Maximal Surface in AdS convex GHM 3-manifold with particles

Jérémy Toulisse

December 19, 2013

Abstract

We prove the existence of a unique maximal surface in an anti-de Sitter (AdS) convex Globally Hyperbolic Maximal (GHM) manifold with particles (i.e. with conical singularities along timelike lines) for cone-angles less than π. We reinterpret this result in terms of Teichmüller theory, and prove the existence of a unique minimal Lagrangian diffeomorphism isotopic to the identity between two hyperbolic structures with conical singularities of the same angles on a closed surface with marked points.

Contents

1 Introduction

For an angle $\theta \in [0, 2\pi]$, consider the space obtained by cutting the hyperbolic disk along two half-lines intersecting at the center and making an angle θ; then by gluing the two half-line bounding the angular sector of angle θ by a rotation of angle $2\pi - \theta$. We denote this singular Riemannian manifold by $\mathbb{H}^2_\theta$. The induced metric is called **hyperbolic metric with conical singularity of angle** θ. This metric is hyperbolic outside the singular point. Let Σ_n be a closed oriented surface of genus g with n marked points $x_1, ..., x_n$.

Definition 1.1. For $\theta := (\theta_1, ..., \theta_n)$, with $\theta_i \in [0, 2\pi]$, a hyperbolic metric with conical singularities of angle θ_i at the points $x_i \in \Sigma$ is a metric on Σ such that each x_i has a neighborhood isometric to a neighborhood of the singular point in $\mathbb{H}^2_{\theta_i}$ and each point $p \in \Sigma \setminus \{x_1, ..., x_n\}$ has a neighborhood isometric to an open set in $\mathbb{H}^2$. We denote by $\mathcal{T}_{g,n,\theta}$ the space of such metrics modulo isotopy fixing each x_i.

For Σ of genus $g \geqslant 2$ without marked points, $\mathcal{T}_{g,0,()} = \mathcal{T}_g$ corresponds to the classical Teichmüller space of Σ, that is, the space of equivalence classes of hyperbolic metric on the surface (where two hyperbolic metrics belong to the same class if and only if they are isotopic).

Minimal Lagrangian diffeomorphism

Definition 1.2. Let $h, h' \in \mathcal{T}_g$, a minimal Lagrangian diffeomorphism $\varphi : (\Sigma, h) \longrightarrow (\Sigma, h')$ is an area preserving diffeomorphism such that its graph is minimal in $(\Sigma \times \Sigma, h \oplus h')$.

In 1992, F. Labourie [Lab92] and R. Schoen [Sch93] proved that for each $h, h' \in \mathcal{T}_g$, there exists a unique minimal Lagrangian diffeomorphism isotopic to the identity $\varphi : (\Sigma, h) \longrightarrow (\Sigma, h')$.

Minimal Lagrangian diffeomorphisms are related to harmonic maps (see [Sam78] and [Wol89]). Namely, let J_0 be a complex structure on Σ, $h \in \mathcal{T}_g$ and $\psi : (\Sigma, J_0) \to (\Sigma, h)$ a harmonic diffeomorphism isotopic to the identity. Let q be the Hopf differential of ψ (that is the $(2, 0)$ part of $\psi^* h^{\mathbb{C}}$ where $h^{\mathbb{C}}$ is the complexified of the metric h). There is a unique harmonic diffeomorphism $\psi' : (\Sigma, J_0) \to (\Sigma, h')$ isotopic to the identity where $h' \in \mathcal{T}_g$ and the Hopf differential of ψ' is $-q$. Moreover, $\varphi := \psi \circ \psi'^{-1}$ is a minimal Lagrangian diffeomorphism isotopic to the identity.

Moreover, for $h, h' \in \mathcal{T}_g$, let $\varphi : (\Sigma, h) \to (\Sigma, h')$ be a minimal Lagrangian diffeomorphism isotopic to the identity. Let (Σ_0, J_0) be the graph of φ together with the complex structure defined by its induced metric in $(\Sigma \times \Sigma, h \oplus h')$. Then the natural projection from Σ_0 to (Σ, h) and to (Σ, h') are harmonic diffeomorphisms isotopic to the identity, and the sum of their Hopf differentials is equal to zero.

That is, minimal Lagrangian diffeomorphisms can be thought as "squares" of harmonic maps.

In his thesis, J. Gell-Redman [GR10] proved the existence of a unique harmonic map from a closed surface with n marked points equipped with a complex structure to a hyperbolic surface of the same genus with n conical singularities of angles less than 2π at the marked points. Thus, a natural question arising, is about the existence of a unique minimal Lagrangian diffeomorphisms isotopic to the identity between hyperbolic surfaces with conical singualrities [BBD$^+$12, Question 6.3]. In this paper, we give an answer. Precisely, we prove the next result:

Theorem 1.1. *For all $h, h' \in \mathcal{T}_{g,n,\theta}$, with $\theta = (\theta_1, ..., \theta_n)$ and $\theta_i < \pi$, $i = 1, ..., n$, there exists a unique minimal Lagrangian diffeomorphism $\varphi : (\Sigma, h) \longrightarrow (\Sigma, h')$ isotopic to the identity.*

In particular, this result extends the result of F. Labourie and R. Schoen to the case of surfaces with conical singularities of angles less than π. In Theorem 6.3, we extend the relations between this unique minimal Lagrangian diffeomorphism and the harmonic maps provided by [GR10]. Namely, we prove that given two metrics $h, h' \in \mathcal{T}_{g,n,\theta}$, there exists a unique conformal structure with marked points J_0 on S such that the Hopf differential $\mathsf{Hopf}(\psi)$ of the unique harmonic map $\psi : (S, J_0) \longrightarrow (S, h)$ is the opposite of the Hopf differential of the unique harmonic map $\psi' : (S, J_0) \longrightarrow (S, h')$ and $\varphi := \psi \circ \psi'^{-1}$ is the unique minimal Lagrangian isotopic to the identity. The proof of this statement uses the deep relations between hyperbolic surfaces and three dimensional anti-de Sitter (AdS) geometry.

AdS geometry The AdS three dimensional space is a Lorentzian symmetric space of constant sectional curvature -1. It can be thought as the Lorentzian analogue of the three dimensional hyperbolic space $\mathbb{H}^3$. In its work on three dimensional Lorentzian geometry, G. Mess [Mes07] parameterized the moduli space of Lorentizan Globally Hyperbolic Maximal (GHM) structures of constant curvature (see below for the definition of a GHM manifold). In particular, he found an analogy between three dimensional AdS GHM geometry and quasi-Fuchsian geometry. In fact, Bers' simultaneous uniformization theorem [Ber60] gives a parameterization of the smooth moduli space $\mathcal{QF}_g$ of quasi-Fuchsian structures on the topological product $M = \Sigma \times \mathbb{R}$. More precisely, he proved that the application associating to every quasi-Fuchsian manifold the conformal class of the metrics of the boundary of M gives a parameterization of $\mathcal{QF}_g$ by $\mathcal{T}_g \times \mathcal{T}_g$.

An AdS GHM manifold is a topological manifold $M = \Sigma \times \mathbb{R}$ equipped with Lorentzian structure of constant curvature -1 such that the manifold contains a Cauchy surface (i.e. a spacelike surface which intersects every inextensible timelike curve exactly once), which is maximal in a certain sense (precised below). G. Mess [Mes07, Section 7] proved that this moduli space is parameterized by two copies of $\mathcal{T}_g$. This result can be thought as an AdS analogue of Bers' theorem.

Several years later, K. Krasnov and J.-M. Schlenker [KS07] proved the existence of a unique maximal surface in each AdS GHM structure on $\Sigma \times \mathbb{R}$. Maximal surfaces are the Lorentzian analogue of minimal surfaces in Riemannian geometry: they are surface of vanishing mean curvature (the name 'maximal' comes from the fact that they maximize the area functional). Moreover, they showed that this result is equivalent to the result of F. Labourie and R. Schoen of the existence of a unique minimal Lagrangian diffeomorphism isotopic to the identity between two hyperbolic surfaces.

A particle in an AdS GHM manifold M is defined as a conical singularity along a timelike line. In this paper, we only consider particles with cone-angles less than π. F. Bonsante and J.-M. Schlenker extended Mess' parameterization to the case of AdS convex GHM manifolds with particles. Precisely, they proved [BS09, Theorem 1.4] that the application from the moduli space of AdS convex GHM structure on $\Sigma_n \times \mathbb{R}$ with n conical singularities of given angles $\theta := (\theta_1, ..., \theta_n) \in (0, \pi)^n$ to the product of two copies of $\mathcal{T}_{g,n,\theta}$ associating to an AdS convex GHM manifold with particles the right and left metrics (see Section 3 below) is one-to-one. So, a natural question is about the existence of a unique maximal surface in each AdS convex GHM manifold with particles [BBD+12, Question 6.2]. In this paper, we give an answer to this question:

Theorem 1.2. *Let (M, g) be an AdS convex GHM 3-manifold with particles of angles less than π, then M contains a unique maximal spacelike surface.*

Moreover, we prove that the existence of a unique maximal surface provides the existence of a unique minimal Lagrangian diffeomorphism isotopic to the identity

$$\varphi : (\Sigma_n, h) \longrightarrow (\Sigma_n, h'),$$

for all $h, h' \in \mathcal{T}_{g,n,\theta}$ (with $\theta_i < \pi$).

The parameterization of the moduli space of AdS convex GHM structure and Theorem 1.2 provide a homeomorphism $\Psi : T^*\mathcal{T}_{g,n,\theta} \longrightarrow \mathcal{T}_{g,n,\theta} \times \mathcal{T}_{g,n,\theta}$. In Theorem 6.3, we give a geometric interpretation of this map: for $h, h' \in \mathcal{T}_{g,n,\theta}$, there exists a unique complex structure J_0 on S such that the Hopf differential of the harmonic map $\psi : (S, J_0) \to (S, h)$ is the opposite of the Hopf differential of the harmonic map $\psi' : (S, J_0) \to (S, h')$, and

$$\Psi^{-1}(h, h') = ([J_0], \mathsf{Hopf}(\psi)),$$

where $[J_0] \in \mathcal{T}_{g,n,\theta}$ is the unique hyperbolic metric whose associated complex structure is J_0.

Acknowledgement. It is a pleasure to thank Jean-Marc Schlenker for its patience while discussing about the paper. I would also thank Francesco Bonsante and Thierry Barbot for helpful and interesting conversations about this subject.

2 AdS GHM 3-manifolds

2.1 Mess parameterization

The AdS 3-space. Let $\mathbb{R}^{2,2}$ be the usual real 4-space with the quadratic form:

$$q(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

The anti-de Sitter (AdS) 3-space is given by:

$$AdS_3 = \{x \in \mathbb{R}^{2,2} \text{ such that } q(x) = -1\}.$$

With the induced metric, it is a Lorentzian symmetric space of dimension 3 with constant curvature -1 diffeomorphic to $\mathbb{D} \times S^1$ (where $\mathbb{D}$ is a disk of dimension 2). In particular, AdS_3 is not simply connected.

In this text, we are going to consider the Klein model of the AdS 3-space: consider the canonical projection

$$\pi : \mathbb{R}^{2,2} \longrightarrow \mathbb{RP}^3.$$

π is a 2-to-1 covering of AdS_3 on $AdS^3 := \pi(AdS_3)$. We call AdS^3 with the metric pushed forward, the Klein model of the AdS 3-space. In this model, AdS^3 is the interior of a hyperboloid of one sheet. Moreover, the geodesics are given by straight lines: spacelike geodesics are the ones which intersect the boundary ∂AdS^3 (that is the hyperboloid) in two points, timelike geodesics are the ones which do not have any intersection and lightlike geodesics are tangents to ∂AdS^3.

Remark 2.1. This model is called Klein model by analogy with the Klein model of the hyperbolic space. In fact, in these models, geodesics are given by straight lines.

The isometry group. As ∂AdS^3 is a hyperboloid of one sheet, it is foliated by two families of straight lines. We call one family the right one and the other, the left one. The group $\mathrm{Isom}_+(AdS^3)$ of orientation and time-orientation preserving isometries of AdS^3 preserves each family of the foliation. Fix a spacelike plane P_0 in AdS^3, its boundary is a spacelike circle in ∂AdS^3 which intersects each line of the right (respectively the left) family exactly once. Then P_0 gives an identification of each family with $\mathbb{RP}^1$ (when changing P_0 to another spacelike plane, the identification changes by a conjugation by an element of $PSL_2(\mathbb{R})$). It is proved in [Mes07, Section 7] that each element of $\mathrm{Isom}_+(AdS^3)$ defines a couple of projective transformation, which uniquely extend to a couple of elements in $PSL(2, \mathbb{R})$. So $\mathrm{Isom}_+(AdS^3) = PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$.

AdS GHM 3-manifold. An AdS GHM (Globally Hyperbolic Maximal) 3-manifold is a manifold M homeomorphic to $M \cong \Sigma \times \mathbb{R}$ (where Σ is a closed oriented surface of genus at least 2), which carries a (G, X)-structure, where $G = \mathrm{Isom}_+(AdS^3)$, $X = AdS^3$ and satisfies two conditions:

1. M contains a spacelike Cauchy surface (that is a closed oriented surface which intersects every inextensible timelike curve exactly once).
2. M cannot be strictly embedded in an AdS manifold satisfying the same properties.

Let Σ be a closed oriented surface of genus $g > 1$, we denote by $\mathcal{M}_{g,AdS}$ the space of AdS GHM structure on $\Sigma \times \mathbb{R}$ considered up to isotopy, and by $\mathcal{T}_g$ the Teichmüller space of Σ.

