

## WEAK FIBER PRODUCTS IN BICATEGORIES OF FRACTIONS

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ABSTRACT. We fix any pair  $(\mathcal{C}, \mathbf{W})$  consisting of a bicategory and a class of morphisms in it, admitting a bicalculus of fractions, i.e. a “localization” of  $\mathcal{C}$  with respect to the class  $\mathbf{W}$ . In the resulting bicategory of fractions, we identify necessary and sufficient conditions for the existence of weak fiber products.

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## INTRODUCTION

In 1967 Pierre Gabriel and Michel Zisman proved in [GZ] that given a category  $\mathcal{C}$  and a class  $\mathbf{W}$  of morphisms in it, satisfying 4 technical conditions (called (CF1) – (CF4), see Appendix B), it is possible to construct a “localization” of  $\mathcal{C}$  with respect to  $\mathbf{W}$ , i.e. a category  $\mathcal{C}[\mathbf{W}^{-1}]$  (called “right category of fractions”) obtained from  $\mathcal{C}$  by formally adding inverses for all the morphisms in  $\mathbf{W}$ . To be more precise, objects of the category of fractions are the same as those of  $\mathcal{C}$ ; a morphism from  $A$  to  $B$  consists of an equivalence class of a triple  $(A', w, f)$  as follows

$$A \xleftarrow{w} A' \xrightarrow{f} B, \quad (0.1)$$

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such that  $w$  belongs to  $\mathbf{W}$  (we refer to Appendix B for the description of the equivalence relation used here). The technical conditions (CF) mentioned before allow to prove that the compositions of such morphisms exists and that it satisfies the usual properties of categories. Such a construction turned out to be very useful in several branches of mathematics, for example homotopy theory and triangulated categories.

In 1997 Dorette Pronk generalized such a construction from categories to bicategories (see [Pr]). To be more precise, given a bicategory  $\mathcal{C}$  and a class  $\mathbf{W}$  of morphisms in it, satisfying 5 technical conditions (called (BF1) – (BF5), see Appendix A), there is a “right bicategory of fractions”  $\mathcal{C}[\mathbf{W}^{-1}]$ . Such a bicategory in general is not unique, but any 2 bicategories of fractions for the same pair  $(\mathcal{C}, \mathbf{W})$  are equivalent using the axiom of choice. Objects in  $\mathcal{C}[\mathbf{W}^{-1}]$  are the same as those of  $\mathcal{C}$ ; morphisms are given by triples  $(A', w, f)$  as in (0.1) (but not quotiented by an equivalence relation, differently from the case of categories of fractions). 2-morphisms consist of classes of equivalence of quintuples of an object, a pair of morphisms and a pair of 2-morphisms, satisfying some technical conditions (for more details, we refer to Appendix A).

In the case when  $\mathcal{C}$  is a category (considered as a trivial bicategory), then the 5 technical conditions (BF) coincide with the 4 technical conditions (CF) and any resulting right bicategory of fractions for  $(\mathcal{C}, \mathbf{W})$  is equivalent to the (trivial bicategory associated to) the right category of fractions for  $(\mathcal{C}, \mathbf{W})$ .

Pronk introduced the notion of bicategory of fractions mainly in order to study certain bicategories of stacks (we refer directly to [Pr] for details). More recently, bicategories of fractions were used intensively mainly in relation with the notion of butterflies; we refer to [AMMV], [MMV] and [R] for some recent interesting development in this area.

The problem that we want to investigate in the present paper is the following: *when do weak fiber products exist in a bicategory of fractions?* We recall that weak fiber products are the natural generalization of (strong) fiber products from categories to bicategories (in the case when the bicategory is a 2-category, they are also called 2-fiber products; we refer to Definition 2.1 for the precise notion of weak fiber product in any bicategory). Weak fiber products are one of the basic tools used whenever one has to deal with a 2-category or bicategory of stacks (on a given site). It is known that weak fiber products of stacks (over a given site) exist because stackification commutes with 2-fiber products. However, very few is known in general about weak fiber products if we restrict to a strict sub-2-category of stacks (for example, the sub-2-category of differentiable stacks in the 2-category of stacks over the site of smooth manifolds, see e.g. [J, Definition 8.1]). Frequently, such sub-2-categories can be described as (equivalent to) bicategories of fractions (see e.g. [Pr, Corollary 43] for a description of the 2-category of differentiable stacks as a bicategory of fractions). So it is interesting to understand under which conditions weak fiber products exist in this framework.

If we try to understand the notion of weak fiber products in the case when we work in a bicategory of fractions, we get soon stuck in a very complicated setup. Roughly speaking (see Definition 2.1 for details), given any bicategory  $\mathcal{D}$ , any triple of objects  $A, B^1, B^2$  and any pair of morphisms  $g^1 : B^1 \rightarrow A$  and  $g^2 : B^2 \rightarrow A$ ,

- (a) a weak fiber product of  $g^1$  and  $g^2$  in  $\mathcal{D}$  is the datum of 1 object  $C$ , 2 of morphisms  $r^1 : C \rightarrow B^1$ ,  $r^2 : C \rightarrow B^2$  and 1 invertible 2-morphism  $\Omega : g^1 \circ r^1 \Rightarrow g^2 \circ r^2$ ;
- (b) in order to verify if a set  $(C, r^1, r^2, \Omega)$  as above gives a weak fiber product, one has to compare it against a set of 1 object  $D$ , 4 morphisms  $s^1, s^2, t, t'$  and 3 2-morphisms  $\Lambda, \Gamma^1, \Gamma^2$  (satisfying some technical conditions);
- (c) the comparison of  $(C, r^1, r^2, \Omega)$  against the set of data in (b) has to give back 1 morphism  $s$  and 3 2-morphisms  $\Lambda^1, \Lambda^2, \Gamma$  (satisfying some technical conditions).

In the special case when  $\mathcal{D}$  is a bicategory of fractions  $\mathcal{C} [\mathbf{W}^{-1}]$ , then the objects of  $\mathcal{D}$  are the same as those of  $\mathcal{C}$ , the morphisms of  $\mathcal{D}$  are triples of an object and a pair of morphisms as in (0.1) and the 2-morphisms of  $\mathcal{D}$  are (classes of equivalence of) quintuples of an object, a pair of morphisms and a pair of 2-morphisms of  $\mathcal{C}$ . Therefore, (a) – (c) above becomes:

- (a)' given any pair of morphisms with the same target in  $\mathcal{C} [\mathbf{W}^{-1}]$ , a weak fiber product of them a priori consists of 4 objects, 6 morphisms and 2 2-morphisms of  $\mathcal{C}$ ;
- (b)' in order to verify if the set of data as in (a)' is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$ , a priori one has to compare it against a set of 8 objects, 14 morphisms and 6 2-morphisms of  $\mathcal{C}$  (satisfying some technical conditions);
- (c)' the comparison of the data of (a)' against the data of (b)' has to give back 4 objects, 8 morphisms and 6 2-morphisms of  $\mathcal{C}$  (satisfying some technical conditions).

This means that having fixed any pair of morphisms with the same target in a bicategory  $\mathcal{D}$ ,

- there are 16 data of  $\mathcal{D}$  (as in (a) – (c)) that we have either to construct (in order to define a weak fiber product) or to consider (in order to prove that what we constructed is actually a weak fiber product);
- if  $\mathcal{D} = \mathcal{C} [\mathbf{W}^{-1}]$ , such data turn out to be given by 58 data of  $\mathcal{C}$ .

As such, the problem of constructing a weak fiber product in a bicategory of fractions apparently is very complicated. In the present paper we prove that such a problem can be considerably simplified by reducing the 58 data mentioned above to only 31. To be more precise, first of all we will show that it is sufficient to find 4 data of  $\mathcal{C}$  in order to define a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$  (instead of the 12 data needed a priori in (a)' above):

**Theorem 0.1.** *Let us fix any bicategory  $\mathcal{C}$  and any class  $\mathbf{W}$  of morphisms in it, satisfying axioms (BF), and let us choose any bicategory of fractions  $\mathcal{C} [\mathbf{W}^{-1}]$  associated to the pair  $(\mathcal{C}, \mathbf{W})$ . Given any pair of morphisms  $f^1 : B^1 \rightarrow A$  and  $f^2 : B^2 \rightarrow A$  in  $\mathcal{C}$ , the following facts are equivalent:*

- (i) *for any pair of morphisms of  $\mathbf{W}$  of the form  $w^1 : B^1 \rightarrow \overline{B}^1$ ,  $w^2 : B^2 \rightarrow \overline{B}^2$ , the pair of morphisms*

$$\begin{array}{ccc} & \overline{B}^1 & \\ & \downarrow (B^1, w^1, f^1) & \\ \overline{B}^2 & \xrightarrow{(B^2, w^2, f^2)} & A \end{array} \quad (0.2)$$

*admits a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$ ;*

- (ii) *there are an object  $C$ , a pair of morphisms  $p^1 : C \rightarrow B^1$ ,  $p^2 : C \rightarrow B^2$  and an invertible 2-morphism  $\omega : f^1 \circ p^1 \Rightarrow f^2 \circ p^2$  in  $\mathcal{C}$ , such that the diagram*

$$\begin{array}{ccc}
C & \xrightarrow{(C, \text{id}_C, p^1)} & B^1 \\
(C, \text{id}_C, p^2) \downarrow & \swarrow \Omega := [C, \text{id}_C, \text{id}_C, i_{(\text{id}_C \circ \text{id}_C) \circ \text{id}_C}, \omega * i_{\text{id}_C}] & \downarrow (B^1, \text{id}_{B^1}, f^1) \\
B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A
\end{array} \tag{0.3}$$

is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$ .

Moreover, given any pair of morphisms  $w^1, w^2$  in  $\mathbf{W}$  as above, a weak fiber product for (0.2) can be obtained easily as a suitable modification of diagram (0.3) (we refer to Corollary 4.2 for details).

In addition, we have the following result, where the 2-morphisms  $\theta_\bullet$  are the associators of the bicategory  $\mathcal{C}$  (they are all trivial if  $\mathcal{C}$  is a 2-category).

**Theorem 0.2.** *Let us fix any bicategory  $\mathcal{C}$  and any class  $\mathbf{W}$  of morphisms in it, satisfying axioms (BF), and let us choose any bicategory of fractions  $\mathcal{C} [\mathbf{W}^{-1}]$  associated to the pair  $(\mathcal{C}, \mathbf{W})$ . Moreover, let us fix any set of data in  $\mathcal{C}$  as in the following diagram*

$$\begin{array}{ccc}
C & \xrightarrow{p^1} & B^1 \\
p^2 \downarrow & \swarrow \omega & \downarrow f^1 \\
B^2 & \xrightarrow{f^2} & A
\end{array} \tag{0.4}$$

Then the induced diagram (0.3) is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$  if and only if the following 3 conditions hold for each object  $D$  of  $\mathcal{C}$ :

- (a) given any pair of morphisms  $q^m : D \rightarrow B^m$  for  $m = 1, 2$  and any invertible 2-morphism  $\lambda : f^1 \circ q^1 \Rightarrow f^2 \circ q^2$  in  $\mathcal{C}$ , there are an object  $E$ , a morphism  $v : E \rightarrow D$  in  $\mathbf{W}$ , a morphism  $q : E \rightarrow C$  and a pair of invertible 2-morphisms  $\lambda^m : q^m \circ v \Rightarrow p^m \circ q$  for  $m = 1, 2$  in  $\mathcal{C}$ , such that:

$$\begin{aligned}
& \theta_{f^2, p^2, q}^{-1} \odot (\omega * i_q) \odot \theta_{f^1, p^1, q} \odot (i_{f^1} * \lambda^1) = \\
& = (i_{f^2} * \lambda^2) \odot \theta_{f^2, q^2, v}^{-1} \odot (\lambda * i_v) \odot \theta_{f^1, q^1, v};
\end{aligned} \tag{0.5}$$

- (b) given any pair of morphisms  $t, t' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\gamma^m : p^m \circ t \Rightarrow p^m \circ t'$  for  $m = 1, 2$  in  $\mathcal{C}$  such that

$$\begin{aligned}
& \theta_{f^2, p^2, t'}^{-1} \odot (\omega * i_{t'}) \odot \theta_{f^1, p^1, t'} \odot (i_{f^1} * \gamma^1) = \\
& = (i_{f^2} * \gamma^2) \odot \theta_{f^2, p^2, t}^{-1} \odot (\omega * i_t) \odot \theta_{f^1, p^1, t},
\end{aligned} \tag{0.6}$$

there are an object  $F$ , a morphism  $u : F \rightarrow D$  in  $\mathbf{W}$  and an invertible 2-morphism  $\gamma : t \circ u \Rightarrow t' \circ u$  in  $\mathcal{C}$ , such that

$$\theta_{p^m, t', u} \odot (i_{p^m} * \gamma) = (\gamma^m * i_u) \odot \theta_{p^m, t, u} \quad \text{for } m = 1, 2; \tag{0.7}$$

(c) given any set of data  $(t, t', \gamma^1, \gamma^2, F, u, \gamma)$  as in (b), if there is another choice of data  $\tilde{F}, \tilde{u} : \tilde{F} \rightarrow D$  in  $\mathbf{W}$  and  $\tilde{\gamma} : t \circ \tilde{u} \Rightarrow t' \circ \tilde{u}$  invertible, such that

$$\theta_{p^m, t', \tilde{u}} \odot (i_{p^m} * \tilde{\gamma}) = (\gamma^m * i_{\tilde{u}}) \odot \theta_{p^m, t, \tilde{u}} \quad \text{for } m = 1, 2, \quad (0.8)$$

then there are an object  $G$ , a morphisms  $z : G \rightarrow F$  in  $\mathbf{W}$ , a morphism  $\tilde{z} : G \rightarrow \tilde{F}$  and an invertible 2-morphism  $\mu : u \circ z \Rightarrow \tilde{u} \circ \tilde{z}$ , such that

$$\begin{aligned} \theta_{t', \tilde{u}, \tilde{z}} \odot (i_{t'} * \mu) \odot \theta_{t', u, z}^{-1} \odot (\gamma * i_z) &= \\ = (\tilde{\gamma} * i_{\tilde{z}}) \odot \theta_{t, \tilde{u}, \tilde{z}} \odot (i_t * \mu) \odot \theta_{t, u, z}^{-1}. \end{aligned} \quad (0.9)$$

As a consequence of Theorem 0.2, we have the following general principle. Suppose that we are working in a given bicategory  $\mathcal{C}$  and that for some reason not all the weak fiber products exist in  $\mathcal{C}$ , or that not all the “interesting” fiber products exist there (for example, the pullbacks along a certain class of “good” maps, etc). Then a possible way to try to solve this problem is the following:

- (1) for each given pair of morphisms  $f^m : B^m \rightarrow A$  for  $m = 1, 2$  (or for each given pair  $(f^1, f^2)$  that is “interesting” as above), try to identify a “candidate” for a weak fiber product in  $\mathcal{C}$ , i.e. a quadruple  $(C, p^1, p^2, \omega)$  as in (0.4);
- (2) given any data as in (1) and any set of data  $(D, q^1, q^2, \lambda, t, t', \gamma^1, \gamma^2)$  as in (a) and (b) above, try to find a set of data  $(E, v, q, \lambda^1, \lambda^2, F, u, \gamma)$  as in (a) and (b), with the only difference that we don't impose that  $v$  and  $u$  belong to some class  $\mathbf{W}$  (since for the moment there is no such class);
- (3) try to identify a class  $\mathbf{W}$  of morphisms in  $\mathcal{C}$ , such that:
  - $\mathbf{W}$  contains all the morphisms  $v$  and  $u$  obtained from the previous procedure, for any set of data  $(A, B^1, B^2, f^1, f^2)$  as in (1) and for any  $(D, q^1, q^2, \lambda, t, t', \gamma^1, \gamma^2)$  as in (2),
  - $\mathbf{W}$  satisfies conditions (BF) for a bicategory of fractions;
- (4) verify if for any data as in (1), condition (c) holds with the associated “candidate”  $(C, p^1, p^2, \omega)$  (with the class  $\mathbf{W}$  constructed in (3));
- (5) if you are successful at each of the previous steps, this means that each pair of morphisms  $(f^1, f^2)$  (or each “interesting” pair of morphisms  $(f^1, f^2)$ ) has a weak fiber product if considered in the right bicategory of fractions  $\mathcal{C} [\mathbf{W}^{-1}]$ .

In other terms, if you are lucky then you are able to construct the desired weak fiber products, provided that you allow some morphisms of  $\mathcal{C}$  to become internal equivalences. Note however that in general there is no guarantee that the bicategory  $\mathcal{C} [\mathbf{W}^{-1}]$  obtained in this way is “interesting”. For example, if we have already managed to solve problem (1) and (2), but a choice for  $\mathbf{W}$  as in (3) is given by the entire class of morphisms of  $\mathcal{C}$ , then in the bicategory of fractions obtained in this way all the morphisms are internal equivalences; so in certain frameworks this procedure could lead to a bicategory that is not useful or interesting to work with.

As a consequence of Theorem 0.2, we are also able to prove:

**Corollary 0.3.** *Let us fix any pair  $(\mathcal{C}, \mathbf{W})$  satisfying axioms (BF) and let us choose any bicategory of fractions  $\mathcal{C} [\mathbf{W}^{-1}]$  associated to the pair  $(\mathcal{C}, \mathbf{W})$ . Let us fix any pair of morphisms  $f^1 : B^1 \rightarrow A$  and  $f^2 : B^2 \rightarrow A$ . Let us suppose that there is a set of data  $(C, p^1, p^2, \omega)$  such that (0.4) is a weak fiber product in  $\mathcal{C}$ . Then conditions (a), (b) and (c) above are satisfied. Therefore, for each pair of morphisms in  $\mathbf{W}$  of the form  $w^1 : B^1 \rightarrow \overline{B}^1$  and  $w^2 : B^2 \rightarrow \overline{B}^2$ , there is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$  for the pair of morphisms  $(B^1, w^1, f^1)$  and  $(B^2, w^2, f^2)$ . In particular,*

if the bicategory  $\mathcal{C}$  is closed under weak fiber products, then also the bicategory  $\mathcal{C}[\mathbf{W}^{-1}]$  is closed under weak fiber products.

As a simple application of Theorems 0.1 and 0.2, in the last part of this paper we will examine the particular case when  $\mathcal{C}$  is a category (considered as a trivial bicategory) and the pair  $(\mathcal{C}, \mathbf{W})$  satisfies conditions (CF) for a right calculus of fractions. As we mentioned before, in this case the pair  $(\mathcal{C}, \mathbf{W})$  satisfies also conditions (BF) for a right bicalculus of fractions and the right category of fractions associated to  $(\mathcal{C}, \mathbf{W})$  (considered as a trivial bicategory) is equivalent to the right bicategory of fractions associated to  $(\mathcal{C}, \mathbf{W})$ . Moreover in this case weak fiber products are simply (strong) fiber products. Then we will prove the following result.

**Proposition 0.4.** *Let us fix any pair  $(\mathcal{C}, \mathbf{W})$  satisfying axioms (CF) for a right calculus of fractions. Given any pair of morphisms  $f^1 : B^1 \rightarrow A$  and  $f^2 : B^2 \rightarrow A$  in  $\mathcal{C}$ , the following facts are equivalent:*

- (iii) *for any pair of morphisms in  $\mathbf{W}$  of the form  $w^1 : B^1 \rightarrow \overline{B}^1$  and  $w^2 : B^2 \rightarrow \overline{B}^2$ , the pair of morphisms*

$$\begin{array}{ccc} & \overline{B}^1 & \\ & \downarrow [B^1, w^1, f^1] & \\ \overline{B}^2 & \xrightarrow{[B^2, w^2, f^2]} & A \end{array} \quad (0.10)$$

*admits a (strong) fiber product in the right category of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$ ;*

- (iv) *there are an object  $C$  in  $\mathcal{C}$  and a pair of morphisms  $p^1 : C \rightarrow B^1$ ,  $p^2 : C \rightarrow B^2$ , such that  $f^1 \circ p^1 = f^2 \circ p^2$  and such that the diagram*

$$\begin{array}{ccc} C & \xrightarrow{[C, \text{id}_C, p^1]} & B^1 \\ [C, \text{id}_C, p^2] \downarrow & \circlearrowleft & \downarrow [B^1, \text{id}_{B^1}, f^1] \\ B^2 & \xrightarrow{[B^2, \text{id}_{B^2}, f^2]} & A \end{array} \quad (0.11)$$

*is a (strong) fiber product in the right category of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$ .*

Moreover, given any set of data  $(C, p^1 : C \rightarrow B^1, p^2 : C \rightarrow B^2)$  such that  $f^1 \circ p^1 = f^2 \circ p^2$ , diagram (0.11) is a (strong) fiber product if and only if the following 2 conditions hold:

- (d) *given any object  $D$  and any pair of morphisms  $q^m : D \rightarrow B^m$  for  $m = 1, 2$ , such that  $f^1 \circ q^1 = f^2 \circ q^2$  in  $\mathcal{C}$ , there are an object  $E$ , a morphism  $v : E \rightarrow D$  in  $\mathbf{W}$  and a morphism  $q : E \rightarrow C$ , such that  $q^m \circ v = p^m \circ q$  for each  $m = 1, 2$ ;*  
(e) *given any set of data  $(D, q^1, q^2, E, v, q)$  as in (d), if there is another choice of data  $\tilde{E}, \tilde{v} : \tilde{E} \rightarrow D$  in  $\mathbf{W}$  and  $\tilde{q} : \tilde{E} \rightarrow C$ , such that  $q^m \circ \tilde{v} = p^m \circ \tilde{q}$  for each  $m = 1, 2$ , then there are an object  $F$ , a morphism  $u : F \rightarrow E$  in  $\mathbf{W}$  and a morphism  $\tilde{u} : F \rightarrow \tilde{E}$ , such that:*
- $v \circ u = \tilde{v} \circ \tilde{u}$ ;
  - $q \circ u = \tilde{q} \circ \tilde{u}$ .

## 1. NOTATIONS

Through all this paper we will use the *axiom of choice*, that we therefore assume without further remarks. The reason for this is twofold: first of all, the construction

of bicategories of fractions in [Pr] in general requires the axiom of choice (except for some special cases described in [T1, Corollary 0.6]); moreover we will use from time to time the universal property of bicategories of fractions, that was proved in [Pr, Theorem 21] implicitly using that axiom.

We mainly refer to [PW, § 1] and [L, § 1.5] for a general overview on bicategories and pseudofunctors. Given any bicategory  $\mathcal{C}$ , we denote its objects by  $A, B, \dots$ , its morphisms by  $f, g, \dots$  and its 2-morphisms by  $\alpha, \beta, \dots$  (we will use  $A_{\mathcal{C}}, f_{\mathcal{C}}, \alpha_{\mathcal{C}}, \dots$  if we have to recall that they belong to  $\mathcal{C}$  when we are using more than one bicategory in the computations). Given any triple of morphisms  $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$  in  $\mathcal{C}$ , we denote by  $\theta_{h,g,f}$  the associator  $h \circ (g \circ f) \Rightarrow (h \circ g) \circ f$  that is part of the structure of the bicategory  $\mathcal{C}$ ; we denote by  $\pi_f : f \circ \text{id}_A \Rightarrow f$  and  $v_f : \text{id}_B \circ f \Rightarrow f$  the right and left unitors for  $\mathcal{C}$  relative to any morphism  $f$  as above. Given another bicategory  $\mathcal{D}$ , we will denote by  $\Theta_{\bullet}, \Pi_{\bullet}$  and  $\Upsilon_{\bullet}$  its associators, right and left unitors respectively. We denote by  $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \Psi_{\bullet}^{\mathcal{F}}, \Sigma_{\bullet}^{\mathcal{F}})$  any pseudofunctor  $\mathcal{C} \rightarrow \mathcal{D}$ . Here for each pair of morphisms  $f, g$  as above,  $\Psi_{g,f}^{\mathcal{F}}$  is the associator from  $\mathcal{F}_1(g \circ f)$  to  $\mathcal{F}_1(g) \circ \mathcal{F}_1(f)$  and for each object  $A$ ,  $\Sigma_A^{\mathcal{F}}$  is the unitor from  $\mathcal{F}_1(\text{id}_A)$  to  $\text{id}_{\mathcal{F}_0(A)}$ .

We recall that a morphism  $e : A \rightarrow B$  in a bicategory  $\mathcal{C}$  is called an *internal equivalence* (or, simply, an *equivalence*) of  $\mathcal{C}$  if and only if there exists a triple  $(d, \delta, \xi)$ , where  $d$  is a morphism from  $B$  to  $A$  and  $\delta : \text{id}_A \Rightarrow d \circ e$  and  $\xi : e \circ d \Rightarrow \text{id}_B$  are invertible 2-morphisms in  $\mathcal{C}$  (in the literature sometimes the name ‘‘internal equivalence’’ is used for denoting the whole quadruple  $(e, d, \delta, \xi)$  instead of the morphism  $e$  alone). In particular, also  $d$  is an internal equivalence and it is usually called a *quasi-inverse* (or *pseudo-inverse*) for  $e$  (in general, the quasi-inverse of an internal equivalence is not unique). An *adjoint equivalence* is a quadruple  $(e, d, \delta, \xi)$  as above, such that

$$v_e \circ (\xi * i_e) \circ \theta_{e,d,e} \circ (i_e * \delta) \circ \pi_e^{-1} = i_e$$

and

$$\pi_d \circ (i_d * \xi) \circ \theta_{d,e,d}^{-1} \circ (\delta * i_d) \circ v_d^{-1} = i_d$$

(this more restrictive definition is actually the original definition of internal equivalence used for example in [Mac, pag. 83]). By [L, Proposition 1.5.7] a morphism  $e$  is (the first component of) an internal equivalence if and only if it is the first component of a (possibly different) adjoint equivalence.

## 2. WEAK FIBER PRODUCTS IN A BICATEGORY

Let us fix any bicategory  $\mathcal{D}$  and any diagram in it as follows:

$$\begin{array}{ccc} C & \xrightarrow{r^1} & B^1 \\ r^2 \downarrow & \swarrow \Omega & \downarrow g^1 \\ B^2 & \xrightarrow{g^2} & A \end{array} \quad (2.1)$$

with  $\Omega$  invertible. Given any object  $D$  in  $\mathcal{D}$ , we define a 1-category  $\text{Iso}_{\mathcal{D}}(D, C)$  whose objects are all the 1-morphisms from  $D$  to  $C$  in  $\mathcal{D}$  and whose morphisms are all the invertible 2-morphisms between such 1-morphisms (as such,  $\text{Iso}_{\mathcal{D}}(D, C)$  is

an internal groupoid in (Sets)). Moreover, we define also a groupoid  $\text{Iso}_{\mathcal{D}}(D, g^1, g^2)$  as follows: its objects are all the triples  $(s^1, s^2, \Lambda)$ , where  $s^1 : D \rightarrow B^1$ ,  $s^2 : D \rightarrow B^2$  are morphisms and  $\Lambda$  is any invertible 2-morphism from  $g^1 \circ s^1$  to  $g^2 \circ s^2$  in  $\mathcal{D}$ . A morphism from a triple  $(s^1, s^2, \Lambda)$  to a triple  $(s'^1, s'^2, \Lambda')$  is any pair  $(\Gamma^1, \Gamma^2)$  of invertible 2-morphisms  $\Gamma^1 : s^1 \Rightarrow s'^1$  and  $\Gamma^2 : s^2 \Rightarrow s'^2$ , such that

$$\Lambda' \circ (i_{g^1} * \Gamma^1) = (i_{g^2} * \Gamma^2) \circ \Lambda : g^1 \circ s^1 \Longrightarrow g^2 \circ s'^2.$$

Then for each object  $D$  in  $\mathcal{D}$ , diagram (2.1) induces a functor

$$\mathcal{F}_D : \text{Iso}_{\mathcal{D}}(D, C) \longrightarrow \text{Iso}_{\mathcal{D}}(D, g^1, g^2)$$

defined on each object  $s : D \rightarrow C$  in  $\text{Iso}_{\mathcal{D}}(D, C)$  by

$$\mathcal{F}_D(s) := \left( r^1 \circ s, r^2 \circ s, \Theta_{g^2, r^2, s}^{-1} \circ (\Omega * i_s) \circ \Theta_{g^1, r^1, s} \right)$$

and on each invertible 2-morphism  $\Gamma : s \Rightarrow s'$  (i.e. each morphism in  $\text{Iso}_{\mathcal{D}}(D, C)$  from  $s$  to  $s'$ ) by

$$\begin{aligned} \mathcal{F}_D(\Gamma) &:= \left( i_{r^1} * \Gamma, i_{r^2} * \Gamma \right) : \left( r^1 \circ s, r^2 \circ s, \Theta_{g^2, r^2, s}^{-1} \circ (\Omega * i_s) \circ \Theta_{g^1, r^1, s} \right) \longrightarrow \\ &\longrightarrow \left( r^1 \circ s', r^2 \circ s', \Theta_{g^2, r^2, s'}^{-1} \circ (\Omega * i_{s'}) \circ \Theta_{g^1, r^1, s'} \right) \end{aligned}$$

(a direct check proves that  $\mathcal{F}_D$  is actually a functor). Then one can give the following definition (see for example [MM, pag. 125] in the case when  $\mathcal{D}$  is a 2-category).

**Definition 2.1.** Let us fix any bicategory  $\mathcal{D}$  and any diagram as (2.1) in it, with  $\Omega$  invertible. We say that such a diagram *has the universal property of weak fiber products* if the functor  $\mathcal{F}_D$  described above is an equivalence of categories (actually, of internal groupoids in (Sets)) for each object  $D$  in  $\mathcal{D}$ . In this case, we say also that (2.1) is a *weak fiber product* (also called *weak pullback* or *2-fiber product* when  $\mathcal{D}$  is a 2-category) of the pair  $(g^1, g^2)$ . Equivalently, (2.1) is a weak fiber product if and only if the following 2 conditions hold for every object  $D$ :

**A1(D):**  $\mathcal{F}_D$  is essentially surjective, i.e. for any set of data  $(s^1, s^2, \Lambda)$  in  $\mathcal{D}$  with  $\Lambda$  invertible as follows

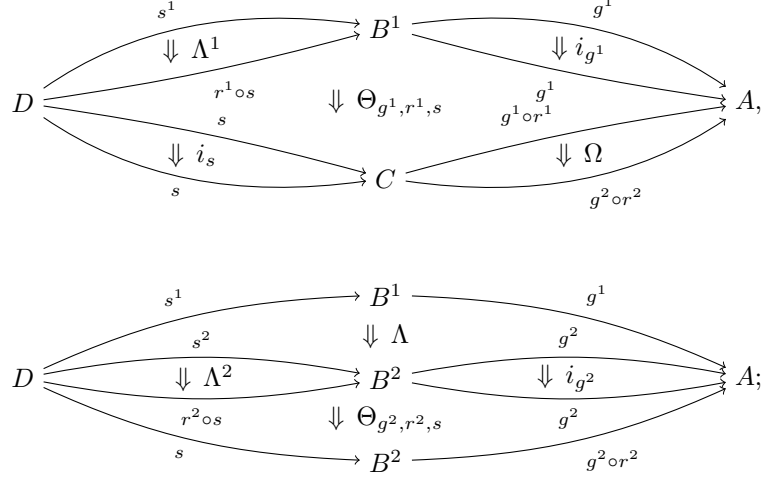
$$\begin{array}{ccc} D & \xrightarrow{s^1} & B^1 \\ s^2 \downarrow & \swarrow \Lambda & \downarrow g^1 \\ B^2 & \xrightarrow{g^2} & A, \end{array}$$

there are a morphism  $s : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that

$$\left( \Omega * i_s \right) \circ \Theta_{g^1, r^1, s} \circ \left( i_{g^1} * \Lambda^1 \right) = \Theta_{g^2, r^2, s} \circ \left( i_{g^2} * \Lambda^2 \right) \circ \Lambda. \quad (2.2)$$

For simplicity of exposition, we write below the 2 diagrams associated to the left and to the right hand side of (2.2):

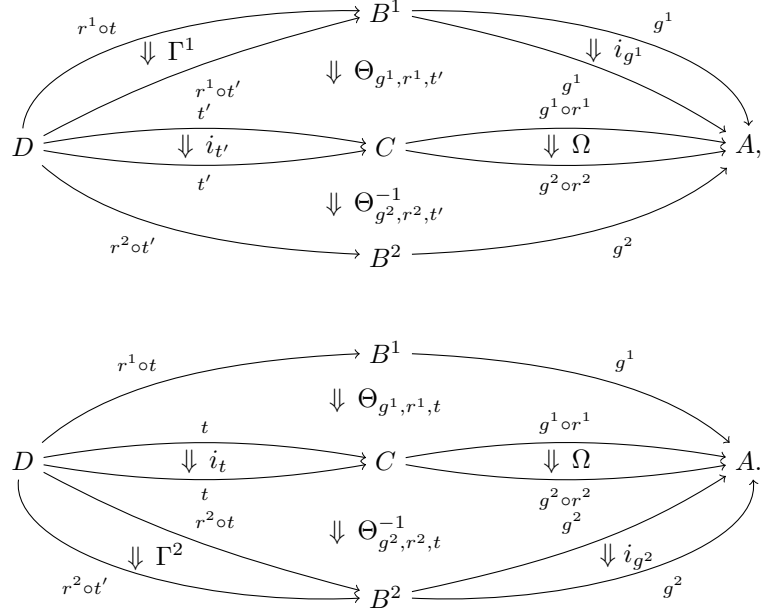




**A2(D):**  $\mathcal{F}_D$  is fully faithful, i.e. for any pair of morphisms  $t, t' : D \rightarrow C$  and for any pair of invertible 2-morphisms  $\Gamma^m : r^m \circ t \Rightarrow r^m \circ t'$  for  $m = 1, 2$ , such that

$$\begin{aligned} & \Theta_{g^2, r^2, t'}^{-1} \odot (\Omega * i_{t'}) \odot \Theta_{g^1, r^1, t'} \odot (i_{g^1} * \Gamma^1) = \\ & = (i_{g^2} * \Gamma^2) \odot \Theta_{g^2, r^2, t}^{-1} \odot (\Omega * i_t) \odot \Theta_{g^1, r^1, t}, \end{aligned} \quad (2.3)$$

there is a *unique* invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . The pair of 2-morphisms in (2.3) is given as follows:



**Remark 2.2.** Equivalently, (2.1) is a weak fiber product in the bicategory  $\mathcal{D}$  if and only if the following conditions are satisfied:

- for each triple  $(D, s^1 : D \rightarrow B^1, s^2 : D \rightarrow B^2)$  in  $\mathcal{D}$  the following property holds:

**B1**( $D, s^1, s^2$ ): for any invertible 2-morphism  $\Lambda : g^1 \circ s^1 \Rightarrow g^2 \circ s^2$ , there are a morphism  $s : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that (2.2) holds;

- for each triple  $(D, t : D \rightarrow C, t' : D \rightarrow C)$  in  $\mathcal{D}$  the following property holds:

**B2**( $D, t, t'$ ): for any pair of invertible 2-morphisms  $\Gamma^m : r^m \circ t \Rightarrow r^m \circ t'$  for  $m = 1, 2$ , such that (2.3) holds, there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r,m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ .

