} \geq \ell = \Phi \otimes \mathbb{T}_{\mathfrak{m}}.$$

Thus, $\mathfrak{n}$ is in the support of Ψ, which means that $\mathfrak{n}$ is a maximal ideal of $\mathbb{T}^{pqrw}$. By the Janquet-Langlands correspondence, $\mathfrak{n}$ is new. □

6.2.1. *Examples.* Consider the easiest case. As in Theorem 6.4, we assume that $p \equiv 1 \pmod{\ell}$ and $\ell \nmid (q-1)(r-1)$. Assume further that $p \not\equiv 1 \pmod{\ell^2}$. In this case, $I_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}$.

Let $f(\tau) = \sum a_n x^n$ be a newform of weight 2 for $\Gamma_0(pqr)$ whose mod ℓ Galois representation is reducible such that $a_p = a_q = a_r = 1$, where $x = e^{2\pi i \tau}$. If $a_w \equiv w + 1 \pmod{\mathfrak{m}^2}$, $\eta_w := T_w - w - 1 \in \mathfrak{m}^2$, so η_w is not a local generator of I at $\mathfrak{m}$.

Moreover, in our examples below, all newforms are defined over $\mathbb{Q}$, i.e., $\mathbb{T}_{pqr} = \mathbb{Z}$ and $\mathfrak{m} = \ell\mathbb{Z}$. Thus, η_q is not a local generator if and only if $\eta_q \equiv 0 \pmod{\ell^2}$. In the examples below, we follow the notation in Stein's table [St].

(1) Admissibility of $(11, 2, 3, w)$ for $s = 4$ when $\ell = 5$.
A newform f of level $pqr = 66$ (as above) is $E[66, 2]$. Let a_n be the eigenvalue of T_n for $E[66, 2]$. Then $a_w \equiv 1 + w \pmod{25}$ when w = 47, 53, 97, 101, 103, 127, 151, 211, 271, 307, 317, and 431 for primes $w < 500$. Thus, a quadruple $(11, 2, 3, w)$ for $s = 4$ is admissible if

$$w = 47, 53, 97, 47, 53, 97, 101, 103, 127, 151, 211, 271, 307, 317, \text{ or } 431$$

for $w < 500$.

(2) Admissibility of $(31, 2, 3, w)$ for $s = 4$ when $\ell = 5$.
A newform f of level $pqr = 186$ (as above) is $E[186, 3]$. Let b_n be the eigenvalue of T_n for $E[186, 3]$. Then $b_w \equiv 1 + w \pmod{25}$ when w = 19, 43, 59, 67, 71, 101, 109, 113, 131, 157, 181, 191, 227, 281, 283, 307, 331, 349, 359, 421, 431, and 443 for primes $w < 500$. Thus, a quadruple $(31, 2, 3, w)$ for $s = 4$ is admissible if

$$w = 19, 43, 59, 67, 71, 101, 109, 113, 131, 157, 181, 191,$$

$$227, 281, 283, 307, 331, 349, 359, 421, 431, \text{ or } 443$$

for $w < 500$.

APPENDIX A. THE COMPONENT GROUP OF $J_0^D(Np)$ OVER $\mathbb{F}_p$

In their paper [DR93], Deligne and Rapoport studied integral models of modular curves. Buzzard extended their result to the case of Shimura curves [Bu97]. In this appendix, we explain the special fiber of $J_0^D(Np)_{/\mathbb{Z}}$ over $\mathbb{F}_p$ for a prime $p \nmid DN$ and the Hecke actions on its component group. Assume that N is a square-free integer prime to D.

A.1. The special fiber $J_0^D(Np)_{/\mathbb{F}_p}$.