We have the fundamental theorem of [Mes07, Proposition 20]:

Theorem 2.1 (Mess)**.** *There is a parameterization Mess:* $\mathcal{M}_{g,AdS} \longrightarrow \mathcal{T}_g \times \mathcal{T}_g$.

Construction of the parameterization. To an AdS GHM structure on M is associated its holonomy representation $\rho : \pi_1(M) = \pi_1(\Sigma) \to \mathrm{Isom}_+(AdS^3)$. Consider the decomposition $\rho := (\rho_l, \rho_r)$, where $\rho_l, \rho_r : \pi_1(\Sigma) \to PSL(2, \mathbb{R})$. G. Mess proved [Mes07, Proposition 19] that these holonomies have maximal Euler class (that is $\mid e(\rho_l) \mid = \mid e(\rho_r) \mid = 2g - 2$, where e designs the Euler class). Using Goldman's criterion [Gol88], we get that these holonomies are holonomies of hyperbolic structures and so define a pair of point in $\mathcal{T}_g$.

Reciprocally, as two holonomy representations ρ_1, ρ_2 of hyperbolic structures are conjugated by an orientation preserving homeomorphism $\phi : \mathbb{RP}^1 \to \mathbb{RP}^1$ and as ∂AdS^3 identifies with $\mathbb{RP}^1 \times \mathbb{RP}^1$ (fixing a totally geodesic spacelike plane P_0), the graph of ϕ defines a nowhere timelike closed curve in ∂AdS^3. Taking the quotient of the convex hull of this curve by the application $\rho := (\rho_1, \rho_2)$, we get a piece of a globally hyperbolic AdS manifold which uniquely embeds in an AdS GHM manifold. So the map Mess is a one-to-one. □

2.2 Surfaces embedded in an AdS GHM 3-manifold

J.-M. Schlenker and K. Krasnov [KS07, Section 3] found results about surfaces embedded in an AdS GHM manifold. We are going to recall some of them, and invite the interested reader to look at [KS07]. Recall that a spacelike surface embedded in a Lorentzian manifold is maximal if its mean curvature vanishes everywhere. They are Lorentzian analogue of minimal surfaces.

Theorem 2.2 (K. Krasnov, J.-M. Schlenker)**.** *Every AdS GHM 3-manifold contains a unique maximal spacelike surface.*

Moreover, they give an explicit formula for the map Mess:

Theorem 2.3 (K. Krasnov, J.-M. Schlenker)**.** *Let S be a spacelike surface embedded in an AdS GHM manifold M whose principal curvatures are in $(-1, 1)$. We denote by E the identity map, J the complex structure on S, B its shape operator and I its first fundamental form. We have:*

$$Mess(M) = (h_l, h_r),$$

where $h_{l,r}(x, y) = I((E \pm JB)x, (E \pm JB)y)$.

Remark 2.2. In particular, the metrics $h_{l,r}$ are hyperbolic and do not depend of the choice of the surface S (up to isotopy).

If we denote by $\mathcal{H}_{g,AdS}$ the space of maximal spacelike surfaces in germs of AdS manifold, it is proved in [KS07] (using the Fundamental Theorem of surfaces embedded in an AdS manifold) that this space is canonically identified with the space of couples (g, h) where g is a smooth metric on Σ and h is a symmetric bilinear form on TS such that:

1. $tr_g(h) = 0$.
2. $d^\nabla h = 0$ (where ∇ is the Levi-Civita connection of g and d^∇ is the covariant derivate).

3. $K_g = -1 - \det_g(h)$ (where K_g is the Gauss curvature). We call this equation **modified Gauss' equation**.

We recall a theorem of Hopf [Hop51]:

Theorem 2.4 (Hopf). *Let g be a Riemannian metric on Σ and h a bilinear symmetric form on $T\Sigma$, then:*

i. $tr_g(h) = 0$ if and only if h is the real part of a quadratic differential q.

ii. If i. holds, then $d^\nabla h = 0$ if and only if q is holomorphic.

iii. if i. and ii. hold, then g (respectively h) is the first (respectively second) fundamental form of a maximal surface if and only if $K_g = -1 - \det_g(h)$.

Moreover, it is proved in [KS07, Lemma 3.6.] that for every conformal class of metric [g] on Σ and every h real part of a holomorphic quadratic differential $q \in T^*_{[g]}\mathcal{T}_g$, there exists a unique metric $g_0 \in [g]$ such that modified Gauss' equation is satisfied. This result allows us to parameterize $\mathcal{H}_{g,AdS}$ by $T^*\mathcal{T}_g$. In this parameterization, h is the real part of a holomorphic quadratic differential, and $g_0 \in [g]$ is the unique metric verifying $K_{g_0} = -1 - \det_{g_0}(h)$. In addition, such a surface has principal curvatures in $(-1, 1)$ [KS07, Lemma 3.11.].

As every AdS GHM manifold contains a unique maximal surface, there is a parameterization $\varphi : T^*\mathcal{T}_g \longrightarrow \mathcal{M}_{g,AdS}$ [KS07, Theorem 3.8]. Hence, we get an application associated to the Mess parameterization:

$$\Psi := \text{Mess} \circ \varphi : T^*\mathcal{T}_g \to \mathcal{T}_g \times \mathcal{T}_g.$$

3 AdS convex GHM 3-manifold with particles

In this section we define the AdS convex GHM manifolds with particles and recall the parameterization of the moduli space of such structures. The proofs of these results can be found in [KS07] and [BS09].

3.1 Extension of Mess parameterization

First, we are going to define the singular AdS space of dimension 3 in order to define the AdS convex GHM manifolds with particles.

Definition 3.1. Let $\theta > 0$, we define AdS^3_θ as the completion of $\mathbb{R} \times \mathbb{R}_{>0} \times (\mathbb{R}/\theta\mathbb{Z})$ with the metric:

$$g = -dt^2 + \cos^2(t)(d\rho^2 + \sinh^2(\rho)d\phi^2)$$

where $t \in \mathbb{R}, \rho \in \mathbb{R}_{>0}$ and $\phi \in (\mathbb{R}/\theta\mathbb{Z})$.

Remark 3.1. AdS^3_θ can be obtained by cutting AdS^3 along two timelike planes intersecting along the line $l := \{\rho = 0\}$, making an angle θ, and gluing the two sides of the angular sector of angle θ by the rotation of angle $2\pi - \theta$ fixing l. A simple computation shows that, outside of the singular line, AdS^3_θ is a Lorentzian manifold of constant curvature -1, and AdS^3_θ has conical singularities of angle θ along the points of l.

Definition 3.2. An AdS cone-manifold is a Lorentzian 3-manifold M in which any point x has a neighborhood isometric to an open subset of AdS^3_θ for some $\theta > 0$. If θ can be taken equal to 2π, x is a smooth point, otherwise θ is uniquely determined.

To define the global hyperbolicity in the singular case, we need to define the orthogonality to the singular locus:

Definition 3.3. Let $S \subset AdS^3_\theta$ be a spacelike surface which intersect the singular line l at a point x. S is said to be orthogonal to l at x if the causal distance (that is the 'distance' along a timelike line) to the totally geodesic plane P orthogonal to the singular line at x is such that:

$$\lim_{y \to x, y \in S} \frac{d(y, P)}{d_S(x, y)} = 0$$

where $d_S(x, y)$ is the distance between x and y along S.

Now, a spacelike surface S in an AdS cone-manifold M which intersects a singular line d at a point y is said to be orthogonal to d if there exists a neighborhood U of y in M isometric to a neighborhood of a singular point in AdS^3_θ such that the isometry sends $S \cap U$ to a surface orthogonal to l in AdS^3_θ.

Now we are able to define the AdS GHM manifold with particles.

Definition 3.4.

- An AdS convex GHM manifold with particles M is an AdS cone-manifold which is homeomorphic to $\Sigma_{g,n} \times \mathbb{R}$ (where $\Sigma_{g,n}$ is a closed oriented surface of genus g with n marked points), such that singularities are along timelike lines $d_1, ..., d_n$ and have fixed angle $\theta_1, .., \theta_n$ with $\theta_i < \pi$; moreover, we impose two conditions:

  1. M contains a spacelike future-convex Cauchy surface orthogonal to the singular locus (condition of convex Global Hyperbolicity).
  2. M cannot be strictly embedded in another manifold satisfying the same conditions (condition of Maximality).

Remark 3.2. The condition of convexity in the definition will allow us to use a convex core. As pointed out by the authors in [BS09], we do not know if every AdS GHM manifold with particles is convex GHM.

Many results known in the non-singular case extend to the singular case (i.e. with particles of angles less than π). We recall some of them here (cf. [BS09], [KS07]):

1. The parameterization *Mess* defined above extends to the singular case; that is, we have a parameterization of the moduli space $\mathcal{M}_{g,n,\theta}$ of AdS convex GHM metrics with conical singularities along timelike lines $d_1, ..., d_n$ with prescribed angle $\theta = (\theta_1, ..., \theta_n) \in (0, \pi)^n$ taking up to isotopy, by two copies of the Teichmüller space $\mathcal{T}_{g,n,\theta}$. Here, $\mathcal{T}_{g,n,\theta}$ is, as in the introduction, the moduli space of hyperbolic metrics with conical singularities at fixed points $x_1, ..., x_n$ of prescribed angles $\theta = (\theta_1, ..., \theta_n)$.

2. Any AdS convex GHM 3-manifold with particles contains a minimal non-empty convex subset called its "convex core" whose boundary is a disjoint union of two pleated spacelike surfaces orthogonal to the singular locus (except in the Fuchsian case which corresponds to the case where the two metrics of the parameterization are equals. In this case, the convex core is a totally geodesic spacelike surface).

Remark 3.3. The analogy between AdS GHM geometry and quasi-Fuchsian geometry explained in the introduction extends to the case with particles. Namely, the parameterization of the moduli space $\mathcal{QF}_{g,n,\theta}$ of quasi-Fuchsian manifolds with particles is given by the product of two copies of $\mathcal{T}_{g,n,\theta}$ (cf. [LS09], [MS09]).

3.2 Maximal surface

Let M be an AdS convex GHM 3-manifold with particles.

Definition 3.5. A maximal surface in M is a spacelike Cauchy surface orthogonal to the singular lines with vanishing mean curvature outside these intersections with the singular lines.

It is proved in [KS07] that, as in the non-singular case, we can define the space $\mathcal{H}_{g,AdS,n,\theta}$ of maximal surfaces in a germ of AdS convex GHM with n particles of angles $\theta = (\theta_1, ..., \theta_n) \in (0, \pi)^n$. This space is again parameterized by the cotangent bundle of $\mathcal{T}_{g,n,\theta}$ (the cotangent space at a point is the space of holomorphic quadratic differentials with at most simple poles at the marked points).

4 Existence of a maximal surface

In this section, we prove the existence part of Theorem 1.2. Note that in the Fuchsian case (that is when the two metrics of the parameterization $Mess$ are equals), the convex core is reduced to a totally geodesic plane orthogonal to the singular locus which is thus maximal (its second fundamental form vanishes). Hence, from now, we suppose that the AdS convex GHM manifold with particles M is not Fuchsian and so contains a convex core whose interior is not empty (see [BS09, Section 5]). The proof will be done in three steps:

1. Approximate the singular metric by a sequence of smooth metrics.

2. Prove the existence of a maximal surface in each manifold with regularized metric. This sequence is convergent (because each maximal surface can be locally seen as a graph of a Lipschitzian map which is contained in a compact set).

3. Prove that the limit surface is a maximal surface.

4.1 Approximation of singular metrics

Take $\theta \in (0, 2\pi)$ and consider $\mathcal{C}_\theta \subset \mathbb{R}^3$ the cone given by the parameterization:

$$\mathcal{C}_\theta := \{(u.\cos(v), u.\sin(v), \text{cotan}(\theta/2).u)\ ,\ (u, v) \in \mathbb{R}_+ \times [0, 2\pi)\}\ .$$

Now, look at the intersection of this cone with the Klein model of the hyperbolic 3-space, and note h_θ the induced metric. Let $\mathbb{H}^2_\theta := (\mathbb{D}, h_\theta)$ (where $\mathbb{D}$ is the unit disk) be the hyperbolic disk equipped with this metric. It is a complete Riemannian manifold of curvature -1 outside the singular point (because it is a convex ruled surface in a constant curvature space) with one conical singularity of angle θ. We call this space **hyperbolic plane with cone singularity**.

Note that the angle of the singularity is given by $\lim\limits_{\rho \to 0} \dfrac{l(C_\rho)}{\rho}$ where $l(C_\rho)$ is the length of the circle of radius ρ centered at the singularity.

Now, to approximate this metric, take $(\epsilon_i)_{i \in \mathbb{N}_{>0}}$, where $\epsilon_i = \dfrac{1}{2i}$ and define a sequence of even functions $(f_i)_{i \in \mathbb{N}_{>0}} \subset \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that:

$$\begin{cases} f_i(0) = -\epsilon_i^2.\text{cotan}(\theta/2) \\ f_i''(x) < 0 \quad \forall x \in (-\epsilon_i, \epsilon_i) \\ f_i(x) = -\text{cotan}(\theta/2).x \text{ if } x \geqslant \epsilon_i \end{cases}$$

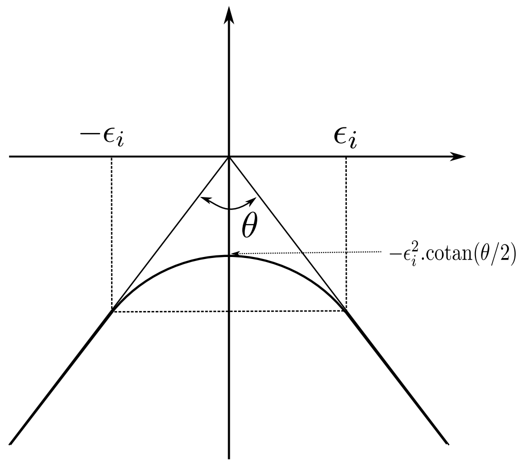

Figure 1: Graph of f_i

Consider the surface $\mathcal{C}_{\theta,i}$ obtained by making a rotation of the graph of f_i around the axis $(0z)$ and consider its intersection with the Klein model of hyperbolic space of dimension three. Denote by $h_{\theta,i}$ the induced metric, by $\mathbb{H}^2_{\theta,i} = (\mathbb{D}, h_{\theta,i})$ the disk equipped with this metric and by $B_i \subset \mathbb{D}$ the smallest ball where the metric is not of constant curvature -1 (note that $B_i \to \{0\}$, where $\{0\}$ is the center of $\mathbb{D}$).