As we will see in Proposition 2.10 and 2.11, in general it is sufficient to verify that condition **B1**, respectively **B2**, holds for a (smaller) subset of triples  $(D, s^1, s^2)$ , respectively  $(D, t, t')$ .

**Remark 2.3.** Given any category  $\mathcal{D}$ , we denote by  $\mathcal{D}^2$  the trivial bicategory obtained from  $\mathcal{D}$ , i.e. the bicategory whose objects and morphisms are the same as those of  $\mathcal{D}$  and whose 2-morphisms are only the 2-identities. Then it is easy to see that a 2-commutative square in  $\mathcal{D}^2$  is a weak fiber product if and only if the same square is a (strong) fiber product in  $\mathcal{D}$ . In other terms, weak fiber products generalize the notion of (strong) fiber products from categories to 2-categories.

In the remaining part of this section we are going to state some useful results about weak fiber products in any bicategory  $\mathcal{D}$ . All such lemmas will play a crucial role when  $\mathcal{D}$  will be a bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$ .

**Proposition 2.4.** *Let us suppose that (2.1) (with  $\Omega$  invertible) is a weak fiber product in a bicategory  $\mathcal{D}$ . Moreover, let us also fix any set of objects, morphisms and 2-morphism as follows for each  $m = 1, 2$ :*

$$\begin{aligned} e^m : \overline{B}^m &\longrightarrow B^m, & d^m : B^m &\longrightarrow \overline{B}^m, \\ \Delta^m : \text{id}_{\overline{B}^m} &\Longrightarrow d^m \circ e^m, & \Xi^m : e^m \circ d^m &\Longrightarrow \text{id}_{B^m}, \end{aligned} \quad (2.4)$$

such that the quadruple  $(e^m, d^m, \Delta^m, \Xi^m)$  is an adjoint equivalence in  $\mathcal{D}$  for each  $m = 1, 2$ . Moreover, let us define

$$\begin{aligned} \overline{\Omega} := & \Theta_{g^2 \circ e^2, d^2, r^2}^{-1} \odot \left( \Theta_{g^2, e^2, d^2} * i_{r^2} \right) \odot \left( \left( i_{g^2} * (\Xi^2)^{-1} \right) * i_{r^2} \right) \odot \left( \Pi_{g^2}^{-1} * i_{r^2} \right) \odot \\ & \odot \Omega \odot \left( \Pi_{g^1} * i_{r^1} \right) \odot \left( \left( i_{g^1} * \Xi^1 \right) * i_{r^1} \right) \odot \left( \Theta_{g^1, e^1, d^1}^{-1} * i_{r^1} \right) \odot \Theta_{g^1 \circ e^1, d^1, r^1} : \\ & (g^1 \circ e^1) \circ (d^1 \circ r^1) \Longrightarrow (g^2 \circ e^2) \circ (d^2 \circ r^2) \end{aligned} \quad (2.5)$$

(where  $\Theta_{\bullet}$  and  $\Pi_{\bullet}$  are the associators and right unitors for  $\mathcal{D}$ ). Then the diagram

$$\begin{array}{ccc} C & \xrightarrow{d^1 \circ r^1} & \overline{B}^1 \\ d^2 \circ r^2 \downarrow & \swarrow \overline{\Omega} & \downarrow g^1 \circ e^1 \\ \overline{B}^2 & \xrightarrow{g^2 \circ e^2} & A \end{array} \quad (2.6)$$

is a weak fiber product in  $\mathcal{D}$ .

Since each internal equivalence is the first component of an adjoint equivalence (see [L, Proposition 1.5.7]), then this result implies at once that:

**Corollary 2.5.** *Let us fix any pair of morphisms  $g^m : B^m \rightarrow A$  for  $m = 1, 2$  that admit a weak fiber product in a bicategory  $\mathcal{D}$ ; then for every pair of internal equivalences  $e^m : \overline{B}^m \rightarrow B^m$  for  $m = 1, 2$ , the morphisms  $g^m \circ e^m : \overline{B}^m \rightarrow A$  for  $m = 1, 2$  have a weak fiber product in  $\mathcal{D}$ .*

*Proof of Proposition 2.4.* For simplicity of exposition, we will give a complete proof only in the case when  $\mathcal{D}$  is a 2-category. In the general case, one has to add associators and unitors of  $\mathcal{D}$  and use the coherence conditions on the bicategory  $\mathcal{D}$  wherever it is necessary. Apart from that, the proofs are exactly the same.

Since the quadruple  $(e^m, d^m, \Delta^m, \Xi^m)$  is an adjoint equivalence, then for each  $m = 1, 2$  we have:

$$\left(\Xi^m * i_{e^m}\right) \odot \left(i_{e^m} * \Delta^m\right) = i_{e^m} \quad \text{and} \quad \left(i_{d^m} * \Xi^m\right) \odot \left(\Delta^m * i_{d^m}\right) = i_{d^m}. \quad (2.7)$$

Let us fix any object  $\overline{D}$  in  $\mathcal{D}$  and let us prove property **A1**( $\overline{D}$ ) for diagram (2.6), so let us fix any set of data  $(\overline{s}^1, \overline{s}^2, \overline{\Lambda})$  in  $\mathcal{D}$  as follows, with  $\overline{\Lambda}$  invertible

$$\begin{array}{ccc} \overline{D} & \xrightarrow{\overline{s}^1} & \overline{B}^1 \\ \overline{s}^2 \downarrow & \swarrow \overline{\Lambda} & \downarrow g^1 \circ e^1 \\ \overline{B}^2 & \xrightarrow{g^2 \circ e^2} & A. \end{array} \quad (2.8)$$

Since  $\mathcal{D}$  is a 2-category, we can consider  $\overline{\Lambda}$  as defined from  $g^1 \circ (e^1 \circ \overline{s}^1)$  to  $g^2 \circ (e^2 \circ \overline{s}^2)$ . Using property **A1**( $\overline{D}$ ) for diagram (2.1), there are a morphism  $\overline{s} : \overline{D} \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : e^m \circ \overline{s}^m \Rightarrow r^m \circ \overline{s}$  for  $m = 1, 2$ , such that

$$\left(\Omega * i_{\overline{s}}\right) \odot \left(i_{g^1} * \Lambda^1\right) = \left(i_{g^2} * \Lambda^2\right) \odot \overline{\Lambda}. \quad (2.9)$$

For each  $m = 1, 2$  we define an invertible 2-morphism

$$\overline{\Lambda}^m := \left(i_{d^m} * \Lambda^m\right) \odot \left(\Delta^m * i_{\overline{s}^m}\right) : \overline{s}^m \Longrightarrow (d^m \circ r^m) \circ \overline{s}.$$

Then using the definitions of  $\overline{\Lambda}^1, \overline{\Lambda}^2$  and  $\overline{\Omega}$  (where we omit associators and unitors of  $\mathcal{D}$  since we are assuming that  $\mathcal{D}$  is a 2-category), we get a series of identities as follows:

$$\begin{aligned} & \left(\overline{\Omega} * i_{\overline{s}}\right) \odot \left(i_{g^1 \circ e^1} * \overline{\Lambda}^1\right) = \\ & = \left(i_{g^2} * (\Xi^2)^{-1} * i_{r^2 \circ \overline{s}}\right) \odot \left(\Omega * i_{\overline{s}}\right) \odot \left(i_{g^1} * \Xi^1 * i_{r^1 \circ \overline{s}}\right) \odot \\ & \quad \odot \left(i_{g^1 \circ e^1 \circ d^1} * \Lambda^1\right) \odot \left(i_{g^1 \circ e^1} * \Delta^1 * i_{\overline{s}^1}\right) \stackrel{(*)}{=} \\ & \stackrel{(*)}{=} \left(i_{g^2} * (\Xi^2)^{-1} * i_{r^2 \circ \overline{s}}\right) \odot \left(\Omega * i_{\overline{s}}\right) \odot \left(i_{g^1} * \Lambda^1\right) \odot \\ & \quad \odot \left(i_{g^1} * \Xi^1 * i_{e^1 \circ \overline{s}^1}\right) \odot \left(i_{g^1 \circ e^1} * \Delta^1 * i_{\overline{s}^1}\right) \stackrel{(**)}{=} \\ & \stackrel{(**)}{=} \left(i_{g^2} * (\Xi^2)^{-1} * i_{r^2 \circ \overline{s}}\right) \odot \left(\Omega * i_{\overline{s}}\right) \odot \left(i_{g^1} * \Lambda^1\right) \stackrel{(2.9)}{=} \\ & \stackrel{(2.9)}{=} \left(i_{g^2} * (\Xi^2)^{-1} * i_{r^2 \circ \overline{s}}\right) \odot \left(i_{g^2} * \Lambda^2\right) \odot \overline{\Lambda} \stackrel{(*)}{=} \\ & \stackrel{(*)}{=} \left(i_{g^2 \circ e^2 \circ d^2} * \Lambda^2\right) \odot \left(i_{g^2} * (\Xi^2)^{-1} * i_{e^2 \circ \overline{s}^2}\right) \odot \overline{\Lambda} \stackrel{(**)}{=} \\ & \stackrel{(**)}{=} \left(i_{g^2 \circ e^2 \circ d^2} * \Lambda^2\right) \odot \left(i_{g^2 \circ e^2} * \Delta^2 * i_{\overline{s}^2}\right) \odot \overline{\Lambda} = \left(i_{g^2 \circ e^2} * \overline{\Lambda}^2\right) \odot \overline{\Lambda}, \quad (2.10) \end{aligned}$$

where the identities of the form  $\stackrel{(*)}{=}$  are a consequence of the interchange law in  $\mathcal{D}$  (see [B, Proposition 1.3.5]) and the identities denoted by  $\stackrel{(**)}{=}$  are obtained using (2.7). Then identity (2.10) proves that diagram (2.6) satisfies property **A1**( $\overline{D}$ ).

Now let us prove also property **A2**( $\overline{D}$ ) for diagram (2.6), so let us fix any pair of morphisms  $\bar{t}, \bar{t}' : \overline{D} \rightarrow C$  and any pair of invertible 2-morphisms  $\overline{\Gamma}^m : (d^m \circ r^m) \circ \bar{t} \Rightarrow (d^m \circ r^m) \circ \bar{t}'$  for  $m = 1, 2$ , such that

$$\left( \overline{\Omega} * i_{\bar{t}'} \right) \odot \left( i_{g^1 \circ e^1} * \overline{\Gamma}^1 \right) = \left( i_{g^2 \circ e^2} * \overline{\Gamma}^2 \right) \odot \left( \overline{\Omega} * i_{\bar{t}} \right). \quad (2.11)$$

Then for each  $m = 1, 2$  we define an invertible 2-morphism

$$\Gamma^m := \left( \Xi^m * i_{r^m \circ \bar{t}'} \right) \odot \left( i_{e^m} * \overline{\Gamma}^m \right) \odot \left( (\Xi^m)^{-1} * i_{r^m \circ \bar{t}} \right) : r^m \circ \bar{t} \Longrightarrow r^m \circ \bar{t}'. \quad (2.12)$$

Then using the interchange law on  $\mathcal{D}$ , we have:

$$\begin{aligned} & \left( \Omega * i_{\bar{t}'} \right) \odot \left( i_{g^1} * \Gamma^1 \right) \stackrel{(2.12)}{=} \\ & \stackrel{(2.12)}{=} \left( \Omega * i_{\bar{t}'} \right) \odot \left( i_{g^1} * \Xi^1 * i_{r^1 \circ \bar{t}'} \right) \odot \left( i_{g^1 \circ e^1} * \overline{\Gamma}^1 \right) \odot \left( i_{g^1} * (\Xi^1)^{-1} * i_{r^1 \circ \bar{t}} \right) \stackrel{(2.5)}{=} \\ & \stackrel{(2.5)}{=} \left( i_{g^2} * \Xi^2 * i_{r^2 \circ \bar{t}'} \right) \odot \left( \overline{\Omega} * i_{\bar{t}'} \right) \odot \left( i_{g^1 \circ e^1} * \overline{\Gamma}^1 \right) \odot \left( i_{g^1} * (\Xi^1)^{-1} * i_{r^1 \circ \bar{t}} \right) \stackrel{(2.11)}{=} \\ & \stackrel{(2.11)}{=} \left( i_{g^2} * \Xi^2 * i_{r^2 \circ \bar{t}'} \right) \odot \left( i_{g^2 \circ e^2} * \overline{\Gamma}^2 \right) \odot \left( \overline{\Omega} * i_{\bar{t}} \right) \odot \left( i_{g^1} * (\Xi^1)^{-1} * i_{r^1 \circ \bar{t}} \right) \stackrel{(2.5)}{=} \\ & \stackrel{(2.5)}{=} \left( i_{g^2} * \Xi^2 * i_{r^2 \circ \bar{t}'} \right) \odot \left( i_{g^2 \circ e^2} * \overline{\Gamma}^2 \right) \odot \left( i_{g^2} * (\Xi^2)^{-1} * i_{r^2 \circ \bar{t}} \right) \odot \\ & \odot \left( \Omega * i_{\bar{t}} \right) \odot \left( i_{g^1} * \Xi^1 * i_{r^1 \circ \bar{t}} \right) \odot \left( i_{g^1} * (\Xi^1)^{-1} * i_{r^1 \circ \bar{t}} \right) \stackrel{(2.12)}{=} \\ & \stackrel{(2.12)}{=} \left( i_{g^2} * \Gamma^2 \right) \odot \left( \Omega * i_{\bar{t}} \right). \end{aligned} \quad (2.13)$$

Since property **A2**( $\overline{D}$ ) holds for diagram (2.1), then (2.13) implies that there is a unique invertible 2-morphism  $\overline{\Gamma} : \bar{t} \Rightarrow \bar{t}'$ , such that

$$i_{r^m} * \overline{\Gamma} = \Gamma^m \quad \text{for } m = 1, 2. \quad (2.14)$$

Then for each  $m = 1, 2$ , by interchange law we have:

$$\begin{aligned} & i_{d^m \circ r^m} * \overline{\Gamma} \stackrel{(2.14)}{=} i_{d^m} * \Gamma^m \stackrel{(2.12)}{=} \\ & \stackrel{(2.12)}{=} \left( i_{d^m} * \Xi^m * i_{r^m \circ \bar{t}'} \right) \odot \left( i_{d^m \circ e^m} * \overline{\Gamma}^m \right) \odot \left( i_{d^m} * (\Xi^m)^{-1} * i_{r^m \circ \bar{t}} \right) \stackrel{(2.7)}{=} \\ & \stackrel{(2.7)}{=} \left( (\Delta^m)^{-1} * i_{d^m \circ r^m \circ \bar{t}'} \right) \odot \left( i_{d^m \circ e^m} * \overline{\Gamma}^m \right) \odot \left( \Delta^m * i_{d^m \circ r^m \circ \bar{t}} \right) = \\ & = \overline{\Gamma}^m \odot \left( (\Delta^m)^{-1} * i_{d^m \circ r^m \circ \bar{t}'} \right) \odot \left( \Delta^m * i_{d^m \circ r^m \circ \bar{t}} \right) = \overline{\Gamma}^m. \end{aligned} \quad (2.15)$$

In order to conclude the proof, we need only to prove that  $\overline{\Gamma}$  is the unique invertible 2-morphism from  $\bar{t}$  to  $\bar{t}'$ , such that (2.15) holds for each  $m = 1, 2$ . So let us suppose that there is another invertible 2-morphism  $\overline{\Gamma}' : \bar{t} \Rightarrow \bar{t}'$ , such that  $i_{d^m \circ r^m} * \overline{\Gamma}' = \overline{\Gamma}^m$  for each  $m = 1, 2$ . Then using again the interchange law, for each  $m = 1, 2$  we have:

$$\begin{aligned} & \Gamma^m \stackrel{(2.12)}{=} \left( \Xi^m * i_{r^m \circ \bar{t}'} \right) \odot \left( i_{e^m} * \overline{\Gamma}^m \right) \odot \left( (\Xi^m)^{-1} * i_{r^m \circ \bar{t}} \right) = \\ & = \left( \Xi^m * i_{r^m \circ \bar{t}'} \right) \odot \left( i_{e^m \circ d^m \circ r^m} * \overline{\Gamma}' \right) \odot \left( (\Xi^m)^{-1} * i_{r^m \circ \bar{t}} \right) = \end{aligned}$$

$$= (i_{r^m} * \bar{\Gamma}') \odot (\Xi^m * i_{r^m \circ \bar{t}}) \odot ((\Xi^m)^{-1} * i_{r^m \circ \bar{t}}) = i_{r^m} * \bar{\Gamma}'.$$

Since  $\bar{\Gamma}$  is the unique invertible 2-morphism from  $\bar{t}$  to  $\bar{t}'$  such that (2.14) holds, we conclude that  $\bar{\Gamma} = \bar{\Gamma}'$ . So we have proved that property **A2**( $\bar{D}$ ) holds for diagram (2.6).  $\square$

**Lemma 2.6.** *Let us suppose that (2.1) (with  $\Omega$  invertible) is a weak fiber product in a bicategory  $\mathcal{D}$  and let us fix any internal equivalence  $e : A \rightarrow \bar{A}$  in  $\mathcal{D}$ . Then the induced square*

$$\begin{array}{ccc} C & \xrightarrow{r^1} & B^1 \\ r^2 \downarrow & \swarrow \bar{\Omega} := \Theta_{e, g^2, r^2} \odot (i_e * \Omega) \odot \Theta_{e, g^1, r^1}^{-1} & \downarrow e \circ g^1 \\ B^2 & \xrightarrow{e \circ g^2} & \bar{A} \end{array} \quad (2.16)$$

is a weak fiber product in  $\mathcal{D}$ .

See Appendix C for a proof.

**Lemma 2.7.** *Let us fix any diagram as (2.1) (with  $\Omega$  invertible) in a bicategory  $\mathcal{D}$ . Moreover, let us fix any pair of morphisms  $\bar{g}^1, \bar{g}^2$  and any pair of invertible 2-morphisms  $\Omega^1$  and  $\Omega^2$  as follows:*

$$\begin{array}{ccc} & & B^1 \\ & & \uparrow \left( \begin{array}{c} \Rightarrow \\ \Omega^1 \end{array} \right) \bar{g}^1 \\ & g^1 & \\ B^2 & \xrightarrow{g^2} & A \\ \downarrow \Omega^2 & & \\ & \bar{g}^2 & \end{array}$$

Then (2.1) is a weak fiber product if and only if the following diagram is a weak fiber product

$$\begin{array}{ccc} C & \xrightarrow{r^1} & B^1 \\ r^2 \downarrow & \swarrow \bar{\Omega} := (\Omega^2 * i_{r^2}) \odot \Omega \odot ((\Omega^1)^{-1} * i_{r^1}) & \downarrow \bar{g}^1 \\ B^2 & \xrightarrow{\bar{g}^2} & A. \end{array} \quad (2.17)$$

See Appendix C for a proof.

**Theorem 2.8.** *Let us fix any bicategory  $\mathcal{D}$ , any pair of morphisms  $g^1 : B^1 \rightarrow A$ ,  $g^2 : B^2 \rightarrow A$  and any triple of internal equivalences*

$$e : A \rightarrow \bar{A}, \quad e^1 : \bar{B}^1 \rightarrow B^1 \quad \text{and} \quad e^2 : \bar{B}^2 \rightarrow B^2.$$

Then the following facts are equivalent:

- (a) the pair  $(g^1, g^2)$  has a weak fiber product;
- (b) the pair  $(e \circ (g^1 \circ e^1), e \circ (g^2 \circ e^2))$  has a weak fiber product.

Moreover, if for each  $m = 1, 2$  we fix a triple  $(d^m, \Delta^m, \Xi^m)$  such that the quadruple  $(e^m, d^m, \Delta^m, \Xi^m)$  is an adjoint equivalence and if we assume that a weak fiber product for (a) is given by diagram (2.1), then a weak fiber product for (b) is given by the following diagram

$$\begin{array}{ccc} C & \xrightarrow{d^1 \circ r^1} & \overline{B}^1 \\ \downarrow d^2 \circ r^2 & \swarrow \overline{\Omega} & \downarrow e \circ (g^1 \circ e^1) \\ \overline{B}^2 & \xrightarrow{e \circ (g^2 \circ e^2)} & \overline{A}, \end{array}$$

where:

$$\begin{aligned} \overline{\Omega} := & \Theta_{e, g^2 \circ e^2, d^2 \circ r^2} \odot \left\{ i_e * \left[ \Theta_{g^2 \circ e^2, d^2, r^2}^{-1} \odot \left( \Theta_{g^2, e^2, d^2} * i_{r^2} \right) \odot \right. \right. \\ & \odot \left( \left( i_{g^2} * (\Xi^2)^{-1} \right) * i_{r^2} \right) \odot \left( \Pi_{g^2}^{-1} * i_{r^2} \right) \odot \Omega \odot \left( \Pi_{g^1} * i_{r^1} \right) \odot \\ & \left. \odot \left( \left( i_{g^1} * \Xi^1 \right) * i_{r^1} \right) \odot \left( \Theta_{g^1, e^1, d^1}^{-1} * i_{r^1} \right) \odot \Theta_{g^1 \circ e^1, d^1, r^1} \right\} \odot \Theta_{e, g^1 \circ e^1, d^1 \circ r^1}^{-1} : \\ & (e \circ (g^1 \circ e^1)) \circ (d^1 \circ r^1) \implies (e \circ (g^2 \circ e^2)) \circ (d^2 \circ r^2). \end{aligned}$$

*Proof.* The implication (a)  $\Rightarrow$  (b) and the last part of the Theorem are given by Proposition 2.4 and Lemma 2.6, so we need only to prove (b)  $\Rightarrow$  (a).

As usual, for simplicity of exposition we assume that  $\mathcal{D}$  is a 2-category. Let us suppose that  $e \circ g^1 \circ e^1$  and  $e \circ g^2 \circ e^2$  have a weak fiber product. Let us fix a pair of triples  $(d^m, \Delta^m, \Xi^m)$  for  $m = 1, 2$  as in (2.4), such that the quadruple  $(e^m, d^m, \Delta^m, \Xi^m)$  is an adjoint equivalence for each  $m = 1, 2$ . In particular, both  $d^1$  and  $d^2$  are internal equivalences. Since the pair  $(e \circ g^1 \circ e^1, e \circ g^2 \circ e^2)$  has a weak fiber product, then by Corollary 2.5 also the pair of morphisms

$$(\overline{g}^1 := e \circ g^1 \circ e^1 \circ d^1 : B^1 \rightarrow \overline{A}, \overline{g}^2 := e \circ g^2 \circ e^2 \circ d^2 : B^2 \rightarrow \overline{A})$$

has a weak fiber product. Then for each  $m = 1, 2$  we define an invertible 2-morphism  $\Omega^m := i_{e \circ g^m} * (\Xi^m)^{-1}$  from  $e \circ g^m$  to  $\overline{g}^m$ . Then by Lemma 2.7 we conclude that the pair of morphisms  $(e \circ g^1, e \circ g^2)$  has a weak fiber product.

Now  $e$  is an internal equivalence, so there are an internal equivalence  $d : \overline{A} \rightarrow A$  and an invertible 2-morphism  $\Delta : \text{id}_A \Rightarrow d \circ e$ . By Lemma 2.6 we get that the pair of morphisms  $(d \circ e \circ g^1, d \circ e \circ g^2)$  has a weak fiber product. Then by Lemma 2.7 applied to the pair of invertible 2-morphisms  $\Delta * i_{g^m} : g^m \Rightarrow d \circ e \circ g^m$  for  $m = 1, 2$ , we get that the pair of morphisms  $(g^1, g^2)$  has a weak fiber product, so we have proved that (b) implies (a).  $\square$

**Lemma 2.9.** *Let us suppose that (2.1) (with  $\Omega$  invertible) is a weak fiber product in a bicategory  $\mathcal{D}$  and let us suppose that  $e : \overline{C} \rightarrow C$  is an internal equivalence in  $\mathcal{D}$ . Then the induced square*

$$\begin{array}{ccc} \overline{C} & \xrightarrow{r^1 \circ e} & B^1 \\ \downarrow r^2 \circ e & \swarrow \overline{\Omega} := \Theta_{g^2, r^2, e}^{-1} \odot \left( \Omega * i_e \right) \odot \Theta_{g^1, r^1, e} & \downarrow g^1 \\ B^2 & \xrightarrow{g^2} & A \end{array} \quad (2.18)$$

is a weak fiber product in  $\mathcal{D}$ .

See Appendix C for a proof.

In Remark 2.2 we described a set of conditions equivalent to conditions **A1** and **A2**. In the following 2 propositions we will show that given a diagram as (2.1), it is sufficient to verify property **B1** for it on a (in general smaller) set of triples  $(D, s^1, s^2)$ ; analogously it is sufficient to verify property **B2** on a (in general smaller) set of triples  $(D, t, t')$ . This will be very useful in order to simplify the computations when  $\mathcal{D}$  is a bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$ .

**Proposition 2.10.** *Let us fix any diagram as (2.1) in a bicategory  $\mathcal{D}$ , with  $\Omega$  invertible, and any triple  $(D, s^1 : D \rightarrow B^1, s^2 : D \rightarrow B^2)$ . Moreover, let us fix any other pair of morphisms  $\bar{s}^m : D \rightarrow B^m$  and any pair of invertible 2-morphisms  $\Omega^m : s^m \Rightarrow \bar{s}^m$  for  $m = 1, 2$ . Then the following facts are equivalent:*

- (a) condition **B1** $(D, s^1, s^2)$  for (2.1) holds;
- (b) condition **B1** $(D, \bar{s}^1, \bar{s}^2)$  for (2.1) holds.

Moreover, given any object  $\bar{D}$  and any internal equivalence  $e : \bar{D} \rightarrow D$ , property (a) is equivalent to:

- (c) condition **B1** $(\bar{D}, s^1 \circ e, s^2 \circ e)$  for (2.1) holds.

*Proof.* As usual, for simplicity of exposition let us suppose that  $\mathcal{D}$  is a 2-category. Let us firstly prove that (a) implies (b), so let us fix any invertible 2-morphism  $\bar{\Lambda} : g^1 \circ \bar{s}^1 \Rightarrow g^2 \circ \bar{s}^2$ . Then we define an invertible 2-morphism

$$\Lambda := \left( i_{g^2} * (\Omega^2)^{-1} \right) \odot \bar{\Lambda} \odot \left( i_{g^1} * \Omega^1 \right) : g^1 \circ s^1 \Longrightarrow g^2 \circ s^2. \quad (2.19)$$

Since we are assuming (a), then there are a morphism  $\bar{s} : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ \bar{s}$  for  $m = 1, 2$ , such that

$$\left( \Omega * i_{\bar{s}} \right) \odot \left( i_{g^1} * \Lambda^1 \right) = \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda. \quad (2.20)$$

Then for each  $m = 1, 2$  we define  $\bar{\Lambda}^m := \Lambda^m \odot (\Omega^m)^{-1} : \bar{s}^m \Rightarrow r^m \circ \bar{s}$ , so:

$$\begin{aligned} \left( \Omega * i_{\bar{s}} \right) \odot \left( i_{g^1} * \bar{\Lambda}^1 \right) &= \left( \Omega * i_{\bar{s}} \right) \odot \left( i_{g^1} * \Lambda^1 \right) \odot \left( i_{g^1} * (\Omega^1)^{-1} \right) \stackrel{(2.20)}{=} \\ &\stackrel{(2.20)}{=} \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda \odot \left( i_{g^1} * (\Omega^1)^{-1} \right) \stackrel{(2.19)}{=} \left( i_{g^2} * \bar{\Lambda}^2 \right) \odot \bar{\Lambda}. \end{aligned}$$

Therefore **B1** $(D, \bar{s}^1, \bar{s}^2)$  holds, i.e. (b) is satisfied.

Since  $\Omega^1$  and  $\Omega^2$  are invertible by hypothesis, then an analogous proof shows that (b) implies (a).

Now let us fix any object  $\bar{D}$  and any internal equivalence  $e : \bar{D} \rightarrow D$  and let us prove that (a) implies (c). Since  $e$  is an internal equivalence, we choose an internal equivalence  $d : D \rightarrow \bar{D}$  and invertible 2-morphisms  $\Delta : \text{id}_{\bar{D}} \Rightarrow d \circ e$  and  $\Xi : e \circ d \Rightarrow \text{id}_D$ , such that

$$\left( \Xi * i_e \right) \odot \left( i_e * \Delta \right) = i_e \quad \text{and} \quad \left( i_d * \Xi \right) \odot \left( \Delta * i_d \right) = i_d. \quad (2.21)$$

In order to prove that (c) holds, let us fix any invertible 2-morphism  $\bar{\Lambda} : g^1 \circ s^1 \circ e \Rightarrow g^2 \circ s^2 \circ e$ . Then we define an invertible 2-morphism

$$\Lambda := \left( i_{g^2 \circ s^2} * \Xi \right) \odot \left( \bar{\Lambda} * i_d \right) \odot \left( i_{g^1 \circ s^1} * \Xi^{-1} \right) : g^1 \circ s^1 \Longrightarrow g^2 \circ s^2. \quad (2.22)$$

Since condition  $\mathbf{B1}(D, s^1, s^2)$  holds for (2.1), then there are a morphism  $s : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that (2.2) holds. Then we set  $\bar{s} := s \circ e : \bar{D} \rightarrow C$  and

$$\bar{\Lambda}^m := \Lambda^m * i_e : s^m \circ e \Longrightarrow r^m \circ s \circ e = r^m \circ \bar{s} \quad \text{for } m = 1, 2. \quad (2.23)$$

Then we have:

$$\begin{aligned} & \left( \Omega * i_{\bar{s}} \right) \odot \left( i_{g^1} * \bar{\Lambda}^1 \right) \stackrel{(2.23)}{=} \left[ \left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) \right] * i_e \stackrel{(2.2)}{=} \\ & \stackrel{(2.2)}{=} \left[ \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda \right] * i_e \stackrel{(2.23)}{=} \left( i_{g^2} * \bar{\Lambda}^2 \right) \odot \left( \Lambda * i_e \right) \stackrel{(2.22), (2.21)}{=} \left( i_{g^2} * \bar{\Lambda}^2 \right) \odot \bar{\Lambda}. \end{aligned}$$

So we have proved that condition  $\mathbf{B1}(\bar{D}, s^1 \circ e, s^2 \circ e)$  holds for (2.1), i.e. we have proved that (c) holds.