Proposition A.1 (Deligne-Rapoport model). *$X_0^D(Np)_{/\mathbb{F}_p}$ consists of two copies of $X_0^D(N)_{/\mathbb{F}_p}$. They meet transversally at supersingular points.*

Let S be the set of supersingular points of $X_0^D(Np)_{/\mathbb{F}_p}$. Then S is isomorphic to the set of isomorphism classes of right ideals of an Eichler order of level N of the definite quaternion algebra over $\mathbb{Q}$ of discriminant Dp (cf. [R90]). By the theory of Raynaud [Ra70], we have the special fiber $J_0^D(Np)_{/\mathbb{F}_p}$ of the Néron model of $J_0^D(Np)_{/\mathbb{Z}}$ at p. It satisfies the following exact sequence

$$0 \longrightarrow J^0 \longrightarrow J_0^D(Np)_{/\mathbb{F}_p} \longrightarrow \Phi_p(J_0^D(Np)) \longrightarrow 0,$$

where J^0 is *the identity component* and $\Phi_p(J_0^D(Np))$ is *the component group*. Moreover, J^0 is an extension of $J_0^D(N) \times J_0^D(N)_{/\mathbb{F}_p}$ by T, *the torus of* $J_0^D(Np)_{/\mathbb{F}_p}$. The Cartier dual of T, $\mathrm{Hom}(T, \mathbb{G}_{\mathrm{m}})$, is called *the character group* $X := X_p(J_0^D(Np))$. It is isomorphic to the group of degree 0 elements in the free abelian group $\mathbb{Z}^S$ generated by the elements of S. (Note that, the degree of an element in $\mathbb{Z}^S$ is the sum of its coefficients.) There is a natural pairing of $\mathbb{Z}^S$ such that

$$\text{for any } s, t \in S, \quad \langle s, t \rangle := \frac{\#\mathrm{Aut}(s)}{2}\delta_{st},$$

where δ_{st} is the Kronecker δ-function. This pairing induces an injection $X \hookrightarrow \mathrm{Hom}(X, \mathbb{Z})$ and the cokernel of it is isomorphic to $\Phi_p(J_0^D(Np))$ by Grothendieck [Gr72]. We call the following exact sequence *the monodromy exact sequence*

$$0 \longrightarrow X \xrightarrow{\ i\ } \mathrm{Hom}(X, \mathbb{Z}) \longrightarrow \Phi_p(J_0^D(Np)) \longrightarrow 0. \tag{A.1}$$

For more details, see [R90].

A.2. Hecke actions on $\Phi_p(J_0^D(Np))$.

By the Proposition 3.8 of [R90], the Frobenius automorphism Frob_p on X is equal to the operator T_p on it. Frob_p sends $s \in S$ to some other $s' \in S$, or might fix s. For elements $s, t \in S$ the above map i sends $s - t$ to $\phi_s - \phi_t$, where

$$\phi_s(x) := \langle s, x \rangle \quad \text{for any } x \in S.$$

Thus in the group $\Phi_p(J_0^D(Np))$, $\phi_s = \phi_t$ for any $s, t \in S$. Since for all $s \in S$, the elements $\frac{2}{\#\mathrm{Aut}(s)}\phi_s$ generate $\mathrm{Hom}(X, \mathbb{Z})$ and $\#\mathrm{Aut}(s)$ is a divisor of 12, $\Phi_p(J_0^D(Np)) \sim \Phi$, where Φ is the cyclic subgroup generated by the image of ϕ_s for some $s \in S$. (cf. Proposition 3.2 of [R90].)

Proposition A.2. *For a prime divisor r of Dp (resp. N), $U_r - 1$ (resp. $U_r - r$) annihilates Φ. Moreover, for a prime r not dividing DNp, $T_r - r - 1$ annihilates Φ.*

Proof. On Φ, $\phi_s = \phi_t$. Thus, $U_p(\phi_s) = \phi_t = \phi_s$, where $t = \mathrm{Frob}_p(s)$. Since S is isomorphic to the set of isomorphism classes of right ideals on an Eichler order of level N in the definite quaternion algebra over $\mathbb{Q}$ of discriminant Dp, the set of supersingular points of $X_0^{Dp/q}(Nq)_{/\mathbb{F}_q}$ is again S for a prime $q \mid D$. In other words, the character group of $J_0^{Dp/q}(Nq)_{/\mathbb{F}_q}$ does not depend on the choice of a prime divisor q of Dp. (Hence the same is true for the component group by the monodromy exact sequence.) Using the same description as above, we have $U_q(\phi_s) = \phi_s$ for a prime divisor q of D.