As $\epsilon_i \to 0$, we get the following result:

Proposition 4.1. *For all compact $K \subset \mathbb{D} \setminus \{0\}$, there exists an $I \in \mathbb{N}$ such that for all $i > I$, $h_{\theta_{|K}} = h_{\theta,i_{|K}}$.*

We define the AdS 3-space with regularized singularity:

Definition 4.1. Let $\theta > 0$, $i \in \mathbb{N}$, we define $AdS^3_{\theta,i}$ as the completion of $\mathbb{R} \times \mathbb{D}$ with the metric:

$$g_i = -dt^2 + \cos^2(t).h_{\theta,i}$$

for $t \in \mathbb{R}$.

Clearly, there exists a smallest tubular neighborhood V^i_θ of $d = \{0\} \times \mathbb{R}$ such that $AdS^3_{\theta,i} \setminus V^i_\theta$ is a Lorentzian manifold of curvature -1.

In this way, we are going to define the regularized AdS convex GHM manifold with particles.

Let $\overline{M} \cong \Sigma \times I$ be a differentiable manifold and $M := (\overline{M}, g)$ be an AdS convex GHM manifold with conical singularities of angle $\theta_1, ..., \theta_n < \pi$ along timelike lines $d_1, ..., d_n$. For all $j \in \{1, ..., n\}$ and $x \in d_j$, there exist a neighborhood of x isometric to a neighborhood of a point on the singular line in $AdS^3_{\theta_j}$. For $i \in \mathbb{N}_{>0}$, we define $M_i := (\overline{M}, g_i)$ as the manifold $\overline{M}$ equipped with the metric g_i such that the neighborhoods of points of d_j are isometric to neighborhoods of points of the central axis in $AdS^3_{\theta_j,i}$. Clearly, M_i is obtained taking the metric of $V^i_{\theta_j}$ in a tubular neighborhood U^i_j of the singular lines d_j for all $j \in \{1, ..., n\}$. In particular, outside these U^i_j, M_i is a regular AdS manifold.

Proposition 4.2. *Let $K \subset \overline{M} \setminus (\bigcup_{j=1}^{n} d_j)$ be a compact set, then there exists $I \in \mathbb{N}$ such that, for all $i > I$, $g_{i_{|K}} = g_{|K}$.*

4.2 Existence of a maximal surface

We are going to show the existence part of Theorem 1.2 by convergence of maximal surfaces in each M_i. A result of Gerhardt [Ger83, theorem 6.2] provides the existence of a maximal surface in M_i given the existence of two smooth barriers, that is, a strictly future-convex smooth (at least $\mathcal{C}^2$) spacelike surface and a strictly past-convex one. This result has been adapted in [ABBZ12, Theorem 4.3] to the case of $\mathcal{C}^0$ barriers. The natural candidates for these barriers are equidistants surfaces from the boundary of the convex core (the geometry of the boundary of the convex core is described in [BS09, Section 5]).

As the differentiable manifolds M and M_i are canonically identified, we can consider the future component of the boundary of the convex core ∂_+ of M as embedded in M_i. For $\epsilon > 0$ fixed, consider the 2ϵ-surface in the future of ∂_+ and denote by $\partial_{+,\epsilon}$ the ϵ-surface in the past of the previous one. As pointed out in [BS09, Proof of Lemma 4.2], this surface differs from the ϵ-surface in the future of ∂_+ (at the pleating locus).

Proposition 4.3. *For i big enough, $\partial_{+,\epsilon} \subset M_i$ is a strictly future-convex $\mathcal{C}^{1,1}$ surface.*

Proof. Outside the open set $U^i := \bigcup_{j=1}^{n} U^i_j$, M_i is isometric to M. Moreover, for each j, $U^i_j \underset{i\to\infty}{\longrightarrow} d_j$. As proved in [BS09, Lemma 5.2], ∂_+ is spacelike and totally geodesic in a neighborhood of $\partial_+ \cap (\bigcup_{j=1}^{n} d_j)$. So, there exists $i_0 \in \mathbb{N}$ such that, for $i > i_0$, $U^j_i \cap \partial_+$ is totally geodesic.

The fact that $\partial_{+,\epsilon}$ is a $\mathcal{C}^{1,1}$ surface is proven in [BS09, Proof of Lemma 4.2].

For the strict convexity outside U^i, the result is proved in [BBZ07, Proposition 6.28]. So it remains to prove that $\partial_{+,\epsilon} \cap U_i$ is a strictly future-convex surface.

Let $d = d_j$ be a singular line which intersect ∂_+ at a point x. As $U := U^j_i \cap \partial_+$ is totally geodesic, $U_\epsilon := U^j_i \cap \partial_{+,\epsilon}$ is the ϵ-surface of U with respect to the metric g_i (in fact, the spacelike surface $\mathcal{P}_0$ given by the equation $\{t = 0\}$ is totally geodesic in $AdS^3_{\theta,i}$, the one given by $\mathcal{P}_\epsilon := \{t = \epsilon\}$ is the ϵ-surface and corresponds to the ϵ-surface in the past of $\mathcal{P}_{2\epsilon}$). Precisely, U_ϵ is obtained by taking the deformation of U along the vector field ϵN, where N is the unit future-pointing vector field normal to ∂_+ extended to the whole M by the condition $\nabla^i_N N = 0$ (where ∇^i is the Levi-Civita connection of g_i).

We are going to prove that the second fundamental form on U_ϵ is positive definite.

Note that in $AdS^3_{\theta_j,i}$, the surfaces $\mathcal{P}_t := \{t = cte.\}$ are equidistant from the totally geodesic spacelike surface $\mathcal{P}_0$. Moreover, the induced metric on $\mathcal{P}_t$ is $I_t = \cos^2(t)h_i$ and so, the variation I'_t under the flow of the unit normal future pointing vector field N is given by

$$I'_t(u,u) = -2\cos(t)\sin(t),$$

for u a unit vector field tangent to $\mathcal{P}_t$. On the other, the variation is given by:

$$I'_t(u,u) = \mathcal{L}_N I(u,u) = -II(u,u),$$

where $\mathcal{L}$ is the Lie derivate and $Bu = -\nabla_u N$.

It follows that $II(u,u) > 0$ for $t > 0$ small enough. So $\partial_{+,\epsilon} \subset M_i$ is strictly future-convex. ☐

So we get a $\mathcal{C}^{1,1}$ barrier. The existence of a $\mathcal{C}^{1,1}$ strictly past-convex surface is analogue. So, by [ABBZ12, Theorem 4.3], we get:

Proposition 4.4. $\forall i > i_0, \exists S_i \subset M_i$ *such that S_i is a maximal Cauchy surface.*

M_i canonically identifies with M, and all the S_i belong to a compact subset of M (the convex core) so, up to passing to a subsequence, $(S_i)_{i>i_0}$ converges $\mathcal{C}^0$ to a surface Σ.

Remark 4.1. Note that, as Σ is a limit of spacelike surfaces, it is nowhere timelike. However, it may contains some lightlike locus. We recall a theorem of C. Gerhardt [Ger83, Theorem 3.1]:

Theorem 4.5. *(C. Gerhardt) Let $\mathcal{S}$ be a limit on compact subsets of a sequence of spacelike surfaces in a globally hyperbolic spacetime. Then if $\mathcal{S}$ contains a segment of a null geodesic, this segment has to be maximal, i.e. it extends to the boundary of M.*

So, if Σ contains a lightlike segment, it would extend to the boundary of M and then Σ would go out of the convex core, which is impossible as Σ is a limit of surfaces embedded in the convex core.

It follows that the only lightlike locus possible is a union of rays such that each ray intersects two singular lines: one in its past, the other in its future. Thus, the lightlike locus of Σ lies in the set of lightlike rays from a particle to another one. Moreover, we prove in Section 4.3 that Σ is orthogonal to the singular lines hence cannot contain lightlike segment going from a particle to another one. So the orthogonality to the singular locus implies that Σ is nowhere lightlike.

Theorem 4.6. *Outside the lightlike and singular loci, Σ has everywhere vanishing mean curvature.*

Proof. To prove this statement, we are going to use the theory of pseudo-holomorphic curves. A good reference for this theory is [LA94]. Recall that a pseudo-holomorphic curve is a map f from a Riemann surface Σ into an almost complex manifold (W, J) such that $df(T\Sigma)$ is stable by J. A sequence of pseudo-holomorphic curves is then a sequence of $(S_n, f_n, J_n)_{n\in\mathbb{N}}$ where for all $n \in \mathbb{N}$, S_n is a Riemann surface, J_n is an almost complex structure on W and $f_n : S_n \to (W, J_n)$ is such that $df_n(TS_n)$ is stable by J_n. If the almost complex structures J_n are integrable, the pseudo-holomorphic curves are holomorphic. Classical results of complex analysis imply that $\mathcal{C}^0$-convergence for holomorphic curves provides $\mathcal{C}^\infty$-convergence and the limit is again a holomorphic curve.

Here we are going to adapt the construction of [Lab94, Section 3] to the case of spacelike surfaces embedded into Lorentzian 3-manifolds. In [Lab94, Section 3], the author lifts a prescribed mean curvature surface embedded in a Riemannian manifold M into a pseudo-holomorphic curve in a vector bundle over a Grassmanian manifold. Precisely, he defines an almost-complex structure J_H on the normal bundle F over G, the Grassmanian of 2-planes in M, such that if a surface $\Sigma \subset M$ has mean curvature H, then its lifting by Gauss' map into F is pseudo-holomorphic with respect to J_H.

Denote by $H_i N$ the mean curvature field of S_i with respect to the metric g, where N is the unit future pointing vector field normal to S_i. As the metrics g and g_i coincide except in the open set $U^i = \bigcup_{j=1}^n U_j^i$, then

$$(H_i)_{|S_i \setminus U^i} = 0. \tag{1}$$

It follows that $H_i \xrightarrow{\mathcal{C}^\infty} 0$ outside the singular and lightlike loci.

Extend H_iN to the whole M by the conditions $\nabla_N H_i = 0$ and $\nabla_N N = 0$ (where ∇ is the Levi-Civita connection with respect to g). Denote by $G_2(M)$ the Grassmanain manifold of spacelike oriented plane in M, that is the vector bundle over M whose fiber at a point m is the Grassmanian of oriented spacelike plane into $T_mM \cong \mathbb{R}^{2,1}$. Define $\pi : F \to G_2(M)$ the line bundle over $G_2(M)$ whose fiber at a spacelike plane $P \subset T_mM$ is the orthogonal to P, denoted by $P^\perp$. It follows that H_iN defines a section of F and a spacelike surface S has mean curvature H_i if and only if its lifting $\varphi(S)$ by Gauss' map into F is tangent to this section.

We are going to define an almost-complex structure $\mathcal{J}_i$ on TF such that $\varphi(S_i)$ is $\mathcal{J}_i$-pseudo-holomorphic. Let $P \in G_2(M)$, where $P \subset T_mM$. We have the following decomposition:

$$T_P(G_2(M)) = \mathrm{Hom}(P, P^\perp) \oplus T_mM.$$

It follows that, using the decomposition $T_mM = P \oplus P^\perp$ that for $x \in F$, with $\pi(x) = P \subset T_mM$ we have:

$$T_xF = T_x\pi^{-1}(P) \oplus T_PG_2(M) = P^\perp \oplus \mathrm{Hom}(P, P^\perp) \oplus P \oplus P^\perp. \tag{2}$$

As the induced metric on a spacelike plane is not degenerate, we have a canonical identification $\mathrm{Hom}(P, P^\perp) \cong P$ (this identification sends $u \in P$ to the map $u^\sharp \in \mathrm{Hom}(P, P^\perp)$ defined by $u^\sharp(v) := g(u, v)N$).

Using (2), we define the subbundle $W \subset TF$ by

$$W(x) = \{0\} \oplus P \oplus P \oplus \{0\} \subset T_x(F),$$

where $\pi(x) = P \subset T_mM$. It follows that the lifting $d\varphi(S_i)$ lies in W.

Define $\mathcal{J}_i$ on W by

$$\mathcal{J}_i(u, v) = (-J_0(u) + 2H_iJ_0(v), J_0(v)),$$

where J_0 is the classical complex structure on P induced by its orientation.

Proposition 4.7. *$\varphi(S_i)$ is a $\mathcal{J}_i$-pseudo holomorphic curve.*

Proof. The tangent bundle to $\varphi(S_i)$ is generated by $(B_i(e_1), e_1)$ and $(B_i(e_2), e_2)$ where (e_1, e_2) is a direct orthonormal framing of S_i diagonalising the shape operator B_i of S_i. It follows that $B_i(e_1) = (k + h_1)e_1$ and $B_i(e_2) = (-k + h_2)e_2$ where $(h_1 + h_2) = 2H_i$.