Conversely, let us suppose that (c) holds and let us prove that (a) holds. Let us choose any internal equivalence  $d$  and any pair of invertible 2-morphisms  $\Delta$  and  $\Xi$  as above. By proceeding as in the proof of (a) $\Rightarrow$ (c) already given, we have that (c) implies that  $\mathbf{B1}(D, s^1 \circ e \circ d, s^2 \circ e \circ d)$  holds for (2.1). Then using the equivalence of (b) with (a) and the pair of invertible 2-morphisms  $\Omega^m := i_{s^m} * \Xi$  for  $m = 1, 2$ , we conclude that  $\mathbf{B1}(D, s^1, s^2)$  holds for (2.1), i.e. (a) is satisfied. This suffices to conclude.  $\square$

**Proposition 2.11.** *Let us fix any diagram as (2.1) in a bicategory  $\mathcal{D}$ , with  $\Omega$  invertible, and any triple  $(D, t : D \rightarrow C, t' : D \rightarrow C)$ . Moreover, let us fix another pair of morphisms  $\bar{t}, \bar{t}' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\Phi : t \Rightarrow \bar{t}$  and  $\Phi' : t' \Rightarrow \bar{t}'$ . Then the following facts are equivalent:*

- (a) condition  $\mathbf{B2}(D, t, t')$  holds for (2.1);
- (b) condition  $\mathbf{B2}(D, \bar{t}, \bar{t}')$  holds for (2.1).

Moreover, given any object  $\bar{D}$  and any internal equivalence  $e : \bar{D} \rightarrow D$ , property (a) is equivalent to:

- (c) condition  $\mathbf{B2}(\bar{D}, t \circ e, t' \circ e)$  holds for (2.1).

*Proof.* Let us firstly prove that (a) implies (b), so let us fix any pair of invertible 2-morphisms  $\bar{\Gamma}^m : r^m \circ \bar{t} \Rightarrow r^m \circ \bar{t}'$  for  $m = 1, 2$ , such that

$$\left( \Omega * i_{\bar{t}'} \right) \odot \left( i_{g^1} * \bar{\Gamma}^1 \right) = \left( i_{g^2} * \bar{\Gamma}^2 \right) \odot \left( \Omega * i_{\bar{t}} \right). \quad (2.24)$$

Then for each  $m = 1, 2$  we define an invertible 2-morphism

$$\Gamma^m := \left( i_{r^m} * (\Phi')^{-1} \right) \odot \bar{\Gamma}^m \odot \left( i_{r^m} * \Phi \right) : r^m \circ t \Longrightarrow r^m \circ t'. \quad (2.25)$$

Then by interchange law we have:

$$\begin{aligned} & \left( \Omega * i_{t'} \right) \odot \left( i_{g^1} * \Gamma^1 \right) \stackrel{(2.25)}{=} \\ & \stackrel{(2.25)}{=} \left( \Omega * i_{t'} \right) \odot \left( i_{g^1 \circ r^1} * (\Phi')^{-1} \right) \odot \left( i_{g^1} * \bar{\Gamma}^1 \right) \odot \left( i_{g^1 \circ r^1} * \Phi \right) = \\ & = \left( i_{g^2 \circ r^2} * (\Phi')^{-1} \right) \odot \left( \Omega * i_{t'} \right) \odot \left( i_{g^1} * \bar{\Gamma}^1 \right) \odot \left( i_{g^1 \circ r^1} * \Phi \right) \stackrel{(2.24)}{=} \\ & \stackrel{(2.24)}{=} \left( i_{g^2 \circ r^2} * (\Phi')^{-1} \right) \odot \left( i_{g^2} * \bar{\Gamma}^2 \right) \odot \left( \Omega * i_{\bar{t}} \right) \odot \left( i_{g^1 \circ r^1} * \Phi \right) = \\ & = \left( i_{g^2 \circ r^2} * (\Phi')^{-1} \right) \odot \left( i_{g^2} * \bar{\Gamma}^2 \right) \odot \left( i_{g^2 \circ r^2} * \Phi \right) \odot \left( \Omega * i_t \right) \stackrel{(2.25)}{=} \end{aligned}$$



$$\stackrel{(2.25)}{=} (i_{g^2} * \Gamma^2) \odot (\Omega * i_t).$$

Since we are assuming (a), then there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$  such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . Then we define an invertible 2-morphism  $\bar{\Gamma} := \Phi' \odot \Gamma \odot \Phi^{-1} : \bar{t} \Rightarrow \bar{t}'$ . Therefore, for each  $m = 1, 2$  we have:

$$\begin{aligned} \bar{\Gamma}^m &\stackrel{(2.25)}{=} (i_{r^m} * \Phi') \odot \Gamma^m \odot (i_{r^m} * \Phi^{-1}) = \\ &= (i_{r^m} * \Phi') \odot (i_{r^m} * \Gamma) \odot (i_{r^m} * \Phi^{-1}) = i_{r^m} * \bar{\Gamma}. \end{aligned}$$

Now let us suppose that there is another invertible 2-morphism  $\bar{\Gamma}' : \bar{t} \Rightarrow \bar{t}'$ , such that  $\bar{\Gamma}^m = i_{r^m} * \bar{\Gamma}'$  for each  $m = 1, 2$ . Then we define an invertible 2-morphism  $\Gamma' := (\Phi')^{-1} \odot \bar{\Gamma}' \odot \Phi : t \Rightarrow t'$ . Therefore, for each  $m = 1, 2$  we have:

$$\begin{aligned} \Gamma^m &\stackrel{(2.25)}{=} (i_{r^m} * (\Phi')^{-1}) \odot \bar{\Gamma}^m \odot (i_{r^m} * \Phi) = \\ &= (i_{r^m} * (\Phi')^{-1}) \odot (i_{r^m} * \bar{\Gamma}') \odot (i_{r^m} * \Phi) = i_{r^m} * \Gamma'. \end{aligned}$$

Since  $\Gamma$  is the unique invertible 2-morphism such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ , then we get that  $\Gamma' = \Gamma$ . Therefore, we conclude that  $\bar{\Gamma}' = \bar{\Gamma}$ , so we have proved that **B2**( $D, \bar{t}, \bar{t}'$ ) holds for (2.1), i.e. that (b) holds. Since  $\Phi$  and  $\Phi'$  are invertible, an analogous proof shows that (b) implies (a).

Now let us fix any object  $\bar{D}$  and any internal equivalence  $e : \bar{D} \rightarrow D$  and let us prove that (a) implies (c). So let us fix any pair of invertible 2-morphisms  $\bar{\Gamma}^m : r^m \circ t \circ e \Rightarrow r^m \circ t' \circ e$  for  $m = 1, 2$ , such that

$$(\Omega * i_{t' \circ e}) \odot (i_{g^1} * \bar{\Gamma}^1) = (i_{g^2} * \bar{\Gamma}^2) \odot (\Omega * i_{t \circ e}). \quad (2.26)$$

Let us choose any internal equivalence  $d$  and any pair of invertible 2-morphisms  $\Delta$  and  $\Xi$  as in the proof of Proposition 2.10 (in particular, let us assume that (2.21) holds). For each  $m = 1, 2$ , let us define an invertible 2-morphism

$$\Gamma^m := (i_{r^m \circ t'} * \Xi) \odot (\bar{\Gamma}^m * i_d) \odot (i_{r^m \circ t} * \Xi^{-1}) : r^m \circ t \Longrightarrow r^m \circ t'. \quad (2.27)$$

Then by interchange law we get:

$$\begin{aligned} &(\Omega * i_{t'}) \odot (i_{g^1} * \Gamma^1) \stackrel{(2.27)}{=} \\ &\stackrel{(2.27)}{=} (\Omega * i_{t'}) \odot (i_{g^1 \circ r^1 \circ t'} * \Xi) \odot (i_{g^1} * \bar{\Gamma}^1 * i_d) \odot (i_{g^1 \circ r^1 \circ t} * \Xi^{-1}) = \\ &= (i_{g^2 \circ r^2 \circ t'} * \Xi) \odot (\Omega * i_{t' \circ e \circ d}) \odot (i_{g^1} * \bar{\Gamma}^1 * i_d) \odot (i_{g^1 \circ r^1 \circ t} * \Xi^{-1}) \stackrel{(2.26)}{=} \\ &\stackrel{(2.26)}{=} (i_{g^2 \circ r^2 \circ t'} * \Xi) \odot (i_{g^2} * \bar{\Gamma}^2 * i_d) \odot (\Omega * i_{t \circ e \circ d}) \odot (i_{g^1 \circ r^1 \circ t} * \Xi^{-1}) = \\ &= (i_{g^2 \circ r^2 \circ t'} * \Xi) \odot (i_{g^2} * \bar{\Gamma}^2 * i_d) \odot (i_{g^2 \circ r^2 \circ t} * \Xi^{-1}) \odot (\Omega * i_t) \stackrel{(2.27)}{=} \\ &\stackrel{(2.27)}{=} (i_{g^2} * \Gamma^2) \odot (\Omega * i_t). \end{aligned} \quad (2.28)$$

By (a), condition **B2**( $D, t, t'$ ) holds for diagram (2.1), so by (2.28) there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . We set  $\bar{\Gamma} := \Gamma * i_e : t \circ e \Rightarrow t' \circ e$ . Then using the interchange law, for each  $m = 1, 2$  we have:

$$\begin{aligned}
i_{r^m} * \bar{\Gamma} &= i_{r^m} * \Gamma * i_e = \Gamma^m * i_e \stackrel{(2.27)}{=} \\
&\stackrel{(2.27)}{=} \left( i_{r^m o t'} * \Xi * i_e \right) \odot \left( \bar{\Gamma}^m * i_{d \circ e} \right) \odot \left( i_{r^m o t} * \Xi^{-1} * i_e \right) \stackrel{(2.21)}{=} \\
&\stackrel{(2.21)}{=} \left( i_{r^m o t' \circ e} * \Delta^{-1} \right) \odot \left( \bar{\Gamma}^m * i_{d \circ e} \right) \odot \left( i_{r^m o t \circ e} * \Delta \right) = \bar{\Gamma}^m.
\end{aligned}$$

Then we need only to prove the uniqueness of  $\bar{\Gamma}$ . Let us suppose that  $\bar{\Gamma}'$  is another invertible 2-morphism from  $t \circ e$  to  $t' \circ e$ , such that  $i_{r^m} * \bar{\Gamma}' = \bar{\Gamma}'^m$  for each  $m = 1, 2$ . We set

$$\Gamma' := \left( i_{t'} * \Xi \right) \odot \left( \bar{\Gamma}' * i_d \right) \odot \left( i_t * \Xi^{-1} \right) : t \Longrightarrow t'. \quad (2.29)$$

Then for each  $m = 1, 2$  we have:

$$\begin{aligned}
i_{r^m} * \Gamma' &\stackrel{(2.29)}{=} \left( i_{r^m o t'} * \Xi \right) \odot \left( i_{r^m} * \bar{\Gamma}' * i_d \right) \odot \left( i_{r^m o t} * \Xi^{-1} \right) = \\
&= \left( i_{r^m o t'} * \Xi \right) \odot \left( \bar{\Gamma}'^m * i_d \right) \odot \left( i_{r^m o t} * \Xi^{-1} \right) \stackrel{(2.27)}{=} \Gamma'^m.
\end{aligned}$$

By uniqueness of  $\Gamma$ , we get that  $\Gamma = \Gamma'$ . Therefore,

$$\bar{\Gamma} = \Gamma * i_e = \Gamma' * i_e \stackrel{(2.29)}{=} \left( i_{t'} * \Xi * i_e \right) \odot \left( \bar{\Gamma}' * i_{d \circ e} \right) \odot \left( i_t * \Xi^{-1} * i_e \right) \stackrel{(2.21)}{=} \bar{\Gamma}'.$$

So  $\mathbf{B2}(\bar{D}, t \circ e, t' \circ e)$  holds for diagram (2.1), i.e. (c) holds. Now let us assume that (c) holds and let us prove that (a) holds. Since  $d$  is an internal equivalence, using the proof that (a) implies (c) we get that  $\mathbf{B2}(\bar{D}, t \circ e \circ d, t' \circ e \circ d)$  holds for (2.1). Then using the equivalence of (a) and (b) and the pair of invertible 2-morphisms  $\Phi := i_t * \Xi$  and  $\Phi' := i_{t'} * \Xi$ , we get that  $\mathbf{B2}(\bar{D}, t, t')$  holds for (2.1), i.e. (a) is satisfied.  $\square$

### 3. WEAK FIBER PRODUCTS IN EQUIVALENT BICATEGORIES

Given any pair  $(\mathcal{C}, \mathbf{W})$  satisfying conditions (BF) for a right bicalculus of fractions (see Appendix A), in general the associated right bicategory of fractions is *unique only up to equivalences of bicategories*, since the “standard” construction (as described in [Pr, § 2.2 and 2.3]) depends on a set of choices  $\mathbf{C}(\mathbf{W})$  involving axioms (BF). In the next pages we will need to do all the computations of weak fiber products in a *chosen* right bicategory of fractions for the pair  $(\mathcal{C}, \mathbf{W})$  and then use this result in order to get a similar result for any other right bicategory of fractions for  $(\mathcal{C}, \mathbf{W})$ .

Therefore, the aim of this section is to prove that *weak fiber products are preserved by equivalences of bicategories*. We recall from [St, (1.33)] that given any pair of bicategories  $\mathcal{A}$  and  $\mathcal{B}$ , a pseudofunctor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  is a *weak equivalence of bicategories* (also known as *weak biequivalence*) if and only if the following conditions hold:

- (X1) for each object  $A_{\mathcal{A}}$ , there are an object  $A_{\mathcal{B}}$  and an internal equivalence from  $\mathcal{F}_0(A_{\mathcal{A}})$  to  $A_{\mathcal{B}}$  in  $\mathcal{B}$ ;
- (X2) for each pair of objects  $A_{\mathcal{A}}, B_{\mathcal{A}}$ , the functor  $\mathcal{F}(A_{\mathcal{A}}, B_{\mathcal{A}})$  is an equivalence of categories from  $\mathcal{A}(A_{\mathcal{A}}, B_{\mathcal{A}})$  to  $\mathcal{B}(\mathcal{F}_0(A_{\mathcal{A}}), \mathcal{F}_0(B_{\mathcal{A}}))$ .

Since we are assuming the axiom of choice, then each weak equivalence of bicategories is a (strong) equivalence of bicategories (see [PW, § 1]), i.e. it admits a quasi-inverse. Conversely, each strong equivalence of bicategories is also a weak

equivalence. So from now on we will simply write “equivalence of bicategories” for any weak, equivalently strong, equivalence of bicategories.

**Lemma 3.1.** *Let us suppose that  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence of bicategories; moreover, let us fix any weak fiber product in  $\mathcal{A}$  as follows:*

$$\begin{array}{ccc}
 C_{\mathcal{A}} & \xrightarrow{r_{\mathcal{A}}^1} & B_{\mathcal{A}}^1 \\
 r_{\mathcal{A}}^2 \downarrow & \swarrow \Omega_{\mathcal{A}} & \downarrow g_{\mathcal{A}}^1 \\
 B_{\mathcal{A}}^2 & \xrightarrow{g_{\mathcal{A}}^2} & A_{\mathcal{A}}.
 \end{array} \tag{3.1}$$

Then the induced diagram

$$\begin{array}{ccc}
 \mathcal{F}_0(C_{\mathcal{A}}) & \xrightarrow{\mathcal{F}_1(r_{\mathcal{A}}^1)} & \mathcal{F}_0(B_{\mathcal{A}}^1) \\
 \mathcal{F}_1(r_{\mathcal{A}}^2) \downarrow & \swarrow \Omega_{\mathcal{B}} := \Psi_{g_{\mathcal{A}}^2, r_{\mathcal{A}}^2}^{\mathcal{F}} \odot \mathcal{F}_2(\Omega_{\mathcal{A}}) \odot \left(\Psi_{g_{\mathcal{A}}^1, r_{\mathcal{A}}^1}^{\mathcal{F}}\right)^{-1} & \downarrow \mathcal{F}_1(g_{\mathcal{A}}^1) \\
 \mathcal{F}_0(B_{\mathcal{A}}^2) & \xrightarrow{\mathcal{F}_1(g_{\mathcal{A}}^2)} & \mathcal{F}_0(A_{\mathcal{A}})
 \end{array} \tag{3.2}$$

is a weak fiber product in  $\mathcal{B}$  (here the 2-morphisms  $\Psi_{\bullet}^{\mathcal{F}}$  are the associators for  $\mathcal{F}$ ).

See Appendix C for a proof.

**Proposition 3.2.** *Let us suppose that  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$  is an equivalence of bicategories; moreover, let us fix any triple of objects  $A_{\mathcal{A}}, B_{\mathcal{A}}^1, B_{\mathcal{A}}^2$  and any pair of morphisms  $g_{\mathcal{A}}^m : B_{\mathcal{A}}^m \rightarrow A_{\mathcal{A}}$  for  $m = 1, 2$ . Then the pair  $(g_{\mathcal{A}}^1, g_{\mathcal{A}}^2)$  has a weak fiber product in  $\mathcal{A}$  if and only if the pair  $(\mathcal{F}_1(g_{\mathcal{A}}^1), \mathcal{F}_1(g_{\mathcal{A}}^2))$  has a weak fiber product in  $\mathcal{B}$ .*

*Proof.* One of the 2 implication is simply Lemma 3.1. So we need only to prove the opposite implication. So let us suppose that the pair  $(\mathcal{F}_1(g_{\mathcal{A}}^1), \mathcal{F}_1(g_{\mathcal{A}}^2))$  has a weak fiber product in  $\mathcal{B}$ . Since  $\mathcal{F}$  is an equivalence of bicategories, then by [L, § 2.2] there are an equivalence of bicategories  $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}$  and a pseudonatural equivalence of pseudofunctors  $\mu : \mathcal{G} \circ \mathcal{F} \Rightarrow \text{id}_{\mathcal{A}}$ . Since  $\mathcal{G}$  is an equivalence of bicategories, then by Lemma 3.1 (with the roles of  $\mathcal{A}$  and  $\mathcal{B}$  reversed, and  $\mathcal{F}$  replaced by  $\mathcal{G}$ ) we have that the pair of morphisms:

$$(\mathcal{G}_1 \circ \mathcal{F}_1(g_{\mathcal{A}}^1), \mathcal{G}_1 \circ \mathcal{F}_1(g_{\mathcal{A}}^2))$$

has a weak fiber product in  $\mathcal{A}$ . Moreover, since  $\mu$  is a pseudonatural equivalence of pseudofunctors, for each  $m = 1, 2$  we have an invertible 2-morphism in  $\mathcal{A}$ :

$$\mu(g_{\mathcal{A}}^m) : \mathcal{G}_1 \circ \mathcal{F}_1(g_{\mathcal{A}}^m) \Longrightarrow g_{\mathcal{A}}^m.$$

So by Lemma 2.7 we conclude that the pair  $(g_{\mathcal{A}}^1, g_{\mathcal{A}}^2)$  has a weak fiber product in  $\mathcal{A}$ .  $\square$

#### 4. WEAK FIBER PRODUCTS IN BICATEGORIES OF FRACTIONS

In this section and in the following ones,  $\mathcal{C}$  will be a fixed bicategory and  $\mathbf{W}$  will be a fixed class of morphisms in it, such that the pair  $(\mathcal{C}, \mathbf{W})$  satisfies conditions (BF) for a right bicalculus of fractions (see Appendix A). We recall that the construction of a bicategory of fractions in [Pr] depends on a set of choices  $C(\mathbf{W})$  involving axioms (BF). For more details on the construction of bicategories of fractions, we refer directly to Appendix A. Then we are ready to give the following:

*Proof of Theorem 0.1.* First of all, let us assume (ii) and let us prove that (i) holds. By (ii), there is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$  for the pair of morphisms  $(B^m, \text{id}_{B^m}, f^m)$  for  $m = 1, 2$ . Now for each  $m = 1, 2$  we consider the morphism  $e^m := (B^m, w^m, \text{id}_{B^m}) : \overline{B}^m \rightarrow B^m$ . By [Pr, Proposition 20] each  $e^m$  is an internal equivalence in  $\mathcal{C}[\mathbf{W}^{-1}]$  because  $w^m$  belongs to  $\mathbf{W}$ . So by Corollary 2.5 for  $\mathcal{D} := \mathcal{C}[\mathbf{W}^{-1}]$ , the morphisms  $g^m \circ e^m$  for  $m = 1, 2$  have a weak fiber product. Now for each  $m = 1, 2$  we consider an invertible 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  as follows:

$$\Omega^m := \left[ B^m, \text{id}_{B^m}, \text{id}_{B^m}, \pi_{w^m \circ \text{id}_{B^m}}, \pi_{f^m \circ \text{id}_{B^m}} \right] :$$

$$g^m \circ e^m = \left( B^m, w^m \circ \text{id}_{B^m}, f^m \circ \text{id}_{B^m} \right) \Longrightarrow \left( B^m, w^m, f^m \right)$$

(where the 2-morphisms  $\pi_\bullet$  are the right unitors of  $\mathcal{C}$ ). Using Lemma 2.7, we conclude that the pair of morphisms in (0.2) has a weak fiber product, i.e. (i) holds.

Now let us assume (i) and let us prove that (ii) holds. If we choose  $w^m := \text{id}_{B^m}$  for each  $m = 1, 2$  and we use the definition of morphisms in a bicategory of fractions, then there are 3 objects  $T, T^1, T^2$ , a pair of morphisms  $v^m : T^m \rightarrow T$  in  $\mathbf{W}$  for  $m = 1, 2$ , a pair of morphisms  $t^m : T^m \rightarrow B^m$  for  $m = 1, 2$  in  $\mathcal{C}$  and an invertible 2-morphism  $\Omega^1$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , such that the following diagram is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc} T & \xrightarrow{(T^1, v^1, t^1)} & B^1 \\ \downarrow (T^2, v^2, t^2) & \Downarrow \Omega^1 & \downarrow (B^1, \text{id}_{B^1}, f^1) \\ B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A. \end{array} \quad (4.1)$$

For simplicity of exposition, from now on we assume that  $\mathcal{C}$  is a 2-category instead of a bicategory. Note that even under this restriction, in general  $\mathcal{C}[\mathbf{W}^{-1}]$  is only a bicategory, with trivial unitors but possibly non-trivial associators. So we will have anyway to explicitly write the associators  $\Theta_\bullet$  for  $\mathcal{C}[\mathbf{W}^{-1}]$ .

As we mentioned above, the bicategory  $\mathcal{C}[\mathbf{W}^{-1}]$  is not unique, but it depends on a set of choices  $C(\mathbf{W})$  (for any pair of morphisms  $(f, v)$  with the same target and such that  $v$  belongs to  $\mathbf{W}$ ). Different sets of choices give equivalent bicategories. Therefore, the proof that (i) implies (ii) will consist of the following 2 steps:

- (a) first of all, we prove that (ii) holds in any bicategory  $\mathcal{C}[\mathbf{W}^{-1}]$  obtained by fixing a set of choices  $C(\mathbf{W})$  satisfying condition (C3) (see Appendix A.1);
- (b) then we use Step (a) in order to prove that (ii) holds for any other set of choices  $C'(\mathbf{W})$ .

So for the moment, we assume that the set of choices  $C(\mathbf{W})$  satisfies condition (C3). In other terms, we assume that the fixed choices in  $C(\mathbf{W})$  are trivial for each pair  $(v, v)$  (with  $v$  in  $\mathbf{W}$ ).

Let us suppose that the fixed choices  $C(\mathbf{W})$  give data as in the following diagram, with  $x^1$  in  $\mathbf{W}$  and  $\rho$  invertible:

$$\begin{array}{ccc}
& R & \\
x^1 \swarrow & \rho & \searrow x^2 \\
T^1 & \xrightarrow{v^1} T \xleftarrow{v^2} & T^2.
\end{array}
\tag{4.2}$$

By condition (C3) the composition of the following morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$

$$T^1 \xleftarrow{\text{id}_{T^1}} T^1 \xrightarrow{v^1} T \quad \text{and} \quad T \xleftarrow{v^1} T^1 \xrightarrow{t^1} B^1$$

is given by

$$T^1 \xleftarrow{\text{id}_{T^1}} T^1 \xrightarrow{t^1} B^1;$$

by (4.2) the composition of

$$T^1 \xleftarrow{\text{id}_{T^1}} T^1 \xrightarrow{v^1} T \quad \text{and} \quad T \xleftarrow{v^2} T^2 \xrightarrow{t^2} B^2$$

is given by

$$T^1 \xleftarrow{x^1} R \xrightarrow{t^2 \circ x^2} B^2.$$

Therefore, if we apply Lemma 2.9 to  $\mathcal{D} := \mathcal{C}[\mathbf{W}^{-1}]$ , to diagram (4.1) and to the internal equivalence  $(T^1, \text{id}_{T^1}, v^1)$  (see again [Pr, Proposition 20]), we get that there is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$  of the form

$$\begin{array}{ccc}
T^1 & \xrightarrow{(T^1, \text{id}_{T^1}, t^1)} & B^1 \\
(R, x^1, t^2 \circ x^2) \downarrow & \swarrow \Omega^2 & \downarrow (B^1, \text{id}_{B^1}, f^1) \\
B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A.
\end{array}
\tag{4.3}$$

Now we apply again Lemma 2.9 to diagram (4.3) and to the internal equivalence  $(R, \text{id}_R, x^1)$ . So using again condition (C3), there is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$  of the form

$$\begin{array}{ccc}
R & \xrightarrow{(R, \text{id}_R, t^1 \circ x^1)} & B^1 \\
(R, \text{id}_R, t^2 \circ x^2) \downarrow & \swarrow \Omega^3 & \downarrow (B^1, \text{id}_{B^1}, f^1) \\
B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A.
\end{array}
\tag{4.4}$$

Since (4.4) is a weak fiber product, then  $\Omega^3$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ . Its source is the morphism  $(R, \text{id}_R, f^1 \circ t^1 \circ x^1)$ , while its target is the morphism  $(R, \text{id}_R, f^2 \circ t^2 \circ x^2)$ . By [T1, Lemma 6.1] applied to  $\alpha := i_{\text{id}_R}$  and to  $\Omega^3$ , there are an object  $C$ , a morphism  $z : C \rightarrow R$  in  $\mathbf{W}$  and a 2-morphism

$$\omega : f^1 \circ t^1 \circ x^1 \circ z \Longrightarrow f^2 \circ t^2 \circ x^2 \circ z$$

in  $\mathcal{C}$ , such that  $\Omega^3 = [C, z, z, i_z, \omega]$ . Using [T1, Proposition 0.8], up to replacing  $C$  and  $z$ , we can assume that  $\omega$  is invertible in  $\mathcal{C}$  since  $\Omega^3$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ .

Now by Lemma 2.9 applied to the weak fiber product (4.4) and to the internal equivalence  $(C, \text{id}_C, z)$  (see again [Pr, Proposition 20]), we have a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$  of the form

$$\begin{array}{ccc}
 C & \xrightarrow{(C, \text{id}_C, p^1)} & B^1 \\
 (C, \text{id}_C, p^2) \downarrow & \swarrow \Omega^4 & \downarrow (B^1, \text{id}_{B^1}, f^1) \\
 B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A,
 \end{array} \tag{4.5}$$

where for each  $m = 1, 2$  we set  $p^m := t^m \circ x^m \circ z$  and where

$$\Omega^4 := \Theta_{(B^2, \text{id}_{B^2}, f^2), (R, \text{id}_R, t^2 \circ x^2), (C, \text{id}_C, z)}^{-1} \odot \left( \Omega^3 * i_{(C, \text{id}_C, z)} \right) \odot$$

$$\odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (R, \text{id}_R, t^1 \circ x^1), (C, \text{id}_C, z)} : \left( C, \text{id}_C, f^1 \circ p^1 \right) \Longrightarrow \left( C, \text{id}_C, f^2 \circ p^2 \right).$$

By Lemma A.3, the associators  $\Theta_\bullet$  in the previous lines are both trivial, so

$$\Omega^4 = \Omega^3 * i_{(C, \text{id}_C, z)} = \left[ C, \text{id}_C, \text{id}_C, i_{\text{id}_C}, \omega \right] : \left( C, \text{id}_C, f^1 \circ p^1 \right) \Rightarrow \left( C, \text{id}_C, f^2 \circ p^2 \right). \tag{4.6}$$

Therefore, we have completely proved Step (a). Now let us fix any other set of choices  $C'(\mathbf{W})$  and let us denote by  $\mathcal{C}'[\mathbf{W}^{-1}]$  the associated bicategory of fractions. This bicategory has the same objects, morphisms and 2-morphisms as those of  $\mathcal{C}[\mathbf{W}^{-1}]$ , but compositions of morphisms and 2-morphisms are (possibly) different (therefore, we cannot conclude directly that (4.5) is a weak fiber product also in  $\mathcal{C}'[\mathbf{W}^{-1}]$ ). By [T2, Corollary 3.6], there is a pseudofunctor

$$\mathcal{Q} : \mathcal{C}[\mathbf{W}^{-1}] \longrightarrow \mathcal{C}'[\mathbf{W}^{-1}]$$

that is the identity on objects, morphisms and 2-morphisms (hence,  $\mathcal{Q}$  is an equivalence of bicategories). Since  $\mathcal{Q}$  is a pseudofunctor, then its associators  $\Psi_\bullet^{\mathcal{Q}}$  (that are induced by the set of choices  $C(\mathbf{W})$  and  $C'(\mathbf{W})$ ) are invertible. So for each  $m = 1, 2$  we can consider the invertible 2-morphism

$$\begin{aligned}
 \Gamma^m &:= \Psi_{(B^m, \text{id}_{B^m}, f^m), (C, \text{id}_C, p^m)}^{\mathcal{Q}} : \left( C, \text{id}_C, f^m \circ p^m \right) = \mathcal{Q} \left( C, \text{id}_C, f^m \circ p^m \right) = \\
 &= \mathcal{Q} \left( \left( B^m, \text{id}_{B^m}, f^m \right) \circ_{\mathcal{C}[\mathbf{W}^{-1}]} \left( C, \text{id}_C, p^m \right) \right) \Longrightarrow \\
 &\Longrightarrow \mathcal{Q} \left( B^m, \text{id}_{B^m}, f^m \right) \circ_{\mathcal{C}'[\mathbf{W}^{-1}]} \mathcal{Q} \left( C, \text{id}_C, p^m \right) = \\
 &= \left( B^m, \text{id}_{B^m}, f^m \right) \circ_{\mathcal{C}'[\mathbf{W}^{-1}]} \left( C, \text{id}_C, p^m \right) = \left( C, \text{id}_C, f^m \circ p^m \right)
 \end{aligned}$$

(in the lines above  $\circ_{\mathcal{C}[\mathbf{W}^{-1}]}$  is the composition in  $\mathcal{C}[\mathbf{W}^{-1}]$ , and analogously for  $\circ_{\mathcal{C}'[\mathbf{W}^{-1}]}$ ). If we apply [T1, Lemma 6.1] for  $\alpha := i_{\text{id}_C}$  and for  $(\Gamma^1)^{-1}$ , we get an object  $C^1$ , a morphism  $z^1 : C^1 \rightarrow C$  in  $\mathbf{W}$  and a 2-morphism

$$\alpha^1 : f^1 \circ p^1 \circ z^1 \Longrightarrow f^1 \circ p^1 \circ z^1$$

in  $\mathcal{C}$ , such that

$$(\Gamma^1)^{-1} = \left[ C^1, z^1, z^1, i_{z^1}, \alpha^1 \right].$$

If we apply [T1, Lemma 6.1] for  $\alpha := i_{z^1}$  and for  $\Gamma^2$ , there are an object  $\overline{C}$ , a morphism  $z^2 : \overline{C} \rightarrow C^1$ , such that  $z^1 \circ z^2$  belongs to  $\mathbf{W}$ , and a 2-morphism

$$\alpha^2 : f^2 \circ p^2 \circ z^1 \circ z^2 \Longrightarrow f^2 \circ p^2 \circ z^1 \circ z^2$$

in  $\mathcal{C}$ , such that

$$\Gamma^2 = \left[ \overline{C}, z^1 \circ z^2, z^1 \circ z^2, i_{z^1 \circ z^2}, \alpha^2 \right].$$

Since (4.5) is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$ , then using Lemma 3.1 we get that the following diagram is a weak fiber product in  $\mathcal{C}' [\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc} C & \xrightarrow{(C, \text{id}_C, p^1)} & B^1 \\ (C, \text{id}_C, p^2) \downarrow & \not\parallel \quad \Omega^5 := \Gamma^2 \odot \Omega^4 \odot (\Gamma^1)^{-1} & \downarrow (B^1, \text{id}_{B^1}, f^1) \\ B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A. \end{array} \quad (4.7)$$

Now for each  $m = 1, 2$  we set  $\overline{p}^m := p^m \circ z^1 \circ z^2 : \overline{C} \rightarrow B^m$  and

$$\overline{\omega} := \alpha^2 \odot \left( \omega * i_{z^1 \circ z^2} \right) \odot \left( \alpha^1 * i_{z^2} \right) : f^1 \circ \overline{p}^1 \Longrightarrow f^2 \circ \overline{p}^2.$$

Then a direct check proves that

$$\Omega^5 = \left[ \overline{C}, z^1 \circ z^2, z^1 \circ z^2, i_{z^1 \circ z^2}, \overline{\omega} \right].$$

Now by Lemma 2.9 applied to (4.7) and to the internal equivalence  $(\overline{C}, \text{id}_{\overline{C}}, z^1 \circ z^2)$  (see [Pr, Proposition 20]), the following diagram is a weak fiber product in  $\mathcal{C}' [\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc} \overline{C} & \xrightarrow{(\overline{C}, \text{id}_{\overline{C}}, \overline{p}^1)} & B^1 \\ (\overline{C}, \text{id}_{\overline{C}}, \overline{p}^2) \downarrow & \not\parallel \quad \Omega^6 & \downarrow (B^1, \text{id}_{B^1}, f^1) \\ B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A, \end{array}$$

where

$$\begin{aligned} \Omega^6 &:= \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (\overline{C}, \text{id}_{\overline{C}}, z^1 \circ z^2)} \odot \\ &\odot \left( \Omega^5 * i_{(\overline{C}, \text{id}_{\overline{C}}, z^1 \circ z^2)} \right) \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (\overline{C}, \text{id}_{\overline{C}}, z^1 \circ z^2)}^{-1}. \end{aligned}$$

By Lemma A.3, the associators  $\Theta_\bullet$  above are both trivial, so

$$\Omega^6 = \Omega^5 * i_{(\overline{C}, \text{id}_{\overline{C}}, z^1 \circ z^2)} = \left[ \overline{C}, \text{id}_{\overline{C}}, \text{id}_{\overline{C}}, i_{\text{id}_{\overline{C}}}, \overline{\omega} \right].$$

So the quadruple  $(\overline{C}, \overline{p}^1, \overline{p}^2, \overline{\omega})$  proves that Step (b) is satisfied in  $\mathcal{C}' [\mathbf{W}^{-1}]$ .  $\square$

**Remark 4.1.** The previous Theorem proves that for each set of choices  $C(\mathbf{W})$  there is a set of data  $(C, p^1, p^2, \omega)$  (a priori depending on  $C(\mathbf{W})$ ), inducing a weak fiber product (0.3) in the bicategory  $\mathcal{C} [\mathbf{W}^{-1}]$ . A priori we don't know whether such a set of data induces a weak fiber product also in the bicategory of fractions associated to a different set of choices  $C'(\mathbf{W})$  or not. Actually, given any set of data  $(C, p^1, p^2, \omega)$ , the following facts are equivalent:

- diagram (0.3) induced by  $(C, p^1, p^2, \omega)$  is a weak fiber product in  $\mathcal{C} [\mathbf{W}^{-1}]$ ;
- the same diagram is a weak fiber product in  $\mathcal{C}' [\mathbf{W}^{-1}]$  for any set of choices  $C'(\mathbf{W})$ .

