An axiomatic approach of the discrete Choquet integral as a tool to aggregate interacting criteria

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Abstract

The most often used operator to aggregate criteria in decision making problems is the classical weighted arithmetic mean. In many problems however, the criteria considered interact, and a substitute to the weighted arithmetic mean has to be adopted. We show that, under rather natural conditions, the discrete Choquet integral is an adequate aggregation operator that extends the weighted arithmetic mean by the taking into consideration of the interaction among criteria. The axiomatic that supports the Choquet integral is presented and an intuitive approach is proposed as well.

Keywords: multicriteria decision making, interacting criteria, Choquet integral.

1 Introduction

Let us consider a finite set of *alternatives* $A = \{a, b, c, \ldots\}$ and a finite set of *criteria* $N = \{1, \ldots, n\}$ in a multicriteria decision making problem. Each alternative $a \in A$ is associated with a *profile* $x^a = (x_1^a, \ldots, x_n^a) \in \mathbb{R}^n$, where, for any $i \in N$, x_i^a represents the partial score of a related to criterion i. We assume that all the partial scores are defined according to the same interval scale, that is, they are defined up to the same positive linear transformation.

From the profile of any alternative a, one can compute a global score $M(x^a)$ by means of an aggregation operator $M : \mathbb{R}^n \to \mathbb{R}$ which takes into account the weights of importance of the criteria. Once the global scores are computed, they can be used to rank the alternatives or select an alternative that best satisfies the given criteria.

Until recently, the most often used aggregation operators were the weighted arithmetic means, that is, operators of the form

$$M_\omega(x) = \sum_{i=1}^n \omega_i \, x_i, \tag{1}$$

with $\sum_i \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N$. However, since these operators are not able to model in any understandable way an interaction among criteria, they can be used only in the presence of independent criteria.

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In order to have a flexible representation of complex interaction phenomena between criteria, it is useful to substitute to the weight vector ω a non-additive set function on N allowing to define a weight not only on each criterion, but also on each subset of criteria. For this purpose the concept of *fuzzy measure* [31] has been introduced.

A *fuzzy measure* (or *Choquet capacity*) on N is a monotonic set function $v : 2^N \to [0, 1]$, with $v(\emptyset) = 0$ and $v(N) = 1$. Monotonicity means that $v(S) \leq v(T)$ whenever $S \subseteq T$. One thinks of $v(S)$ as the *weight* related to the subset S of criteria. Throughout this paper, we will denote by $\mathcal{F}_N$ the set of all fuzzy measures on N.

Now, a suitable aggregation operator, which generalizes the weighted arithmetic mean, is the discrete Choquet integral, whose use was proposed by many authors (see e.g. [7] and the references there). Given $v \in \mathcal{F}_N$, the Choquet integral of $x \in \mathbb{R}^n$ with respect to v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^n x_{(i)} \, [v(A_{(i)}) - v(A_{(i+1)})], \tag{2}$$

where $(\cdot)$ indicates a permutation of N such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Also $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \emptyset$.

We see that the Choquet integral (2) is a linear expression, up to a reordering of the elements. Moreover, it identifies with the weighted arithmetic mean (discrete Lebesgue integral) as soon as the fuzzy measure v is additive, that is such that $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$.

The main aim of this paper is to present the Choquet integral as an appropriate extension to the weighted arithmetic mean for the aggregation of criteria. This operator offers indeed a large flexibility while keeping in some sense a linear form. Although its definition is not very intuitive, we will show that the Choquet integral can be characterized axiomatically by means of rather natural properties.

The outline of this paper is as follows. In Section 2 we examine three types of dependence between criteria: correlation, substitutiveness/complementarity, and preferential dependence. In Section 3 we set the framework of our study by introducing some properties often required for aggregation. In Section 4 we present the Choquet integral in a rather intuitive way and we propose an axiomatic characterization. Finally, in Section 5 we introduce the importance and interaction indices which enable to interpret the behavior of aggregation.

In order to avoid a heavy notation we will often omit braces for singletons, e.g. writing $a(i)$, $N \setminus i$ instead of $a(\{i\})$, $N \setminus \{i\}$. Also, for pairs, we will often write ij instead of $\{i,j\}$, as for example $a(ij)$.