Since the degree of the map U_r is r for a prime divisor r of N, $U_r(\phi_s) = \sum a_i \phi_{s_i}$ and $\sum a_i = r$. Therefore $U_r(\phi_s) = r\phi_s$ because in Φ, $\phi_s = \phi_{s_i}$ for all i. Similarly, $T_r(\phi_s) = (r+1)\phi_s$ for a prime r not dividing DNp. ☐

A.3. **The order of Φ.**

Proposition A.3. *The order of Φ is equal to $\phi(Dp)\psi(N)$ up to products of powers of 2 and 3.*

Proof. Let n be the order of Φ. Therefore, for any degree 0 divisor $t = \sum a_i s_i$, $n\phi_s(t) = 0$. We decompose n as a sum $\sum n_i$ for non-negative integers n_i. Then,

$$n\phi_s(t) = \left(\sum_j n_j \phi_s\right)\left(\sum_i a_i s_i\right) = \sum_j n_j \left(\phi_{s_j}\left(\sum_i a_i s_i\right)\right) = \sum_j n_j a_j \frac{\#\mathrm{Aut}(s_j)}{2} = 0.$$

Therefore, for any $i \neq j$, we have $n_i \frac{\#\mathrm{Aut}(s_i)}{2} = n_j \frac{\#\mathrm{Aut}(s_j)}{2} = c$ by taking $t = s_i - s_j$. Since $n > 0$, each n_i is positive and it is equal to $\frac{2c}{\#\mathrm{Aut}(s_i)}$, where $2c$ is the smallest positive integer which makes $\frac{2c}{\#\mathrm{Aut}(s_i)}$ an integer for all i. Since $\#\mathrm{Aut}(s_i)$ divides 12, $2c$ is a divisor of 12. Thus,

$$n = \sum_{s \in S} \frac{12}{\#\mathrm{Aut}(s)}$$

up to products of powers of 2 and 3.

Recall Eichler's mass formula. (cf. Corollary 5.2.3 of [Vi80].)

Proposition A.4 (mass formula)**.** *Let S be the set of isomorphism classes of right ideals of an Eichler order of level N in a definite quaternion algebra of discriminant Dp over a number field K. Then,*

$$\sum_{s \in S} \frac{\#R^\times}{\#\mathrm{Aut}(s)} = 2^{1-d} \times |\zeta_K(-1)| \times h_K \times \phi(Dp)\psi(N),$$

where R is the ring of integers of K, ζ_K is the Dedekind zeta function of K, d is the degree of K over $\mathbb{Q}$, and h_K is the class number of K.

In our case $K = \mathbb{Q}$, so $|\zeta_K(-1)| = \frac{1}{12}$, $d_K = h_K = 1$, and $\#R^\times = 2$. Thus, the result follows. □

A.4. **Degeneracy maps between component groups.** Let q be a prime not dividing DNp. Let Φ (resp. Φ') be the cyclic subgroup of the component group of $J_0^D(Np)$ (resp. $J_0^D(Npq)$) at p generated by the image of ϕ_s for some s as above.

Let $\gamma_q : J_0^D(Np) \times J_0^D(Np) \to J_0^D(Npq)$ be the map defined by $\gamma_q(a,b) = \alpha_q^*(a) + \beta_q^*(b)$, where α_q, β_q are the two degeneracy maps $X_0^D(Npq) \to X_0^D(Np)$. Then, γ_q induces a map $\gamma : \Phi \times \Phi \to \Phi'$.