This will be an obvious consequence of the fact that conditions (a), (b) and (c) in Theorem 0.2 (that we are going to prove below) do not depend on a set of choices  $C(\mathbf{W})$ , but only on the pair  $(\mathcal{C}, \mathbf{W})$ , hence they are verified for every bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$  associated to  $(\mathcal{C}, \mathbf{W})$ .

As we mentioned in the Introduction, Theorem 0.1 gives an explicit form for a weak fiber product (0.3) (whenever it exists) in the case when the pair of fixed morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  has the special form  $(B^m, \text{id}_{B^m}, f^m)$  for  $m = 1, 2$ . The same Theorem shows that whenever such a special pair of morphisms have a weak fiber product, then also those pairs with  $(\text{id}_{B^1}, \text{id}_{B^2})$  replaced by any pair of morphisms in  $\mathbf{W}$  have a weak fiber product. However in Theorem 0.1 we did not give any explicit description of a weak fiber product in that case. The next Corollary fills this gap (as in the previous pages, the 2-morphisms  $\theta_\bullet$  and  $\nu_\bullet$  are the associators, respectively the left unitors of  $\mathcal{C}$ ).

**Corollary 4.2.** *Let us fix any pair  $(\mathcal{C}, \mathbf{W})$  satisfying conditions (BF), any bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$  (i.e. any set of choices  $C(\mathbf{W})$ ), any pair of morphisms  $f^m : B^m \rightarrow A$  for  $m = 1, 2$  and any pair of morphisms  $w^m : B^m \rightarrow \overline{B}^m$  in  $\mathbf{W}$  for  $m = 1, 2$ . Moreover, let us fix any object  $C$ , any pair of morphisms  $p^m : C \rightarrow B^m$  for  $m = 1, 2$  and any invertible 2-morphism  $\omega : f^1 \circ p^1 \Rightarrow f^2 \circ p^2$  in  $\mathcal{C}$ , such that diagram (0.3) is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ . In addition, let us suppose that for each  $m = 1, 2$  the fixed choices  $C(\mathbf{W})$  give data as in upper part of the following diagram, with  $v^m$  in  $\mathbf{W}$  and  $\sigma^m$  invertible:*

$$\begin{array}{ccc}
 & C^m & \\
 v^m \swarrow & & \searrow q^m \\
 C & \xrightarrow{w^m \circ p^m} \overline{B}^m & \xleftarrow{w^m} B^m \\
 & \sigma^m \Rightarrow & 
 \end{array} \tag{4.8}$$

(this implies that  $(B^m, w^m, f^m) \circ (C, \text{id}_C, w^m \circ p^m) = (C^m, \text{id}_C \circ v^m, f^m \circ q^m)$  for each  $m = 1, 2$ ). Then let us choose any set of data as follows (the existence of such data is a consequence of axioms (BF), see Appendix A):

- (i) for each  $m = 1, 2$ , an object  $C'^m$ , a morphism  $u^m : C'^m \rightarrow C^m$  in  $\mathbf{W}$  and an invertible 2-morphism  $\tau^m : (p^m \circ v^m) \circ u^m \Rightarrow q^m \circ u^m$ , such that:

$$i_{w^m} * \tau^m = \theta_{w^m, q^m, u^m}^{-1} \odot \left( \left( \sigma^m \odot \theta_{w^m, p^m, v^m} \right) * i_{u^m} \right) \odot \theta_{w^m, p^m \circ v^m, u^m};$$

- (ii) an object  $C''$ , a pair of morphisms  $z^m : C'' \rightarrow C'^m$  for  $m = 1, 2$ , with  $z^1$  in  $\mathbf{W}$ , and an invertible 2-morphism  $\mu : v^1 \circ (u^1 \circ z^1) \Rightarrow v^2 \circ (u^2 \circ z^2)$ .

Then the following diagram is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$

$$\begin{array}{ccc}
 C & \xrightarrow{(C, \text{id}_C, w^1 \circ p^1)} & \overline{B}^1 \\
 (C, \text{id}_C, w^2 \circ p^2) \downarrow & \Downarrow \overline{\Omega} & \downarrow (B^1, w^1, f^1) \\
 \overline{B}^2 & \xrightarrow{(B^2, w^2, f^2)} & A,
 \end{array} \tag{4.9}$$

where

$$\overline{\Omega} := \left[ C'', u^1 \circ z^1, u^2 \circ z^2, \left( v_{v^2}^{-1} * i_{u^2 \circ z^2} \right) \odot \mu \odot \left( v_{v^1} * i_{u^1 \circ z^1} \right), \delta \right] :$$





$$\begin{array}{ccccc}
& & \overline{B}^m & & \\
& \swarrow \text{id}_{\overline{B}^m} & & \searrow \text{id}_{\overline{B}^m} & \\
\overline{B}^m & & B^m & & \overline{B}^m; \\
& \swarrow w^m & \downarrow i_{w^m} & \searrow i_{w^m} & \\
& & B^m & & \\
& \swarrow w^m & \downarrow \text{id}_{B^m} & \searrow w^m & \\
& & B^m & & 
\end{array}$$

Moreover, for each  $m = 1, 2$  we define an invertible 2-morphism  $\Xi^m : e^m \circ d^m \Rightarrow \text{id}_{B^m}$  in  $\mathcal{C}[\mathbf{W}^{-1}]$  as the 2-morphism represented by the following diagram:

$$\begin{array}{ccccc}
& & \tilde{B}^m & & \\
& \swarrow t^m & & \searrow s^m & \\
B^m & & B^m & & B^m. \\
& \swarrow \text{id}_{B^m} & \downarrow \varepsilon^m & \searrow i_{s^m \circ r^m} & \\
& & B^m & & \\
& \swarrow \text{id}_{B^m} & \downarrow s^m \circ r^m & \searrow \text{id}_{B^m} & \\
& & B^m & & 
\end{array} \tag{4.12}$$

Following the proof of [Pr, Proposition 20], the quadruple  $(e^m, d^m, \Delta^m, \Xi^m)$  is an *adjoint* equivalence in  $\mathcal{C}[\mathbf{W}^{-1}]$  for each  $m = 1, 2$ . For each such  $m$ , let us set:

$$g^m := \left( B^m \xleftarrow{\text{id}_{B^m}} B^m \xrightarrow{f^m} A \right) \quad \text{and} \quad r^m := \left( C \xleftarrow{\text{id}_C} C \xrightarrow{p^m} B^m \right).$$

Since we assumed that  $\mathcal{C}$  is a 2-category, then the 2-morphism  $\Omega$  of  $\mathcal{C}[\mathbf{W}^{-1}]$  appearing in (0.3) is given by

$$\Omega = \left[ C, \text{id}_C, \text{id}_C, i_{\text{id}_C}, \omega \right] : g^1 \circ r^1 \Longrightarrow g^2 \circ r^2. \tag{4.13}$$

Then we define an invertible 2-morphism

$$\overline{\Omega} : (g^1 \circ e^1) \circ (d^1 \circ r^1) \Longrightarrow (g^2 \circ e^2) \circ (d^2 \circ r^2)$$

as the following composition, where the 2-morphisms  $\Theta_\bullet$  are the associators of  $\mathcal{C}[\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc}
& (g^1 \circ e^1) \circ (d^1 \circ r^1) & \\
& \downarrow \Theta_{g^1 \circ e^1, d^1, r^1} & \\
C & \begin{array}{l} \xrightarrow{((g^1 \circ e^1) \circ d^1) \circ r^1} \\ \xrightarrow{(g^1 \circ (e^1 \circ d^1)) \circ r^1} \\ \xrightarrow{(g^1 \circ \text{id}_{B^1}) \circ r^1 = g^1 \circ r^1} \\ \xrightarrow{(g^2 \circ \text{id}_{B^2}) \circ r^2 = g^2 \circ r^2} \\ \xrightarrow{(g^2 \circ (e^2 \circ d^2)) \circ r^2} \\ \xrightarrow{((g^2 \circ e^2) \circ d^2) \circ r^2} \\ \xrightarrow{(g^2 \circ e^2) \circ (d^2 \circ r^2)} \end{array} & \begin{array}{l} \downarrow \Theta_{g^1, e^1, d^1}^{-1} * i_{r^1} \\ \downarrow (i_{g^1} * \Xi^1) * i_{r^1} \\ \downarrow \Omega \\ \downarrow (i_{g^2} * (\Xi^2)^{-1}) * i_{r^2} \\ \downarrow \Theta_{g^2, e^2, d^2} * i_{r^2} \\ \downarrow \Theta_{g^2 \circ e^2, d^2, r^2}^{-1} \end{array} & A. \\
& & (4.14)
\end{array}$$

Since we are working in the case when  $\mathcal{C}$  is a 2-category, then it is easy to prove that the unitors of  $\mathcal{C}[\mathbf{W}^{-1}]$  are trivial. Therefore, the 2-morphism  $\bar{\Omega}$  above coincides with the 2-morphism  $\bar{\Omega}$  defined in Proposition 2.4 for  $\mathcal{D} := \mathcal{C}[\mathbf{W}^{-1}]$ . By hypothesis, diagram (0.3) is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ , so by Proposition 2.4 we get that also the following diagram is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc}
C & \xrightarrow{d^1 \circ r^1 = (C, \text{id}_C, w^1 \circ p^1)} & \bar{B}^1 \\
\downarrow d^2 \circ r^2 = (C, \text{id}_C, w^2 \circ p^2) & \Downarrow \bar{\Omega} & \downarrow g^1 \circ e^1 = (B^1, w^1, f^1) \\
\bar{B}^2 & \xrightarrow{g^2 \circ e^2 = (B^2, w^2, f^2)} & A.
\end{array}$$

Then in order to prove the claim we need only to compute all the 2-morphisms in (4.14) and to prove that their composition is equal to the 2-morphism in (4.9).

Since we are assuming that  $\mathcal{C}$  is a 2-category, then for each  $m = 1, 2$  we have

$$\begin{aligned}
(g^m \circ e^m) \circ (d^m \circ r^m) &= (B^m, w^m, f^m) \circ (C, \text{id}_C, w^m \circ p^m) \stackrel{(4.8)}{=} \\
&\stackrel{(4.8)}{=} (C^m, v^m, f^m \circ q^m).
\end{aligned} \tag{4.15}$$

Let us suppose that for each  $m = 1, 2$  the fixed choices  $C(\mathbf{W})$  give data as in the upper part of the following diagram, with  $k^m$  in  $\mathbf{W}$  and  $\eta^m$  invertible:

$$\begin{array}{ccccc}
& & F^m & & \\
& k^m & \swarrow & h^m & \\
C & \xrightarrow{p^m} & B^m & \xleftarrow{t^m} & \tilde{B}^m. \\
& & \eta^m & & \\
& & \Rightarrow & & 
\end{array} \tag{4.16}$$

Then for each  $m = 1, 2$  we have:

$$\begin{aligned}
((g^m \circ e^m) \circ d^m) \circ r^m &= \left( (B^m, w^m, f^m) \circ (B^m, \text{id}_{B^m}, w^m) \right) \circ (C, \text{id}_C, p^m) \stackrel{(4.10)}{=} \\
&\stackrel{(4.10)}{=} (\tilde{B}^m, t^m, f^m \circ s^m) \circ (C, \text{id}_C, p^m) \stackrel{(4.16)}{=}
\end{aligned}$$

$$\stackrel{(4.16)}{=} \left( F^m, k^m, f^m \circ s^m \circ h^m \right). \quad (4.17)$$

Now for each  $m = 1, 2$ , we want to compute the associators  $\Theta_{g^m \circ e^m, d^m, r^m}$  from (4.15) to (4.17) appearing in (4.14). As a preliminary step, for each  $m = 1, 2$  we use axiom (BF3) in order to get a set of data as in the upper part of the following diagram, with  $a^m$  in  $\mathbf{W}$  and  $\gamma^m$  invertible:

$$\begin{array}{ccc} & G^m & \\ a^m \swarrow & & \searrow b^m \\ C'' & \xrightarrow{v^m \circ u^m \circ z^m} C & \xleftarrow{k^m} F^m \end{array} \quad \begin{array}{c} \gamma^m \\ \Downarrow \\ \cong \end{array}$$

Then we use (BF4a) and (BF4b) in order to get an object  $T^m$ , a morphism  $j^m : T^m \rightarrow G^m$  in  $\mathbf{W}$  and an invertible 2-morphism

$$\rho^m : q^m \circ u^m \circ z^m \circ a^m \circ j^m \Downarrow s^m \circ h^m \circ b^m \circ j^m,$$

such that  $i_{w^m} * \rho^m$  coincides with the following composition:

$$\begin{array}{ccccccc} & & C^m & \xrightarrow{q^m} & B^m & & \\ & & \downarrow \gamma^m & & \downarrow (\sigma^m)^{-1} & & \\ T^m & \xrightarrow{j^m} & G^m & \xrightarrow{u^m \circ z^m \circ a^m} & C^m & \xrightarrow{v^m} & B^m \\ & & \downarrow b^m & & \downarrow k^m & & \downarrow w^m \\ & & F^m & \xrightarrow{k^m} & C & \xrightarrow{p^m} & B^m \\ & & & & \downarrow \eta^m & & \downarrow \xi^m \\ & & & & \tilde{B}^m & \xrightarrow{s^m} & B^m \\ & & & & & & \downarrow w^m \\ & & & & & & \bar{B}^m \end{array} \quad (4.18)$$

Then we compute the associator from (4.15) to (4.17) using [T1, Proposition 0.1] for  $\underline{f} := r^m$ ,  $\underline{g} := d^m$  and  $\underline{h} := g^m \circ e^m$ . Using the previous choices, we have that the 2-morphisms appearing in [T1, Proposition 0.1(0.4)] are given as follows:

$$\delta := i_{p^m}, \quad \sigma := \sigma^m, \quad \xi := \xi^m, \quad \eta := \eta^m.$$

Then in [T1, Proposition 0.1] we choose

$$\gamma := \gamma^m * i_{j^m}, \quad \omega := \left( \eta^m * i_{b^m \circ j^m} \right) \odot \left( i_{p^m} * \gamma^m * i_{j^m} \right), \quad \rho := \rho^m.$$

Then we get that the associator  $\Theta_{\underline{g}^m \circ e^m, \underline{d}^m, \underline{r}^m}$  from (4.15) to (4.17) is represented by the following diagram:

$$\begin{array}{ccc} & C^m & \\ v^m \swarrow & & \searrow f^m \circ q^m \\ C & & A \\ \downarrow \gamma^m * i_{j^m} & & \downarrow i_{f^m} * \rho^m \\ & T^m & \\ & \uparrow u^m \circ z^m \circ a^m \circ j^m & \\ & F^m & \\ & \downarrow b^m \circ j^m & \\ & & f^m \circ s^m \circ h^m \end{array} \quad (4.19)$$

Now using (4.10), it is easy to prove that

$$g^m \circ (e^m \circ d^m) = \left( \tilde{B}^m, t^m, f^m \circ s^m \right) = (g^m \circ e^m) \circ d^m$$

and that  $\Theta_{g^m, e^m, d^m}$  is the 2-identity of this morphism; hence for each  $m = 1, 2$ , in diagram (4.14) we have:

$$\Theta_{g^m, e^m, d^m} * i_{r^m} = i_{((g^m \circ e^m) \circ d^m) \circ r^m} \stackrel{(4.16)}{=} i_{(F^m, k^m, f^m \circ s^m \circ h^m)}. \quad (4.20)$$

Now a direct check using (4.12) proves that for each  $m = 1, 2$  the 2-morphism  $i_{g^m} * \Xi^m$  is represented by the following diagram:

$$\begin{array}{ccccc} & & \tilde{B}^m & & \\ & \swarrow t^m & \uparrow r^m & \searrow f^m \circ s^m & \\ B^m & & B'^m & & A. \\ & \swarrow \text{id}_{B^m} & \downarrow s^m \circ r^m & \searrow f^m & \\ & & B^m & & \end{array}$$

$\Downarrow \varepsilon^m$        $\Downarrow i_{f^m \circ s^m \circ r^m}$

Then we need to compute the 2-morphisms  $(i_{g^m} * \Xi^m) * i_{r^m}$  appearing in (4.14). For each  $m = 1, 2$  we use axiom (BF3) in order to get data as in the upper part of the following diagram, with  $c^m$  in  $\mathbf{W}$  and  $\phi^m$  invertible:

$$\begin{array}{ccccc} & & L^m & & \\ & \swarrow c^m & \downarrow \phi^m & \searrow o^m & \\ T^m & \xrightarrow{h^m \circ b^m \circ j^m} & \tilde{B}^m & \xleftarrow{r^m} & B'^m \end{array}$$

Then we use [T1, Proposition 0.3] in order to compute  $(i_{g^m} * \Xi^m) * i_{r^m}$ . In the case under exam, the 2-morphisms  $\alpha$  and  $\beta$  appearing in that Proposition are given by  $\varepsilon^m$  and  $i_{f^m \circ s^m \circ r^m}$  respectively; moreover, the 2-morphisms  $\rho^1, \rho^2$  of that Proposition are given by  $\eta^m$  and  $i_{p^m}$  respectively. Then we choose the 2-morphisms  $\sigma^1, \sigma^2$  and  $\alpha'$  appearing in that Proposition as follows: we set  $\sigma^1 := \phi^m$ ,  $\alpha' := i_{k^m \circ b^m \circ j^m \circ c^m}$  and we define  $\sigma^2$  as the following composition:

$$\begin{array}{ccccccc} & & & F^m & \xrightarrow{k^m} & C & \\ & \swarrow b^m \circ j^m \circ c^m & & \downarrow \phi^m & \searrow h^m & \downarrow \eta^m & \\ L^m & & & & & \tilde{B}^m & \xrightarrow{t^m} B^m. \\ & \searrow o^m & & & \swarrow r^m & \downarrow \varepsilon^m & \\ & & B'^m & \xrightarrow{r^m} & \tilde{B}^m & \swarrow s^m & \end{array}$$

Then replacing all these choices in [T1, Proposition 0.3(0.12)] we get the 2-identity over  $t^m \circ r^m \circ o^m$ . So in that Proposition we can choose  $\delta := i_{o^m}$ . Therefore, replacing in the definition of  $\beta'$  in that Proposition, we conclude that for each  $m = 1, 2$  the 2-morphism  $(i_{g^m} * \Xi^m) * i_{r^m}$  appearing in (4.14) is represented by the following diagram

$$\begin{array}{ccccc}
& & F^m & & \\
& \swarrow & \uparrow & \searrow & \\
& C & L^m & A & \\
& \swarrow & \downarrow & \searrow & \\
& & C & & 
\end{array}
\begin{array}{l}
k^m \\
\Downarrow i_{k^m} \circ b^m \circ j^m \circ c^m \\
id_C \\
b^m \circ j^m \circ c^m \\
\Downarrow i_{f^m} * \zeta^m \\
k^m \circ b^m \circ j^m \circ c^m \\
f^m \circ s^m \circ h^m \\
f^m \circ p^m
\end{array}
\tag{4.21}$$

where  $\zeta^m$  is the following composition:

$$\begin{array}{ccccccc}
& & F^m & \xrightarrow{h^m} & \tilde{B}^m & & \\
& \swarrow & \downarrow \phi^m & \searrow & \downarrow (\varepsilon^m)^{-1} & \searrow & \\
L^m & \xrightarrow{\sigma^m} & B'^m & \xrightarrow{r^m} & \tilde{B}^m & \xrightarrow{t^m} & B^m \\
& \swarrow & \downarrow (\phi^m)^{-1} & \searrow & \downarrow (\eta^m)^{-1} & \searrow & \\
& & F^m & \xrightarrow{h^m} & C & \xrightarrow{p^m} & 
\end{array}
\begin{array}{l}
b^m \circ j^m \circ c^m \\
\downarrow \phi^m \\
(\phi^m)^{-1} \\
b^m \circ j^m \circ c^m \\
r^m \\
r^m \\
h^m \\
(\eta^m)^{-1} \\
k^m
\end{array}
\tag{4.22}$$

Now using (4.19), (4.20) and (4.21) we get that for each  $m = 1, 2$  the 2-morphism

$$F^m := \left( (i_{g^m} * \Xi^m) * i_{r^m} \right) \odot \left( \Theta_{g^m, e^m, d^m}^{-1} * i_{r^m} \right) \odot \Theta_{g^m \circ e^m, d^m, r^m} \tag{4.23}$$

is represented by the following diagram

$$\begin{array}{ccccc}
& & C^m & & \\
& \swarrow & \uparrow & \searrow & \\
& C & L^m & A & \\
& \swarrow & \downarrow & \searrow & \\
& & C & & 
\end{array}
\begin{array}{l}
v^m \\
\Downarrow \gamma^m * i_{j^m} \circ c^m \\
id_C \\
u^m \circ z^m \circ a^m \circ j^m \circ c^m \\
\Downarrow i_{f^m} * (\zeta^m \odot (\rho^m * i_{c^m})) \\
k^m \circ b^m \circ j^m \circ c^m \\
f^m \circ q^m \\
f^m \circ p^m
\end{array}$$

Using the definition of 2-morphism in a bicategory of fractions (see Appendix A.2), we get easily that  $F^m$  is also represented by the following diagram

$$\begin{array}{ccccc}
& & C^m & & \\
& \swarrow & \uparrow & \searrow & \\
& C & L^m & A & \\
& \swarrow & \downarrow & \searrow & \\
& & C & & 
\end{array}
\begin{array}{l}
v^m \\
\Downarrow i_{v^m} \circ u^m \circ z^m \circ a^m \circ j^m \circ c^m \\
id_C \\
u^m \circ z^m \circ a^m \circ j^m \circ c^m \\
\Downarrow i_{f^m} * \chi^m \\
v^m \circ u^m \circ z^m \circ a^m \circ j^m \circ c^m \\
f^m \circ q^m \\
f^m \circ p^m
\end{array}
\tag{4.24}$$

where

$$\chi^m := \left( i_{p^m} * (\gamma^m)^{-1} * i_{j^m} \circ c^m \right) \odot \zeta^m \odot \left( \rho^m * i_{c^m} \right).$$

Now we want to write  $F^m$  in a shorter form. As a preliminary step, we want to compute  $i_{w^m} * \chi^m$ ; for that, we replace  $i_{w^m} * \rho^m$  with (4.18) and  $\zeta^m$  with (4.22). So we get that  $i_{w^m} * \chi^m$  coincides with the following composition:

$$\begin{array}{ccccccc}
 & & C^m & \xrightarrow{q^m} & B^m & & \\
 & \swarrow u^m \circ z^m \circ a^m & \downarrow \gamma^m & \searrow v^m & \downarrow (\sigma^m)^{-1} & & \\
 G^m & \xrightarrow{b^m} & F^m & \xrightarrow{k^m} & C & \xrightarrow{p^m} & B^m \xrightarrow{w^m} \bar{B}^m \\
 & \downarrow j^m \circ c^m & \downarrow \phi^m & \downarrow h^m & \downarrow \eta^m & \downarrow t^m & \downarrow \xi^m \\
 L^m & \xrightarrow{o^m} & B'^m & \xrightarrow{r^m} & \tilde{B}^m & \xrightarrow{s^m} & B^m \\
 & \downarrow j^m \circ c^m & \downarrow (\phi^m)^{-1} & \downarrow r^m & \downarrow (\varepsilon^m)^{-1} & \downarrow t^m & \downarrow \xi^m \\
 G^m & \xrightarrow{b^m} & F^m & \xrightarrow{k^m} & C & & \\
 & \swarrow u^m \circ z^m \circ a^m & \downarrow (\gamma^m)^{-1} & \searrow v^m & & & \\
 & & C^m & & & & 
 \end{array}
 \tag{4.25}$$

In such a diagram, using (4.11) we can replace the composition of  $\xi^m$  and  $(\varepsilon^m)^{-1}$  with a 2-identity. Then we can simply the terms  $\phi^m$ ,  $\eta^m$  and  $\gamma^m$  (in this order) with their inverses. So we get:

$$i_{w^m} * \chi^m = (\sigma^m)^{-1} * i_{u^m \circ z^m \circ a^m \circ j^m \circ c^m} = i_{w^m} * (\tau^m)^{-1} * i_{z^m \circ a^m \circ j^m \circ c^m},$$

where the last identity is a consequence of hypothesis (i). So by [T1, Lemma 1.1] there are an object  $H^m$  and a morphism  $y^m : H^m \rightarrow L^m$ , such that

$$\chi^m * i_{y^m} = (\tau^m)^{-1} * i_{z^m \circ a^m \circ j^m \circ c^m \circ y^m}. \tag{4.26}$$

So for each  $m = 1, 2$  we have:

$$\begin{aligned}
 F^m &\stackrel{(4.24), (4.26)}{=} \left[ H^m, u^m \circ z^m \circ a^m \circ j^m \circ c^m \circ y^m, v^m \circ u^m \circ z^m \circ a^m \circ j^m \circ c^m \circ y^m, \right. \\
 &\quad \left. i_{v^m \circ u^m \circ z^m \circ a^m \circ j^m \circ c^m \circ y^m}, i_{f^m} * (\tau^m)^{-1} * i_{z^m \circ a^m \circ j^m \circ c^m \circ y^m} \right] = \\
 &= \left[ C''^m, u^m \circ z^m, v^m \circ u^m \circ z^m, i_{v^m \circ u^m \circ z^m}, i_{f^m} * (\tau^m)^{-1} * i_{z^m} \right]. \tag{4.27}
 \end{aligned}$$

Now using (4.23) together with (4.14), we get that  $\bar{\Omega} = (F^2)^{-1} \odot \Omega \odot F^1$ . Using (4.27) for  $m = 1$  and (4.13), we get that  $\Omega \odot F^1$  is represented by the following diagram:

$$\begin{array}{ccccc}
 & & C^1 & & \\
 & \swarrow v^1 & \uparrow u^1 \circ z^1 & \searrow f^1 \circ q^1 & \\
 C & \xleftarrow{\downarrow i_{v^1 \circ u^1 \circ z^1}} & C'' & \xrightarrow{\downarrow \kappa^1} & A \\
 & \swarrow \text{id}_C & \downarrow v^1 \circ u^1 \circ z^1 & \searrow f^2 \circ p^2 & \\
 & & C & & 
 \end{array}
 \tag{4.28}$$

where

$$\kappa^1 := \left( \omega * i_{v^1 \circ u^1 \circ z^1} \right) \odot \left( i_{f^1} * (\tau^1)^{-1} * i_{z^1} \right).$$

Using the inverse of (4.27) for  $m = 2$  and the choices in (ii) in the claim, we get that  $F^2$  is represented by the following diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & \swarrow & \uparrow & \searrow & \\
 C & \xleftarrow{\text{id}_C} & C & \xrightarrow{f^1 \circ p^2} & A \\
 & \searrow & \downarrow \mu & \downarrow \kappa^2 & \\
 & & C'' & & \\
 & \swarrow & \downarrow & \searrow & \\
 & & C^2 & & \\
 & \xleftarrow{v^2} & & \xrightarrow{f^2 \circ q^2} & 
 \end{array}
 \tag{4.29}$$

where

$$\kappa^2 := \left( i_{f^2} * \tau^2 * i_{z^2} \right) \odot \left( i_{f^2 \circ p^2} * \mu \right).$$

Lastly, using (4.28), (4.29) and [T1, Proposition 0.2], we get that the 2-morphism  $\overline{\Omega} = (F^2)^{-1} \odot \Omega \odot F^1$  coincides with (4.9), so we conclude.  $\square$

Now we want to prove Theorem 0.2, so the problem that we have to solve is the following: given any set of data in  $\mathcal{C}$  as follows

$$\begin{array}{ccc}
 C & \xrightarrow{p^1} & B^1 \\
 p^2 \downarrow & \swarrow \omega & \downarrow f^1 \\
 B^2 & \xrightarrow{f^2} & A
 \end{array}
 \tag{4.30}$$

with  $\omega$  invertible, when is the associated diagram (0.3) a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ ? In the next 2 sections we will consider separately conditions **A1** and **A2** for (0.3) and we will manage to give equivalent but simple conditions for both properties.