For any subset $S \subseteq N$, e_S will denote the characteristic vector of S in $\{0,1\}^n$, i.e. the vector of $\{0,1\}^n$ whose i-th component is 1 if and only if $i \in S$. We also introduce the notation

$$xSy := \sum_{i\in S} x_i\, e_i + \sum_{i\in N\setminus S} y_i\, e_i, \qquad x, y \in \mathbb{R}^n.$$

Finally, $\wedge$ and $\vee$ will denote the minimum and maximum operations, respectively.

2 What does "interacting criteria" mean ?

In many practical applications the decision criteria present some interaction. However, the problem of modeling such an interaction remains a difficult question, often overlooked. Although everybody agrees that interaction phenomena do exist in real situations, the lack of a suitable tool to model them frequently causes the practitioner to assume that his criteria are independent and exhaustive. This comes primarily from the absence of a precise definition of interaction.

The interaction phenomena among criteria can be very complex and difficult to identify. In this section we present three types of dependence, quite different from each other: correlation, substitutiveness/complementarity, and preferential dependence.

2.1 Correlation

Correlation is probably the best known and most intuitive type of dependence, see e.g. [7] and [27, Sect. 10.3]. Two criteria $i, j \in N$ are positively correlated if one can observe a positive correlation between the partial scores related to i and those related to j.

For example, consider the problem of evaluating students with respect to three mathematical subjects (criteria) : statistics, probability, and algebra. Clearly, the first two criteria are correlated since, usually, students good at statistics are also good at probability, and vice versa. Thus, these two criteria present some degree of redundancy.

Suppose that a weighted arithmetic mean is used to evaluate the students and assume that the third criterion is more important than the first two, so that the weights could be 0.3, 0.3, 0.4, respectively. Since the first two criteria somewhat overlap, the global evaluation will be overestimated (resp. underestimated) for students good (resp. bad) at statistics and/or probability.

This undesirable phenomenon can be easily overcome by using a suitable fuzzy measure v and the Choquet integral $\mathcal{C}_v$. A positive correlation between criteria i and j must then be modeled by the following inequality:

$$v(ij) < v(i) + v(j),$$

which expresses a *negative interaction* or a *negative synergy* between i and j. More exactly, if i and j are positively correlated then the marginal contribution of j to every combination of criteria that contains i is strictly less than the marginal contribution of j to the same combination when i is excluded, i.e.,

$$v(T \cup ij) - v(T \cup i) < v(T \cup j) - v(T), \qquad T \subseteq N \setminus ij.$$

In case of equality, criteria i and j are not correlated.

Now, suppose that i and j are negatively correlated, that is, high partial scores along i usually imply low partial scores along j, and vice versa. In that case the simultaneous satisfaction of both criteria is rather uncommon, and the alternatives that present such a satisfaction profile should be favored (for example, students good at both law and algebra). Thus, a negative correlation between criteria i and j must be modeled by the following inequality:

$$v(ij) > v(i) + v(j),$$

which expresses a *positive interaction* or a *positive synergy* between i and j. These two criteria then present some degree of opposition or complementarity.

Here again, a proper modeling of the correlation between i and j requires the taking into consideration of the other combinations. We then write

$$v(T \cup ij) - v(T \cup i) > v(T \cup j) - v(T), \qquad T \subseteq N \setminus ij.$$

2.2 Substitutiveness/complementarity

Another type of dependence is that of substitutiveness between criteria, see e.g. [19]. Consider again two criteria $i, j \in N$, and suppose that the decision maker demands that the satisfaction of only one criterion produces almost the same effect than the satisfaction of both. For example, it is important that students be good at scientific *or* literary subjects. Of course it is better that they be good at both directions, but it is less important.

Clearly, such a behavior cannot be expressed by a weighted arithmetic mean. Here, the importance of the pair $\{i,j\}$ is close to the importance of the single criteria i and j, even in the presence of the other criteria. This condition can be easily expressed by a fuzzy measure v such that

$$v(T) < \left\{\begin{matrix} v(T \cup i) \\ v(T \cup j) \end{matrix}\right\} \approx v(T \cup ij), \qquad T \subseteq N \setminus ij,$$

where "$\approx$" means approximately equal. In that case, we observe that criteria i and j are almost *substitutive* or *interchangeable*. In the extreme case of equality, they can be merged.