Proposition A.5. *Let K (resp. C) be the kernel (resp. cokernel) of γ.*

$$0 \longrightarrow K \longrightarrow \Phi \times \Phi \xrightarrow{\gamma} \Phi' \longrightarrow C \longrightarrow 0.$$

Then, $K \sim \Phi$ and $C \simeq \mathbb{Z}/(q+1)\mathbb{Z}$.

Proof. Let S (resp. S') be the set of supersingular points of $X_0^D(Np)_{/\mathbb{F}_p}$ (resp. $X_0^D(Npq)_{/\mathbb{F}_p}$). Let $\Phi = \langle \phi_s \rangle$ (resp. $\Phi' = \langle \phi_t \rangle$) for some $s \in S$ (resp. $t \in S'$). Since the degree of α_q^* is $q+1$, $\alpha_q^*(s) = \sum a_i t_i$ for some $t_i \in S'$, where $\sum a_i = q+1$. Because $\phi_{t_i} = \phi_t$ in Φ', $\alpha_q^*(\phi_s) = (q+1)\phi_t$. By the same argument as above, $\beta_q^*(\phi_s) = (q+1)\phi_t$. Thus, the image of γ is generated by $(q+1)\phi_t$ and $C \simeq \mathbb{Z}/(q+1)\mathbb{Z}$. Since $\alpha_q^*(\phi_s) = \beta_q^*(\phi_s)$, $(a, -a) \in K$ for any $a \in \Phi$. By comparing orders, we have that the orders of K and Φ are equal up to products of powers of 2 and 3. □

Corollary A.6. *For primes $r \mid Dp$ (resp. $r \mid N$, $r \nmid DNpq$), $U_r - 1$ (resp. $U_r - r$, $T_r - r - 1$) annihilates C. Moreover, $U_q + 1$ annihilates C.*

Proof. This follows from A.2. □

Remark A.7. If $\ell > 3$ is prime, then the ℓ-primary part of K (resp. C) is equal to the one of the kernel K' (resp. the cokernel C') of

$$0 \longrightarrow K' \longrightarrow \Phi_p(J_0^D(Np)) \times \Phi_p(J_0^D(Np)) \xrightarrow{\gamma_q} \Phi_p(J_0^D(Npq)) \longrightarrow C' \longrightarrow 0,$$

because $\Phi_p(J_0^D(Np))$ and Φ are equal up to 2-, 3- primary subgroups.

Appendix B. Multiplicity one theorems

B.1. **The dimension of $J_0^{pr}(1)[\mathfrak{m}]$.** Let $J := J_0^{pr}(1)$ be the Jacobian of the Shimura curve $X_0^{pr}(1)$ and $\mathbb{T} := \mathbb{T}^{pr}$ be the Hecke ring in $\mathrm{End}(J)$. Assume that $p \not\equiv 1 \pmod{\ell}$ and $r \equiv -1 \pmod{\ell}$. Then, the corresponding ideal to $\mathfrak{m} := (\ell,\ U_p - 1,\ U_r + 1,\ T_w - w - 1\ :\ \text{for primes } w \nmid pr) \subset \mathbb{T}$ in $\mathbb{T}_{pr}$ is neither p-old nor r-old. In this case we can prove multiplicity one theorem for $J[\mathfrak{m}]$.

Theorem B.1 (Ribet)**.** *$J[\mathfrak{m}]$ is of dimension 2.*

For a proof, we need the following proposition.