## 5. CONDITION A1 IN A BICATEGORY OF FRACTIONS

**Lemma 5.1.** *Let us fix any pair  $(\mathcal{C}, \mathbf{W})$  satisfying conditions (BF) and any bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$  associated to it (i.e. any set of choices  $\mathbf{C}(\mathbf{W})$ ). Moreover, let us also fix any set of data in  $\mathcal{C}$  as in diagram (4.30) with  $\omega$  invertible. Then the following facts are equivalent:*

- (i1) for any object  $D$ , condition **A1**( $D$ ) holds for diagram (0.3) in  $\mathcal{C}[\mathbf{W}^{-1}]$ ;
- (i2) for any object  $D$  and for any pair of morphisms  $q^1 : D \rightarrow B^1$ ,  $q^2 : D \rightarrow B^2$  in  $\mathcal{C}$ , condition **B1**( $D, (D, \text{id}_D, q^1), (D, \text{id}_D, q^2)$ ) holds for diagram (0.3).

*Proof.* For simplicity of exposition, let us suppose that  $\mathcal{C}$  is a 2-category.

We recall from Remark 2.2 that (i1) is equivalent to

- (i1)' for any object  $D$ , and for any pair of morphisms  $s^1 : D \rightarrow B^1$  and  $s^2 : D \rightarrow B^2$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , property **B1**( $D, s^1, s^2$ ) holds for diagram (0.3).



Clearly (i1)' implies (i2): it is simply the case when  $s^m := (D, \text{id}_D, q^m)$  for  $m = 1, 2$ . Let us assume that (i2) holds and let us prove (i1)'. So let us fix any object  $D$  in  $\mathcal{C}$  and any pair of morphisms  $s^1 : D \rightarrow B^1$  and  $s^2 : D \rightarrow B^2$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ . By definition of morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$ , for each  $m = 1, 2$  there are an object  $D^m$ , a morphism  $w^m : D^m \rightarrow D$  in  $\mathbf{W}$  and a morphism  $t^m : D^m \rightarrow B^m$  in  $\mathcal{C}$ , such that  $s^m = (D^m, w^m, t^m)$ . Now we use (BF3) in order to get data as in the upper part of the following diagram, with  $v^2$  in  $\mathbf{W}$  and  $\alpha$  invertible:

$$\begin{array}{ccc} & D^3 & \\ v^2 \swarrow & & \searrow v^1 \\ D^2 & \xrightarrow{w^2} D & \xleftarrow{w^1} D^1 \end{array} \quad \begin{array}{c} \alpha \\ \Rightarrow \end{array}$$

Moreover, let us suppose that the fixed choices  $C(\mathbf{W})$  give data as in the upper part of the following diagram, with  $r$  in  $\mathbf{W}$  and  $\varepsilon$  invertible:

$$\begin{array}{ccc} & D^4 & \\ r \swarrow & & \searrow q \\ D^3 & \xrightarrow{w^2 \circ v^2} D & \xleftarrow{w^2 \circ v^2} D^3 \end{array} \quad \begin{array}{c} \varepsilon \\ \Rightarrow \end{array} \quad (5.1)$$

(since the choices  $C(\mathbf{W})$  are arbitrary, then we cannot assume that condition (C3) holds, so we cannot say anything more about the data above). By construction and (BF2), the morphism  $w^2 \circ v^2$  belongs to  $\mathbf{W}$ . So using (BF4a) and (BF4b), there are an object  $D^5$ , a morphism  $h : D^5 \rightarrow D^4$  in  $\mathbf{W}$  and an invertible 2-morphism  $\eta : r \circ h \Rightarrow q \circ h$ , such that  $\varepsilon * i_h = i_{w^2 \circ v^2} * \eta$ . Since we are assuming (i2), then condition  $\mathbf{B1}(D^3, (D^3, \text{id}_{D^3}, t^1 \circ v^1), (D^3, \text{id}_{D^3}, t^2 \circ v^2))$  holds for (0.3). Then for each  $m = 1, 2$  we consider the invertible 2-morphism

$$\Omega^m : (D^3, \text{id}_{D^3}, t^m \circ v^m) \Rightarrow (D^4, r, t^m \circ v^m \circ q) \quad (5.2)$$

represented by the following diagram:

$$\begin{array}{ccc} & D^3 & \\ \text{id}_{D^3} \swarrow & & \searrow t^m \circ v^m \\ D^3 & & B^m \\ \downarrow \eta^{-1} & \uparrow q \circ h & \downarrow i_{t^m \circ v^m \circ q \circ h} \\ & D^5 & \\ & \downarrow h & \\ & D^4 & \\ r \swarrow & & \searrow t^m \circ v^m \circ q \end{array}$$

Then using the equivalence of (a) and (b) in Proposition 2.10 and (5.2), we get that condition  $\mathbf{B1}(D^3, (D^4, r, t^1 \circ v^1 \circ q), (D^4, r, t^2 \circ v^2 \circ q))$  holds for (0.3). Now using (5.1), for each  $m = 1, 2$  we have

$$(D^3, w^2 \circ v^2, t^m \circ v^m) \circ (D^3, \text{id}_{D^3}, w^2 \circ v^2) = (D^4, r, t^m \circ v^m \circ q). \quad (5.3)$$

Since  $w^2 \circ v^2$  belongs to  $\mathbf{W}$ , then the morphism  $e := (D^3, \text{id}_{D^3}, w^2 \circ v^2)$  is an internal equivalence in  $\mathcal{C}[\mathbf{W}^{-1}]$  (see [Pr, Proposition 20]). Therefore, using the equivalence of (a) and (c) in Proposition 2.10, and (5.3), we get that  $\mathbf{B1}(D, (D^3, w^2 \circ v^2, t^1 \circ v^1), (D^3, w^2 \circ v^2, t^2 \circ v^2))$  holds for (0.3).

$v^1), (D^3, w^2 \circ v^2, t^2 \circ v^2))$  holds for (0.3). Now we consider the pair of invertible 2-morphisms

$$\begin{aligned}\tilde{\Omega}^1 &:= \left[ D^3, \text{id}_{D^3}, v^1, \alpha, i_{t^1 \circ v^1} \right] : \left( D^3, w^2 \circ v^2, t^1 \circ v^1 \right) \Longrightarrow \left( D^1, w^1, t^1 \right) = s^1, \\ \tilde{\Omega}^2 &:= \left[ D^3, \text{id}_{D^3}, v^2, i_{w^2 \circ v^2}, i_{t^2 \circ v^2} \right] : \left( D^3, w^2 \circ v^2, t^2 \circ v^2 \right) \Longrightarrow \left( D^2, w^2, t^2 \right) = s^2.\end{aligned}$$

Using the equivalence of (a) and (b) in Proposition 2.10, we get that  $\mathbf{B1}(D, s^1, s^2)$  holds for (0.3), i.e. (i1)' holds.  $\square$

**Lemma 5.2.** *Let us fix the same notations of Lemma 5.1. Then the following facts are equivalent:*

- (i2) for any object  $D$  and for any pair of morphisms  $q^1 : D \rightarrow B^1$ ,  $q^2 : D \rightarrow B^2$  in  $\mathcal{C}$ , condition  $\mathbf{B1}(D, (D, \text{id}_D, q^1), (D, \text{id}_D, q^2))$  holds for diagram (0.3);
- (i3) for any object  $D$  of  $\mathcal{C}$  the following condition holds:
  - (a) given any pair of morphisms  $q^m : D \rightarrow B^m$  for  $m = 1, 2$  and any invertible 2-morphism  $\lambda : f^1 \circ q^1 \Rightarrow f^2 \circ q^2$  in  $\mathcal{C}$ , there are an object  $E$ , a morphism  $v : E \rightarrow D$  in  $\mathbf{W}$ , a morphism  $q : E \rightarrow C$  and a pair of invertible 2-morphisms  $\lambda^m : q^m \circ v \Rightarrow p^m \circ q$  for  $m = 1, 2$  in  $\mathcal{C}$ , such that:

$$\begin{aligned}\theta_{f^2, p^2, q}^{-1} \odot (\omega * i_q) \odot \theta_{f^1, p^1, q} \odot (i_{f^1} * \lambda^1) &= \\ = (i_{f^2} * \lambda^2) \odot \theta_{f^2, q^2, v}^{-1} \odot (\lambda * i_v) \odot \theta_{f^1, q^1, v}.\end{aligned}$$

*Proof.* As usual, we assume for simplicity that  $\mathcal{C}$  is a 2-category. Let us suppose that (i2) holds and let us fix any quadruple  $(D, q^1, q^2, \lambda)$  as in (i3). Then we can consider a diagram as follows in  $\mathcal{C}[\mathbf{W}^{-1}]$ :

$$\begin{array}{ccc} D & \xrightarrow{(D, \text{id}_D, q^1)} & B^1 \\ \downarrow (D, \text{id}_D, q^2) & \swarrow \Lambda := [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \lambda] & \downarrow (B^1, \text{id}_{B^1}, f^1) \\ B^2 & \xrightarrow{(B^2, \text{id}_{B^2}, f^2)} & A. \end{array} \quad (5.4)$$

Since  $\lambda$  is invertible in  $\mathcal{C}$ , then we get easily that  $\Lambda$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ , so by (i2) there are a morphism

$$s := \left( D \xleftarrow{w} \overline{D} \xrightarrow{r} C \right) : D \longrightarrow C$$

in  $\mathcal{C}[\mathbf{W}^{-1}]$  and a pair of invertible 2-morphisms

$$\Lambda^m : \left( D, \text{id}_D, q^m \right) \Longrightarrow \left( C, \text{id}_C, p^m \right) \circ \left( \overline{D}, w, r \right)$$

for  $m = 1, 2$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , such that

$$\begin{aligned}\left( \Omega * i_{(\overline{D}, w, r)} \right) \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (\overline{D}, w, r)} \odot \left( i_{(B^1, \text{id}_{B^1}, f^1)} * \Lambda^1 \right) &= \\ = \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (\overline{D}, w, r)} \odot \left( i_{(B^2, \text{id}_{B^2}, f^2)} * \Lambda^2 \right) \odot \Lambda.\end{aligned} \quad (5.5)$$

For each  $m = 1, 2$ ,  $\Lambda^m$  is defined from  $(D, \text{id}_D, q^m)$  to  $(\overline{D}, w, p^m \circ r)$ . Therefore by [T1, Lemma 6.1] applied to  $\alpha := i_w$  and to  $\Lambda^1$ , there are an object  $D^1$ , a morphism  $u^1 : D^1 \rightarrow \overline{D}$  such that  $w \circ u^1$  belongs to  $\mathbf{W}$ , and a 2-morphism

$$\alpha^1 : q^1 \circ w \circ u^1 \Longrightarrow p^1 \circ r \circ u^1$$

in  $\mathcal{C}$ , such that

$$\Lambda^1 = \left[ D^1, w \circ u^1, u^1, i_{w \circ u^1}, \alpha^1 \right].$$

By [T1, Proposition 0.8], we can assume that  $\alpha^1$  is invertible in  $\mathcal{C}$  since  $\Lambda^1$  is invertible in  $\mathcal{C}$  [ $\mathbf{W}^{-1}$ ]. By [T1, Lemma 6.1] applied to  $\alpha := i_{w \circ u^1}$  and to  $\Lambda^2$ , there are an object  $D^2$ , a morphism  $u^2 : D^2 \rightarrow D^1$  such that  $w \circ u^1 \circ u^2$  belongs to  $\mathbf{W}$ , and a 2-morphism

$$\alpha^2 : q^2 \circ w \circ u^1 \circ u^2 \Longrightarrow p^2 \circ r \circ u^1 \circ u^2$$

in  $\mathcal{C}$ , such that

$$\Lambda^2 = \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, \alpha^2 \right].$$

As above, we can assume that  $\alpha^2$  is invertible in  $\mathcal{C}$  since  $\Lambda^2$  is invertible in  $\mathcal{C}$  [ $\mathbf{W}^{-1}$ ]. Now by Lemma A.5 (in the special case when  $\mathcal{C}$  is a 2-category), we have:

$$i_{(B^1, \text{id}_{B^1}, f^1)} * \Lambda^1 = \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, i_{f^1} * \alpha^1 * i_{u^2} \right] \quad (5.6)$$

and

$$i_{(B^2, \text{id}_{B^2}, f^2)} * \Lambda^2 = \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, i_{f^2} * \alpha^2 \right]. \quad (5.7)$$

Moreover, using (0.3) and Lemma A.4 (in the special case when  $\mathcal{C}$  is a 2-category), we have:

$$\Omega * i_{(\overline{D}, w, r)} = \left[ C, \text{id}_C, \text{id}_C, i_{\text{id}_C}, \omega \right] * i_{(\overline{D}, w, r)} = \left[ \overline{D}, \text{id}_{\overline{D}}, \text{id}_{\overline{D}}, i_w, \omega * i_r \right]. \quad (5.8)$$

In addition, by Lemma A.3 each 2-morphism of the form  $\Theta_\bullet$  in (5.5) is trivial. Therefore, by replacing (5.4), (5.6), (5.7) and (5.8) in (5.5), we get:

$$\begin{aligned} & \left[ \overline{D}, \text{id}_{\overline{D}}, \text{id}_{\overline{D}}, i_w, \omega * i_r \right] \odot \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, i_{f^1} * \alpha^1 * i_{u^2} \right] = \\ & = \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, i_{f^2} * \alpha^2 \right] \odot \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \lambda \right]. \end{aligned} \quad (5.9)$$

This is equivalent to saying that

$$\begin{aligned} & \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, \left( \omega * i_{r \circ u^1 \circ u^2} \right) \odot \left( i_{f^1} * \alpha^1 * i_{u^2} \right) \right] = \\ & = \left[ D^2, w \circ u^1 \circ u^2, u^1 \circ u^2, i_{w \circ u^1 \circ u^2}, \left( i_{f^2} * \alpha^2 \right) \odot \left( \lambda * i_{w \circ u^1 \circ u^2} \right) \right]. \end{aligned}$$

So by [T1, Proposition 0.7] there are an object  $E$  and a morphisms  $u^3 : E \rightarrow D^2$ , such that  $w \circ u^1 \circ u^2 \circ u^3$  belongs to  $\mathbf{W}$  and such that

$$\left( \omega * i_{r \circ u^1 \circ u^2 \circ u^3} \right) \odot \left( i_{f^1} * \alpha^1 * i_{u^2 \circ u^3} \right) = \left( i_{f^2} * \alpha^2 * i_{u^3} \right) \odot \left( \lambda * i_{w \circ u^1 \circ u^2 \circ u^3} \right). \quad (5.10)$$

Now we define

$$\begin{aligned} v & := w \circ u^1 \circ u^2 \circ u^3 : E \longrightarrow D, & q & := r \circ u^1 \circ u^2 \circ u^3 : E \longrightarrow C, \\ \lambda^1 & := \alpha^1 * i_{u^2 \circ u^3} : q^1 \circ v \Longrightarrow p^1 \circ q, & \lambda^2 & := \alpha^2 * i_{u^3} : q^2 \circ v \Longrightarrow p^2 \circ q. \end{aligned}$$

So (5.10) reads as follows:

$$\left(\omega * i_q\right) \odot \left(i_{f^1} * \lambda^1\right) = \left(i_{f^2} * \lambda^2\right) \odot \left(\lambda * i_v\right),$$

hence we have proved that (i2) implies (i3).

Conversely, let us assume that (i3) holds. Let us fix any object  $D$ , any pair of morphisms  $q^1 : D \rightarrow B^1$ ,  $q^2 : D \rightarrow B^2$  in  $\mathcal{C}$ ; then we have to prove that condition **B1**( $D, (D, \text{id}_D, q^1), (D, \text{id}_D, q^2)$ ) holds for diagram (0.3). So let us fix any invertible 2-morphism

$$\Lambda : \left(B^1, \text{id}_{B^1}, f^1\right) \circ \left(D, \text{id}_D, q^1\right) \Longrightarrow \left(B^2, \text{id}_{B^2}, f^2\right) \circ \left(D, \text{id}_D, q^2\right) \quad (5.11)$$

in  $\mathcal{C}[\mathbf{W}^{-1}]$ . By [T1, Lemma 6.1] applied to  $\alpha := i_{\text{id}_D}$  and  $\Lambda$ , there are an object  $\overline{D}$ , a morphism  $w : \overline{D} \rightarrow D$  in  $\mathbf{W}$  and an invertible 2-morphism  $\lambda : f^1 \circ q^1 \circ w \Rightarrow f^2 \circ q^2 \circ w$  in  $\mathcal{C}$ , such that

$$\Lambda = \left[\overline{D}, w, w, i_w, \lambda\right] : \left(D, \text{id}_D, f^1 \circ q^1\right) \Longrightarrow \left(D, \text{id}_D, f^2 \circ q^2\right). \quad (5.12)$$

Now we apply condition (i3) for the set of data  $(\overline{D}, q^1 \circ w, q^2 \circ w, \lambda)$ . Then there are an object  $E$ , a morphism  $v : E \rightarrow \overline{D}$  in  $\mathbf{W}$ , a morphism  $q : E \rightarrow C$  and a pair of invertible 2-morphisms

$$\lambda^m : q^m \circ w \circ v \Longrightarrow p^m \circ q \quad \text{for } m = 1, 2,$$

in  $\mathcal{C}$ , such that

$$\left(\omega * i_q\right) \odot \left(i_{f^1} * \lambda^1\right) = \left(i_{f^2} * \lambda^2\right) \odot \left(\lambda * i_v\right). \quad (5.13)$$

Now we consider the morphism  $s := (E, w \circ v, q) : D \rightarrow C$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ ; moreover, for each  $m = 1, 2$  we consider the invertible 2-morphism

$$\Lambda^m : \left(D, \text{id}_D, q^m\right) \Longrightarrow \left(C, \text{id}_C, p^m\right) \circ s = \left(E, w \circ v, p^m \circ q\right),$$

represented by the data in the internal part of the following diagram

$$\begin{array}{ccccc}
 & & D & & \\
 & \swarrow \text{id}_D & & \searrow q^m & \\
 D & & & & B^m \\
 & \swarrow w \circ v & \uparrow w \circ v & \searrow \lambda^m & \\
 & & E & & \\
 & \swarrow w \circ v & \downarrow \text{id}_E & \searrow p^m \circ q & \\
 & & E & & 
 \end{array} \quad (5.14)$$

By Lemma A.4 we have:

$$\Omega * i_s \stackrel{(0.3)}{=} \left[C, \text{id}_C, \text{id}_C, i_{\text{id}_C}, \omega\right] * i_{(E, w \circ v, q)} = \left[E, \text{id}_E, \text{id}_E, i_{w \circ v}, \omega * i_q\right]; \quad (5.15)$$

moreover, by Lemma A.5 we have the following formula for each  $m = 1, 2$ :

$$\begin{aligned}
 & i_{(B^m, \text{id}_{B^m}, f^m)} * \Lambda^m \stackrel{(5.14)}{=} i_{(B^m, \text{id}_{B^m}, f^m)} * \left[E, w \circ v, \text{id}_E, i_{w \circ v}, \lambda^m\right] = \\
 & = \left[E, w \circ v, \text{id}_E, i_{w \circ v}, i_{f^m} * \lambda^m\right] : \left(D, \text{id}_D, f^m \circ q^m\right) \Longrightarrow \left(E, w \circ v, f^m \circ p^m \circ q\right).
 \end{aligned} \quad (5.16)$$

Therefore, using (5.15), (5.16) for  $m = 1$  and Lemma A.3, we get:

$$\begin{aligned} & \left( \Omega * i_s \right) \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), s} \odot \left( i_{(B^1, \text{id}_{B^1}, f^1)} * \Lambda^1 \right) = \\ & = \left[ E, \text{id}_E, \text{id}_E, i_{w \circ v}, \omega * i_q \right] \odot i_{(E, w \circ v, f^1 \circ p^1 \circ q)} \odot \left[ E, w \circ v, \text{id}_E, i_{w \circ v}, i_{f^1} * \lambda^1 \right] = \\ & = \left[ E, w \circ v, \text{id}_E, i_{w \circ v}, \left( \omega * i_q \right) \odot \left( i_{f^1} * \lambda^1 \right) \right]. \end{aligned} \quad (5.17)$$

Using (5.16) for  $m = 2$ , (5.12) and Lemma A.3, we have

$$\begin{aligned} & \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), s} \odot \left( i_{(B^2, \text{id}_{B^2}, f^2)} * \Lambda^2 \right) \odot \Lambda = \\ & = i_{(E, w \circ v, f^2 \circ p^2 \circ q)} \odot \left[ E, w \circ v, \text{id}_E, i_{w \circ v}, i_{f^2} * \lambda^2 \right] \odot \left[ E, w \circ v, w \circ v, i_{w \circ v}, \lambda * i_v \right] = \\ & = \left[ E, w \circ v, \text{id}_E, i_{w \circ v}, \left( i_{f^2} * \lambda^2 \right) \odot \left( \lambda * i_v \right) \right]. \end{aligned} \quad (5.18)$$

Then using (5.13) we get that (5.17) and (5.18) coincide. So we conclude that condition **B1**( $D, (D, \text{id}_D, q^1), (D, \text{id}_D, q^2)$ ) holds for diagram (0.3) in  $\mathcal{C} [\mathbf{W}^{-1}]$ , i.e. property (i2) is verified.  $\square$

## 6. CONDITION A2 IN A BICATEGORY OF FRACTIONS

**Lemma 6.1.** *Let us fix the same notations of Lemma 5.1. Then the following facts are equivalent:*

- (ii1) *for any object  $D$ , condition **A2**( $D$ ) holds for diagram (0.3) in  $\mathcal{C} [\mathbf{W}^{-1}]$ ;*
- (ii2) *for any object  $D$  and for any pair of morphisms  $t, t' : D \rightarrow C$  in  $\mathcal{C}$ , condition **B2**( $D, (D, \text{id}_D, t), (D, \text{id}_D, t')$ ) holds for diagram (0.3).*

The proof follows the same lines of the proof of Lemma 5.1, using Proposition 2.11 instead of Proposition 2.10, so we omit the details.

**Lemma 6.2.** *Let us fix the same notations of Lemma 5.1. Then the following facts are equivalent:*

- (ii2) *for any object  $D$  and for any pair of morphisms  $t, t' : D \rightarrow C$  in  $\mathcal{C}$ , condition **B2**( $D, (D, \text{id}_D, t), (D, \text{id}_D, t')$ ) holds for diagram (0.3);*
- (ii3) *for any object  $D$ , the following 2 conditions hold:*
  - (b) *given any pair of morphisms  $t, t' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\gamma^m : p^m \circ t \Rightarrow p^m \circ t'$  for  $m = 1, 2$  in  $\mathcal{C}$ , such that*

$$\begin{aligned} & \theta_{f^2, p^2, t'}^{-1} \odot \left( \omega * i_{t'} \right) \odot \theta_{f^1, p^1, t'} \odot \left( i_{f^1} * \gamma^1 \right) = \\ & = \left( i_{f^2} * \gamma^2 \right) \odot \theta_{f^2, p^2, t}^{-1} \odot \left( \omega * i_t \right) \odot \theta_{f^1, p^1, t}, \end{aligned} \quad (6.1)$$

*there are an object  $F$ , a morphism  $u : F \rightarrow D$  in  $\mathbf{W}$  and an invertible 2-morphism  $\gamma : t \circ u \Rightarrow t' \circ u$  in  $\mathcal{C}$ , such that*

$$\theta_{p^m, t', u} \odot \left( i_{p^m} * \gamma \right) = \left( \gamma^m * i_u \right) \odot \theta_{p^m, t, u} \quad \text{for } m = 1, 2; \quad (6.2)$$

- (c) *given any set of data  $(t, t', \gamma^1, \gamma^2, F, u, \gamma)$  as in (b), if there is another choice of data  $\tilde{F}, \tilde{u} : \tilde{F} \rightarrow D$  in  $\mathbf{W}$  and  $\tilde{\gamma} : t \circ \tilde{u} \Rightarrow t' \circ \tilde{u}$  invertible, such that*

$$\theta_{p^m, t', \tilde{u}} \odot \left( i_{p^m} * \tilde{\gamma} \right) = \left( \gamma^m * i_{\tilde{u}} \right) \odot \theta_{p^m, t, \tilde{u}} \quad \text{for } m = 1, 2, \quad (6.3)$$

then there are an object  $G$ , a morphism  $z : G \rightarrow F$  in  $\mathbf{W}$ , a morphism  $\tilde{z} : G \rightarrow \tilde{F}$  and an invertible 2-morphism  $\mu : u \circ z \Rightarrow \tilde{u} \circ \tilde{z}$ , such that

$$\begin{aligned} & \theta_{t', \tilde{u}, \tilde{z}} \odot (i_{t'} * \mu) \odot \theta_{t', u, z}^{-1} \odot (\gamma * i_z) = \\ & = (\tilde{\gamma} * i_{\tilde{z}}) \odot \theta_{t, \tilde{u}, \tilde{z}} \odot (i_t * \mu) \odot \theta_{t, u, z}^{-1}. \end{aligned} \quad (6.4)$$

*Proof.* Again, we give a complete proof in the case when  $\mathcal{C}$  is a 2-category. Let us suppose that (ii2) holds, let us fix any object  $D$  and let us prove that (b) holds. So let us fix any tuple  $(t, t', \gamma^1, \gamma^2)$  as in (b), such that (6.1) is satisfied. Then for each  $m = 1, 2$  we define an invertible 2-morphism  $\Gamma^m$  from

$$(C, \text{id}_C, p^m) \circ (D, \text{id}_D, t) = (D, \text{id}_D, p^m \circ t)$$

to

$$(C, \text{id}_C, p^m) \circ (D, \text{id}_D, t') = (D, \text{id}_D, p^m \circ t')$$

in  $\mathcal{C}[\mathbf{W}^{-1}]$  as the 2-morphism represented by the following diagram:

$$\begin{array}{ccccc} & & D & & \\ & \swarrow \text{id}_D & \uparrow \text{id}_D & \searrow p^m \circ t & \\ D & & D & & B^m. \\ & \nwarrow \text{id}_D & \downarrow \text{id}_D & \swarrow p^m \circ t' & \\ & & D & & \end{array} \quad (6.5)$$

Using (0.3) and (6.5) together with Lemmas A.3, A.4 and A.5, we have

$$\begin{aligned} & \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (D, \text{id}_D, t')}^{-1} \odot \left( \Omega * i_{(D, \text{id}_D, t')} \right) \odot \\ & \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (D, \text{id}_D, t')} \odot \left( i_{(B^1, \text{id}_{B^1}, f^1)} * \Gamma^1 \right) = \\ & = i_{(D, \text{id}_D, f^2 \circ p^2 \circ t')} \odot \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_{t'} \right] \odot \\ & \odot i_{(D, \text{id}_D, f^1 \circ p^1 \circ t')} \odot \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, i_{f^1} * \gamma^1 \right] = \\ & = \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \left( \omega * i_{t'} \right) \odot \left( i_{f^1} * \gamma^1 \right) \right] \stackrel{(6.1)}{=} \\ & \stackrel{(6.1)}{=} \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \left( i_{f^2} * \gamma^2 \right) \odot \left( \omega * i_t \right) \right] = \\ & = \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, i_{f^2} * \gamma^2 \right] \odot i_{(D, \text{id}_D, f^2 \circ p^2 \circ t')} \odot \\ & \odot \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_t \right] \odot i_{(D, \text{id}_D, f^1 \circ p^1 \circ t')} = \\ & = \left( i_{(B^2, \text{id}_{B^2}, f^2)} * \Gamma^2 \right) \odot \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (D, \text{id}_D, t')}^{-1} \odot \\ & \odot \left( \Omega * i_{(D, \text{id}_D, t)} \right) \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (D, \text{id}_D, t)}. \end{aligned} \quad (6.6)$$

Since we are assuming (ii2), then (6.6) implies that there is a unique invertible 2-morphism  $\Gamma : (D, \text{id}_D, t) \Rightarrow (D, \text{id}_D, t')$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , such that

$$\Gamma^m = i_{(C, \text{id}_C, p^m)} * \Gamma \quad \text{for } m = 1, 2. \quad (6.7)$$

By [T1, Lemma 6.1] for  $\alpha := i_{\text{id}_D}$  and  $\Gamma$ , there are an object  $T$ , a morphism  $q : T \rightarrow D$  in  $\mathbf{W}$  and a 2-morphism  $\eta : t \circ q \Rightarrow t' \circ q$ , such that  $\Gamma = [T, q, q, i_q, \eta]$ . Since  $\Gamma$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ , then by [T1, Proposition 0.8] we can assume that  $\eta$  is invertible. Then by Lemma A.5 we have:

$$\begin{aligned} [T, q, q, i_q, \gamma^1 * i_q] &= [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \gamma^1] \stackrel{(6.5)}{=} \\ &\stackrel{(6.5)}{=} \Gamma^1 \stackrel{(6.7)}{=} i_{(C, \text{id}_C, p^1)} * \Gamma = [T, q, q, i_q, i_{p^1} * \eta]. \end{aligned}$$

By [T1, Proposition 0.7], the previous identity implies that there are an object  $R$  and a morphism  $x^1 : R \rightarrow T$ , such that  $q \circ x^1$  belongs to  $\mathbf{W}$  and such that

$$\gamma^1 * i_{q \circ x^1} = i_{p^1} * \eta * i_{x^1}. \quad (6.8)$$

By Lemma A.5 we have:

$$\begin{aligned} [R, q \circ x^1, q \circ x^1, i_{q \circ x^1}, \gamma^2 * i_{q \circ x^1}] &= [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \gamma^2] \stackrel{(6.5)}{=} \Gamma^2 \stackrel{(6.7)}{=} \\ &\stackrel{(6.7)}{=} i_{(C, \text{id}_C, p^2)} * \Gamma = [T, q, q, i_q, i_{p^2} * \eta] = [R, q \circ x^1, q \circ x^1, i_{q \circ x^1}, i_{p^2} * \eta * i_{x^1}]. \end{aligned}$$

Again by [T1, Proposition 0.7], the previous identity implies that there are an object  $F$  and a morphism  $x^2 : F \rightarrow R$ , such that  $q \circ x^1 \circ x^2$  belongs to  $\mathbf{W}$  and such that

$$\gamma^2 * i_{q \circ x^1 \circ x^2} = i_{p^2} * \eta * i_{x^1 \circ x^2}. \quad (6.9)$$

We set  $u := q \circ x^1 \circ x^2 : F \rightarrow D$  and

$$\gamma := \eta * i_{x^1 \circ x^2} : t \circ u \Longrightarrow t' \circ u. \quad (6.10)$$

Then from (6.8) and (6.9) we get that  $i_{p^m} * \gamma = \gamma^m * i_u$  for each  $m = 1, 2$ ; moreover  $\gamma$  is invertible because  $\eta$  is so by construction. So we have proved that (ii2) implies condition (b) for each object  $D$  of  $\mathcal{C}$ .

Let us also prove that (ii2) implies (c). So let us fix any set of data  $(t, t', \gamma^1, \gamma^2, F, u, \gamma)$  as in (b) and any set of data  $(\tilde{F}, \tilde{u}, \tilde{\gamma})$  as in (c). In particular, we assume that (6.2) and (6.3) hold. Then we define a pair of invertible 2-morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$  as follows:

$$\Gamma := [F, u, u, i_u, \gamma], \tilde{\Gamma} := [\tilde{F}, \tilde{u}, \tilde{u}, i_{\tilde{u}}, \tilde{\gamma}] : (D, \text{id}_D, t) \Longrightarrow (D, \text{id}_D, t').$$

Then by Lemma A.5, for each  $m = 1, 2$  we have

$$\begin{aligned} i_{(C, \text{id}_C, p^m)} * \Gamma &= [F, u, u, i_u, i_{p^m} * \gamma] \stackrel{(6.2)}{=} \\ &\stackrel{(6.2)}{=} [F, u, u, i_u, \gamma^m * i_u] = [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \gamma^m] = [\tilde{F}, \tilde{u}, \tilde{u}, i_{\tilde{u}}, \gamma^m * i_{\tilde{u}}] \stackrel{(6.3)}{=} \\ &\stackrel{(6.3)}{=} [\tilde{F}, \tilde{u}, \tilde{u}, i_{\tilde{u}}, i_{p^m} * \tilde{\gamma}] = i_{(C, \text{id}_C, p^m)} * \tilde{\Gamma}. \end{aligned}$$

Then by the uniqueness part of condition **B2**( $D, (D, \text{id}_D, t), (D, \text{id}_D, t')$ ) we conclude that  $\Gamma = \tilde{\Gamma}$ . Then by Lemma A.7 there are an object  $G$ , a morphism  $z : G \rightarrow F$  in  $\mathbf{W}$ , a morphism  $\tilde{z} : G \rightarrow \tilde{F}$  and an invertible 2-morphism  $\mu : u \circ z \Rightarrow \tilde{u} \circ \tilde{z}$ , such that

$$(i_{t'} * \mu) \odot (\gamma * i_z) \odot (i_t * \mu^{-1}) = \tilde{\gamma} * i_{\tilde{z}}.$$

Such an identity is equivalent to (6.4) (in the case when  $\mathcal{C}$  is a 2-category), so we have proved that (ii2) implies condition (c) for each object  $D$ , hence (ii3) holds.