Alternatively, the decision maker can demand that the satisfaction of only one criterion produces a very weak effect compared with the satisfaction of both. We then speak of *complementarity*, which is modeled by a fuzzy measure v such that

$$v(T) \approx \left\{ \begin{matrix} v(T \cup i) \\ v(T \cup j) \end{matrix} \right\} < v(T \cup ij), \qquad T \subseteq N \setminus ij.$$

Notice that, contrary to the correlation phenomena, the substitutiveness and complementarity between criteria cannot be detected by observing the score table. They just represent the opinion of the decision maker on the relative importance of criteria, independently of the partial scores obtained by the alternatives along these criteria.

2.3 Preferential dependence

The last type of dependence we present is the *preferential dependence*, and its opposite, the *preferential independence*, well-known in multiattribute utility theory (MAUT), see e.g. [6, 13, 32].

Suppose that the preferences over A of the decision maker are known and expressed by a weak order $\succeq$ (i.e., a strongly complete and transitive binary relation). Through the natural identification of alternatives with their profiles in $\mathbb{R}^n$, this preference relation can be considered as a preference relation on $\mathbb{R}^n$.

Definition 2.1 *The subset S of criteria is said to be* preferentially independent *of $N \setminus S$ if, for all $x, x', y, z \in \mathbb{R}^n$, we have*

$$xSy \succeq x'Sy \quad \Leftrightarrow \quad xSz \succeq x'Sz.$$

The whole set of criteria N is said to be mutually preferentially independent *if S is preferentially independent of $N \setminus S$ for every $S \subseteq N$.*

Roughly speaking, the preference of xSy over $x'Sy$ is not influenced by the common part y. The following example shows that it can be natural for a subset of criteria to be preferentially independent of its complementary.

Example 2.1 *Consider again the problem of evaluating students with respect to statistics (St), probability (Pr), and algebra (Al). Assume this time that the first two subjects are more important than the third. Four students a, b, c, d have been evaluated as follows (marks are expressed on a scale from 0 to 20):*

student	*St*	*Pr*	*Al*
a	*19*	*15*	*18*
b	*19*	*18*	*15*
c	*11*	*15*	*18*
d	*11*	*18*	*15*

The examiner is asked to express its advice by giving a ranking over $A = \{a, b, c, d\}$. Evidently, the preferences $a \succ c$ and $b \succ d$ are immediately suggested. Next, the examiner realizes that the other comparisons are not so obvious since the associated profiles interlace. Since statistics and probability are somewhat substitutive, the following reasoning is proposed: when a student is good at statistics, it is preferable that he/she is better at algebra than probability, so $a \succ b$. However, when a student is not good at statistics, it is more important that he/she is better at probability than algebra, and so $d \succ c$. Thus, the last two criteria are not preferentially independent of the first.

Now, let us assume the existence of an aggregation operator $M : \mathbb{R}^n \to \mathbb{R}$ which represents $\succeq$, that is such that

$$a \succ b \quad \Leftrightarrow \quad M(x^a) > M(x^b),$$

for any $a, b \in A$. Such an operator is called a utility function in MAUT.

It is known [6, 29] that the mutual preferential independence among the criteria is a necessary condition (but not sufficient) for the aggregation operator M to be additive, that is of the form:

$$M(x) = \sum_{i=1}^{n} u_i(x_i),$$

where the functions $u_i : \mathbb{R} \to \mathbb{R}$ are defined up to a positive linear transformation.

In other terms, if some criteria are preferentially dependent of the others, then no additive aggregation operator can model the preferences of the decision maker. In particular, this excludes the use of the weighted arithmetic mean.

A study of the preferential independence when using the Choquet integral can be found in [10, Sect. 10.1.4] and [22, 24]. Note that in Example 2.1 the Choquet integral is able to represent the preferences expressed by the examiner, see [17] for more details.

3 Some preliminary assumptions

To motivate the use of the Choquet integral as an aggregation operator, we will adopt an axiomatic approach based on some selected properties: *increasing monotonicity*, *idempotence*, and *stability with respect to the same interval scales.* These properties can be desirable and even required in many practical situations.

3.1 Increasing monotonicity

Definition 3.1 (In) *$M : \mathbb{R}^n \to \mathbb{R}$ is increasing (in each argument) if, for all $x, x' \in \mathbb{R}^n$, we have*

$$x_i \leq x'_i \quad \forall i \in N \quad \Rightarrow \quad M(x) \leq M(x').$$

An increasing aggregation operator presents a nonnegative response to any increase of the arguments. In other terms, increasing a partial score cannot decrease the result. This seems to be a quite acceptable property.