So one sees that:

$$\mathcal{J}_i^2(u, v) = (-J_0(-J_0(u)+2H_iJ_0(v)), J_0^2(v)) = (-u+2H_iv-2H_iv, -v) = -(u, v),$$

and

$$\mathcal{J}_i(B(e_1), e_1) = \mathcal{J}_i((k + h_1)e_1, e_1) = (-(k + h_1)e_2 + 2H_ie_2, e_2) = (B_ie_2, e_2).$$

□

Moreover, as $U_i^j \underset{i\to\infty}{\longrightarrow} d_j$ for all j, then from (1), on each compact $K \subset M$ which intersects Σ but does not intersect any singular lines, there exists $i_K \in \mathbb{N}$ such that, for all $i > i_K$,

$$\mathcal{J}_{i_{|\varphi(K\cap S_i)}}(u,v) = (-J_0(u), J_0(v)).$$

Its almost-complex structure corresponds to the multiplication by $\sqrt{-1}$ for an identification between P and $\mathbb{C}$ (which exists as P is an oriented plane). So there exists a chart such that $\mathcal{J}_i$ is given by the multiplication by $\sqrt{-1}$. It follows that $\mathcal{J}_i$ is integrable and $\varphi(K \cap S_i)$ is a holomorphic curve. It follows that, on each compact $K \subset M$ which does not intersect the singular lines nor the lightlike locus of Σ (one needs $K \cap \Sigma$ to be spacelike to define $\varphi(K \cap \Sigma)$), $\varphi(K \cap \Sigma)$ is a limit of holomorphic curves. As it is true for each compact K satisfying the above properties, it follows

$$\varphi(S_i) \underset{i\to\infty}{\overset{\mathcal{C}^\infty}{\longrightarrow}} \mathcal{S}_\infty$$

and $\mathcal{S}_\infty = \varphi(\Sigma)$ is a holomorphic curve for the complex structure defined on the lifting of Σ outside the lighlike and singular loci by:

$$\mathcal{J}_\infty(u,v) = (-J_0(u), J_0(v)).$$

Moreover, as $\mathcal{S}_\infty$ projects on Σ,

$$S_i \underset{i\to\infty}{\overset{\mathcal{C}^\infty}{\longrightarrow}} \Sigma.$$

Note that the arguments of [Lab94, Introduction] can be adapted in the Lorentzian case (in the same way as before) to show that a $\mathcal{J}_\infty$-holomorphic curve is either a curve contained in the fiber of the projection, or a lifting of a nowhere timelike surface in M with vanishing mean curvature outside the lightlike and singular loci (or a union of the two). Hence, the surface Σ has everywhere vanishing mean curvature outside its lightlike and singular loci.

Remark 4.2. If a piece of $\mathcal{S}_\infty$ is contained in the fiber of the projection, it follows that the fiber should contain at least a subspace of complex 1. Such a phenomenon corresponds to a singularity of a principal curvature of Σ and, because Σ has vanishing mean curvature, to a singularity of Gauss' curvature. Hence, if Σ is orthogonal to the singular locus and its curvature is nowhere infinite, it is a maximal surface.

□

4.3 Σ is orthogonal to the singular locus

The orthogonality to the singular locus will be proved considering the link of the surface at a singular point $p = d \cap \Sigma$, that is essentially the set of rays from p that are tangents to the surface. In this section, we see locally the surface as the graph of a function u over $D_\alpha = D_\alpha(0,r) = ((0,r) \times [0,\alpha)) \cup \{0\}$, a small disk contained in the totally geodesic plane orthogonal to d passing through p (it follows that $u(0) = 0$).

First, we describe the link at a regular point of an AdS convex GHM manifold, then the link at a singular point. The link of a surface at a smooth point is a circle in a sphere with an angular metric (called **HS-surface** in [Sch98]). However, as the intersections of Σ with the singular lines are not necessary smooth, we will define the link as the domain contained between the two curves given by the limsup and liminf at zero of $\dfrac{u(\rho,\theta)}{\rho}$.

The link of a point Consider $p \in M$ a regular point of an AdS convex GHM manifold, so T_pM is identified with the Minkowski 3-space $\mathbb{R}^{2,1}$. As the link of p (denoted by $\mathcal{L}_p$) is the set of rays from p, it is the set of half-lines in $\mathbb{R}^{2,1}$. Geometrically, $\mathcal{L}_p$ is a 2-sphere, and the metric is given by the angle "distance". So one can see that $\mathcal{L}_p$ is divided into five subsets (depending if the ray is timelike, lightlike or spacelike and if it is future or past pointed):

- The set of future and past pointed timelike rays that carry a hyperbolic metric.
- The projectivisation of the light cone, which defines two circles called **past and future lightlike circles**.
- The set of spacelike rays which carries a de Sitter metric.

Now, to obtain the link of a singular point of angle $\alpha \leq 2\pi$, we just have to cut $\mathcal{L}_p$ along two meridian separated by an angle α and glue by a rotation. We get a surface denoted $\mathcal{L}_{p,\alpha}$ (see Figure 2).

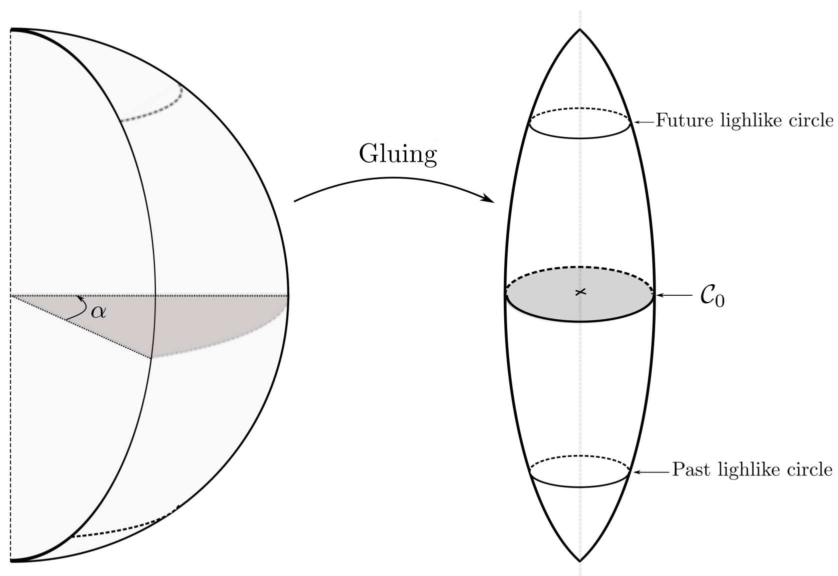

Figure 2: Link at a singular point

The link of a surface Let S be a smooth surface in M and $p \in M$ a point of S. The space of rays from p tangent to S is just the projection of the tangent plane and so describe a circle in $\mathcal{L}_p$. We denote this circle by $\mathcal{C}_{S,p}$. Obviously, if S is a spacelike surface, $\mathcal{C}_{S,p}$ is a spacelike geodesic in the de Sitter domain of $\mathcal{L}_p$ and if S is timelike or lightlike, $\mathcal{C}_{S,p}$ intersects one of the timelike circle in $\mathcal{L}_p$.

Now, if $p \in S$ belongs to a singular line, we define the link of S at p as the domain $\mathcal{C}_{S,p}$ delimited by the limsup and the liminf of $\dfrac{u(\rho,\theta)}{\rho}$. When $\mathcal{C}_{S,p}$ is a smooth curve in $\mathcal{L}_{p,\alpha}$, an important fact is that the angle of the conical singularity of S at p is given by the length of $\mathcal{C}_{S,p}$. In particular, we have the following result:

Proposition 4.8. *If the link of a spacelike surface at an intersection with a singular line is regular, then at this point, the surface carries a conical singularity of angle at most the angle of the singular line.*

Proof. The angle of the cone singularity of S at p is given by $\lim_{\rho\to 0}\frac{l(C_\rho)}{\rho}$ where $C_\rho \subset S$ is the circle of radius $\rho > 0$ centered at p. Hence, $\lim_{\rho\to 0}\frac{l(C_\rho)}{\rho} = l(\mathcal{C}_{S,p})$: the length of the curve $\mathcal{C}_{S,p}$ with respect to the metric of $\mathcal{L}_{p,\alpha}$.

However, $\alpha = l(\mathcal{C}_{0,p})$ where $\mathcal{C}_{0,p}$ is the link of a surface orthogonal to the singular locus. So, $\mathcal{C}_{0,p}$ is the orthogonal projection of $\mathcal{C}_{S,p}$. As in de Sitter domains, the orthogonal projection increases length, we get that the length of $l(\mathcal{C}_{S,p}) \leq l(\mathcal{C}_{0,p}) = \alpha$. □

Another important result is the following one:

Proposition 4.9. *Let S be a nowhere timelike surface which intersects a singular line of angle $\alpha < \pi$ at a point p. If $\mathcal{C}_{S,p}$ intersects a lightlike circle in $\mathcal{L}_{p,\alpha}$, then $\mathcal{C}_{S,p}$ does not cross $\mathcal{C}_{0,p}$. That is, $\mathcal{C}_{S,p}$ remains strictly in one hemisphere (where a hemisphere is a connected component of $\mathcal{L}_{p,\alpha} \setminus \mathcal{C}_{0,p}$).*

Proof. Fix a non-zero vector $u \in T_p(S)$ and for $\theta \in [0, \alpha)$, denote by v_θ the vector making an angle θ with u. Suppose that v_{θ_0} corresponds to the direction where $\mathcal{C}_{S,p}$ intersects a lightlike circle, for example, the future lightlike circle. As the surface is nowhere timelike, S remains in the future of the lightlike plane containing v_{θ_0}. But the link of a lightlike plane at a non singular point p is a great circle in $\mathcal{L}_p$ which intersects the two differents lightlike circles at the directions given by v_{θ_0} and $v_{\theta_0+\pi}$. So it intersects $\mathcal{C}_{0,p}$ at the directions corresponding to $v_{\theta_0 \pm \pi/2}$.

Now, if p belongs to a singular line of angle $\alpha < \pi$, the link of the lightlike plane which contains v_{θ_0} is obtained by cutting the link $T_p(M)$ along the direction of $v_{\theta_0 \pm \alpha/2}$ and gluing the two wedges by a rotation (see the Figure 2). So, the link of our lightlike plane remains in the upper hemisphere, which implies the same for $\mathcal{C}_{S,p}$. □

Remark 4.3. By contraposition, we get that if $\mathcal{C}_{S,p}$ intersects $\mathcal{C}_{0,p}$, it does not intersect a lightlike circle.

If $u \in \mathcal{C}^1(D_\alpha)$, we get that there exists $\eta > 0$ (depending of α) such that:

- If $\partial_\rho u(0, \theta_0) = 1$ (and so θ_0 corresponds to the direction of the lightlike vector), then
$$u(\rho, \theta) \geq \eta.\rho \ \forall \theta \in [0, \alpha), \ \rho \ll 1. \tag{3}$$
- If $\partial_\rho u(0, \theta_1) = 0$ (that is $\mathcal{C}_{S,p}$ intersects $\mathcal{C}_{0,p}$), then
$$u(\rho, \theta) \leq (1 - \eta).\rho \ \forall \theta \in [0, \alpha) \ \rho \ll 1. \tag{4}$$

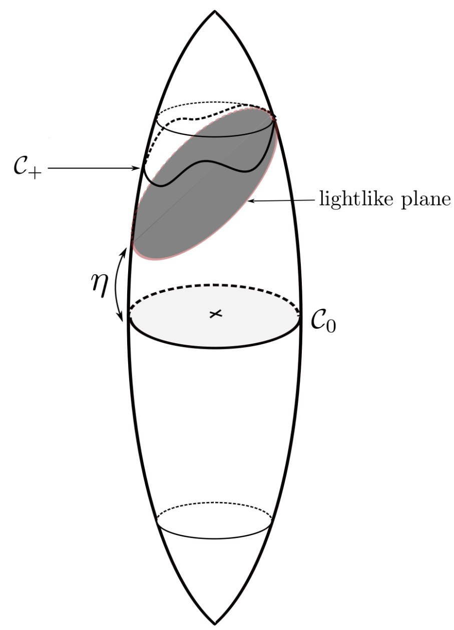

Figure 3: The link remains in the upper hemisphere

These two results will be used in the next part.

Link of Σ and orthogonality. As said above, our maximal surface is not necessary smooth at its intersections with the singular locus. Let $p \in \Sigma$ be an intersection with a singular line d of angle $\alpha < \pi$. As before, consider Σ as the local graph of a function
$$u : D_\alpha \to \mathbb{R}$$
in a neighborhood of p. We consider the "augmented" link of p, that is, the connected domain contained between the curves $\mathcal{C}_\pm$, where $\mathcal{C}_+$ is the curve corresponding to $\limsup_{\rho \to 0} \frac{u(\rho, \theta)}{\rho}$, and $\mathcal{C}_-$ corresponding to the liminf. We denote $\mathcal{C}_{\Sigma,p} \subset \mathcal{L}_{p,\alpha}$ the compact connected domain between $\mathcal{C}_+$ and $\mathcal{C}_-$.