Conversely, let us suppose that (ii3) holds and let us prove that (ii2) holds. So let us fix any object  $D$  and any pair of morphisms  $t, t' : D \rightarrow C$  in  $\mathcal{C}$ ; we have to prove that condition **B2**( $D, (D, \text{id}_D, t), (D, \text{id}_D, t')$ ) holds for diagram (0.3). In order to do that, let us fix any pair of invertible 2-morphisms

$$\Gamma^m : (C, \text{id}_C, p^m) \circ (D, \text{id}_D, t) \Longrightarrow (C, \text{id}_C, p^m) \circ (D, \text{id}_D, t') \quad \text{for } m = 1, 2$$

in  $\mathcal{C}[\mathbf{W}^{-1}]$ , such that

$$\begin{aligned} & \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (D, \text{id}_D, t')}^{-1} \odot (\Omega * i_{(D, \text{id}_D, t')}) \odot \\ & \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (D, \text{id}_D, t')} \odot (i_{(B^1, \text{id}_{B^1}, f^1)} * \Gamma^1) = \\ & = (i_{(B^2, \text{id}_{B^2}, f^2)} * \Gamma^2) \odot \Theta_{(B^2, \text{id}_{B^2}, f^2), (C, \text{id}_C, p^2), (D, \text{id}_D, t)}^{-1} \odot \\ & \odot (\Omega * i_{(D, \text{id}_D, t)}) \odot \Theta_{(B^1, \text{id}_{B^1}, f^1), (C, \text{id}_C, p^1), (D, \text{id}_D, t)}. \end{aligned} \quad (6.11)$$

By [T1, Lemma 6.1] applied to  $\alpha := i_{\text{id}_D}$  and to  $\Gamma^1$ , there are an object  $K$ , a morphism  $x^1 : K \rightarrow D$  in  $\mathbf{W}$  and a 2-morphism  $\alpha^1 : p^1 \circ t \circ x^1 \Rightarrow p^1 \circ t' \circ x^1$  in  $\mathcal{C}$ , such that

$$\Gamma^1 = [K, x^1, x^1, i_{x^1}, \alpha^1] : (D, \text{id}_D, p^1 \circ t) \Longrightarrow (D, \text{id}_D, p^1 \circ t'). \quad (6.12)$$

Since  $\Gamma^1$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ , then by [T1, Proposition 0.8] we can assume that  $\alpha^1$  is invertible in  $\mathcal{C}$ . Now we apply [T1, Lemma 6.1] to  $\alpha := i_{x^1}$  and to  $\Gamma^2$ . Then there are an object  $M$ , a morphism  $x^2 : M \rightarrow K$  such that  $x^1 \circ x^2$  belongs to  $\mathbf{W}$ , and a 2-morphism

$$\tilde{\alpha}^2 : p^2 \circ t \circ x^1 \circ x^2 \Longrightarrow p^2 \circ t' \circ x^1 \circ x^2,$$

such that

$$\Gamma^2 = [M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, \tilde{\alpha}^2]. \quad (6.13)$$

As above, we can assume that  $\tilde{\alpha}^2$  is invertible in  $\mathcal{C}$  since  $\Gamma^2$  is invertible in  $\mathcal{C}[\mathbf{W}^{-1}]$ . If we set  $\tilde{\alpha}^1 := \alpha^1 * i_{x^2}$ , then from (6.12) we get

$$\Gamma^1 = [M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, \tilde{\alpha}^1]. \quad (6.14)$$

So using Lemma A.5, for each  $m = 1, 2$  we have

$$i_{(B^m, \text{id}_{B^m}, f^m)} * \Gamma^m = [M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, i_{f^m} * \tilde{\alpha}^m]. \quad (6.15)$$

Moreover, using (0.3) and Lemma A.4, we have  $\Omega * i_{(D, \text{id}_D, t)} = [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_t]$  and analogously  $\Omega * i_{(D, \text{id}_D, t')} = [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_{t'}]$ . Using such identities together with Lemma A.3, we get that

$$\begin{aligned} & [M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, (\omega * i_{t' \circ x^1 \circ x^2}) \odot (i_{f^1} * \tilde{\alpha}^1)] = \\ & = [D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_{t'}] \odot \\ & \odot [M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, i_{f^1} * \tilde{\alpha}^1] \stackrel{(6.15)}{=} \end{aligned}$$



$$\begin{aligned}
& \stackrel{(6.15)}{=} i_{(D, \text{id}_D, f^2 \circ p^2 \circ t')} \odot \left( \Omega * i_{(D, \text{id}_D, t')} \right) \odot i_{(D, \text{id}_D, f^1 \circ p^1 \circ t')} \odot \left( i_{(B^1, \text{id}_{B^1}, f^1)} * \Gamma^1 \right) \stackrel{(6.11)}{=} \\
& \stackrel{(6.11)}{=} \left( i_{(B^2, \text{id}_{B^2}, f^2)} * \Gamma^2 \right) \odot i_{(D, \text{id}_D, f^2 \circ p^2 \circ t)} \odot \left( \Omega * i_{(D, \text{id}_D, t)} \right) \odot i_{(D, \text{id}_D, f^1 \circ p^1 \circ t)} \stackrel{(6.15)}{=} \\
& \stackrel{(6.15)}{=} \left[ M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, i_{f^2} * \tilde{\alpha}^2 \right] \odot \\
& \quad \odot \left[ D, \text{id}_D, \text{id}_D, i_{\text{id}_D}, \omega * i_t \right] = \\
& = \left[ M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, \left( i_{f^2} * \tilde{\alpha}^2 \right) \odot \left( \omega * i_{t \circ x^1 \circ x^2} \right) \right]. \quad (6.16)
\end{aligned}$$

Using (6.16) and [T1, Proposition 0.7], there are an object  $N$  and a morphism  $y : N \rightarrow M$ , such that  $x^1 \circ x^2 \circ y$  belongs to  $\mathbf{W}$  and

$$\begin{aligned}
& \left( \left( \omega * i_{t' \circ x^1 \circ x^2} \right) \odot \left( i_{f^1} * \tilde{\alpha}^1 \right) \right) * i_y = \\
& = \left( \left( i_{f^2} * \tilde{\alpha}^2 \right) \odot \left( \omega * i_{t \circ x^1 \circ x^2} \right) \right) * i_y. \quad (6.17)
\end{aligned}$$

Then we set

$$n := t \circ x^1 \circ x^2 \circ y : N \longrightarrow C, \quad n' := t' \circ x^1 \circ x^2 \circ y : N \longrightarrow C \quad (6.18)$$

and

$$\gamma^m := \tilde{\alpha}^m * i_y : p^m \circ n \Longrightarrow p^m \circ n' \quad \text{for } m = 1, 2. \quad (6.19)$$

Then (6.17) implies that:

$$\left( \omega * i_{n'} \right) \odot \left( i_{f^1} * \gamma^1 \right) = \left( i_{f^2} * \gamma^2 \right) \odot \left( \omega * i_n \right).$$

We recall that we assumed that (ii3) holds. This implies that (b) holds for  $D$  replaced by  $N$  and  $(t, t')$  replaced by  $(n, n')$ . So there are an object  $F$ , a morphism  $u : F \rightarrow N$  in  $\mathbf{W}$  and an invertible 2-morphism  $\gamma : n \circ u \Rightarrow n' \circ u$ , such that

$$i_{p^m} * \gamma = \gamma^m * i_u \quad \text{for } m = 1, 2. \quad (6.20)$$

Then we set  $a := x^1 \circ x^2 \circ y \circ u : F \rightarrow D$  (so that  $\gamma$  is defined from  $t \circ a$  to  $t' \circ a$ ) and

$$\Gamma := \left[ F, a, a, i_a, \gamma \right] : \left( D, \text{id}_D, t \right) \Longrightarrow \left( D, \text{id}_D, t' \right). \quad (6.21)$$

Then for each  $m = 1, 2$  we have:

$$i_{p^m} * \gamma \stackrel{(6.20)}{=} \gamma^m * i_u \stackrel{(6.19)}{=} \tilde{\alpha}^m * i_{y \circ u}. \quad (6.22)$$

Then using Lemma A.5, for each  $m = 1, 2$  we have

$$\begin{aligned}
i_{(C, \text{id}_C, p^m)} * \Gamma &= i_{(C, \text{id}_C, p^m)} * \left[ F, a, a, i_a, \gamma \right] = \left[ F, a, a, i_a, i_{p^m} * \gamma \right] \stackrel{(6.22)}{=} \\
& \stackrel{(6.22)}{=} \left[ F, x^1 \circ x^2 \circ y \circ u, x^1 \circ x^2 \circ y \circ u, i_{x^1 \circ x^2 \circ y \circ u}, \tilde{\alpha}^m * i_{y \circ u} \right] = \\
& = \left[ M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, \tilde{\alpha}^m \right] \stackrel{(6.13), (6.14)}{=} \Gamma^m.
\end{aligned}$$

This proves that the existence part of condition  $\mathbf{B2}(D, (D, \text{id}_D, t), (D, \text{id}_D, t'))$  is satisfied. Then we need only to prove that the 2-morphism  $\Gamma$  defined above is the unique invertible 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  such that  $i_{(C, \text{id}_C, p^m)} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . So let us suppose that there is another invertible 2-morphism  $\tilde{\Gamma} : (D, \text{id}_D, t) \Rightarrow (D, \text{id}_D, t')$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , such that  $i_{(C, \text{id}_C, p^m)} * \tilde{\Gamma} = \Gamma^m$  for each

$m = 1, 2$ . Then we apply [T1, Lemma 6.1] to  $\alpha := i_{x^1 \circ x^2 \circ y \circ u}$  and to  $\tilde{\Gamma}$ . Then there are an object  $L$ , a morphism  $b : L \rightarrow F$  such that  $x^1 \circ x^2 \circ y \circ u \circ b$  belongs to  $\mathbf{W}$ , and a 2-morphism

$$\beta : t \circ x^1 \circ x^2 \circ y \circ u \circ b \Longrightarrow t' \circ x^1 \circ x^2 \circ y \circ u \circ b,$$

such that

$$\tilde{\Gamma} = \left[ L, x^1 \circ x^2 \circ y \circ u \circ b, x^1 \circ x^2 \circ y \circ u \circ b, i_{x^1 \circ x^2 \circ y \circ u \circ b}, \beta \right]. \quad (6.23)$$

Then by Lemma A.5, for each  $m = 1, 2$  we have

$$\begin{aligned} & \left[ L, x^1 \circ x^2 \circ y \circ u \circ b, x^1 \circ x^2 \circ y \circ u \circ b, i_{x^1 \circ x^2 \circ y \circ u \circ b}, i_{p^m} * \beta \right] = \\ & = i_{(C, \text{id}_C, p^m)} * \tilde{\Gamma} = \Gamma^m \stackrel{(6.13), (6.14)}{=} \left[ M, x^1 \circ x^2, x^1 \circ x^2, i_{x^1 \circ x^2}, \tilde{\alpha}^m \right] \stackrel{(6.19)}{=} \\ & \stackrel{(6.19)}{=} \left[ L, x^1 \circ x^2 \circ y \circ u \circ b, x^1 \circ x^2 \circ y \circ u \circ b, i_{x^1 \circ x^2 \circ y \circ u \circ b}, \gamma^m * i_{u \circ b} \right]. \quad (6.24) \end{aligned}$$

Now we apply [T1, Proposition 0.7] to (6.24) for  $m = 1$ . So there are an object  $H$  and a morphism  $s : H \rightarrow L$ , such that  $x^1 \circ x^2 \circ y \circ u \circ b \circ s$  belongs to  $\mathbf{W}$  and such that

$$i_{p^1} * \beta * i_s = \gamma^1 * i_{u \circ b \circ s}. \quad (6.25)$$

Moreover, from (6.24) for  $m = 2$ , we get:

$$\begin{aligned} & \left[ H, x^1 \circ x^2 \circ y \circ u \circ b \circ s, x^1 \circ x^2 \circ y \circ u \circ b \circ s, i_{x^1 \circ x^2 \circ y \circ u \circ b \circ s}, i_{p^2} * \beta * i_s \right] = \\ & = \left[ H, x^1 \circ x^2 \circ y \circ u \circ b \circ s, x^1 \circ x^2 \circ y \circ u \circ b \circ s, i_{x^1 \circ x^2 \circ y \circ u \circ b \circ s}, \gamma^2 * i_{u \circ b \circ s} \right]. \end{aligned}$$

So again by [T1, Proposition 0.7], there are an object  $I$  and a morphism  $r : I \rightarrow H$ , such that  $x^1 \circ x^2 \circ y \circ u \circ b \circ s \circ r$  belongs to  $\mathbf{W}$  and

$$i_{p^2} * \beta * i_{s \circ r} = \gamma^2 * i_{u \circ b \circ s \circ r}. \quad (6.26)$$

Since also  $x^1 \circ x^2 \circ y$  belongs to  $\mathbf{W}$  by construction, then by Lemma A.6 there are an object  $\tilde{F}$  and a morphism  $c : \tilde{F} \rightarrow I$ , such that the morphism  $\tilde{u} := u \circ b \circ s \circ r \circ c : \tilde{F} \rightarrow N$  belongs to  $\mathbf{W}$ . Then we define  $\tilde{\gamma} := \beta * i_{s \circ r \circ c}$ , so from (6.25) and (6.26) we get that

$$i_{p^m} * \tilde{\gamma} = \gamma^m * i_{\tilde{u}} \quad \text{for } m = 1, 2. \quad (6.27)$$

We recall that we already used (b) (for the data  $(N, n, n', \gamma^1, \gamma^2)$ ) in order to get a set of data  $(F, u, \gamma)$  such that (6.20) holds. Since (6.27) holds, then we can apply (c) for the data  $(\tilde{F}, \tilde{u}, \tilde{\gamma})$ , so there are an object  $G$ , a morphism  $z : G \rightarrow F$  in  $\mathbf{W}$ , a morphism  $\tilde{z} : G \rightarrow \tilde{F}$  and an invertible 2-morphism

$$\mu : u \circ z \Longrightarrow \tilde{u} \circ \tilde{z} = u \circ b \circ s \circ r \circ c \circ \tilde{z},$$

such that

$$(i_{n'} * \mu) \odot (\gamma * i_z) = (\tilde{\gamma} * i_{\tilde{z}}) \odot (i_n * \mu).$$

If we replace  $\tilde{\gamma}$  with  $\beta * i_{s \circ r \circ c}$ , and  $n, n'$  with their definition in (6.18), then the previous identity implies that:

$$\gamma * i_z = (i_{t' \circ x^1 \circ x^2 \circ y} * \mu^{-1}) \odot (\beta * i_{s \circ r \circ c \circ \tilde{z}}) \odot (i_{t \circ x^1 \circ x^2 \circ y} * \mu). \quad (6.28)$$

So using [Pr, § 2.3] we have

$$\Gamma \stackrel{(6.21)}{=} \left[ F, a, a, i_a, \gamma \right] \stackrel{(6.28)}{=} \stackrel{(6.28)}{=} \left[ L, x^1 \circ x^2 \circ y \circ u \circ b, x^1 \circ x^2 \circ y \circ u \circ b, i_{x^1 \circ x^2 \circ y \circ u \circ b}, \beta \right] \stackrel{(6.23)}{=} \tilde{\Gamma},$$

so we have proved also the uniqueness part of  $\mathbf{B2}(D, (D, \text{id}_D, t), (D, \text{id}_D, t'))$ , i.e. we have proved that (ii3) implies (ii2).  $\square$

Therefore, we have:

*Proof of Theorem 0.2.* Given an object  $C$ , a pair of morphisms  $p^m : C \rightarrow B^m$  for  $m = 1, 2$  and an invertible 2-morphism  $\omega : f^1 \circ p^1 \Rightarrow f^2 \circ p^2$  in  $\mathcal{C}$ , the induced diagram (0.3) is a weak fiber product if and only if it satisfies conditions  $\mathbf{A1}(D)$  and  $\mathbf{A2}(D)$  for each object  $D$  of  $\mathcal{C} [\mathbf{W}^{-1}]$ , i.e. for each object  $D$  of  $\mathcal{C}$ . Using Lemmas 5.1 and 5.2, condition  $\mathbf{A1}(D)$  holds for each object  $D$  if and only if property (a) of Theorem 0.2 is satisfied for each  $D$ . Using Lemmas 6.1 and 6.2, condition  $\mathbf{A2}(D)$  holds for each object  $D$  if and only if properties (b) and (c) are satisfied for each  $D$ . This suffices to conclude.  $\square$

Moreover, we are ready to give also the following proof.

*Proof of Corollary 0.3.* As usual, for simplicity of exposition we give the proof assuming that  $\mathcal{C}$  is a 2-category. Let us fix any object  $D$  in  $\mathcal{C}$  and let us start by proving that condition (a) of Theorem 0.2 is satisfied. So let us suppose that we have fixed any pair of morphisms  $q^m : D \rightarrow B^m$  for  $m = 1, 2$  and any invertible 2-morphism  $\lambda : f^1 \circ q^1 \Rightarrow f^2 \circ q^2$  in  $\mathcal{C}$ . By hypothesis, (0.4) is a weak fiber product in the bicategory  $\mathcal{C}$ ; so by  $\mathbf{A1}(D)$  there are a morphism  $q : D \rightarrow C$  and a pair of invertible 2-morphisms  $\lambda^m : q^m \Rightarrow p^m \circ q$  for  $m = 1, 2$ , such that

$$(\omega * i_q) \odot (i_{f^1} * \lambda^1) = (i_{f^2} * \lambda^2) \odot \lambda.$$

Then condition (a) holds if we set  $E := D$  and  $v := \text{id}_D$ . Now let us prove (b), so let us fix any pair of morphisms  $t, t' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\gamma^m : p^m \circ t \Rightarrow p^m \circ t'$  for  $m = 1, 2$  in  $\mathcal{C}$ , such that

$$(\omega * i_t) \odot (i_{f^1} * \gamma^1) = (i_{f^2} * \gamma^2) \odot (\omega * i_{t'}). \quad (6.29)$$

Since (0.4) is a weak fiber product in  $\mathcal{C}$ , then by  $\mathbf{A2}(D)$  there is a unique invertible 2-morphism  $\gamma : t \Rightarrow t'$ , such that

$$i_{p^m} * \gamma = \gamma^m \quad \text{for } m = 1, 2. \quad (6.30)$$

So condition (b) is satisfied if we set  $F := D$  and  $u := \text{id}_D$ .

Hence, we only need to prove condition (c). So let us fix any pair of morphisms  $t, t' : D \rightarrow C$ , any pair of invertible 2-morphisms  $\gamma^m : p^m \circ t \Rightarrow p^m \circ t'$  for  $m = 1, 2$  such that (6.29) holds, any pair of objects  $F, \tilde{F}$ , any pair of morphisms  $u : F \rightarrow D$  and  $\tilde{u} : \tilde{F} \rightarrow D$ , both in  $\mathbf{W}$ , and any pair of invertible 2-morphisms  $\gamma : t \circ u \Rightarrow t' \circ u$  and  $\tilde{\gamma} : t \circ \tilde{u} \Rightarrow t' \circ \tilde{u}$ , such that

$$i_{p^m} * \gamma = \gamma^m * i_u \quad \text{and} \quad i_{p^m} * \tilde{\gamma} = \gamma^m * i_{\tilde{u}} \quad \text{for } m = 1, 2. \quad (6.31)$$

Using axiom (BF3) there is a set of data as in the upper part of the following diagram, with  $z$  in  $\mathbf{W}$  and  $\mu$  invertible.

$$\begin{array}{ccc}
& G & \\
z \swarrow & & \searrow \tilde{z} \\
F & \xrightarrow{\mu} & D \\
\downarrow u & & \downarrow \tilde{u} \\
& D & \xleftarrow{\tilde{u}} \tilde{F}
\end{array}$$

For each  $m = 1, 2$ , we consider the invertible 2-morphism

$$\phi^m := \gamma^m * i_{u \circ z} : p^m \circ t \circ u \circ z \Longrightarrow p^m \circ t' \circ u \circ z.$$

Then using (6.29) we get

$$\left( \omega * i_{t' \circ u \circ z} \right) \odot \left( i_{f^1} * \phi^1 \right) = \left( i_{f^2} * \phi^2 \right) \odot \left( \omega * i_{t \circ u \circ z} \right). \quad (6.32)$$

From the first part of (6.31), for each  $m = 1, 2$  we have

$$i_{p^m} * \left( \gamma * i_z \right) = \phi^m. \quad (6.33)$$

Moreover, from the second part of (6.31) and interchange law, for each  $m = 1, 2$  we have:

$$\begin{aligned}
& i_{p^m} * \left( \left( i_{t'} * \mu^{-1} \right) \odot \left( \tilde{\gamma} * i_{\tilde{z}} \right) \odot \left( i_t * \mu \right) \right) = \\
& = \left( i_{p^m \circ t'} * \mu^{-1} \right) \odot \left( \gamma^m * i_{\tilde{u} \circ \tilde{z}} \right) \odot \left( i_{p^m \circ t} * \mu \right) = \gamma^m * i_{u \circ z} = \phi^m. \quad (6.34)
\end{aligned}$$

Since (0.4) is a weak fiber product in  $\mathcal{C}$ , then using condition **A2**( $G$ ) together with (6.32), (6.33) and (6.34), we get that

$$\gamma * i_z = \left( i_{t'} * \mu^{-1} \right) \odot \left( \tilde{\gamma} * i_{\tilde{z}} \right) \odot \left( i_t * \mu \right).$$

This equation is equivalent to (0.9) when  $\mathcal{C}$  is a 2-category, so condition (c) holds for each object  $D$ .

So Theorem 0.2 implies that diagram (0.3) is a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ . Then by Theorem 0.1 for every pair of morphisms in  $\mathbf{W}$  of the form  $w^1 : B^1 \rightarrow \overline{B}^1$  and  $w^2 : B^2 \rightarrow \overline{B}^2$ , the pair of morphisms  $(B^1, w^1, f^1)$  and  $(B^2, w^2, f^2)$  has a weak fiber product in  $\mathcal{C}[\mathbf{W}^{-1}]$ .  $\square$

## 7. (STRONG) PULLBACKS IN CATEGORIES OF FRACTIONS

As we mentioned in the Introduction, the right bicalculus of fractions developed by Dorette Pronk generalizes the usual right calculus of fractions described by Pierre Gabriel and Michel Zisman (see [GZ]). We refer to Appendix B for more details on axioms (CF) for a right calculus of fractions and on the construction of a right category of fractions. Then we can give a proof of the last result mentioned in the Introduction.

*Proof of Proposition 0.4.* We recall (see Proposition B.1) that given any category  $\mathcal{C}$  and any class  $\mathbf{W}$  of morphisms in it, the pair  $(\mathcal{C}, \mathbf{W})$  satisfies the axioms for a right calculus of fractions if and only if the pair  $(\mathcal{C}^2, \mathbf{W})$  satisfies the axioms for a right bicalculus of fractions (here given any category  $\mathcal{C}$ , we denote by  $\mathcal{C}^2$  the associated trivial bicategory). If any of such conditions is satisfied, then there is an equivalence of bicategories

$$\mathcal{E} : \mathcal{C}^2[\mathbf{W}^{-1}] \longrightarrow (\mathcal{C}[\mathbf{W}^{-1}])^2$$

given on objects as the identity and on any morphism  $(A', w, f) : A \rightarrow B$  as  $\mathcal{E}(A', w, f) := [A', w, f]$ .

If we fix any pair of morphisms  $g^1 : B^1 \rightarrow A$  and  $g^2 : B^2 \rightarrow A$  in  $\mathcal{C}^2 [\mathbf{W}^{-1}]$ ; then the following facts are equivalent:

- the pair  $(g^1, g^2)$  has a weak fiber product in the bicategory  $\mathcal{C}^2 [\mathbf{W}^{-1}]$ ;
- the pair  $(\mathcal{E}(g^1), \mathcal{E}(g^2))$  has a weak fiber product in the bicategory  $(\mathcal{C} [\mathbf{W}^{-1}])^2$ ;
- the pair  $(\mathcal{E}(g^1), \mathcal{E}(g^2))$  has a (strong) fiber product in the category  $\mathcal{C} [\mathbf{W}^{-1}]$ .

The equivalence of the first 2 conditions follows from the existence of  $\mathcal{E}$  and Proposition 3.2; the equivalence of the last 2 conditions is simply Remark 2.3.

Since  $\mathcal{C}$  is a category, considered as a trivial bicategory, then the 2-morphism  $\omega$  appearing in Theorems 0.1 is a 2-identity, i.e.  $f^1 \circ p^1 = f^2 \circ p^2$ . Using the previous set of equivalent conditions and the equivalence of (i) and (ii) in Theorem 0.1, this implies at once the equivalence of (iii) and (iv) in Proposition 0.4.

Now also all the 2-morphisms appearing in Theorem 0.2 are 2-identities, hence saying that there is a 2-morphism joining a pair of morphisms is equivalent to saying that such a pair of morphisms coincide. Moreover, all the identities from (0.5) to (0.9) are simply of the form  $i_a = i_a$  for some morphism  $a$  in  $\mathcal{C}$ , hence they are automatically satisfied, so they will be ignored in the following lines. So let us fix any set of data  $(C, p^1, p^2)$  such that  $f^1 \circ p^1 = f^2 \circ p^2$ . Then the following facts are equivalent:

- (1) for any object  $D$ , condition (a) of Theorem 0.2 holds;
- (2) condition (d) of Proposition 0.4 holds.

Moreover, also the following facts are equivalent:

- (3) for any object  $D$ , condition (b) of Theorem 0.2 holds;
- (4) given any object  $R$  and any pair of morphisms  $r, r' : R \rightarrow C$  such that  $p^m \circ r = p^m \circ r'$  for each  $m = 1, 2$ , there are an object  $S$  and a morphism  $h : S \rightarrow R$  in  $\mathbf{W}$ , such that  $r \circ h = r' \circ h$ .

In addition, the following facts are equivalent:

- (5) for any object  $D$ , condition (c) of Theorem 0.2 holds;
- (6) given any set of data  $(R, r, r', S, h)$  as in (4), any object  $\tilde{S}$  and any morphism  $\tilde{h} : \tilde{S} \rightarrow R$  in  $\mathbf{W}$  such that  $r \circ \tilde{h} = r' \circ \tilde{h}$ , there are an object  $M$  and a pair of morphisms  $d : M \rightarrow S$  in  $\mathbf{W}$  and  $\tilde{d} : M \rightarrow \tilde{S}$ , such that  $h \circ d = \tilde{h} \circ \tilde{d}$ .

Using condition (CF3) (with  $f := h$  and  $w := \tilde{h}$ ), there are an object  $M$  and a pair of morphisms  $d : M \rightarrow S$  in  $\mathbf{W}$  and  $\tilde{d} : M \rightarrow \tilde{S}$ , such that  $h \circ d = \tilde{h} \circ \tilde{d}$ . So (6) is automatically satisfied, hence also (5) is true. So using Theorem 0.2 we have that (0.11) is a (strong) fiber product if and only if conditions (2) and (4) holds.

Now we claim that if we assume (2), then (4) is equivalent to:

- (7) condition (e) of Proposition 0.4 holds.

So first of all, let us assume (2) and (4) and let us prove that (7) holds. So let us fix any set of data  $(D, q^1, q^2, E, v, q, \tilde{E}, \tilde{v}, \tilde{q})$  as in Proposition 0.4 (d) and (e) (in particular, such that  $q^m \circ v = p^m \circ q$  and  $q^m \circ \tilde{v} = p^m \circ \tilde{q}$  for each  $m = 1, 2$ ). Let us apply axiom (CF3) to the pair of morphisms  $(v, \tilde{v})$ . Then there are an object  $R$ , a pair of morphisms  $w : R \rightarrow E$  in  $\mathbf{W}$  and  $\tilde{w} : R \rightarrow \tilde{E}$ , such that  $v \circ w = \tilde{v} \circ \tilde{w}$ . Then we set

$$r := q \circ w : R \longrightarrow C \quad \text{and} \quad r' := \tilde{q} \circ \tilde{w} : R \longrightarrow C.$$

Then for each  $m = 1, 2$  we have:

$$p^m \circ r = p^m \circ q \circ w = q^m \circ v \circ w = q^m \circ \tilde{v} \circ \tilde{w} = p^m \circ \tilde{q} \circ \tilde{w} = p^m \circ r'.$$

So by (4) there are an object  $F$  and a morphism  $z : F \rightarrow R$  in  $\mathbf{W}$ , such that  $r \circ z = r' \circ z$ . We set  $u := w \circ z$  and  $\tilde{u} := \tilde{w} \circ z$ . So we have

$$v \circ u = v \circ w \circ z = \tilde{v} \circ \tilde{w} \circ z = \tilde{v} \circ \tilde{u}$$

and

$$q \circ u = q \circ w \circ z = r \circ z = r' \circ z = \tilde{q} \circ \tilde{w} \circ z = \tilde{q} \circ \tilde{u},$$

so (7) is satisfied.

Conversely, let us suppose that (2) and (7) hold and let us prove (4). So let us fix any object  $R$  and any pair of morphisms  $r, r' : R \rightarrow C$ , such that  $p^m \circ r = p^m \circ r'$  for each  $m = 1, 2$ . Then let us set  $q^m := p^m \circ r : R \rightarrow B^m$  for  $m = 1, 2$ . Then condition (d) of Proposition 0.4 is satisfied if we choose  $E := R$ ,  $v := \text{id}_R$  and  $q := r$ . Moreover, (d) is also satisfied by choosing  $\tilde{E} := R$ ,  $\tilde{v} := \text{id}_R$  and  $\tilde{q} := r'$ . Hence, by (7) there are an object  $S$  and a pair of morphisms  $h, \tilde{h} : S \rightarrow R$  in  $\mathbf{W}$ , such that  $\text{id}_R \circ h = \text{id}_R \circ \tilde{h}$  and  $r \circ h = r' \circ \tilde{h}$ . This implies that  $r \circ h = r' \circ h$ , so (4) holds. So (0.11) is a (strong) fiber product if and only if (2) and (4) hold, if and only if (2) and (7) hold. This is sufficient to conclude.  $\square$

Proposition 0.4 can also be obtained directly working in the category of fractions, i.e. not relying on Theorems 0.1 and 0.2. This gives a check of correctness for the mentioned 2 Theorems.

## APPENDIX A. BICATEGORIES OF FRACTIONS

In this and in the next appendix we will recall some basic notions about categories and bicategories of fractions and we will list a series of lemmas used often in this paper.