Proposition B.2. *$\mathbb{T}_{\mathfrak{m}}$ is Gorenstein.*

Proof. Let Y be the character group of $J_{/\mathbb{F}_p}$. Then by Ribet [R90], there is an exact sequence;

$$0 \longrightarrow Y \longrightarrow L \longrightarrow X \oplus X \longrightarrow 0,$$

where $L := X_r(J_0(pr))$ (resp. $X := X_r(J_0(r))$) is the character group of $J_0(pr)_{/\mathbb{F}_r}$ (resp. $J_0(r)_{/\mathbb{F}_r}$). Since $\mathfrak{m}$ is not old, $(\mathbb{T}_{pr})_{\mathfrak{a}} \simeq \mathbb{T}_{\mathfrak{m}}$ and $X_{\mathfrak{b}} = 0$, where $\mathfrak{a}$ (resp. $\mathfrak{b}$) is the corresponding Eisenstein ideal to $\mathfrak{m}$ in $\mathbb{T}_{pr}$ (resp. $\mathbb{T}_r$). Thus, we have $Y_{\mathfrak{m}} \simeq L_{\mathfrak{a}}$. Since for $\mathfrak{a}$, multiplicity one theorem holds [Y14], it implies that $L_{\mathfrak{a}}$ is free of rank 1 over $(\mathbb{T}_{pr})_{\mathfrak{a}}$, i.e., $Y_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$. By Grothendieck [Gr72], there is a monodromy exact sequence,

$$0 \longrightarrow Y \longrightarrow \mathrm{Hom}(Y, \mathbb{Z}) \longrightarrow \Phi \longrightarrow 0,$$

where $\Phi := \Phi_p(J)$ is the component group of $J_{/\mathbb{F}_p}$. After tensoring with $\mathbb{Z}_\ell$ over $\mathbb{Z}$,

$$0 \longrightarrow Y \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}(Y \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell) \longrightarrow \Phi_\ell \longrightarrow 0.$$

Using an idempotent $e_{\mathfrak{m}} \in \mathbb{T}_\ell := \mathbb{T} \otimes \mathbb{Z}_\ell$, we get

$$0 \longrightarrow Y_{\mathfrak{m}} \longrightarrow \mathrm{Hom}(Y_{\mathfrak{m}}, \mathbb{Z}_\ell) \longrightarrow \Phi_{\mathfrak{m}} \longrightarrow 0.$$

By the Ribet's exact sequence in §3.3, we have

$$0 \longrightarrow K \longrightarrow (X \oplus X)/(\mu_p(X \oplus X)) \longrightarrow \Phi \longrightarrow C \longrightarrow 0.$$

Since first, second, and fourth terms vanish after localizing at $\mathfrak{m}$ (resp. $\mathfrak{a}$), $\Phi_{\mathfrak{m}} = 0$, which implies that $Y_{\mathfrak{m}} \simeq \mathrm{Hom}(Y_{\mathfrak{m}}, \mathbb{Z}_\ell)$ is self-dual. Therefore $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein. ☐

Now we prove the theorem above.

Proof of Theorem B.1. Let $J_{\mathfrak{m}} := \cup_n J[\mathfrak{m}^n]$ be the $\mathfrak{m}$-divisible group of J and let $T_{\mathfrak{m}}J$ be the Tate module of J at $\mathfrak{m}$, which is $\mathrm{Hom}(J_{\mathfrak{m}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Then $T_{\mathfrak{m}}J$ is free of rank 2 if and only if $J[\mathfrak{m}]$ is of dimension 2 over $\mathbb{T}/\mathfrak{m}$. Since J has purely toric reduction at p, there is an exact sequence for any $n \geq 1$ (c.f. [R76])

$$0 \longrightarrow \mathrm{Hom}(Y/\ell^n Y, \mu_{\ell^n}) \longrightarrow J[\ell^n] \longrightarrow Y/\ell^n Y \longrightarrow 0.$$

By taking projective limit, we have

$$0 \longrightarrow \mathrm{Hom}(Y \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell J \longrightarrow Y \otimes \mathbb{Z}_\ell \longrightarrow 0,$$

where $\mathbb{Z}_\ell(1)$ is the Tate twist. By applying idempotent $e_{\mathfrak{m}}$, we get

$$0 \longrightarrow \mathrm{Hom}(Y_{\mathfrak{m}}, \mathbb{Z}_\ell(1)) \longrightarrow T_{\mathfrak{m}}J \longrightarrow Y_{\mathfrak{m}} \longrightarrow 0.$$

Since $Y_{\mathfrak{m}}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}}$, $T_{\mathfrak{m}}J$ is free of rank 2 over $\mathbb{T}_{\mathfrak{m}}$. ☐

Remark B.3. By Mazur (appendix of [Ti97]), $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein if and only if $J[\mathfrak{m}]$ is of dimension 2.