3.2 Idempotence

Basically, the Choquet integral is the integral of a real function with respect to a fuzzy measure, by analogy with the Lebesgue integral which is defined with respect to an ordinary (i.e. additive) measure. As the integral of a function in a sense represents its average value, the discrete Choquet integral can be viewed as a mean or an averaging aggregation operator.

Cauchy [3] considered in 1821 the mean of n independent variables $x_1, \ldots, x_n$ as a function $M(x_1, \ldots, x_n)$ which should be internal to the set of x_i values:

$$\min x_i \le M(x_1, \ldots, x_n) \le \max x_i. \qquad (3)$$

Such means satisfy trivially the property of *idempotence*, i.e., if all x_i are identical, $M(x_1, \ldots, x_n)$ restitutes the common value.

Definition 3.2 (Id) *$M : \mathbb{R}^n \to \mathbb{R}$ is idempotent if*

$$M(x, \ldots, x) = x, \qquad x \in E.$$

This property seems natural enough. Besides, one can readily see that, for increasing operators, it is equivalent to Cauchy's internality (3), and both are accepted by all statisticians as requisites for means and typical values.

3.3 Stability with respect to the same interval scales

Assume that each criterion $i \in N$ is an *interval scale* (see e.g. [14]), that is, an homomorphism $x_i : A \to \mathbb{R}$ defined up to a positive linear transformation $\phi_i(x_i) = r_i\, x_i + s_i$ with $r_i > 0$. For example, marks obtained by students in a given course often define an interval scale in the sense that it is possible to express the marks on a $[0, 20]$ scale or on a $[-1, 1]$ scale while representing the same information.

It is clear that the aggregation of the partial scores of a given alternative over all the criteria does not make any sense if criteria do not represent the same scale. For example, suppose that a student has taken two exams: statistics and algebra (marks obtained are supposed to be given on a scale from 0 to 20). The corresponding examiners have observed that the student is excellent so that the first one gives 20 in statistics, and the second, which is known as less tolerant, gives 18 in algebra. Clearly, both examiners have used different interval scales, bounded from above by 20 and 18, respectively.

From a theoretical viewpoint, it is known that aggregating values defined on independent interval scales leads to a dictatorial aggregation. Indeed, assuming that the set of aggregated values also define an interval scale, a meaningful aggregation operator $M : \mathbb{R}^n \to \mathbb{R}$ should satisfy the following functional equation (see [1, 2])

$$M(r_1\, x_1 + s_1, \ldots, r_n\, x_n + s_n) = R(r, s)\, M(x) + S(r, s)$$

for all $x, s \in \mathbb{R}^n$, $r \in]0, +\infty[^n$, $R(r, s) > 0$, and $S(r, s) \in \mathbb{R}$. Now the solutions of this equation are of the form (see [2, case #11])

$$M(x) = a\, x_j + b \qquad (x \in \mathbb{R}^n),$$

where $j \in N$ and $a, b \in \mathbb{R}$. When **Id** is assumed, these solutions simply become $M(x) = x_j$ with $j \in N$.

Therefore, we shall assume that all the partial scores are given according to the same interval scale, so that any partial score on a criterion can be compared with any other score on another criterion. The criteria are then said to be *commensurable*, and the operator M must fulfil a less restrictive equation, that is

$$M(r\, x_1 + s, \ldots, r\, x_n + s) = R(r, s)\, M(x) + S(r, s)$$

for all $x \in \mathbb{R}^n$, $r > 0$, $s \in \mathbb{R}$, $R(r, s) > 0$, and $S(r, s) \in \mathbb{R}$. Moreover, assuming **Id** leads to $R(r, s) = r$ and $S(r, s) = s$ for all $r > 0$ and $s \in \mathbb{R}$. Indeed, for any $x \in \mathbb{R}$, we have

$$\begin{aligned} r\, x + s &= M(r\, x + s, \ldots, r\, x + s) \\ &= R(r, s)\, M(x, \ldots, x) + S(r, s) \\ &= R(r, s)\, x + S(r, s). \end{aligned}$$

Definition 3.3 (SPL) *$M : \mathbb{R}^n \to \mathbb{R}$ is stable for the same positive linear transformations if*

$$M(r\, x_1 + s, \ldots, r\, x_n + s) = r\, M(x) + s$$

for all $x \in \mathbb{R}^n$, $r > 0$, $s \in \mathbb{R}$.