Let us fix any bicategory  $\mathcal{C}$  (with the notations already mentioned in § 1) and any class  $\mathbf{W}$  of morphisms in it. We recall that  $\mathbf{W}$  is said to *admit a right bicalculus of fractions* if and only if the following conditions are satisfied (see [Pr, § 2.1]):

- (BF1) for every object  $A$  of  $\mathcal{C}$ , the 1-identity  $\text{id}_A$  belongs to  $\mathbf{W}$ ;
- (BF2)  $\mathbf{W}$  is closed under compositions;
- (BF3) for every morphism  $w : A \rightarrow B$  in  $\mathbf{W}$  and for every morphism  $f : C \rightarrow B$ , there are an object  $D$ , a morphism  $w' : D \rightarrow C$  in  $\mathbf{W}$ , a morphism  $f' : D \rightarrow A$  and an invertible 2-morphism  $\alpha : f \circ w' \Rightarrow w \circ f'$ ;
- (BF4) (a) given any morphism  $w : B \rightarrow A$  in  $\mathbf{W}$ , any pair of morphisms  $f^1, f^2 : C \rightarrow B$  and any 2-morphism  $\alpha : w \circ f^1 \Rightarrow w \circ f^2$ , there are an object  $D$ , a morphism  $v : D \rightarrow C$  in  $\mathbf{W}$  and a 2-morphism  $\beta : f^1 \circ v \Rightarrow f^2 \circ v$ , such that

$$\alpha * i_v = \theta_{w, f^2, v} \circ (i_w * \beta) \circ \theta_{w, f^1, v}^{-1};$$

- (b) if  $\alpha$  in (a) is invertible, then so is  $\beta$ ;
- (c) if  $(D', v' : D' \rightarrow C, \beta' : f^1 \circ v' \Rightarrow f^2 \circ v')$  is another triple with the same properties of  $(D, v, \beta)$  in (a), then there are an object  $E$ , a pair of morphisms  $u : E \rightarrow D$ ,  $u' : E \rightarrow D'$  and an invertible 2-morphism  $\zeta : v \circ u \Rightarrow v' \circ u'$ , such that  $v \circ u$  belongs to  $\mathbf{W}$  and

$$\begin{aligned} & \theta_{f^2, v', u'}^{-1} \odot (\beta' * i_{u'}) \odot \theta_{f^1, v', u'} \odot (i_{f^1} * \zeta) = \\ & = (i_{f^2} * \zeta) \odot \theta_{f^2, v, u}^{-1} \odot (\beta * i_u) \odot \theta_{f^1, v, u}. \end{aligned}$$

(BF5) if  $w : A \rightarrow B$  is a morphism in  $\mathbf{W}$ ,  $v : A \rightarrow B$  is any morphism and if there exists an invertible 2-morphism  $\alpha : v \Rightarrow w$ , then also  $v$  belongs to  $\mathbf{W}$ .

We recall the following fundamental result:

**Theorem A.1.** [Pr, Theorem 21] *Given any pair  $(\mathcal{C}, \mathbf{W})$  satisfying conditions (BF), there are a bicategory  $\mathcal{C}[\mathbf{W}^{-1}]$  (called (right) bicategory of fractions) and a pseudofunctor  $\mathcal{U}_{\mathbf{W}} : \mathcal{C} \rightarrow \mathcal{C}[\mathbf{W}^{-1}]$  that sends each element of  $\mathbf{W}$  to an internal equivalence and that is universal with respect to such property.*

In the notations of [Pr],  $\mathcal{U}_{\mathbf{W}}$  is called *bifunctor*, but this notation is no more in use; for the precise meaning of “universal” above, we refer directly to [Pr].

**Remark A.2.** In [Pr] the theorem above is stated with (BF1) replaced by the slightly stronger hypothesis

(BF1)' all the internal equivalences of  $\mathcal{C}$  are in  $\mathbf{W}$ .

By looking carefully at the proofs in [Pr], it is easy to see that the only part of axiom (BF1)' that is really used in all the computations is (BF1), so we are allowed to state the theorem of [Pr] under such less restrictive hypothesis.

In order to describe explicitly  $\mathcal{C}[\mathbf{W}^{-1}]$ , one has to make some choices as below. By [Pr, Theorem 21], *different choices will give equivalent bicategories of fractions* where objects, 1-morphisms and 2-morphisms are the same, but compositions of 1-morphisms and 2-morphisms are (possibly) different.

**A.1. Choices in a bicategory of fractions.** Following [Pr, § 2.2 and 2.3] in order to construct a bicategory of fractions, we have to fix a set of choices as follows:

$C(\mathbf{W})$ : for every set of data in  $\mathcal{C}$  as follows

$$A' \xrightarrow{f} B \xleftarrow{v} B' \tag{A.1}$$

with  $v$  in  $\mathbf{W}$ , using (BF3) we *choose* an object  $A''$ , a pair of morphisms  $v' : A'' \rightarrow A'$  in  $\mathbf{W}$  and  $f' : A'' \rightarrow B'$  and an invertible 2-morphism  $\rho : f \circ v' \Rightarrow v \circ f'$  in  $\mathcal{C}$ .

The choices using (BF3) in general are not unique; following [Pr, § 2.2] we have only to impose the following conditions:

- (C1) whenever (A.1) is such that  $B = A'$  and  $f = \text{id}_B$ , then we choose the data of  $C(\mathbf{W})$  to be given by  $A'' := B'$ ,  $f' := \text{id}_B$ ,  $v' := v$  and  $\rho := \pi_v^{-1} \odot v_v$ ;
- (C2) whenever (A.1) is such that  $B = B'$  and  $v = \text{id}_B$ , then we choose the data of  $C(\mathbf{W})$  to be given by  $A'' := A'$ ,  $f' := f$ ,  $v' := \text{id}_{A'}$  and  $\rho := v_f^{-1} \odot \pi_f$ .

For simplicity of computations, in some of the proofs of this paper we will consider a set of choices  $C(\mathbf{W})$  satisfying also the following additional condition:

- (C3) whenever (A.1) is such that  $A' = B'$  and  $f = v$  (with  $v$  in  $\mathbf{W}$ ), then we choose the data of  $C(\mathbf{W})$  to be given by  $A'' := A'$ ,  $f' := \text{id}_{A'}$ ,  $v' := \text{id}_{A'}$  and  $\rho := i_{f \text{oid}_{A'}}$ .

Condition (C3) is not strictly necessary in order to do a right bicalculus of fractions, but it simplifies lots of the computations in the present paper. We have only to check that it is compatible with conditions (C1) and (C2) required by [Pr], but this is obvious using the axioms of a bicategory. In other terms, for each pair  $(\mathcal{C}, \mathbf{W})$  satisfying condition (BF3), there is always a set of choices  $C(\mathbf{W})$  satisfying (C1),

(C2) and (C3). According to [Pr, § 2.3] one should also fix an additional set of choices depending on axiom (BF4), but actually such additional set of choices is not necessary (see [T1, Theorem 0.5]).

**A.2. Morphisms and 2-morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$ .** We recall (see [Pr]) that the objects of  $\mathcal{C}[\mathbf{W}^{-1}]$  are the same as those of  $\mathcal{C}$ . A morphism from  $A$  to  $B$  in  $\mathcal{C}[\mathbf{W}^{-1}]$  is any triple  $(A', w, f)$ , where  $A'$  is an object of  $\mathcal{C}$ ,  $w : A' \rightarrow A$  is an element of  $\mathbf{W}$  and  $f : A' \rightarrow B$  is a morphism of  $\mathcal{C}$ . Given any pair of morphisms from  $A$  to  $B$  and from  $B$  to  $C$  in  $\mathcal{C}[\mathbf{W}^{-1}]$  as follows

$$A \xleftarrow{w} A' \xrightarrow{f} B \quad \text{and} \quad B \xleftarrow{v} B' \xrightarrow{g} C$$

(with both  $w$  and  $v$  in  $\mathbf{W}$ ), one has to use choices  $C(\mathbf{W})$  for the pair  $(f, v)$  in order to get data  $(A'', v', f')$  as above and then define the composition of the previous morphisms of  $\mathcal{C}[\mathbf{W}^{-1}]$  as  $(A'', w \circ v', g \circ f')$ .

Given any pair of objects  $A, B$  and any pair of morphisms  $(A^m, w^m, f^m) : A \rightarrow B$  for  $m = 1, 2$ , a 2-morphism from  $(A^1, w^1, f^1)$  to  $(A^2, w^2, f^2)$  is an equivalence class of data  $(A^3, v^1, v^2, \alpha, \beta)$  in  $\mathcal{C}$  as follows

$$\begin{array}{ccccc}
 & & A^1 & & \\
 & \swarrow & \uparrow & \searrow & \\
 A & & A^3 & & B \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & A^2 & & 
 \end{array}
 \quad \begin{array}{l}
 \text{w}^1 \\
 \text{v}^1 \\
 \text{f}^1 \\
 \Downarrow \alpha \\
 \text{w}^2 \\
 \text{v}^2 \\
 \text{f}^2
 \end{array}
 \quad (A.2)$$

such that  $w^1 \circ v^1$  belongs to  $\mathbf{W}$  and such that  $\alpha$  is invertible in  $\mathcal{C}$  (in [Pr, § 2.3] it is also required that  $w^2 \circ v^2$  belongs to  $\mathbf{W}$ , but this follows from (BF5)). Any other set of data

$$\begin{array}{ccccc}
 & & A^1 & & \\
 & \swarrow & \uparrow & \searrow & \\
 A & & A'^3 & & B \\
 & \nwarrow & \downarrow & \nearrow & \\
 & & A^2 & & 
 \end{array}
 \quad \begin{array}{l}
 \text{w}^1 \\
 \text{v}'^1 \\
 \text{f}^1 \\
 \Downarrow \alpha' \\
 \text{w}^2 \\
 \text{v}'^2 \\
 \text{f}^2
 \end{array}$$

(such that  $w^1 \circ v'^1$  belongs to  $\mathbf{W}$  and  $\alpha'$  is invertible) represents the same 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  if and only if there is a set of data  $(A^4, z, z', \sigma^1, \sigma^2)$  in  $\mathcal{C}$  as in the following diagram



$$\begin{array}{ccccc}
& & A^1 & & \\
& \nearrow^{v'^1} & & \nwarrow^{v^1} & \\
A'^3 & \xleftarrow{z'} & A^4 & \xrightarrow{z} & A^3, \\
& \searrow_{v'^2} & & \swarrow_{v^2} & \\
& & A^2 & & 
\end{array}
\begin{array}{c}
\sigma^1 \\
\Rightarrow \\
\sigma^2 \\
\Leftarrow
\end{array}$$

such that  $(w^1 \circ v^1) \circ z$  belongs to  $\mathbf{W}$ ,  $\sigma^1$  and  $\sigma^2$  are both invertible,

$$\begin{aligned}
& (i_{w^2} * \sigma^2) \odot \theta_{w^2, v^2, z}^{-1} \odot (\alpha * i_z) \odot \theta_{w^1, v^1, z} \odot (i_{w^1} * \sigma^1) = \\
& = \theta_{w^2, v'^2, z'}^{-1} \odot (\alpha' * i_{z'}) \odot \theta_{w^1, v'^1, z'}
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}
& (i_{f^2} * \sigma^2) \odot \theta_{f^2, v^2, z}^{-1} \odot (\beta * i_z) \odot \theta_{f^1, v^1, z} \odot (i_{f^1} * \sigma^1) = \\
& = \theta_{f^2, v'^2, z'}^{-1} \odot (\beta' * i_{z'}) \odot \theta_{f^1, v'^1, z'}
\end{aligned} \tag{A.4}$$

(in [Pr, § 2.3] it is also required that  $(w^1 \circ v^1) \circ z'$  belongs to  $\mathbf{W}$ , but this follows from (BF5)). We denote by

$$[A^3, v^1, v^2, \alpha, \beta] : (A^1, w^1, f^1) \Longrightarrow (A^2, w^2, f^2)$$

the class of any data as in (A.2). We refer to [Pr] for the description of associators and compositions of 2-morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$ . A simplified description can be found in [T1, Propositions 0.1, 0.2, 0.3 and 0.4].

**A.3. Useful lemmas in a bicategory of fractions.** We denote by  $\Theta_\bullet$  the associators of a bicategory of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$  (constructed as in [Pr, Appendix A.2]). Then we have:

**Lemma A.3.** [T1, Corollary 2.2 and Remark 2.3] *Let us fix any triple of morphisms  $h : D \rightarrow C, g : C \rightarrow B, f : B \rightarrow A$  in  $\mathcal{C}$  and any morphism  $w : D \rightarrow D'$  in  $\mathbf{W}$ . If  $\mathcal{C}$  is a 2-category, then the associator  $\Theta_{(B, \text{id}_B, f), (C, \text{id}_C, g), (D, w, h)}$  coincides with the 2-identity of the morphism  $(D, w, f \circ g \circ h) : D' \rightarrow A$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ .*

The following is a special case of [T1, Proposition 0.3].

**Lemma A.4.** *Let us fix any morphism and any representative of a 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  as follows.*

$$\begin{array}{c}
A \xleftarrow{w} A' \xrightarrow{g} B, \quad \begin{array}{ccccc}
& & B & & \\
& \nearrow^{\text{id}_B} & & \nwarrow^{h^1} & \\
B & \xrightarrow{\text{id}_B} & B & \xrightarrow{\text{id}_B} & C. \\
& \searrow_{\text{id}_B} & & \swarrow_{h^2} & \\
& & B & & 
\end{array}
\end{array}$$

$\Downarrow i_{\text{id}_B \circ \text{id}_B}$        $\Downarrow \gamma$

Then the 2-morphism

$$\left[ B, \text{id}_B, \text{id}_B, i_{\text{id}_B}, \gamma \right] * i_{(A', w, g)} : \left( A', w \circ \text{id}_{A'}, h^1 \circ g \right) \Longrightarrow \left( A', w \circ \text{id}_{A'}, h^2 \circ g \right)$$

is equal to  $[A', \text{id}_{A'}, \text{id}_{A'}, i_{(w \circ \text{id}_{A'}) \circ \text{id}_{A'}}, \pi_{h^2 \circ g}^{-1} \odot ((\pi_{h^2} \odot \gamma \odot \pi_{h^1}^{-1}) * i_g) \odot \pi_{h^1 \circ g}]$ .

The following is a special case of [T1, Proposition 0.4].

**Lemma A.5.** *Let us fix any morphism and any representative of a 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$  in  $\mathcal{C}[\mathbf{W}^{-1}]$  as follows:*

$$\begin{array}{ccc} & A^1 & \\ w^1 \swarrow & & \searrow f^1 \\ A & & B, \\ \downarrow \alpha & A^3 & \downarrow \beta \\ w^2 \swarrow & & \searrow f^2 \\ & A^2 & \end{array}, \quad B \xleftarrow{\text{id}_B} B \xrightarrow{g} C.$$

Then the 2-morphism

$$i_{(B, \text{id}_B, g)} * [A^3, u^1, u^2, \alpha, \beta] : \left( A^1, w^1 \circ \text{id}_{A^1}, g \circ f^1 \right) \Longrightarrow \left( A^2, w^2 \circ \text{id}_{A^2}, g \circ f^2 \right)$$

is equal to  $[A^3, u^1, u^2, (\pi_{w^2}^{-1} * i_{u^2}) \odot \alpha \odot (\pi_{w^1} * i_{u^1}), \theta_{g, f^2, u^2} \odot (i_g * \beta) \odot \theta_{g, f^1, u^1}^{-1}]$ .

The following is a simple application of the definition of right saturation (see [T2, Definition 2.11]) together with [T2, Proposition 2.11].

**Lemma A.6.** *Let us fix any triple of objects  $A, B, C$ , and any pair of morphisms  $w : B \rightarrow A$  and  $v : C \rightarrow B$ , such that both  $w$  and  $w \circ v$  belong to  $\mathbf{W}$ . Then there are an object  $D$  and a morphism  $z : D \rightarrow C$ , such that  $v \circ z$  belongs to  $\mathbf{W}$ .*

**Lemma A.7.** *Let us fix any set of objects  $A, A', B$ , any morphism  $w : A' \rightarrow A$  in  $\mathbf{W}$  and any pair of morphisms  $f^1, f^2 : A' \rightarrow B$ . Let us also fix any pair of 2-morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$*

$$\Gamma, \Gamma' : \left( A', w, f^1 \right) \Longrightarrow \left( A', w, f^2 \right)$$

and let us suppose that  $\Gamma = [C, v, v, i_{w \circ v}, \gamma]$  and  $\Gamma' = [C', v', v', i_{w \circ v'}, \gamma']$  (for some choice of  $C, C', v, v', \gamma$  and  $\gamma'$ ). Then  $\Gamma = \Gamma'$  if and only if there are an object  $D$ , a morphism  $z : D \rightarrow C$  in  $\mathbf{W}$ , a morphism  $z' : D \rightarrow C'$  and an invertible 2-morphism  $\mu : v \circ z \Rightarrow v' \circ z'$ , such that

$$\begin{aligned} (i_{f^2} * \mu) \odot \theta_{f^2, v, z}^{-1} \odot (\gamma * i_z) \odot \theta_{f^1, v, z} \odot (i_{f^1} * \mu^{-1}) &= \\ &= \theta_{f^2, v', z'}^{-1} \odot (\gamma' * i_{z'}) \odot \theta_{f^1, v', z'}. \end{aligned}$$

*Proof.* The ‘‘if’’ part is a direct consequence of the definition of 2-morphism in  $\mathcal{C}[\mathbf{W}^{-1}]$ . Conversely, let us suppose that  $\Gamma = \Gamma'$ . Then there is a set of data  $(\overline{C}, t, t', \sigma^1, \sigma^2)$  in  $\mathcal{C}$  as in the following diagram

$$\begin{array}{ccccc}
& & A' & & \\
& \nearrow^{v'} & & \nwarrow^v & \\
C' & \xleftarrow{t'} & \tilde{C} & \xrightarrow{t} & C, \\
& \searrow_{v'} & & \swarrow_v & \\
& & A' & & 
\end{array}
\begin{array}{c}
\sigma^1 \\
\Rightarrow \\
\sigma^2 \\
\Leftarrow
\end{array}$$

such that  $(w \circ v) \circ t$  belongs to  $\mathbf{W}$ ,  $\sigma^1$  and  $\sigma^2$  are both invertible,

$$\begin{aligned}
& (i_w * \sigma^2) \odot \theta_{w,v,t}^{-1} \odot (i_{w \circ v} * i_t) \odot \theta_{w,v,t} \odot (i_w * \sigma^1) = \\
& = \theta_{w,v',t'}^{-1} \odot (i_{w \circ v'} * i_{t'}) \odot \theta_{w,v',t'}
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
& (i_{f_2} * \sigma^2) \odot \theta_{f_2,v,t}^{-1} \odot (\gamma * i_t) \odot \theta_{f_1,v,t} \odot (i_{f_1} * \sigma^1) = \\
& = \theta_{f_2,v',t'}^{-1} \odot (\gamma' * i_{t'}) \odot \theta_{f_1,v',t'}
\end{aligned} \tag{A.6}$$

Since  $\Gamma$  is a 2-morphism in a bicategory of fractions, then  $w \circ v$  belongs to  $\mathbf{W}$ . Since also  $(w \circ v) \circ t$  belongs to  $\mathbf{W}$ , then by Lemma A.6 there are an object  $\tilde{C}$  and a morphism  $r : \tilde{C} \rightarrow \tilde{C}$ , such that  $t \circ r$  belongs to  $\mathbf{W}$ .

From (A.5) we get that  $i_w * (\sigma^2 * i_r) = i_w * (\sigma^1 * i_r)^{-1}$ . So using [T1, Lemma 1.1] there are an object  $D$  and a morphism  $s : D \rightarrow \tilde{C}$  in  $\mathbf{W}$ , such that

$$(\sigma^2 * i_r) * i_s = (\sigma^1 * i_r)^{-1} * i_s. \tag{A.7}$$

By construction and (BF2) the morphism  $(t \circ r) \circ s$  belongs to  $\mathbf{W}$ , so by (BF5) we conclude that also the morphism  $z := t \circ (r \circ s) : D \rightarrow C$  belongs to  $\mathbf{W}$ . We define also  $z' := t' \circ (r \circ s) : D \rightarrow C'$  and

$$\mu := \theta_{v',t',r \circ s}^{-1} \odot (\sigma^2 * i_{r \circ s}) \odot \theta_{v,t,r \circ s} : v \circ z \implies v' \circ z'.$$

Then we conclude using (A.7) and (A.6).  $\square$

## APPENDIX B. CATEGORIES OF FRACTIONS

We recall (see [GZ]) that given a category  $\mathcal{C}$  and a class  $\mathbf{W}$  of morphisms in it, the pair  $(\mathcal{C}, \mathbf{W})$  is said to *admit a right calculus of fractions* if and only if the following properties hold:

- (CF1)  $\mathbf{W}$  contains all the identities of  $\mathcal{C}$ ;
- (CF2)  $\mathbf{W}$  is closed under compositions;
- (CF3) (“right Ore condition”) for every morphism  $w : A \rightarrow B$  in  $\mathbf{W}$  and any morphism  $f : C \rightarrow B$ , there are an object  $D$ , a morphism  $w' : D \rightarrow C$  in  $\mathbf{W}$  and a morphism  $f' : D \rightarrow A$ , such that  $f \circ w' = w \circ f'$ ;
- (CF4) (“right cancellability”) given any morphism  $w : B \rightarrow A$  in  $\mathbf{W}$  and any pair of morphisms  $f^1, f^2 : C \rightarrow B$  such that  $w \circ f^1 = w \circ f^2$ , there are an object  $D$  and a morphism  $v : D \rightarrow C$  in  $\mathbf{W}$ , such that  $f^1 \circ v = f^2 \circ v$ .

Given any pair  $(\mathcal{C}, \mathbf{W})$  satisfying this set of axioms, the right category of fractions  $\mathcal{C}[\mathbf{W}^{-1}]$  associated to it is described as follows. Its objects are the same as those of  $\mathcal{C}$ ; a morphism from  $A$  to  $B$  is any equivalence class  $[A', w, f]$  of a triple  $(A', w, f)$  as follows:

$$A \xleftarrow{w} A' \xrightarrow{f} B$$

with  $w$  in  $\mathbf{W}$ . Any 2 triples  $(A^1, w^1, f^1)$  and  $(A^2, w^2, f^2)$  (both defined from  $A$  to  $B$ ) are declared equivalent if and only if there are an object  $A^3$ , a pair of morphisms  $v^1 : A^3 \rightarrow A^1$  and  $v^2 : A^3 \rightarrow A^2$ , such that:

- $w^1 \circ v^1$  belongs to  $\mathbf{W}$ ;
- $w^1 \circ v^1 = w^2 \circ v^2$ ;
- $f^1 \circ v^1 = f^2 \circ v^2$ ;

(the fact that this is an equivalence relation is obvious using the axioms). The composition of morphisms in  $\mathcal{C}[\mathbf{W}^{-1}]$  is obtained by choosing representatives, then using (CF3) and then taking the class of the resulting composition. As such, composition is well-defined and associative.

Now given any category  $\mathcal{C}$ , we denote by  $\mathcal{C}^2$  the trivial bicategory obtained from  $\mathcal{C}$ , i.e. the bicategory whose objects and morphisms are the same as those of  $\mathcal{C}$  and whose 2-morphisms are only the 2-identities. Then a direct check proves the following fact.

**Proposition B.1.** *Let us fix any category  $\mathcal{C}$  and any class  $\mathbf{W}$  of morphisms in it. Then the pair  $(\mathcal{C}, \mathbf{W})$  satisfies the axioms for a right calculus of fractions if and only if the pair  $(\mathcal{C}^2, \mathbf{W})$  satisfies the axioms for a right bicalculus of fractions. If any of such conditions is satisfied, then:*

- (a) *given any pair of objects  $A, B$  in  $\mathcal{C}$  and any pair of morphisms  $(A^m, w^m, f^m) : A \rightarrow B$  for  $m = 1, 2$  in  $\mathcal{C}^2[\mathbf{W}^{-1}]$ , if  $[A^1, w^1, f^1] = [A^2, w^2, f^2]$  in  $\mathcal{C}[\mathbf{W}^{-1}]$  then there is exactly one 2-morphism  $\Gamma$  from  $(A^1, w^1, f^1)$  to  $(A^2, w^2, f^2)$  in  $\mathcal{C}^2[\mathbf{W}^{-1}]$ ; moreover such a  $\Gamma$  is invertible;*
- (b) *given any set of data  $(A, B, A^m, w^m, f^m)$  as before, if  $[A^1, w^1, f^1] \neq [A^2, w^2, f^2]$  in  $\mathcal{C}[\mathbf{W}^{-1}]$ , then there are no 2-morphisms from  $(A^1, w^1, f^1)$  to  $(A^2, w^2, f^2)$  in  $\mathcal{C}^2[\mathbf{W}^{-1}]$ ;*
- (c) *there is an equivalence of bicategories*

$$\mathcal{E} : \mathcal{C}^2[\mathbf{W}^{-1}] \longrightarrow (\mathcal{C}[\mathbf{W}^{-1}])^2$$

*given on objects as the identity, on any morphism  $(A', w, f) : A \rightarrow B$  as  $\mathcal{E}(A', w, f) := [A', w, f]$  and induced on 2-morphisms by (a) and (b).*

This makes precise the informal concept (stated in the Introduction) that the right bicalculus of fractions generalizes the right calculus of fractions.

#### APPENDIX C. PROOFS OF SOME TECHNICAL LEMMAS

*Proof of Lemma 2.6.* As usual, we give a complete proof assuming for simplicity that  $\mathcal{D}$  is a 2-category; in this case  $\bar{\Omega} = i_e * \Omega$ . Since  $e$  is an internal equivalence, then by [L, Proposition 1.5.7]  $e$  is the first component of an adjoint equivalence. So there are an internal equivalence  $d : \bar{A} \rightarrow A$  and invertible 2-morphisms  $\Delta : \text{id}_A \Longrightarrow d \circ e$  and  $\Xi : e \circ d \Longrightarrow \text{id}_{\bar{A}}$  such that

$$(\Xi * i_e) \circ (i_e * \Delta) = i_e \quad \text{and} \quad (i_d * \Xi) \circ (\Delta * i_d) = i_d. \quad (\text{C.1})$$

Let us fix any object  $D$  in  $\mathcal{D}$  and let us prove conditions **A1**( $D$ ) and **A2**( $D$ ) for diagram (2.16). First of all, we prove **A1**( $D$ ), so we fix any set of data  $(s^1, s^2, \bar{\Lambda})$  with  $\bar{\Lambda}$  invertible as follows

$$\begin{array}{ccc} D & \xrightarrow{s^1} & B^1 \\ s^2 \downarrow & \swarrow \bar{\Lambda} & \downarrow eog^1 \\ B^2 & \xrightarrow{eog^2} & \bar{A}. \end{array}$$

Then we consider the invertible 2-morphism

$$\Lambda := \left( \Delta^{-1} * i_{g^2 \circ s^2} \right) \odot \left( i_d * \bar{\Lambda} \right) \odot \left( \Delta * i_{g^1 \circ s^1} \right) : g^1 \circ s^1 \Longrightarrow g^2 \circ s^2. \quad (\text{C.2})$$

Since (2.1) satisfies condition **A1**( $D$ ), then there are a morphism  $s : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that

$$\left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) = \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda. \quad (\text{C.3})$$

Now by interchange law we have

$$\begin{aligned} i_e * \Lambda &\stackrel{(\text{C.2})}{=} \left( i_e * \Delta^{-1} * i_{g^2 \circ s^2} \right) \odot \left( i_{eod} * \bar{\Lambda} \right) \odot \left( i_e * \Delta * i_{g^1 \circ s^1} \right) \stackrel{(\text{C.1})}{=} \\ &\stackrel{(\text{C.1})}{=} \left( \Xi * i_{eog^2 \circ s^2} \right) \odot \left( i_{eod} * \bar{\Lambda} \right) \odot \left( \Xi^{-1} * i_{eog^1 \circ s^1} \right) = \bar{\Lambda}, \end{aligned} \quad (\text{C.4})$$

so

$$\begin{aligned} \left( \bar{\Omega} * i_s \right) \odot \left( i_{eog^1} * \Lambda^1 \right) &= i_e * \left( \left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) \right) \stackrel{(\text{C.3})}{=} \\ &\stackrel{(\text{C.3})}{=} \left( i_{eog^2} * \Lambda^2 \right) \odot \left( i_e * \Lambda \right) \stackrel{(\text{C.4})}{=} \left( i_{eog^2} * \Lambda^2 \right) \odot \bar{\Lambda}. \end{aligned}$$

This proves that condition **A1**( $D$ ) holds for diagram (2.16). Let us also prove condition **A2**( $D$ ), so let us fix any pair of morphisms  $t, t' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\Gamma^m : r^m \circ t \Rightarrow r^m \circ t'$  for  $m = 1, 2$ , such that

$$\left( \bar{\Omega} * i_{t'} \right) \odot \left( i_{eog^1} * \Gamma^1 \right) = \left( i_{eog^2} * \Gamma^2 \right) \odot \left( \bar{\Omega} * i_t \right). \quad (\text{C.5})$$

Then by interchange law we have

$$\begin{aligned} &\left( \Omega * i_{t'} \right) \odot \left( i_{g^1} * \Gamma^1 \right) = \\ &= \left( \Delta^{-1} * i_{g^2 \circ r^2 \circ t'} \right) \odot \left\{ i_{doe} * \left[ \left( \Omega * i_{t'} \right) \odot \left( i_{g^1} * \Gamma^1 \right) \right] \right\} \odot \left( \Delta * i_{g^1 \circ r^1 \circ t} \right) \stackrel{(\text{C.5})}{=} \\ &= \left( \Delta^{-1} * i_{g^2 \circ r^2 \circ t'} \right) \odot \left\{ i_{doe} * \left[ \left( i_{g^2} * \Gamma^2 \right) \odot \left( \Omega * i_t \right) \right] \right\} \odot \left( \Delta * i_{g^1 \circ r^1 \circ t} \right) = \\ &= \left( i_{g^2} * \Gamma^2 \right) \odot \left( \Omega * i_t \right). \end{aligned}$$

Since (2.1) satisfies condition **A2**( $D$ ), then there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . This proves that condition **A2**( $D$ ) holds also for diagram (2.16).  $\square$

*Proof of Lemma 2.7.* As usual, we give the proof in the case when  $\mathcal{D}$  is a 2-category. Since  $\Omega^1$  and  $\Omega^2$  are invertible, then the roles of  $(g^1, g^2)$  and  $(\bar{g}^1, \bar{g}^2)$  are interchangeable. Hence, we will only prove that if (2.1) is a weak fiber product, then (2.17) is

also a weak fiber product.