B.2. **The dimension of $J_0(pqr)[\mathfrak{m}]$.** Let $J := J_0(pqr)$ and $\mathbb{T} := \mathbb{T}_{pqr}$. Let $L := X_p(J)$ be the character group of J at p and $\mathfrak{m} := (\ell,\ U_p - 1,\ U_q - 1,\ U_r + 1,\ T_s - s - s\ :\ \text{for primes } s \nmid pqr) \subset \mathbb{T}$.

Assume that $p \not\equiv 1 \pmod{\ell}$, $q \not\equiv 1 \pmod{\ell}$, and $r \equiv -1 \pmod{\ell}$. Then,

Theorem B.4. *$L/\mathfrak{m}L$ is of dimension 1 over $\mathbb{T}/\mathfrak{m}$ and $J[\mathfrak{m}]$ is of dimension 2.*

Proof. By Theorem 4.2(2) of [Y14], $\dim J[\mathfrak{m}] = 2$. Let T be the torus of J at p. Note that $J[\mathfrak{m}]$ is a non-trivial extension of μ_ℓ by $\mathbb{Z}/\ell\mathbb{Z}$ that ramified only at r. Since Frob_p acts by pU_p on T, $\mathbb{Z}/\ell\mathbb{Z}$ cannot be in $T[\mathfrak{m}]$. Since the dimension of $J[\mathfrak{m}]$ is 2, $T[\mathfrak{m}]$ is at most of dimension 1. On the other hand, $\mathbb{T}^{p\text{-new}}$ acts faithfully on T and $\mathfrak{m}$ is p-new because a pair (p, r) for $s = 1$ is admissible. Accordingly, the dimension of $T[\mathfrak{m}]$ is at least 1. Therefore $L/\mathfrak{m}L$, which is the dual space of $T[\mathfrak{m}]$, is of dimension 1. □

Appendix C. The Skorobogatov subgroup of $J_0^{pD}(N)$ at p

In his paper [Sk05], Skorobogatov introduced "Shimura coverings" of Shimura curves. Let B be a quaternion algebra over $\mathbb{Q}$ of discriminant pD such that $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$. Let $\mathcal{O}$ be an Eichler order of B of level N, and set $\Gamma_0^{pD}(N) := \mathcal{O}^{\times,1}$, the set of reduced norm 1 elements in $\mathcal{O}$. Let I_p be the unique two-sided ideal of $\mathcal{O}$ of reduced norm p. Then, $1 + I_p \subset \Gamma_0^{pD}(N)$ and it defines a covering $X \to X_0^{pD}(N)$. By Jordan [Sk05], there is an unramified subcovering $X \to X_p \to X_0^{pD}(N)$ whose Galois group is $\mathbb{Z}/((p+1)/\epsilon(p))$, where $\epsilon(p)$ is 1, 2, 3, or 6. (About $\epsilon(p)$, see page 781 of [Sk05].) Since unramifed abelian coverings of $X_0^{pD}(N)$ correspond to subgroups of $J_0^{pD}(N)$, we define the Skorobogatov subgroup of $J_0^{pD}(N)$ from X_p.

Definition C.1. The *Skorobogatov subgroup* Σ_p of $J_0^{pD}(N)$ at p is the subgroup of $J_0^{pD}(N)$ which corresponds to the above unramified covering X_p of $X_0^{pD}(N)$.