We readily see that any operator $M : \mathbb{R}^n \to \mathbb{R}$ fulfilling **SPL** also fulfils **Id**. Moreover, the following proposition shows that such an operator can be considered only on $[0, 1]^n$ without loss of generality (see also [16])

Proposition 3.1 *Any aggregation operator $M : \mathbb{R}^n \to \mathbb{R}$ fulfilling* **SPL** *is completely defined by its restriction to $[0, 1]^n$.*

Proof. Let $M' : [0, 1]^n \to \mathbb{R}$ denote the restriction to $[0, 1]^n$ of M, and let $x \in \mathbb{R}^n$. By **SPL**, we have

$$M(x) = x_{(1)} + (x_{(n)} - x_{(1)})\, M'\big(\tfrac{x_1 - x_{(1)}}{x_{(n)} - x_{(1)}}, \ldots, \tfrac{x_n - x_{(1)}}{x_{(n)} - x_{(1)}}\big)$$

if $x_{(1)} < x_{(n)}$, and $M(x) = x_{(1)}$ if $x_{(1)} = x_{(n)}$. We then can conclude. ■

Thus, if the profile x is expressed in $[\alpha, \beta]^n$ for some $\alpha < \beta$, then its representative x' in $[0, 1]^n$, defined by

$$x'_i = \frac{x_i - \alpha}{\beta - \alpha}, \qquad i \in N,$$

has the global score $M(x')$ and, by **SPL**, we have

$$M(x) = (\beta - \alpha)\, M(x') + \alpha.$$

4 The Choquet integral

In this section, we intend to present the Choquet integral in an intuitive way. Moreover, we propose an axiomatic characterization to motivate the use of this operator in applications.

It is worth mentioning that the Choquet integral was first introduced in capacity theory [5]. Its use as a (fuzzy) integral with respect to a fuzzy measure was then proposed by Höhle [12] and rediscovered later by Murofushi and Sugeno [20, 21].

4.1 Intuitive approach

Given any $v \in \mathcal{F}_N$, we are searching for a suitable aggregation operator $M_v : \mathbb{R}^n \to \mathbb{R}$ which generalizes the weighted arithmetic mean in the sense that it identifies with this latter one as soon as v is additive.

First, we can easily observe that any fuzzy measure $v \in \mathcal{F}_N$ can be expressed in a unique way as:

$$v(S) = \sum_{T \subseteq S} a(T), \qquad S \subseteq N,$$

where $a(T) \in \mathbb{R}$ for every $T \subseteq N$. In combinatorics, a viewed as a set function on N is called the *Möbius transform* of v (see e.g. Rota [26]), which is given by

$$a(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, v(T), \qquad S \subseteq N.$$

For example, we have $a(\emptyset) = 0$ and $a(i) = v(i)$ for all $i \in N$. For a pair of criteria $i, j \in N$, $a(ij)$ represents the difference between the weight of the pair $\{i, j\}$ and the sum of the weights of i and j:

$$a(ij) = v(ij) - [v(i) + v(j)].$$

This difference is positive (resp. negative) in case of positive (resp. negative) interaction. It is zero if i and j add up without interfering. Thus, $a(ij)$ somewhat reflects the degree of interaction between i and j. Notice however that an appropriate definition of interaction has been proposed, see Section 5.2.

We can also observe that if v is additive then $a(S) = 0$ for all subsets $S \subseteq N$ such that $|S| \geq 2$. In that case, the aggregation operator should be the weighted arithmetic mean:

$$M_v(x) = \sum_{i \in N} v(i)\, x_i = \sum_{i \in N} a(i)\, x_i.$$

When v is not additive, we must take into account the interaction among criteria. For this purpose, we can start from the weighted arithmetic mean, which is a linear expression, and then add terms of the "second order" that involve the corrective coefficients $a(ij)$, then terms of the "third order", etc., so that we have

$$M_v(x) = \sum_{i \in N} a(i)\, x_i + \sum_{\{i,j\} \subseteq N} a(ij)\, [x_i \wedge x_j] + \ldots$$

that is,

$$M_v(x) = \sum_{T \subseteq N} a(T) \bigwedge_{i \in T} x_i. \tag{4}$$

This expression is nothing else than the Choquet integral (2) expressed in terms of the Möbius representation, see [4, 9]. Note that in the non-linear terms we have used the minimum operation instead of the product as we want the operator to satisfy the stability property **SPL**.