So let us fix any object  $D$  in  $\mathcal{D}$  and let us prove condition **A1**( $D$ ) for (2.17), so let us consider any set of data  $(s^1, s^2, \bar{\Lambda})$  in  $\mathcal{D}$  as follows, with  $\bar{\Lambda}$  invertible:

$$\begin{array}{ccc} D & \xrightarrow{s^1} & B^1 \\ s^2 \downarrow & \swarrow \bar{\Lambda} & \downarrow \bar{g}^1 \\ B^2 & \xrightarrow{\bar{g}^2} & A. \end{array}$$

Then we define an invertible 2-morphism

$$\Lambda := \left( (\Omega^2)^{-1} * i_{s^2} \right) \odot \bar{\Lambda} \odot \left( \Omega^1 * i_{s^1} \right) : g^1 \circ s^1 \Longrightarrow g^2 \circ s^2. \quad (\text{C.6})$$

Since (2.1) is a weak fiber product, then by **A1**( $D$ ) for (2.1) there are a morphism  $s : D \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : s^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that:

$$\left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) = \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda. \quad (\text{C.7})$$

Therefore, by interchange law we have:

$$\begin{aligned} & \left( \bar{\Omega} * i_s \right) \odot \left( i_{\bar{g}^1} * \Lambda^1 \right) \stackrel{(2.17)}{=} \\ & \stackrel{(2.17)}{=} \left( \Omega^2 * i_{r^2 \circ s} \right) \odot \left( \Omega * i_s \right) \odot \left( (\Omega^1)^{-1} * i_{r^1 \circ s} \right) \odot \left( i_{\bar{g}^1} * \Lambda^1 \right) = \\ & = \left( \Omega^2 * i_{r^2 \circ s} \right) \odot \left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) \odot \left( (\Omega^1)^{-1} * i_{s^1} \right) \stackrel{(\text{C.7})}{=} \\ & \stackrel{(\text{C.7})}{=} \left( \Omega^2 * i_{r^2 \circ s} \right) \odot \left( i_{g^2} * \Lambda^2 \right) \odot \Lambda \odot \left( (\Omega^1)^{-1} * i_{s^1} \right) = \\ & = \left( i_{\bar{g}^2} * \Lambda^2 \right) \odot \left( \Omega^2 * i_{s^2} \right) \odot \Lambda \odot \left( (\Omega^1)^{-1} * i_{s^1} \right) \stackrel{(\text{C.6})}{=} \left( i_{\bar{g}^2} * \Lambda^2 \right) \odot \bar{\Lambda}. \end{aligned}$$

Therefore, property **A1**( $D$ ) holds for diagram (2.17). Let us prove also condition **A2**( $D$ ) for (2.17), so let us fix any pair of morphisms  $t, t' : D \rightarrow C$  and any pair of invertible 2-morphisms  $\Gamma^m : r^m \circ t \Rightarrow r^m \circ t'$  for  $m = 1, 2$ , such that:

$$\left( \bar{\Omega} * i_{t'} \right) \odot \left( i_{\bar{g}^1} * \Gamma^1 \right) = \left( i_{\bar{g}^2} * \Gamma^2 \right) \odot \left( \bar{\Omega} * i_t \right). \quad (\text{C.8})$$

By interchange law we have:

$$\begin{aligned} & \left( \Omega * i_{t'} \right) \odot \left( i_{g^1} * \Gamma^1 \right) = \\ & = \left( \Omega * i_{t'} \right) \odot \left( (\Omega^1)^{-1} * i_{r^1 \circ t'} \right) \odot \left( i_{\bar{g}^1} * \Gamma^1 \right) \odot \left( \Omega^1 * i_{r^1 \circ t} \right) \stackrel{(2.17)}{=} \\ & \stackrel{(2.17)}{=} \left( (\Omega^2)^{-1} * i_{r^2 \circ t'} \right) \odot \left( \bar{\Omega} * i_{t'} \right) \odot \left( i_{\bar{g}^1} * \Gamma^1 \right) \odot \left( \Omega^1 * i_{r^1 \circ t} \right) \stackrel{(\text{C.8})}{=} \\ & \stackrel{(\text{C.8})}{=} \left( (\Omega^2)^{-1} * i_{r^2 \circ t'} \right) \odot \left( i_{\bar{g}^2} * \Gamma^2 \right) \odot \left( \bar{\Omega} * i_t \right) \odot \left( \Omega^1 * i_{r^1 \circ t} \right) = \\ & = \left( i_{g^2} * \Gamma^2 \right) \odot \left( (\Omega^2)^{-1} * i_{r^2 \circ t} \right) \odot \left( \bar{\Omega} * i_t \right) \odot \left( \Omega^1 * i_{r^1 \circ t} \right) \stackrel{(2.17)}{=} \\ & \stackrel{(2.17)}{=} \left( i_{g^2} * \Gamma^2 \right) \odot \left( \Omega * i_t \right). \end{aligned}$$

Since (2.1) is a weak fiber product, then by **A2**( $D$ ) for (2.1) there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r^m} * \Gamma = \Gamma^m$  for each  $m = 1, 2$ . Therefore, property **A2**( $D$ ) holds also for diagram (2.17).  $\square$

*Proof of Lemma 2.9.* We give a proof in the case when  $\mathcal{D}$  is a 2-category; this implies that  $\bar{\Omega} = \Omega * i_e$ . Since  $e$  is an internal equivalence, we can choose a morphism  $d : C \rightarrow \bar{C}$  and invertible 2-morphisms  $\Delta : \text{id}_{\bar{C}} \Rightarrow d \circ e$  and  $\Xi : e \circ d \Rightarrow \text{id}_C$ , such that

$$\left( \Xi * i_e \right) \odot \left( i_e * \Delta \right) = i_e \quad \text{and} \quad \left( i_d * \Xi \right) \odot \left( \Delta * i_d \right) = i_d. \quad (\text{C.9})$$

We fix any object  $\bar{D}$  in  $\mathcal{D}$  and we prove conditions **A1**( $\bar{D}$ ) and **A2**( $\bar{D}$ ) for diagram (2.18). In order to prove the first property, let us fix any set of data  $(\bar{s}^1, \bar{s}^2, \bar{\Lambda})$  in  $\mathcal{D}$  as in the following diagram, with  $\bar{\Lambda}$  invertible:

$$\begin{array}{ccc} \bar{D} & \xrightarrow{\bar{s}^1} & B^1 \\ \bar{s}^2 \downarrow & \swarrow \bar{\Lambda} & \downarrow g^1 \\ B^2 & \xrightarrow{g^2} & A. \end{array}$$

Since (2.1) is a weak fiber product, then by **A1**( $\bar{D}$ ) for (2.1) there are a morphism  $s : \bar{D} \rightarrow C$  and a pair of invertible 2-morphisms  $\Lambda^m : \bar{s}^m \Rightarrow r^m \circ s$  for  $m = 1, 2$ , such that:

$$\left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) = \left( i_{g^2} * \Lambda^2 \right) \odot \bar{\Lambda}. \quad (\text{C.10})$$

Then we set  $\bar{s} := d \circ s : \bar{D} \rightarrow \bar{C}$ ; for each  $m = 1, 2$  we define

$$\bar{\Lambda}^m := \left( i_{r^m} * \Xi^{-1} * i_s \right) \odot \Lambda^m : \bar{s}^m \Longrightarrow r^m \circ e \circ d \circ s = r^m \circ e \circ \bar{s}. \quad (\text{C.11})$$

By definition of  $\bar{s}$  and  $\bar{\Omega}$  and interchange law, we have:

$$\begin{aligned} & \left( \bar{\Omega} * i_{\bar{s}} \right) \odot \left( i_{g^1} * \bar{\Lambda}^1 \right) \stackrel{(\text{C.11})}{=} \\ & \stackrel{(\text{C.11})}{=} \left( \Omega * i_{e \circ d \circ s} \right) \odot \left( i_{g^1 \circ r^1} * \Xi^{-1} * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) = \\ & = \left( i_{g^2 \circ r^2} * \Xi^{-1} * i_s \right) \odot \left( \Omega * i_s \right) \odot \left( i_{g^1} * \Lambda^1 \right) \stackrel{(\text{C.10})}{=} \\ & \stackrel{(\text{C.10})}{=} \left( i_{g^2 \circ r^2} * \Xi^{-1} * i_s \right) \odot \left( i_{g^2} * \Lambda^2 \right) \odot \bar{\Lambda} \stackrel{(\text{C.11})}{=} \left( i_{g^2} * \bar{\Lambda}^2 \right) \odot \bar{\Lambda}. \end{aligned}$$

Therefore diagram (2.18) satisfies property **A1**( $\bar{D}$ ).

Let us prove also property **A2**( $\bar{D}$ ) for (2.18), so let us fix any pair of morphisms  $\bar{t}, \bar{t}' : \bar{D} \rightarrow \bar{C}$  and any pair of invertible 2-morphisms  $\bar{\Gamma}^m : (r^m \circ e) \circ \bar{t} \Rightarrow (r^m \circ e) \circ \bar{t}'$  for  $m = 1, 2$ , such that

$$\left( \bar{\Omega} * i_{\bar{t}'} \right) \odot \left( i_{g^1} * \bar{\Gamma}^1 \right) = \left( i_{g^2} * \bar{\Gamma}^2 \right) \odot \left( \bar{\Omega} * i_{\bar{t}} \right). \quad (\text{C.12})$$

Let us consider the morphisms  $t := e \circ \bar{t}$  and  $t' := e \circ \bar{t}'$ , both defined from  $\bar{D}$  to  $C$ . Then by (C.12) we have  $(\Omega * i_{t'}) \odot (i_{g^1} * \bar{\Gamma}^1) = (i_{g^2} * \bar{\Gamma}^2) \odot (\Omega * i_t)$ . Since (2.1) is a weak fiber product, then by **A2**( $\bar{D}$ ) there is a unique invertible 2-morphism  $\Gamma : t \Rightarrow t'$ , such that  $i_{r^m} * \Gamma = \bar{\Gamma}^m$  for each  $m = 1, 2$ . Then we define

$$\bar{\Gamma} := \left( \Delta^{-1} * i_{\bar{t}'} \right) \odot \left( i_d * \Gamma \right) \odot \left( \Delta * i_{\bar{t}} \right) : \bar{t} \Longrightarrow \bar{t}'.$$

A direct computation using (C.9) and the interchange law shows that  $i_e * \bar{\Gamma} = \Gamma$ . Therefore

$$i_{r^m \circ e} * \bar{\Gamma} = i_{r^m} * \Gamma = \bar{\Gamma}^m \quad \text{for } m = 1, 2. \quad (\text{C.13})$$

So in order to conclude we need only to prove that  $\bar{\Gamma}$  is the unique invertible 2-morphism  $\bar{t} \Rightarrow \bar{t}'$  such that (C.13) holds. So let us fix another invertible 2-morphism  $\bar{\Gamma}' : \bar{t} \Rightarrow \bar{t}'$ , such that  $i_{r^m \circ e} * \bar{\Gamma}' = \bar{\Gamma}^m$  for each  $m = 1, 2$ . Then we have  $i_{r^m} * (i_e * \bar{\Gamma}') = \bar{\Gamma}^m$  for each  $m = 1, 2$ ; by uniqueness of  $\Gamma$  we conclude that  $i_e * \bar{\Gamma}' = \Gamma$ . So we get  $i_e * \bar{\Gamma} = \Gamma = i_e * \bar{\Gamma}'$ , hence by interchange law we have:

$$\begin{aligned} \bar{\Gamma} &= \left( \Delta^{-1} * i_{\bar{t}'} \right) \odot \left( i_{d \circ e} * \bar{\Gamma} \right) \odot \left( \Delta * i_{\bar{t}} \right) = \\ &= \left( \Delta^{-1} * i_{\bar{t}'} \right) \odot \left( i_{d \circ e} * \bar{\Gamma}' \right) \odot \left( \Delta * i_{\bar{t}} \right) = \bar{\Gamma}'. \end{aligned}$$

This suffices to conclude.  $\square$

*Proof of Lemma 3.1.* For simplicity of exposition, in this proof we assume that both  $\mathcal{A}$  and  $\mathcal{B}$  are 2-categories and that  $\mathcal{F}$  is a strict pseudofunctor between them, i.e. a 2-functor (in other terms, we assume that  $\mathcal{F}$  preserves compositions and identities). So in particular  $\Omega_{\mathcal{B}} = \mathcal{F}_2(\Omega_{\mathcal{A}})$ . In the more general case the proof is analogous: it suffices to add unitors and associators for  $\mathcal{F}$  wherever it is necessary and to use the coherence conditions on the pseudofunctor  $\mathcal{F}$ .

So let us fix any object  $D_{\mathcal{B}}$  in  $\mathcal{B}$  and let us prove conditions **A1**( $D_{\mathcal{B}}$ ) and **A2**( $D_{\mathcal{B}}$ ) for diagram (3.2). By property (X1) for  $\mathcal{F}$ , there are an object  $D_{\mathcal{A}}$  and an internal equivalence  $e_{\mathcal{B}} : \mathcal{F}_0(D_{\mathcal{A}}) \rightarrow D_{\mathcal{B}}$ . Since  $e_{\mathcal{B}}$  is an internal equivalence then there are an internal equivalence  $d_{\mathcal{B}} : D_{\mathcal{B}} \rightarrow \mathcal{F}_0(D_{\mathcal{A}})$  and a pair of invertible 2-morphisms  $\Xi_{\mathcal{B}} : e_{\mathcal{B}} \circ d_{\mathcal{B}} \Rightarrow \text{id}_{D_{\mathcal{B}}}$  and  $\Delta_{\mathcal{B}} : \text{id}_{\mathcal{F}_0(D_{\mathcal{A}})} \Rightarrow d_{\mathcal{B}} \circ e_{\mathcal{B}}$ , such that

$$\left( \Xi_{\mathcal{B}} * i_{e_{\mathcal{B}}} \right) \odot \left( i_{e_{\mathcal{B}}} * \Delta_{\mathcal{B}} \right) = i_{e_{\mathcal{B}}} \quad \text{and} \quad \left( i_{d_{\mathcal{B}}} * \Xi_{\mathcal{B}} \right) \odot \left( \Delta_{\mathcal{B}} * i_{d_{\mathcal{B}}} \right) = i_{d_{\mathcal{B}}}. \quad (\text{C.14})$$

In order to prove condition **A1**( $D_{\mathcal{B}}$ ) for (3.2), let us fix any set  $(s_{\mathcal{B}}^1, s_{\mathcal{B}}^2, \Lambda_{\mathcal{B}})$  in  $\mathcal{B}$  as follows, with  $\Lambda_{\mathcal{B}}$  invertible

$$\begin{array}{ccc} D_{\mathcal{B}} & \xrightarrow{s_{\mathcal{B}}^1} & \mathcal{F}_0(B_{\mathcal{A}}^1) \\ s_{\mathcal{B}}^2 \downarrow & \swarrow \Lambda_{\mathcal{B}} & \downarrow \mathcal{F}_1(g_{\mathcal{A}}^1) \\ \mathcal{F}_0(B_{\mathcal{A}}^2) & \xrightarrow{\mathcal{F}_1(g_{\mathcal{A}}^2)} & \mathcal{F}_0(A_{\mathcal{A}}). \end{array}$$

By property (X2) for  $\mathcal{F}$ , for each  $m = 1, 2$  there are a morphism  $s_{\mathcal{A}}^m : D_{\mathcal{A}} \rightarrow B_{\mathcal{A}}^m$  and an invertible 2-morphism  $\chi_{\mathcal{B}}^m : \mathcal{F}_1(s_{\mathcal{A}}^m) \Rightarrow s_{\mathcal{B}}^m \circ e_{\mathcal{B}}$ . Again by (X2) there is a (unique) invertible 2-morphism  $\Lambda_{\mathcal{A}} : g_{\mathcal{A}}^1 \circ s_{\mathcal{A}}^1 \Rightarrow g_{\mathcal{A}}^2 \circ s_{\mathcal{A}}^2$ , such that

$$\begin{aligned} \mathcal{F}_2(\Lambda_{\mathcal{A}}) &= \left( i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * (\chi_{\mathcal{B}}^2)^{-1} \right) \odot \left( \Lambda_{\mathcal{B}} * i_{e_{\mathcal{B}}} \right) \odot \left( i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \chi_{\mathcal{B}}^1 \right) : \\ &\mathcal{F}_1(g_{\mathcal{A}}^1 \circ s_{\mathcal{A}}^1) \Longrightarrow \mathcal{F}_1(g_{\mathcal{A}}^2 \circ s_{\mathcal{A}}^2). \end{aligned} \quad (\text{C.15})$$

Since (3.1) satisfies property **A1**( $D_{\mathcal{A}}$ ), then there are a morphism  $s_{\mathcal{A}} : D_{\mathcal{A}} \rightarrow C_{\mathcal{A}}$  and a pair of invertible 2-morphisms  $\Lambda_{\mathcal{A}}^m : s_{\mathcal{A}}^m \Rightarrow r_{\mathcal{A}}^m \circ s_{\mathcal{A}}$  for  $m = 1, 2$ , such that



$$\left(\Omega_{\mathcal{A}} * i_{s_{\mathcal{A}}}\right) \odot \left(i_{g_{\mathcal{A}}^1} * \Lambda_{\mathcal{A}}^1\right) = \left(i_{g_{\mathcal{A}}^2} * \Lambda_{\mathcal{A}}^2\right) \odot \Lambda_{\mathcal{A}}. \quad (\text{C.16})$$

Since we are assuming that  $\mathcal{F}$  is a strict pseudofunctor, then by applying  $\mathcal{F}_2$  to (C.16) and using (C.15), we get:

$$\begin{aligned} & \left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{\mathcal{F}_1(s_{\mathcal{A}})}\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \mathcal{F}_2(\Lambda_{\mathcal{A}}^1)\right) = \\ & = \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \mathcal{F}_2(\Lambda_{\mathcal{A}}^2)\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * (\chi_{\mathcal{B}}^2)^{-1}\right) \odot \left(\Lambda_{\mathcal{B}} * i_{e_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \chi_{\mathcal{B}}^1\right). \end{aligned}$$

This implies that:

$$\begin{aligned} & \left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{\mathcal{F}_1(s_{\mathcal{A}})}\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \left(\mathcal{F}_2(\Lambda_{\mathcal{A}}^1) \odot (\chi_{\mathcal{B}}^1)^{-1}\right)\right) = \\ & = \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \left(\mathcal{F}_2(\Lambda_{\mathcal{A}}^2) \odot (\chi_{\mathcal{B}}^2)^{-1}\right)\right) \odot \left(\Lambda_{\mathcal{B}} * i_{e_{\mathcal{B}}}\right). \quad (\text{C.17}) \end{aligned}$$

Then by interchange law we have:

$$\begin{aligned} & \left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{\mathcal{F}_1(s_{\mathcal{A}}) \circ d_{\mathcal{B}}}\right) \odot \\ & \odot \left\{ i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \left[ \left(\mathcal{F}_2(\Lambda_{\mathcal{A}}^1) * i_{d_{\mathcal{B}}}\right) \odot \left((\chi_{\mathcal{B}}^1)^{-1} * i_{d_{\mathcal{B}}}\right) \odot \left(i_{s_{\mathcal{B}}^1} * \Xi_{\mathcal{B}}^{-1}\right) \right] \right\} \stackrel{(\text{C.17})}{=} \\ & \stackrel{(\text{C.17})}{=} \left\{ i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \left[ \left(\mathcal{F}_2(\Lambda_{\mathcal{A}}^2) * i_{d_{\mathcal{B}}}\right) \odot \left((\chi_{\mathcal{B}}^2)^{-1} * i_{d_{\mathcal{B}}}\right) \odot \left(i_{s_{\mathcal{B}}^2} * \Xi_{\mathcal{B}}^{-1}\right) \right] \right\} \odot \Lambda_{\mathcal{B}}. \quad (\text{C.18}) \end{aligned}$$

Then we set  $s_{\mathcal{B}} := \mathcal{F}_1(s_{\mathcal{A}}) \circ d_{\mathcal{B}} : D_{\mathcal{B}} \rightarrow \mathcal{F}_0(C_{\mathcal{A}})$  and for each  $m = 1, 2$ :

$$\Lambda_{\mathcal{B}}^m := \left(\mathcal{F}_2(\Lambda_{\mathcal{A}}^m) * i_{d_{\mathcal{B}}}\right) \odot \left((\chi_{\mathcal{B}}^m)^{-1} * i_{d_{\mathcal{B}}}\right) \odot \left(i_{s_{\mathcal{B}}^m} * \Xi_{\mathcal{B}}^{-1}\right) : s_{\mathcal{B}}^m \Longrightarrow \mathcal{F}_1(r_{\mathcal{A}}^m) \circ s_{\mathcal{B}},$$

so (C.18) reads as follows:

$$\left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{s_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \Lambda_{\mathcal{B}}^1\right) = \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \Lambda_{\mathcal{B}}^2\right) \odot \Lambda_{\mathcal{B}}.$$

This shows that condition **A1**( $D_{\mathcal{B}}$ ) holds for diagram (3.2).

Now let us also prove **A2**( $D_{\mathcal{B}}$ ) for (3.2), so let us fix any pair of morphisms  $t_{\mathcal{B}}, t'_{\mathcal{B}} : D_{\mathcal{B}} \rightarrow \mathcal{F}_0(C_{\mathcal{A}})$  any pair of invertible 2-morphisms  $\Gamma_{\mathcal{B}}^m : \mathcal{F}_1(r_{\mathcal{A}}^m) \circ t_{\mathcal{B}} \Rightarrow \mathcal{F}_1(r_{\mathcal{A}}^m) \circ t'_{\mathcal{B}}$ , such that

$$\left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{t'_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \Gamma_{\mathcal{B}}^1\right) = \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \Gamma_{\mathcal{B}}^2\right) \odot \left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{t_{\mathcal{B}}}\right). \quad (\text{C.19})$$

By property (X2) there are a pair of morphisms  $t_{\mathcal{A}}, t'_{\mathcal{A}} : D_{\mathcal{A}} \rightarrow C_{\mathcal{A}}$  and a pair of invertible 2-morphisms

$$\Phi_{\mathcal{B}} : \mathcal{F}_1(t_{\mathcal{A}}) \Longrightarrow t_{\mathcal{B}} \circ e_{\mathcal{B}} \quad \text{and} \quad \Phi'_{\mathcal{B}} : \mathcal{F}_1(t'_{\mathcal{A}}) \Longrightarrow t'_{\mathcal{B}} \circ e_{\mathcal{B}}.$$

Then using the interchange law we get:

$$\begin{aligned} & \left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{\mathcal{F}_1(t'_{\mathcal{A}})}\right) \odot \\ & \odot \left\{ i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \left[ \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^1)} * (\Phi'_{\mathcal{B}})^{-1}\right) \odot \left(\Gamma_{\mathcal{B}}^1 * i_{e_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^1)} * \Phi_{\mathcal{B}}\right) \right] \right\} = \\ & = \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^2 \circ r_{\mathcal{A}}^2)} * (\Phi'_{\mathcal{B}})^{-1}\right) \odot \left\{ \left[\left(\mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{t'_{\mathcal{B}}}\right) \odot \right. \right. \\ & \left. \left. \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1)} * \Gamma_{\mathcal{B}}^1\right) \right] * i_{e_{\mathcal{B}}}\right\} \odot \left(i_{\mathcal{F}_1(g_{\mathcal{A}}^1 \circ r_{\mathcal{A}}^1)} * \Phi_{\mathcal{B}}\right) \stackrel{(\text{C.19})}{=} \end{aligned}$$

$$\begin{aligned}
& \stackrel{(C.19)}{=} \left( i_{\mathcal{F}_1(g_{\mathcal{A}}^2 \circ r_{\mathcal{A}}^2)} * (\Phi'_{\mathcal{B}})^{-1} \right) \odot \left\{ \left[ \left( i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \Gamma_{\mathcal{B}}^2 \right) \odot \right. \right. \\
& \quad \left. \left. \odot \left( \mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{t_{\mathcal{B}}} \right) \right] * i_{e_{\mathcal{B}}} \right\} \odot \left( i_{\mathcal{F}_1(g_{\mathcal{A}}^1 \circ r_{\mathcal{A}}^1)} * \Phi_{\mathcal{B}} \right) = \\
& = \left\{ i_{\mathcal{F}_1(g_{\mathcal{A}}^2)} * \left[ \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^2)} * (\Phi'_{\mathcal{B}})^{-1} \right) \odot \left( \Gamma_{\mathcal{B}}^2 * i_{e_{\mathcal{B}}} \right) \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^2)} * \Phi_{\mathcal{B}} \right) \right] \right\} \odot \\
& \quad \odot \left( \mathcal{F}_2(\Omega_{\mathcal{A}}) * i_{\mathcal{F}_1(t_{\mathcal{A}})} \right). \tag{C.20}
\end{aligned}$$

Again by property (X2) for  $\mathcal{F}$ , for each  $m = 1, 2$  there is a (unique) invertible 2-morphism  $\Gamma_{\mathcal{A}}^m : r_{\mathcal{A}}^m \circ t_{\mathcal{A}} \Rightarrow r_{\mathcal{A}}^m \circ t'_{\mathcal{A}}$ , such that

$$\begin{aligned}
\mathcal{F}_2(\Gamma_{\mathcal{A}}^m) &= \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * (\Phi'_{\mathcal{B}})^{-1} \right) \odot \left( \Gamma_{\mathcal{B}}^m * i_{e_{\mathcal{B}}} \right) \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi_{\mathcal{B}} \right) : \\
& \mathcal{F}_1(r_{\mathcal{A}}^m \circ t_{\mathcal{A}}) \Longrightarrow \mathcal{F}_1(r_{\mathcal{A}}^m \circ t'_{\mathcal{A}}). \tag{C.21}
\end{aligned}$$

Then (C.20) reads as follows:

$$\mathcal{F}_2 \left[ \left( \Omega_{\mathcal{A}} * i_{t'_{\mathcal{A}}} \right) \odot \left( i_{g_{\mathcal{A}}^1} * \Gamma_{\mathcal{A}}^1 \right) \right] = \mathcal{F}_2 \left[ \left( i_{g_{\mathcal{A}}^2} * \Gamma_{\mathcal{A}}^2 \right) \odot \left( \Omega_{\mathcal{A}} * i_{t_{\mathcal{A}}} \right) \right].$$

Then again by property (X2) we conclude that

$$\left( \Omega_{\mathcal{A}} * i_{t'_{\mathcal{A}}} \right) \odot \left( i_{g_{\mathcal{A}}^1} * \Gamma_{\mathcal{A}}^1 \right) = \left( i_{g_{\mathcal{A}}^2} * \Gamma_{\mathcal{A}}^2 \right) \odot \left( \Omega_{\mathcal{A}} * i_{t_{\mathcal{A}}} \right).$$

Since diagram (3.1) satisfies property **A2**( $D_{\mathcal{A}}$ ), then the previous identity implies that there is a unique invertible 2-morphism  $\Gamma_{\mathcal{A}} : t_{\mathcal{A}} \Rightarrow t'_{\mathcal{A}}$ , such that  $i_{r_{\mathcal{A}}^m} * \Gamma_{\mathcal{A}} = \Gamma_{\mathcal{A}}^m$  for each  $m = 1, 2$ . So by interchange law, for each  $m = 1, 2$  we have:

$$\begin{aligned}
\Gamma_{\mathcal{B}}^m &= \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m) \circ t'_{\mathcal{B}}} * \Xi_{\mathcal{B}} \right) \odot \left( \Gamma_{\mathcal{B}}^m * i_{e_{\mathcal{B}} \circ d_{\mathcal{B}}} \right) \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m) \circ t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1} \right) \stackrel{(C.21)}{=} \\
& \stackrel{(C.21)}{=} \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m) \circ t'_{\mathcal{B}}} * \Xi_{\mathcal{B}} \right) \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi'_{\mathcal{B}} * i_{d_{\mathcal{B}}} \right) \odot \left( \mathcal{F}_2(\Gamma_{\mathcal{A}}^m) * i_{d_{\mathcal{B}}} \right) \odot \\
& \quad \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi_{\mathcal{B}}^{-1} * i_{d_{\mathcal{B}}} \right) \odot \left( i_{\mathcal{F}_1(r_{\mathcal{A}}^m) \circ t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1} \right) = \\
& = i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \left\{ \left( i_{t'_{\mathcal{B}}} * \Xi_{\mathcal{B}} \right) \odot \left[ \left( \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1} \right) * i_{d_{\mathcal{B}}} \right] \odot \left( i_{t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1} \right) \right\}. \tag{C.22}
\end{aligned}$$

Hence, if we set

$$\Gamma_{\mathcal{B}} := \left( i_{t'_{\mathcal{B}}} * \Xi_{\mathcal{B}} \right) \odot \left[ \left( \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1} \right) * i_{d_{\mathcal{B}}} \right] \odot \left( i_{t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1} \right) : t_{\mathcal{B}} \Rightarrow t'_{\mathcal{B}}, \tag{C.23}$$

then (C.22) implies that  $i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Gamma_{\mathcal{B}} = \Gamma_{\mathcal{B}}^m$  for each  $m = 1, 2$ .

Now in order to conclude that (3.2) satisfies property **A2**( $D_{\mathcal{B}}$ ), we need only to prove that  $\Gamma_{\mathcal{B}}$  is the unique invertible 2-morphism with such a property. So let us fix another invertible 2-morphism  $\Gamma'_{\mathcal{B}} : t_{\mathcal{B}} \Rightarrow t'_{\mathcal{B}}$  such that  $i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Gamma'_{\mathcal{B}} = \Gamma_{\mathcal{B}}^m$  for each  $m = 1, 2$ . By (X2) there is a unique invertible 2-morphism  $\Gamma'_{\mathcal{A}} : t_{\mathcal{A}} \Rightarrow t'_{\mathcal{A}}$ , such that

$$\mathcal{F}_2(\Gamma'_{\mathcal{A}}) = (\Phi'_{\mathcal{B}})^{-1} \odot \left( \Gamma'_{\mathcal{B}} * i_{e_{\mathcal{B}}} \right) \odot \Phi_{\mathcal{B}} : \mathcal{F}_1(t_{\mathcal{A}}) \Longrightarrow \mathcal{F}_1(t'_{\mathcal{A}}). \tag{C.24}$$

Now by interchange law, we have:

$$\begin{aligned}
\Gamma_{\mathcal{B}} * i_{e_{\mathcal{B}}} & \stackrel{(C.23), (C.14)}{=} \left( i_{t'_{\mathcal{B}} \circ e_{\mathcal{B}}} * \Delta_{\mathcal{B}}^{-1} \right) \odot \left( \left( \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1} \right) * i_{d_{\mathcal{B}} \circ e_{\mathcal{B}}} \right) \odot \\
& \quad \odot \left( i_{t_{\mathcal{B}} \circ e_{\mathcal{B}}} * \Delta_{\mathcal{B}} \right) = \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1}. \tag{C.25}
\end{aligned}$$

Therefore, for each  $m = 1, 2$  we have:

$$\begin{aligned}
\mathcal{F}_2\left(i_{r_{\mathcal{A}}^m} * \Gamma'_{\mathcal{A}}\right) &= i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \mathcal{F}_2(\Gamma'_{\mathcal{A}}) \stackrel{(C.24)}{=} \\
\stackrel{(C.24)}{=} \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * (\Phi'_{\mathcal{B}})^{-1}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Gamma'_{\mathcal{B}} * i_{e_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi_{\mathcal{B}}\right) &= \\
= \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * (\Phi'_{\mathcal{B}})^{-1}\right) \odot \left(\Gamma_{\mathcal{B}}^m * i_{e_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi_{\mathcal{B}}\right) &= \\
= \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * (\Phi'_{\mathcal{B}})^{-1}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Gamma_{\mathcal{B}} * i_{e_{\mathcal{B}}}\right) \odot \left(i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \Phi_{\mathcal{B}}\right) &= \\
= i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \left((\Phi'_{\mathcal{B}})^{-1} \odot \left(\Gamma_{\mathcal{B}} * i_{e_{\mathcal{B}}}\right) \odot \Phi_{\mathcal{B}}\right) \stackrel{(C.25)}{=} & \\
\stackrel{(C.25)}{=} i_{\mathcal{F}_1(r_{\mathcal{A}}^m)} * \mathcal{F}_2(\Gamma_{\mathcal{A}}) = \mathcal{F}_2\left(i_{r_{\mathcal{A}}^m} * \Gamma_{\mathcal{A}}\right) = \mathcal{F}_2(\Gamma_{\mathcal{A}}^m). &
\end{aligned}$$

By (X2), this implies that  $i_{r_{\mathcal{A}}^m} * \Gamma'_{\mathcal{A}} = \Gamma_{\mathcal{A}}^m$  for each  $m = 1, 2$ . By construction,  $\Gamma_{\mathcal{A}}$  is the unique invertible 2-morphism from  $t_{\mathcal{A}}$  to  $t'_{\mathcal{A}}$  such that  $i_{r_{\mathcal{A}}^m} * \Gamma_{\mathcal{A}} = \Gamma_{\mathcal{A}}^m$  for each  $m = 1, 2$ , hence  $\Gamma'_{\mathcal{A}} = \Gamma_{\mathcal{A}}$ . Therefore,

$$\begin{aligned}
\Gamma'_{\mathcal{B}} * i_{e_{\mathcal{B}}} &\stackrel{(C.24)}{=} \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma'_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1} = \\
&= \Phi'_{\mathcal{B}} \odot \mathcal{F}_2(\Gamma_{\mathcal{A}}) \odot \Phi_{\mathcal{B}}^{-1} \stackrel{(C.25)}{=} \Gamma_{\mathcal{B}} * i_{e_{\mathcal{B}}}. \tag{C.26}
\end{aligned}$$

Hence by interchange law we have:

$$\begin{aligned}
\Gamma'_{\mathcal{B}} &= \left(i_{t'_{\mathcal{B}}} * \Xi_{\mathcal{B}}\right) \odot \left(\Gamma'_{\mathcal{B}} * i_{e_{\mathcal{B}} \circ d_{\mathcal{B}}}\right) \odot \left(i_{t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1}\right) \stackrel{(C.26)}{=} \\
&\stackrel{(C.26)}{=} \left(i_{t'_{\mathcal{B}}} * \Xi_{\mathcal{B}}\right) \odot \left(\Gamma_{\mathcal{B}} * i_{e_{\mathcal{B}} \circ d_{\mathcal{B}}}\right) \odot \left(i_{t_{\mathcal{B}}} * \Xi_{\mathcal{B}}^{-1}\right) = \Gamma_{\mathcal{B}},
\end{aligned}$$

so we have proved also the uniqueness part of condition **A2**( $D_{\mathcal{B}}$ ). Therefore diagram (3.2) is a weak fiber product in  $\mathcal{B}$ .  $\square$

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