Lemma 4.10. *The curves $\mathcal{C}_+$ and $\mathcal{C}_-$ are $\mathcal{C}^{0,1}$.*

Proof. We give the proof for $\mathcal{C}_-$ (the one for $\mathcal{C}_+$ is analogue). For $\theta \in [0, \alpha)$, denote by

$$k(\theta) := \liminf_{\rho \to 0} \frac{u(\rho, \theta)}{\rho}.$$

Fix $\theta_0, \theta \in [0, \alpha)$. By definition, there exists a decreasing sequence $(\rho_k)_{k \in \mathbb{N}} \subset \mathbb{R}_{>0}$ such that $\lim_{k \to \infty} \rho_k = 0$ and

$$\lim_{k \to \infty} \frac{u(\rho_k, \theta_0)}{\rho_k} = k(\theta_0).$$

As Σ is nowhere timelike, for each $k \in \mathbb{N}$, Σ remains in the cone of spacelike and lightlike geodesic from $((\rho_k, \theta_0), u(\rho_k, \theta_0)) \in \Sigma$. That is,

$$|u(\rho_k, \theta) - u(\rho_k, \theta_0)| \leq d_a(\theta, \theta_0)\rho_k,$$

where d_a is the angular distance between two directions. So we get

$$\lim_{k \to \infty} \frac{u(\rho_k, \theta)}{\rho_k} \leq k(\theta_0) + d_a(\theta, \theta_0),$$

and so

$$k(\theta) \leq k(\theta_0) + d_a(\theta, \theta_0).$$

On the other hand, for all $\epsilon > 0$ small enough, there exists $R > 0$ such that, for all $\rho \in (0, R)$ we have:

$$u(\rho, \theta_0) > (k(\theta_0) - \epsilon)\rho.$$

By the same argument as before, because Σ is nowhere timelike, we get

$$|u(\rho, \theta) - u(\rho, \theta_0)| \leq d_a(\theta, \theta_0)\rho,$$

that is

$$u(\rho, \theta) \geq u(\rho, \theta_0) - d_a(\theta, \theta_0)\rho.$$

So

$$u(\rho, \theta) > (k(\theta_0) - \epsilon)\rho - d_a(\theta, \theta_0)\rho,$$

taking $\epsilon \to 0$, we obtain

$$k(\theta) \geq k(\theta_0) - d_a(\theta, \theta_0).$$

So the function k is 1-Lipschitz □

Theorem 4.11. *$\mathcal{C}_{\Sigma,p} = \mathcal{C}_{0,p}$, and so Σ is orthogonal to the singular line d.*

To prove this statement, we distinguish these two cases:

If $\mathcal{C}_{\Sigma,p}$ intersects a lightlike circle. For example $\mathcal{C}_+$ intersects the upper lightlike circle (the proof is analogue if $\mathcal{C}_-$ intersects the lower lightlike circle). The proof in this case is based on the following lemma:

Lemma 4.12. *In the above situation,* $\liminf_{\rho\to 0}\frac{u(\rho,\theta)}{\rho}\geq \eta$ *for all* $\theta\in[0,\alpha)$.

Proof. As $\mathcal{C}_+$ intersects the upper timelike circle, there exist $\theta_0\in[0,\alpha)$, and $(\rho_k)_{k\in\mathbb{N}}\subset\mathbb{R}_{>0}$ a strictly decreasing sequence, converging to zero, such that

$$\lim_{k\to\infty}\frac{u(\rho_k,\theta_0)}{\rho_k}=1.$$

From (3), for a fixed $\eta>\widetilde{\eta}$, there exist $k_0\in\mathbb{N}$ such that:

$$\forall k>k_0,\ u(\rho_k,\theta)\geq\widetilde{\eta}\rho_k\ \forall\theta\in[0,\alpha[$$

.

As Σ has vanishing mean curvature outside its intersections with the singular locus, we can use a maximum principle. Namely, if a strictly future-convex surface S intersects Σ at a point x outside the singular locus, then Σ lies locally in the future of S (the case is analogue for past-convex surfaces). It follows that on an open set $V\subset D_\alpha$, $\sup_{x\in V}u(x)=\sup_{x\in\partial V}u(x)$ and $\inf_{x\in V}u(x)=\inf_{x\in\partial V}u(x)$

Now, consider the open annulus $A_k:=D_k\setminus\overline{D}_{k+1}\subset D_\alpha$ where D_k is the open disk of center 0 and radius ρ_k. As Σ is a maximal surface, we can apply the maximal principle to u on A_k, we get:

$$\inf_{A_k}u=\min_{\partial A_k}u\geq\widetilde{\eta}\rho_{k+1}.$$

So, for all $\rho\in[0,r)$, there exists $k\in\mathbb{N}$ such that $\rho\in[\rho_{k+1},\rho_k]$ and

$$u(\rho,\theta)\geq\widetilde{\eta}\rho_{k+1}. \tag{5}$$

We obtain that, $\forall\theta\in[0,\alpha)$, $u(\rho,\theta)>0$ and so $\liminf_{\rho\to 0}\frac{u(\rho,\theta)}{\rho}\geq 0$.

Now, suppose that

$$\exists\theta_1\in[0,\alpha)\text{ such that }\liminf_{\rho\to 0}\frac{u(\rho,\theta_1)}{\rho}=0,$$

then there exists $(r_k)_{k\in\mathbb{N}}\subset\mathbb{R}_{>0}$ a strictly decreasing sequence converging to zero with

$$\lim_{k\to\infty}\frac{u(r_k,\theta_1)}{r_k}=0.$$

Moreover, we can choose a subsequence of $(\rho_k)_{k\in\mathbb{N}}$ and $(r_k)k\in\mathbb{N}$ such that $r_k\in[\rho_{k+1},\rho_k[\ \forall k\in\mathbb{N}$.

This implies, by (4) that there exist $k_1\in\mathbb{N}$ such that

$$\forall k>k_1,\ u(r_k,\theta)\leq(1-\widetilde{\eta})r_k\ \forall\theta\in[0,\alpha).$$

Now, applying the maximum principle to the open annulus $B_k := \mathcal{D}'_k \setminus \overline{\mathcal{D}'}_{k+1} \subset D_\alpha$ where $\mathcal{D}'_k$ is the open disk of center 0 and radius r_k, we get:

$$\sup_{B_k} u = \max_{\partial B_k} u \leq (1-\widetilde{\eta})r_k.$$

And so we get that for all $\rho \in [0,r)$ there exists $k \in \mathbb{N}$ with $\rho \in [r_{k+1}, r_k]$ and we have:

$$u(\rho,\theta) \leq (1-\widetilde{\eta})r_k \leq (1-\widetilde{\eta})\rho_k. \tag{6}$$

Now we are able to prove the lemma:

Take $\epsilon < 1$, as $\lim \dfrac{u(\rho_k,\theta_0)}{\rho_k} = 1$, there exist $k_3 \in \mathbb{N}$ such that:

$$\forall k > k_3,\ u(\rho_k,\theta_0) \geq (1-\epsilon\widetilde{\eta})\rho_k.$$

Using (6) we get:

$$(1-\epsilon.\widetilde{\eta}).\rho_k \leq u(\rho_k,\theta_0) \leq (1-\widetilde{\eta})\rho_{k+1}, \text{ and so } \frac{\rho_{k+1}}{\rho_k} \leq \frac{1-\epsilon.\widetilde{\eta}}{1-\widetilde{\eta}}. \tag{7}$$

Now, as $\lim \dfrac{u(r_k,\theta_0)}{r_k} = 0$, there exist $N' \in \mathbb{N}$ such that, for all k bigger than N' we have:

$$u(r_k,\theta_1) \leq \epsilon.\widetilde{\eta}.r_k \leq \epsilon.\widetilde{\eta}.\rho_k.$$

Using (5), we get:

$$\widetilde{\eta}.\rho_{k+1} \leq u(r_k,\theta_0) \leq \epsilon.\widetilde{\eta}.\rho_k, \text{ and so } \frac{\rho_{k+1}}{\rho_k} \leq \epsilon. \tag{8}$$

But as $\epsilon < 1$, the conditions (7) and (8) are incompatibles. □

Using this lemma, the proof of the theorem in this case follows: as the curve $\mathcal{C}_-$ does not cross $\mathcal{C}_{0,p}$ and is contained in the de Sitter domain, we obtain $l(\mathcal{C}_-) < l(\mathcal{C}_{0,p})$ (where l is the length). For $D_r \subset D_\alpha$ the disk of radius r and center 0 and $A_g(u(D_r))$ the area of the graph of $u_{|D_r}$, we get:

$$\begin{aligned} A_g(u(D_r)) &\leq \int_0^r l(\mathcal{C}_-)\rho d\rho \\ &< \int_0^r l(\mathcal{C}_{0,p})\rho d\rho. \end{aligned}$$

The first inequality comes from the fact that $\int_0^r l(\mathcal{C}_-)\rho d\rho$ corresponds to the area of a flat piece of surface with link $\mathcal{C}_-$ which is bigger than the area of a curved surface (because we are in a Lorentzian manifold).

If we consider the local deformation of Σ sending a neighborhood $U \cap \Sigma$ of p into the totally geodesic disk orthogonal to the singular line (making the resulting surface connected). The last inequality shows that this deformation would strictly increase the area of Σ. However, Σ is a limit of a sequence of maximal surfaces and so maximize the area. So there exists no such deformation and $\mathcal{C}_{\Sigma,p}$ cannot cross the lightlike circles.

□

Corollary 4.13. Σ *is nowhere lightlike.*

Proof. The proof directly follow from Remark 4.1. In fact, if the link of Σ at a singular point does not contain any lightlike direction, Σ cannot contain any lightlike ray going from a particle to another one. □

Now, consider the second case for the orthogonality.

If $\mathcal{C}_{\Sigma,p}$ does not cross the lightlike circle. Consider

$$H := \{x \in \mathbb{R}^{2,2} \text{ s.t. } q(x) < 0\}.$$

This manifold is foliated by the submanifolds $H_\tau := \{x \in \mathbb{R}^{2,2} \text{ s.t. } q(x) = -\tau^2\}$ for $\tau \in \mathbb{R}_{>0}$ wich are Lorentzian homogeneous manifolds of constant scalar curvature $\kappa(H_\tau) = -\dfrac{1}{\tau^2}$. Moreover, we have a canonical embedding $i : AdS^3 \to H$ and all the H_τ are homothetic to $i(AdS^3) = H_1$ by the application

$$\begin{array}{cccc} \phi_\tau & : & \mathbb{R}^{2,2} & \to & \mathbb{R}^{2,2} \\ & & x & \mapsto & \tau x. \end{array}$$

Now, consider two timelike half-planes P_1 and P_2 in AdS^3 making an angle $\alpha < \pi$ and intersecting along a timelike line l. For $j = 1, 2$, define $\mathcal{P}_j := \{\phi_t(i(P_j)),\ t \in \mathbb{R}_{>0}\}$. The $\mathcal{P}_j$ are timelike half-hyperplanes in H intersecting along a timelike 2-plane and making an angle α. We define H_α as the space obtained by cutting H along the $\mathcal{P}_j$ and gluing the angular sector of angle α by the rotation of angle $2\pi - \alpha$ fixing $\mathcal{P}_1 \cap \mathcal{P}_2$. By construction AdS^3_α canonically embeds into H_α (by an application again denoted by i) and the slices $H_{\tau,\alpha} := H_\alpha \cap q^{-1}(-\tau)$ are all homothetic to $i(AdS^3_\alpha) = H_{1,\alpha}$ by the application ϕ_τ (here, there is an abuse of notation because we consider ϕ_τ and q as defined on H_α). We note $g'_{\tau,\alpha} = \phi_\tau^*(\mathsf{h}_{\tau,\alpha})$ where $\mathsf{h}_{\tau,\alpha}$ is the induced metric on $H_{\tau,\alpha}$. Then, define $AdS^3_{\alpha,\tau}$ as the completion of $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}/\alpha\mathbb{Z}$ with the metric

$$g_\tau = -dt^2 + \cos^2(t/\tau)(d\rho^2 + \tau^2 \sinh^2(\rho/\tau) d\phi^2).$$

Here $t \in \mathbb{R}$, $\rho \in \mathbb{R}_{>0}$, $\phi \in \mathbb{R}/\alpha\mathbb{Z}$. One sees from the expression of g_τ that $AdS^3_{\alpha,\tau}$ is a Lorentzian manifold with constant scalar curvature $-\frac{1}{\tau^2}$ (it is obtained from the metric od AdS^3_α by a dilatation of factor τ) with one conical singularity of angle α at the line $\{\rho = 0\}$. It follows that $AdS^3_{\alpha,\tau}$ is isometric to $H_{\alpha,\tau}$.

Let p be the intersection of Σ with a singular line of angle α in M. By definition, there exist a neighborhood $V \subset M$ of p and an isometry ψ which sends V to a neighborhood of a point in AdS^3_α lying on the singular line l. Let $U = \psi(V \cap \Sigma)$. It is a piece of surface in AdS^3_α which intersects l at a point 0. Denote by $l_\tau = \phi_\tau(i(l))$, $0_\tau = \phi_\tau(i(0))$ and $U_\tau = \phi_\tau(i(U))$. As $H_{\alpha,\tau}$ is isometric to $AdS^3_{\alpha,\tau}$, we can consider that l_τ, 0_τ and U_τ are lying in $AdS^3_{\alpha,\tau}$.

Thanks to the identification (as manifolds) of $AdS^3_{\tau,\alpha}$ for all $\tau > 0$ by the coordinate system $(t,\rho,\phi) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}/\alpha\mathbb{Z}$, we can take a compact set K containing 0 in $\mathbb{D} \times \mathbb{R}$ and consider the sequence $(U_n \cap K)_{n\in\mathbb{N}}$. It is a sequence of graph of Lipschitzian functions contained in a compact set. Then, up to a subsequence, $U_n \xrightarrow{\mathcal{C}^0} U_\infty$.

For each compact set $K' \subset K$ which does not intersect the singular line, lift the sequence $(U_n \cap K')_{n\in\mathbb{N}}$ into a sequence of holomorphic curve in the normal bundle of the Grassmanian of oriented spacelike 2 planes in $\mathbb{D} \times \mathbb{R}$ (as in the proof of Theorem 4.6). Note that it is possible since $U_n \cap K'$ is a piece of spacelike surface in $AdS^3_{n,\tau}$ with vanishing mean curvature for all $n \in \mathbb{N}$. It follows from the same arguments that for each compact $K' \subset K$ which does not intersect the particle, $U_n \cap K' \xrightarrow{\mathcal{C}^\infty} U_\infty \cap K'$. So $U_n \xrightarrow{\mathcal{C}^\infty} U_\infty$, outside the singular line.