These subgroups have similar properties to Shimura subgroups. For example,

Proposition C.2. *On Σ_p, U_p (resp. U_q, U_r, T_s) acts by -1 (resp. 1, r, $s+1$) for primes $q \mid D$, $r \mid N$, and $s \nmid pDN$.*

Proof. The proof is similar to the action of Hecke operators on Shimura subgroups. By using moduli theoretic description of $X_0^{pD}(N)$, the complex points of X classifies (A, P) where A is a false elliptic curve with level N structure and P is a generator of $A[I_p]$. Since the level structures at primes r dividing DN are compatible with the level structure at p, which gives rise to our covering X, the Atkin-Lehner involution w_r acts trivially on the covering group. This gives the action of U_q when q divides D because $U_q = w_q$. Since for primes r dividing N, $U_r + w_r = \beta_r^*(\alpha_r)_*$ and $\beta_r^* = w_r\alpha_r^*$, $U_r = w_r\alpha_r^*(\alpha_r)_* - w_r = w_r(r+1) - w_r = r$ on Σ_p, where α_r and β_r are two degeneracy maps from $X_0^{pD}(N)$ to $X_0^{pD}(N/r)$. For primes $s \nmid pDN$, $T_s = (\beta_s)_*\alpha_s^* = (\beta_s)_* w_s \beta_s^* = (\beta_s)_*\beta_s^* = s + 1$ since the image of Skorobogatov subgroups by degeneracy maps lies in Skorobogatov subgroups and w_s acts trivially on it.

Consider U_p on Σ_p. The map U_p sends (A, P) to $(A/A[I_p], Q)$, where $\langle P, Q\rangle = \zeta_p$ for some fixed primitive p-th root of unity ζ_p and the pairing $\langle -,\ -\rangle$ on $A[I_p] \times A[p]/A[I_p]$. (About the above pairing, see [Bu97].) For σ in the covering group of $X \to X_0^{pD}(N)$, it sends (A, P) to $(A, \sigma P)$. Thus $U_p\sigma U_p^{-1} = \sigma^{-1}$, which implies U_p acts by -1 on Σ_p. □

Remark C.3. It might be easier than above if you consider the actions of w_p on the group of 2×2 matrices as in Calegari and Venkatesh. See page 29 of [CV12].

Proposition C.4. *Let K be the kernel of the map*

$$J_0^{pr}(1) \times J_0^{pr}(1) \to J_0^{pr}(q)$$

by the degeneracy maps α_q^* and β_q^*. Then K contains an antidiagonal embedding of Σ_r.

Proof. Let Σ_r (resp. Σ) be the Skorobogatov subgroup of $J_0^{pr}(1)$ (resp. $J_0^{pr}(q)$) at r. Since w_q acts trivially on Σ and the image of Σ_r by α_q^* lies in Σ, $\alpha_q^*(a) + \beta_q^*(-a) = \alpha_q^*(a) + w_q(\alpha_q^*(-a)) = \alpha_q^*(a) - \alpha_q^*(a) = 0$. Thus, K contains $\{(a, -a) \in J_0^{pr}(1) \times J_0^{pr}(1)\ :\ a \in \Sigma_r\}$. □

Remark C.5. Since K contains an antidiagonal embedding of Σ_r, $K[\mathfrak{m}] \neq 0$, where $\mathfrak{m} := (\ell,\ U_p - 1,\ U_r + 1,\ T_s - s - 1\ :\ \text{for primes } s \nmid pr)$ if $r \equiv -1 \pmod{\ell}$.

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E-mail address: `hwajong@gmail.com`

UNIVERSITÉ DU LUXEMBOURG, FACULTÉ DES SCIENCES, DE LA TECHNOLOGIE ET DE LA COMMUNICATION, 6, RUE RICHARD COUDENHOVE-KALERGI, L-1359 LUXEMBOURG, LUXEMBOURG