Moreover, one sees from the expression of g_n that $g_n \xrightarrow{\mathcal{C}^\infty} \mathsf{g}$, where

$$\mathsf{g} = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

is the Minkowski metric with cone singularity (recall that $t \in \mathbb{R}$, $\rho \in \mathbb{R}_{>0}$, $\phi \in \mathbb{R}/\alpha\mathbb{Z}$); that is the metric obtained by cutting $\mathbb{R}^{2,1}$ along two timelike halfplanes making an angle α and intersecting along the timelike line $d := \{\rho = 0\}$, then gluing by a rotation. We denote the singular axis of $\mathbb{R}^{2,1}_\alpha$ by l_∞.

Let $\mathcal{N} : U_\infty \setminus \{0_\infty\} \longrightarrow U^i\mathbb{R}^{2,1}_\alpha$ where $0_\infty := d \cap U_\infty$ and $U^1\mathbb{R}^{2,1}_\alpha$ is the unit tangent bundle of $\mathbb{R}^{2,1}_\alpha$. We have the following lemma:

Lemma 4.14. $\mathcal{N} : U_\infty \setminus \{0_\infty\} \longrightarrow \mathbb{H}^2_\alpha$ *and is holomorphic (with respect to the complex structure on* $\mathbb{H}^2_\alpha$ *defined by the opposite orientation).*

Proof. Fix a point $p \in U_\infty \setminus \{0_\infty\}$ and a simple closed loop $\gamma : [0,1] \longrightarrow U_\infty \setminus \{0_\infty\}$ based at p. Note that, by construction, $\mathbb{R}^{2,1}_\alpha$ contains $\mathbb{H}^2_\alpha$ as an embedded spacelike surface orthogonal to l_∞ (here $\mathbb{H}^2_\alpha$ is obtained by gluing the intersection of the angular sector of angle α in $\mathbb{R}^{2,1}$ with the future component of the hyperboloid by the rotation φ_α of angle α preserving the central axis).

Fix $\hat{p} \in \widetilde{\mathbb{R}}^{2,1}_\alpha$ a lifting of p in the universal cover of $\mathbb{R}^{2,1}_\alpha \setminus \{l_\infty\}$ and denote by $\widetilde{\gamma} : [0,1] \longrightarrow \widetilde{U}_\infty \subset \widetilde{\mathbb{R}}^{2,1}_\alpha$ a piece of the lifting of $\gamma([0,1])$ with $\widetilde{\gamma}(0) = \hat{p}$. Note that $\mathbb{R}^{2,1}_\alpha \setminus \{0_\infty\}$ is obtained by taking the quotient of $\widetilde{\mathbb{R}}^{2,1}_\alpha$ by the holonomy representation ρ which sends $[\gamma]$, the homotopy class of γ, to φ_α (in fact, $\widetilde{\mathbb{R}}^{2,1}_\alpha$ is obtained by gluing together infinitely many angular sector of angle α, so the action of φ_α extends naturally to $\widetilde{\mathbb{R}}^{2,1}_\alpha$).

It suffices to show that the set $\mathcal{N}(\widetilde{\gamma}([0,1]))$ lies in $\widetilde{\mathbb{H}}^2_\alpha$ and so $\mathcal{N}(\gamma([0,1]) \subset \widetilde{\mathbb{H}}^2_\alpha/\rho([\gamma]) = \mathbb{H}^2_\alpha \setminus \{0_\alpha\}$ (where $0_\alpha = l_\infty \cap \mathbb{H}^2_\alpha$). As γ does not intersect 0_∞, for each point $x \in \widetilde{\gamma}([0,1])$ there is an isometry Ψ from a neighborhood $\mathcal{U}$ of x in $\widetilde{\mathbb{R}}^{2,1}_\alpha$ to an open set of $\mathbb{R}^{2,1}$. Let $V := \Psi(\mathcal{U} \cap \widetilde{U}_\infty)$; it is a piece of spacelike surface in $\mathbb{R}^{2,1}$ whose set of unit future pointing normal vector $N(V)$ lies in $\mathbb{H}^2$. However, if one wants to send $N(V)$ by Ψ^{-1} to a subset of $\widetilde{\mathbb{H}}^2_\alpha$ corresponding to unit future pointing vectors normal to $\mathcal{U} \cap \widetilde{U}_\infty$, we have to be sure that

$N(V) \cap 0_\alpha = \emptyset$. It is not true in general. However, denote by $K \subset V$ the set of points x such that the unit future pointing normal vector to V at x lies at 0_α; we have

Lemma 4.15. *K is discrete.*

Proof. Let $K = K_1 \cup K_2$ where $K_1 := \{x \in K,\ \det(B(x)) = 0\}$ and $K_2 := \{x \in K,\ \det(B(x)) \neq 0\}$ for B the shape operator of V. It follows that K_2 is the inverse image of 0_α by a regular map hence is discrete. Now, if $x \in K_1$, then for each $y \in V$ in a neighborhood of x, the unit future pointing normal vector to V at y is given by parallel translation along the unique geodesic joining x to y hence is different from 0_α. So K_1 is discrete. □

Now we can lift $N(V \setminus K)$ to a subset of $\widetilde{\mathbb{H}}^2_\alpha$. Applying this construction to a finite open covering of $\widetilde{\gamma}([0,1])$ glued in a good way, we obtain that the set of unit future pointing vector normal to $\widetilde{U}_\infty$ at $\widetilde{\gamma}([0,1])$ is a curve in $\widetilde{\mathbb{H}}^2_\alpha$ (except on a discrete subset) whose quotient by $\rho([\gamma])$ lies in $\mathbb{H}^2_\alpha \setminus \{0_\alpha\}$. That is $\mathcal{N}(\gamma([0,1])) \subset \mathbb{H}^2_\alpha \setminus \{0_\alpha\}$, except for a discrete set of points.

Hence, there exists a discrete set $\mathcal{K} \subset U_\infty$ such that

$$\mathcal{N}(U_\infty \setminus \mathcal{K}) \subset \mathbb{H}^2_\alpha \setminus \{0_\alpha\}.$$

However, as U_∞ is smooth at each $x \in \mathcal{K} \setminus \{0_\infty\}$, it admits a normal vector at x obtained by taking the limit of normal vectors to U_∞ at y when y tends to x. On sees by construction that $\mathcal{N}(K \setminus \{0_\infty\}) = 0_\alpha$. Hence $\mathcal{N} : U_\infty \setminus \{0_\infty\} \longrightarrow \mathbb{H}^2_\alpha$.

Now, $U_\infty \setminus \{0_\infty\}$ has everywhere vanishing mean curvature. So we can choose an orthonormal framing on $U_\infty \setminus \{0_\infty\}$ such that, with respect to this framing, the shape operator B of $U_\infty \setminus \{0_\infty\}$ as expression

$$B = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

So $\mathcal{N}^* I(x,y) = III(x,y) = I(B^2 x, y) = k^2 I(x,y)$ (where I and III are the first and third fundamental forms of U_∞ respectively). That is $\mathcal{N}$ is conformal and reverses the orientation and so is holomorphic with respect to the holomorphic structure defined by the opposite orientation of $\mathbb{H}^2_\alpha$. □

Lemma 4.16. *U_∞ is smooth at $\{0_\infty\}$ and orthogonal to l_∞.*

Proof. The induced metric g on U_∞ is Riemannian and carries a conical singularity of angle $\alpha' \leq \alpha$ at 0_∞ (from Proposition 4.8). Hence, there exists a unique hyperbolic metric g_0 with conical singularity of angle α' at 0_∞ in the conformal class of g. For such a g_0, there is an isometry from an open neighborhood of $0_{\alpha'} \in \mathbb{H}^2_{\alpha'}$ (where $0_{\alpha'}$ is the center of $\mathbb{H}^2_{\alpha'}$ in the disk model) to an open neighborhood of 0_∞ in U_∞. Such an isometry sends $0_{\alpha'}$ to 0_∞.

Note that $\mathbb{H}^2_{\alpha'}$ is obtained by gluing an angular sector $\mathbb{D}_{\alpha'}$ of angle α' between two half-lines in $\mathbb{D}^2$ (endowed with the hyperbolic metric) by a rotation. So it provides an isometric parameterization w of a neighborhood of $0_\infty \in U_\infty$ by $\mathbb{D}_{\alpha'}$. In the same way, we have an isometric parameterization w' of a neighborhood of

$0_\alpha \in \mathbb{H}^2_\alpha$ by $\mathbb{D}_\alpha$. Now, fix two uniformizations $\Psi_{\alpha'}$ and Ψ_α of $\mathbb{D}_{\alpha'}$ (respectively $\mathbb{D}_\alpha$) by the unit disk $\mathbb{D}$ which sends the center of $\mathbb{D}$ to $0_{\alpha'}$ (respectively 0_α). See Figure 4.

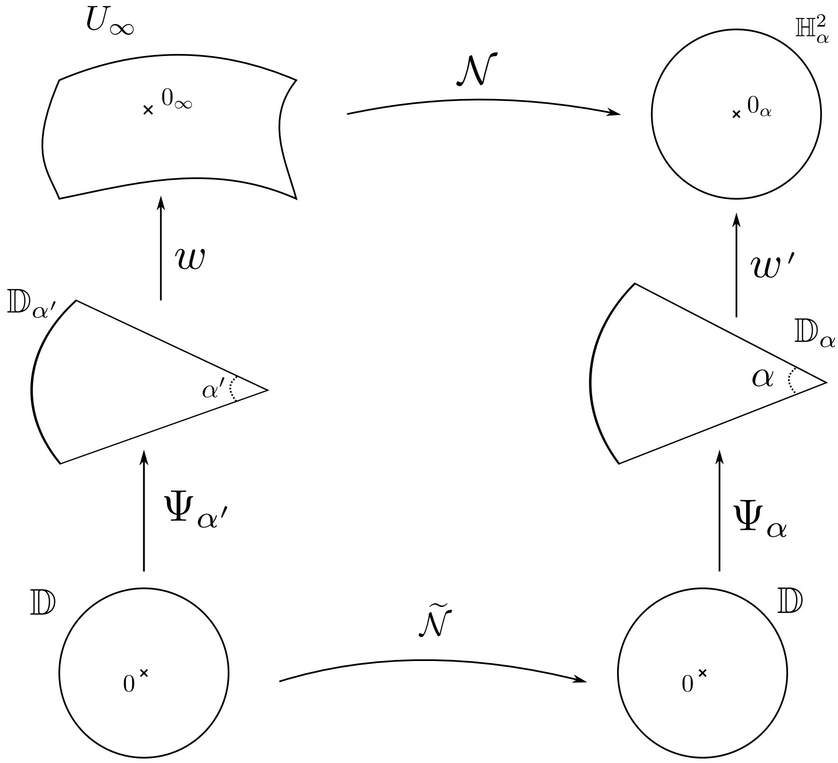

Figure 4: Holomorphic parameterization

Note that $|\Psi_{\alpha'}(z)|_{\mathbb{D}_{\alpha'}} \underset{0}{\sim} |z|^{\alpha'/2\pi}$.

Gauss' equation implies that Gauss' curvature K of U_∞ is given by $K = |w'^{-1}\mathcal{N}'(w)|^2_{\mathbb{D}_\alpha}$. And the expression of Gauss' map in the charts $(\mathbb{D}, \Psi_{\alpha'})$ and $(\mathbb{D}, \Psi_\alpha)$, denoted by $\widetilde{\mathcal{N}} : \mathbb{D} \setminus \{0\} \longrightarrow \mathbb{D}$ extends to a holomorphic map.

The singularity of $\widetilde{\mathcal{N}}$ at 0 can be of 3 differents types:

- removable (that is $\widetilde{\mathcal{N}}$ is holomorphic),
- a pole of order $k_0 > 0$ (that is $z^k\widetilde{\mathcal{N}}(z)$ tends to 0 at 0 for all $k > k_0$),
- essential (that is the Laurent series of $\mathcal{N}(z)$ has infinitely many non zeros negatives coefficients).

Suppose that $\widetilde{\mathcal{N}}$ as a pole of order $k_0 > 0$ at 0. Then,

$$|\widetilde{\mathcal{N}}'(z)| \underset{0}{\sim} \frac{a}{|z|^{k_0+1}}, \text{ for } a \in \mathbb{R}_{>0}.$$

Hence,

$$|w'^{-1}\mathcal{N}'(w)|_{\mathbb{D}_\alpha} \underset{0_{\alpha'}}{\sim} \left(\frac{a}{|w|^{k_0+1}}\right)^{\alpha/\alpha'}.$$

However, by Gauss-Bonnet, K is a L^1-function, and so $\widetilde{\mathcal{N}}$ cannot have a pole of order $k_0 > 0$. The same argument shows that $\widetilde{\mathcal{N}}$ cannot have an essential singularity at 0. So $\widetilde{\mathcal{N}}$ must have a removable singularity at 0.

In particular, $\mathcal{N}$ is defined at 0_∞, so U_∞ admits a unit future pointing normal vector at 0_∞ denoted n. As U_∞ is obtained by gluing a piece of surface with boundary $\hat{U}_\infty \subset \mathbb{R}^{2,1}$ by the rotation φ_α of angle $2\pi - \alpha$ fixing the central axis d, n is obtained by gluing $\hat{n}$, the unit future pointing vector normal to $\widetilde{U}_\infty$ at $\widetilde{U}_\infty \cap d$ by φ_α. The only way for gluing $\widetilde{n}$ onto itself is having $\widetilde{n}$ tangent to d, which implies that U_∞ is orthogonal to l_∞. □

Remark 4.4. As $U_n \xrightarrow{\mathcal{C}^\infty} U_\infty$, then $K_n \xrightarrow{\mathcal{C}^\infty} K_\infty$. In particular, it implies that K_1 is everywhere finite, which prevents the apparition of bubbles in the limit of $\varphi(S_i)$ in Theorem 4.6 (see Remark 4.2).

Now, we can prove the following statement:

Proposition 4.17. *Σ is orthogonal to d.*

Proof. For all $\tau \in \mathbb{R}_{>0}$ we define a function ψ_τ on a neighborhood of 0_τ in U_τ as follow: let $u_\tau \in T_{0_\tau} H_\tau$ be a future pointing vector tangent to d_τ. For x in a neighborhood $V_\tau \subset U_\tau$ of 0_τ, let $u_\tau(x)$ be the parallel transport of u_τ with respect to the Levi-Civita connection of $\mathsf{h}_{\tau,\alpha}$ and $\mathcal{N}_\tau(x)$ be a future pointing vector normal to U_τ at x. Set

$$\psi_\tau(x) := \frac{\mathsf{h}_{\tau,\alpha}(u_\tau(x), \mathcal{N}_\tau(x))}{\|u_\tau(x)\|_{\mathsf{h}_{\tau,\alpha}} . \|\mathcal{N}_\tau(x)\|_{\mathsf{h}_{\tau,\alpha}}}.$$

We note that ψ_τ does not depend on the choice of u_τ and $\mathcal{N}_\tau$. Moreover, as the surfaces U_τ differs by homothety which preserve $\{0_\tau\}$, the function $\psi_\tau(0)$ is constant. As U_∞ is orthogonal to l_∞, $\lim_{\tau\to\infty} \psi_\tau(0) = 0$ and so $\psi_1(0) = 0$. □

5 Uniqueness

In this section, we show the uniqueness part of Theorem 1.2:

Theorem 5.1. *$\Sigma \subset M$ is the unique maximal surface.*

Before, we recall the Jacobi field equation in semi-Riemannian manifolds of constant sectional survature:

Lemma 5.2. *Let (N, g) be a semi-Riemannian manifold of constant sectional curvature K and γ a spacelike or timelike geodesic of N. Then, a Jacobi field J over γ satisfies the equation:*

$$J'' - \epsilon K J = 0,$$

where $\epsilon = +1$ if γ is spacelike or $\epsilon = -1$ if γ is timelike.

Proof. A vector field J over γ is a Jacobi field if and only if, it satisfies:

$$J'' + R(J, \gamma')\gamma' = 0.$$

Taking the scalar product with J, we get

$$g(J'', J) = -g(R(J, \gamma')\gamma', J).$$

However, the sectionnal curvature is given by $K = \dfrac{g(R(J,\gamma')\gamma', J)}{g(\gamma', \gamma')g(J, J)}$. So we get:

$$g(J'' + \epsilon K J, J) = 0.$$

□

Proof of Theorem 5.1. For $\gamma : [0,1] \to M$ a timelike geodesic segment, we define the length of γ by

$$l(\gamma) := \int_0^1 (-g(\gamma'(t), \gamma'(t)))^{1/2} dt.$$

Suppose that there exist two differents maximal surfaces Σ_1 and Σ_2 in (M, g). Let

$$C := \sup_{\gamma \in \Gamma} \{l(\gamma)\}$$

where Γ is the set of timelike geodesic segments $\gamma : [0,1] \to M$ with $\gamma(0) \in \Sigma_1$ and $\gamma(1) \in \Sigma_2$. Consider $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ such that

$$\lim \int_0^1 (-g(\gamma_n'(t), \gamma_n'(t)))^{1/2} dt = C.$$

Suppose for example that γ_n is future directed for n big enough, and denote by $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ the sequences defined by $(\gamma_n(0))_{n\in\mathbb{N}}$ and $(\gamma_n(1))_{n\in\mathbb{N}}$ respectively. For $n \in \mathbb{N}$, choose a lifting $\widetilde{x}_n$ of x_n in the universal cover $\widetilde{M}$ of M. This choice fixes a lifting of the sequence $(x_n)_{n\in\mathbb{N}}$ which converges to $\widetilde{x}_0 \in \widetilde{\Sigma_1} \subset \widetilde{M}$. Moreover, it fixes a lifting of $(\gamma_n)_{n\in\mathbb{N}}$ and allows us to define $\widetilde{y}_n := \widetilde{\gamma}_n(1)$ (that is, it fixes a lifting of $(y_n)_{n\in\mathbb{N}}$). As the future of $\widetilde{x}_0$ intersects an open subset of $\widetilde{\Sigma}_2$ in a compact set which contains an infinite number of points, $(\widetilde{y}_n)$ converges to $\widetilde{y}_0$ (up to passing to a subsequence). Then, $\widetilde{y}_0$ projects to $y_0 \in \Sigma_2$ and C is equal to the length of the projection of the timelike geodesic segment joining $\widetilde{x}_0$ to $\widetilde{y}_0$. We denote by $\gamma \in \Gamma$ this geodesic segment.

There exists a tubular neighborhood U of γ and an isometry Ψ from U to a tubular neighborhood V of the central axis in AdS^3_α (for some α maybe equal to 2π if γ does not lie on a particle) which sends γ to the central axis. From now, denote Σ_1, Σ_2 and γ their image by Ψ in AdS^3_α. Choose Ψ such that it sends the tangent plane to Σ_1 at x_0 on $\mathcal{P}_1 := \{(\rho, \theta, t) \in AdS^3_\alpha,\ t = 0\}$. Recall that $AdS^3_\alpha := (\mathbb{D} \times \mathbb{R}, g_\alpha)$ where $g_\alpha = -dt^2 + \cos^2(t) h_\alpha$ and h_α is the hyperbolic metric on $\mathbb{D}$ with cone singularity of angle α defined in Section 4.

Lemma 5.3. *In this model of AdS^3_α, timelike goedesics orthogonal to $\mathcal{P}_1$ are given by $\{\rho = cte., \ \theta = cte.\}$ and totally geodesic planes orthogonal to γ at $\gamma(t_0)$ are given by the equation $\mathcal{P}_{\gamma(t_0)} := \{t = \gamma(t_0)\cosh(\rho)\}$.*

Proof. The first point is obvious from the expression of g_α. For the second point, note that $\mathcal{P}_{\gamma(t_0)}$ is obtained by revolution around the central axis of the deformation of a radial geodesic contained in $\mathcal{P}_1$ by the flow of a vector field normal to $\mathcal{P}_1$. Take a radial geodesic $\gamma_\theta := \{\theta = cte.\}$ (with $\gamma_\theta(0) = 0$) contained in $\mathcal{P}_1$ and, for N the unit future pointing vector field normal to $\mathcal{P}_1$, consider the Jacobi field given by

$$\begin{cases} J(0) = & \gamma(t_0)N \\ J'(0) = & 0 \end{cases}$$

It follows by Lemma 5.2 that the deformation of γ_θ is given by the equation $\{\theta = cte.,\ t = \gamma(t_0)\cosh(\rho)\}$, which gives the equation of $\mathcal{P}_{\gamma(t_0)}$ by revolution around the central axis. □

Denote by $k \geq 0$ (resp. $k' \geq 0$) the principal curvature of Σ_1 (resp. Σ_2) at x_0 (resp. y_0). We can suppose, without loss of generality, that $k \geq k'$. Take $u \in U^1_{x_0}\Sigma_1$ the principal direction corresponding to $-k$.

Fix $\epsilon > 0$ and consider the deformation $\gamma_\epsilon(t) := \exp(J(\gamma(t)))$ for all $t \in [0, 1]$ where J is the Jacobi field given by $J(0) = \epsilon u$ and $J'(0) = 0$ (note that $\gamma_\epsilon \in \Gamma$). It follows that γ_ϵ is orthogonal to $\mathcal{P}_1$ and so it is given by a straight line in the radial plane containing u. Finally, denote by x_ϵ and y_ϵ the intersections of γ_ϵ with Σ_1 and Σ_2 respectively, by p_i its intersection with $\mathcal{P}_i$, $i = 1, 2$ (where $\mathcal{P}_2$ is the totally geodesic plane tangent to Σ_2 at y_0) and by p_C its intersection with P_C: the equidistant surface at a distance C in the future of $\mathcal{P}_1$ (see Figure 5). We get

$$l(\gamma_\epsilon) - l(\gamma) = d_{\gamma_\epsilon}(x_\epsilon, p_1) + d_{\gamma_\epsilon}(p_C, p_2) \pm d_{\gamma_\epsilon}(p_2, y_\epsilon),$$

where d_{γ_ϵ} is the causal distance along γ_ϵ and the sign $\pm$ depends if y_ϵ lies in the future or in the past of p_2.

However, $d_{\gamma_\epsilon}(x_\epsilon, p_1) = k\epsilon^2 + o(\epsilon^2)$, $d_{\gamma_\epsilon}(p_C, p_2) > 0$ and $\pm d_{\gamma_\epsilon}(p_2, y_\epsilon) \geq -k'\epsilon^2$. So $l(\gamma_\epsilon) > C$, which is impossible because $\gamma_\epsilon \in \Gamma$ and $l(\gamma_\epsilon) > C = \sup_{\beta\in\Gamma}\{l(\beta)\}$. □

6 Consequences

The map *Mess* parameterizes the moduli space $\mathcal{M}_{g,n,\theta}$ of AdS convex GHM metrics with particles of angles less than π on M by $\mathcal{T}_{g,n,\theta} \times \mathcal{T}_{g,n,\theta}$ for $\theta := (\theta_1, ..., \theta_n) \in (0, \pi)^n$. Moreover, it is proved in [KS07, Theorem 5.11] that the space $\mathcal{H}_{g,n,\theta}$ of maximal surfaces in a germ of AdS convex GHM manifold with particles of angles $\theta \in (0, \pi)^n$ is parameterized by the cotangent bundle $T^*\mathcal{T}_{g,n,\theta}$. Given such a surface, it is possible to reconstruct locally the AdS convex globally hyperbolic manifold which then uniquely embeds in a convex GHM one.

As each AdS convex GHM manifold with particles of angles less than π contains a unique maximal surface, then $\mathcal{M}_{g,n,\theta}$ is identified with $\mathcal{H}_{g,n,\theta}$, and so,

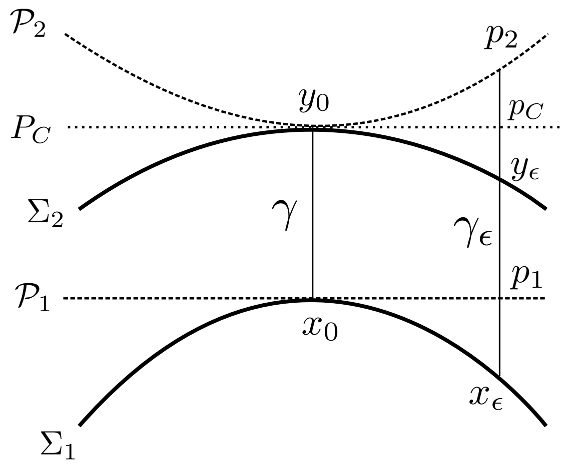

Figure 5: Uniqueness

Theorem 6.1. *The parameterization Mess induces a homeomorphism*

$$\Psi : T^*\mathcal{T}_{g,n,\theta} \longrightarrow \mathcal{T}_{g,n,\theta} \times \mathcal{T}_{g,n,\theta}.$$

6.1 Interpretation in terms of minimal Lagrangian diffeomorphisms

In this paragraph, we prove Theorem 1.1. Before, we recall a result of [Lab92]:

Proposition 6.2. *Let (S, g) be a Riemannian surface and $A : TS \longrightarrow TS$ a smooth bundle morphism such that A is everywhere invertible and $d^D A = 0$ (here d^D is the differential of a vector valued form which is defined thanks to the Levi-Civita connection D of TS, see e.g. [Bes08] for the construction). Let h be defined by $h(u, v) = g(Au, Av)$, then the Levi-Civita connection ∇ of h is given by:*

$$\nabla_u v = A^{-1} D_u(Av),$$

and its curvature is:

$$K_h = \frac{K_g}{\det(A)}.$$

Remark 6.1. For $(g, g') \in \mathcal{T}_{g,n,\theta} \times \mathcal{T}_{g,n,\theta}$, suppose that there exists a bundle morphism

$$b : (TS, g) \longrightarrow (TS, g')$$

such that

1. b is self-adjoint for g with positive eigenvalues.
2. $d^\nabla b = 0$ (where ∇ is the Levi-Civita connection associated to g).
3. $\det(b) = 1$.
4. $g(b., b.)$ is the pull-back of g' by a diffeomorphism φ fixing each marked point and isotopic to the identity.

Then the diffeomorphism φ is minimal Lagrangian outside the singular locus.

Proof of Theorem 1.1.

Existence: Let (h_l, h_r) be a pair of hyperbolic metrics on S with conical singularities at the x_i of angle $\theta_i < \pi$; it defines two points in $\mathcal{T}_{g,n,\theta}$, and thanks to the homeomorphism $Mess$, it defines a unique AdS convex GHM manifold with particles M. Let Σ be its maximal surface with shape operator B, first fundamental form I and identity map E. By definition of the map $Mess$, We have

$$\begin{cases} h_l(x,y) = & I((E+JB)x,(E+JB)y) \\ h_r(x,y) = & I((E-JB)x,(E-JB)y) \end{cases}$$

We define the bundle morphism $b : TS \longrightarrow TS$, defined outside the x_i by:

$$b = (E+JB)^{-1}(E-JB),$$

(note that it is well defined as from [KS07, Lemma 5.15], the eigenvalues of B are in $(-1,1)$). We are going to prove that b verifies the four properties of Remark 6.1:

1. $$\begin{aligned} h_l(bx,y) &= I((E-JB)x,(E+JB)y) = I((E+JB)(E-JB)x,y) \\ &= I((E-JB)(E+JB)x,y) = I((E+JB)x,(E-JB)y) \\ &= h_l(x,by). \end{aligned}$$

   Moreover, it is proved in [KS07, Lemma 5.15] that B has eigenvalues in $(-1,1)$, so $(E \pm JB)$ has strictly positives eigenvalues and the same for b.

2. $\det(b) = \dfrac{\det(E-JB)}{\det(E+JB)} = \dfrac{1+\det(JB)}{1+\det(JB)} = 1$, (as $tr(JB) = 0$).

3. Denote by D the Levi-Civita connection associated to I, and consider the morphism $A = (E+JB)$. Note that $d^D A = 0$ (by Codazzi's equation) and so A verifies the hypotheses of Proposition 6.2. The Levi-Civita connection ∇ of $g(A.,A.)$ is then given by:

   $$\nabla_u v = A^{-1} D_u(Av)$$

   and so $d^\nabla b = A^{-1} d^D(E-JB) = 0$.

4. Obviously, we have that $h_l(b.,b.) = h_r(.,.)$.

Uniqueness: Suppose that there exist b_1, b_2 as in Remark 6.1, define:

$$4g_i(.,.) = g((E+b_i).,(E+b_i).) \text{ and } B_i = -J_i(E+b_i)^{-1}(E-b_i) \ i = 1,2,$$

where J_i is the complex structure associated to g_i. As for $i = 1,2$, b_i has positive eigenvalues, B_i is well defined and its eigenvalues are in $(-1,1)$. Moreover, $\det(b_i) = 1$ and $\mathrm{tr}_{g_i}(B_i) = 0$, so B_i is self-adjoint for g_i. A simple computation gives:

$$b_i = (E+J_iB_i)^{-1}(E-J_iB_i), \ E+J_iB_i = 2(E+b_i)^{-1}, \ i = 1,2.$$

So, the Levi-Civita connection $\widetilde{\nabla}^i_x y$ of g_i is given by:

$$\widetilde{\nabla}^i_x y = (E+b_i)^{-1}\nabla_x((E+b_i)y)$$

(by Proposition 6.2) where ∇ is the Levi-Civita connection of g. Therefore, we get:

$$\begin{aligned}\widetilde{\nabla}^i B_i(x,y) &= (E+b_i)^{-1}\nabla_y((E+b_i)By) - (E+b_i)^{-1}\nabla_y((E+b_i)x) - B_i[x,y] \\ &= (E+b_i)^{-1}(\nabla(E+b_i))(x,y) \\ &= 0.\end{aligned}$$

Moreover, as g has curvature -1, we obtain:

$$K_{g_i} = -\det(E+JB_i) = -1-\det(B_i).$$

So B_i is traceless, self-adjoint and satisfies the Codazzi and Gauss equation, which implies that setting $h_i = g_i(B_i.,.)$, we get $(g_i,h_i)\in\mathcal{H}_{g,AdS,n,\theta}$ and we obtain:

$$\begin{aligned}h_{r,i}(x,y) &= g_i((E+JB_i)x,(E+JB_i)y) \\ &= g_i(2(E+b_i)^{-1}x, 2(E+b_i)^{-1}y) \\ &= g(x,y).\end{aligned}$$

And in the same way:

$$h_{r,i}(x,y) = g'(x,y).$$

That is, $\mathrm{Mess}^{-1}(h_{l,1},h_{r,1}) = \mathrm{Mess}^{-1}(h_{l,2},h_{r,2})$ so $h_{l,1}=h_{l,2}$ and $h_{r,1}=h_{r,2}$. This implies that $B_1=B_2$ and so $b_1=b_2$.

Hence such a bundle morphism is unique and define a unique minimal Lagrangian difffeomorphism $\Psi:(\Sigma,g)\longrightarrow(\Sigma,g')$ isotopic to the identity (cf. [Lab92, Section 2]). Note that here, the hyperbolic metrics are normalized in such a way that $\varphi = Id$. □

6.2 Interpretation of the homeomorphism Ψ

Here we prove a nice geometric interpretation of the homeomorphism Ψ which extends the relations between harmonic maps and minimal Lagrangian diffeomorphisms of [Sam78] and [Wol89]:

Theorem 6.3. *Let h_1 and h_2 be two hyperbolic metrics on S with cone singularities of angles $\theta_i<\pi$ at the marked points x_i. There exists a unique complex structure J_0 with marked points at the x_i on S such that*

$$\mathit{Hopf}(\phi_1) = -\mathit{Hopf}(\phi_2)$$

where $\phi_i:(S,J_0)\longrightarrow(S,h_i)$ is the unique harmonic map isotopic to the identity provided by [GR10] and $\mathit{Hopf}(\phi_i)$ is the Hopf differential of ϕ_i, $i=1,2$. Moreover, $\Psi^{-1}(h_1,h_2) = (J_0, i\mathit{Hopf}(\phi_1))\in T^*\mathcal{T}_{g,n,\theta}$.

Proof. Let $h_l, h_r \in \mathcal{T}_{g,n,\theta}$ and let I, B, E and J be the first fundamental form, shape operator, identity and complex structure associated to the unique maximal surface Σ of the AdS convex GHM manifold with particles $Mess(h_l, h_r)$. It follows that

$$\begin{cases} h_l(.,.) = I((E+JB).,(E+JB).) \\ h_r(.,.) = I((E-JB).,(E-JB).) \end{cases}$$

Take $\varphi : (S, h_l) \longrightarrow (S, h_r)$ be the unique minimal Lagrangian diffeomorphism isotopic to the identity. Note that, here $\Psi = Id$ because $h_l = h_r(b.,b.)$ for $b = (E+JB)(E-JB)^{-1}$ (see the proof of Theorem 1.1).

Denote by Γ the graph of φ in $(S\times S, h_l \oplus h_r)$ and by h_Γ the induced metric on Γ. And easy computation shows that $h_\Gamma = 2(I+III)$, where $III = I(B.,B.)$ is the third fundamental form of Σ. In fact, tangent vectors to Γ have the form $(u, d\varphi(u)) = (u,u)$ (and will be denoted by u when no confusion will be possible) where $u \in TS$. Hence,

$$h_\Gamma(u,v) = h_l(u,v) + h_r(u,v) = 2I(u,v) + 2I(JBu, JBv) = 2(I+III)(u,v).$$

Note that $III = k^2 I$, so the conformal class of h_Γ is equal to the conformal class of I, and so J is the complex structure of Γ.

Consider $\pi_1 : \Gamma \longrightarrow (S, h_l)$ and $\pi_2 : \Gamma \longrightarrow (S, h_r)$ the projection of the first and second factor respectively. Note that, as Γ is minimal in $(S\times S, h_l \oplus h_r)$, these projections are harmonics. By the main theorem of [GR10], these maps are the unique harmonic maps isotopic to the identity from (S,J) to (S,h_l) and from (S,J) to (S,h_r) respectively.

Now, we are going to compute $\mathsf{Hopf}(\pi_1)$. By definition,

$$\mathsf{Hopf}(\pi_1) = (\pi_1^* h_l^{\mathbb{C}})^{2,0},$$

that is, $\mathsf{Hopf}(\pi_1)$ is the $(2,0)$ part (with respect to J) of the pull-back by π_1 of the complexified metric $h_l^{\mathbb{C}}$.

Take (e_1, e_2) an orthonormal framing of principal direction of Σ. That is $I(e_i, e_j) = \delta_{ij}$, $Be_1 = ke_1$ and $Be_2 = -ke_2$.

Denote by $T^{\mathbb{C}}\Gamma = T\Gamma \underset{\mathbb{R}}{\otimes} \mathbb{C}$ the complexified tangent bundle, and set as usually:

$$\begin{cases} Z = \frac{1}{2}(e_1 - iJe_1) = \frac{1}{2}(e_1 - ie_2) \\ \overline{Z} = \frac{1}{2}(e_1 + iJe_1) = \frac{1}{2}(e_1 + ie_2) \end{cases}$$

And

$$\begin{cases} dz = \frac{1}{2}(dx + idy) \\ d\overline{z} = \frac{1}{2}(dx - idy) \end{cases}$$

(where dx and dy are the dual of e_1 and e_2 respectively). It follows that Z is a generator of the holomorphic tangent bundle $T^{1,0}\Gamma$ and $d\overline{z}(Z) = 0$.

Moreover, $\pi_1^* h_l^{\mathbb{C}} = \phi dz^2 + \psi dz d\overline{z} + \varphi d\overline{z}^2$, so $\mathsf{Hopf}(\pi_1) = \phi dz^2$. One gets that:

$$\phi = \pi_1^* h_l^{\mathbb{C}}(Z,Z) = \frac{1}{4} I((E+JB)(e_1 - ie_2), (E+JB)(e_1 - ie_2)) = -iI(JBe_1, e_2) = -ik.$$

So $\mathsf{Hopf}(\pi_1) = -ikdz^2$. An analogue computation shows that $\mathsf{Hopf}(\pi_2) = ikdz^2$.

Now, it remains to show that $\Re(i\mathsf{Hopf}(\pi_1)) = II$. Using $dz = dx + idy$, one gets that

$$i\mathsf{Hopf}(\pi_i) = kdz^2 = kdx^2 - kdy^2 + ik(dxdy + dydx),$$

hence $\Re(i\mathsf{Hopf}(\pi_1)) = kdx^2 - kdy^2 = II$.

The uniqueness comes from the uniqueness of a minimal Lagrangian diffeomorphism isotopic to the identity. In fact, suppose that there exists J_1 and J_2 two complex structures with marked points and $\phi_1^i : (S, J_i) \longrightarrow (S, h_1)$, $\phi_2^i : (S, J_i) \longrightarrow (S, h_2)$ harmonic maps isotopic to the identity for $i = 1, 2$ satisfying the condition of the theorem, then $\phi_2^i \circ (\phi_1^i)^{-1}$ are minimal Lagrangian diffeomorphisms isotopic to the identity and so are equals. Then $J_1 = J_2$ is the complex structure of the metric induced on the graph of its minimal Lagrangian diffeomorphism. □

References

[ABBZ12] L. Andersson, T. Barbot, F. Béguin, and A. Zeghib. Cosmological time versus CMC time in spacetimes of constant curvature. *Asian J. Math.*, 16(1):37–87, 2012.

[BBD+12] T. Barbot, F. Bonsante, J. Danciger, W.M. Goldman, F. Guéritaud, F. Kassel, K. Krasnov, J-M. Schlenker, and A. Zeghib. Some open questions on Anti-de Sitter geometry. *arXiv preprint arXiv:1205.6103*, 2012.

[BBZ07] T. Barbot, F. Béguin, and A. Zeghib. Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3. *Geometriae Dedicata*, 126(1):71–129, 2007.

[Ber60] L. Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc.*, 66:94–97, 1960.

[Bes08] A.L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[BS09] F. Bonsante and J-M. Schlenker. AdS manifolds with particles and earthquakes on singular surfaces. *Geom. Funct. Anal.*, 19(1):41–82, 2009.

[Ger83] C. Gerhardt. H-surfaces in Lorentzian manifolds. *Comm. Math. Phys.*, 89(4):523–553, 1983.

[Gol88] W. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.

[GR10] J. Gell-Redman. Harmonic maps into conic surfaces with cone angles less than 2π. *arXiv preprint arXiv:1010.4156*, 2010.

[Hop51] H. Hopf. Über Flächen mit einer Relation zwischen den Hauptkrümmungen. *Math. Nachr.*, 4:232–249, 1951.

[KS07] K. Krasnov and J-M. Schlenker. Minimal surfaces and particles in 3-manifolds. *Geom. Dedicata*, 126:187–254, 2007.

[LA94] J. Lafontaine and M. Audin. Introduction: applications of pseudo-holomorphic curves to symplectic topology. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 1–14. Birkhäuser, Basel, 1994.

[Lab92] F. Labourie. Surfaces convexes dans l'espace hyperbolique et $\mathbf{C}P^1$-structures. *J. London Math. Soc. (2)*, 45(3):549–565, 1992.

[Lab94] F. Labourie. Exemples de courbes pseudo-holomorphes en géométrie riemannienne. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 251–269. Birkhäuser, Basel, 1994.

[LS09] C. Lecuire and J.-M. Schlenker. The convex core of quasifuchsian manifolds with particles. arXiv:0909.4182. To appear, *Geometry & Topology*, 2009.

[Mes07] G. Mess. Lorentz spacetimes of constant curvature. *Geom. Dedicata*, 126:3–45, 2007.

[MS09] S. Moroianu and J.-M. Schlenker. Quasi-Fuchsian manifolds with particles. *J. Differential Geom.*, 83(1):75–129, 2009.

[Sam78] J. H. Sampson. Some properties and applications of harmonic mappings. *Ann. Sci. École Norm. Sup. (4)*, 11(2):211–228, 1978.

[Sch93] R. M. Schoen. The role of harmonic mappings in rigidity and deformation problems. In *Complex geometry (Osaka, 1990)*, volume 143 of *Lecture Notes in Pure and Appl. Math.*, pages 179–200. Dekker, New York, 1993.

[Sch98] J.-M. Schlenker. Métriques sur les polyèdres hyperboliques convexes. *J. Differential Geom.*, 48(2):323–405, 1998.

[Wol89] M. Wolf. The Teichmüller theory of harmonic maps. *J. Differential Geom.*, 29(2):449–479, 1989.

Jérémy Toulisse
University of Luxembourg, Campus Kirchberg
Mathematics Research Unit, BLG
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg
Grand Duchy of Luxembourg
jeremy.toulisse@uni.lu