4.2 Axiomatic characterization

Although the Choquet integrals have become popular in the field of fuzzy sets and multicriteria decision making, there exist few axiomatic characterizations of this family in the literature. The most representative one is given by Schmeidler [28], using the concept of comonotonic additivity. However, such a characterization is not very attractive in the context of multicriteria decision making.

We present here a characterization of the class of Choquet integrals with n arguments on the basis of four properties. This characterization can be found in the Ph.D. dissertation of the author [15, Sect. 6.1].

Definition 4.1 (LM) *The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ ($v \in \mathcal{F}_N$) are* linear with respect to the fuzzy measure *if there exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ ($T \subseteq N$) such that*

$$M_v = \sum_{T \subseteq N} v(T)\, f_T, \qquad v \in \mathcal{F}_N.$$

This definition is motivated by the following observation. We know that the operator $M_v(x)$ we want to characterize is not linear with respect to its argument $x \in \mathbb{R}^n$. However, we can ask it to be linear at least with respect to the fuzzy measure v, in order to keep the aggregation model as simple as possible. This is a rather natural assumption.

Since the conversion formulas between v and a are linear, M_v fulfils **LM** if and only if there exist 2^n functions $g_T : \mathbb{R}^n \to \mathbb{R}$ ($T \subseteq N$) such that

$$M_v = \sum_{T \subseteq N} a(T)\, g_T, \qquad v \in \mathcal{F}_N. \tag{5}$$

Define $v_T \in \mathcal{F}_N$ by $v_T(S) = 1$ if and only if $S \supseteq T$, and 0 otherwise. In game theory, v_T is called the *unanimity game* for T. The Möbius representation of v_T is given by $a_T(S) = 1$ if and only if $S = T$, and 0 otherwise. From (5) it follows immediately that

$$g_T = M_{v_T}, \qquad T \subseteq N.$$

To identify M_v with the Choquet integral $\mathcal{C}_v$, we have yet to impose that (see (4))

$$M_{v_T}(x) = \bigwedge_{i \in T} x_i, \qquad x \in \mathbb{R}^n.$$

This can be done by using **In** and **SPL**, but also by giving an appropriate definition of the weights of subsets of criteria. Such a definition has not been given so far. In fact, the interpretation of $v(S)$ as the weight or the degree of importance of subset S should appear clearly in the definition of the Choquet integral.

By Proposition 3.1, we can assume that all the partial scores are given in $[0,1]$. Regarding the weighted arithmetic mean (1), we clearly have

$$M_\omega(e_i) = \omega_i, \qquad i \in N,$$

showing that the weight ω_i of criterion i is actually the global evaluation of the alternative that fully satisfies criterion i and does not satisfy the others at all. More generally, we have

$$M_\omega(e_S) = \sum_{i\in S} \omega_i, \qquad S \subseteq N,$$

indicating that the weight of a group S of independent criteria is defined as the global evaluation of the alternative that completely satisfies criteria S and totally fails to satisfy the remaining ones. Adapting this observation to the case of dependent criteria leads to the following definition.

Definition 4.2 (PW) *Let $v \in \mathcal{F}_N$. $M_v : \mathbb{R}^n \to \mathbb{R}$ is* properly weighted by v *if $M_v(e_S) = v(S)$ for all $S \subseteq N$.*

Back to the example of Section 2.1, we define the weight of statistics and probability together by the global evaluation of a student who presents the profile $(1,1,0)$ in $[0,1]^3$:

$$M_v(1,1,0) = v(\mathrm{St},\mathrm{Pr}).$$

If the criteria were not correlated, we would have

$$M_v(1,1,0) = v(\mathrm{St}) + v(\mathrm{Pr}),$$

which shows that the inequality

$$v(\mathrm{St},\mathrm{Pr}) < v(\mathrm{St}) + v(\mathrm{Pr})$$

must hold in case of positive correlation.

We are now able to present the characterization of the class of all Choquet integrals with n arguments. The statement is the following.

Theorem 4.1 *The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ ($v \in \mathcal{F}_N$) satisfy* **LM**, **In**, **SPL**, **PW** *if and only if $M_v = \mathcal{C}_v$ for all $v \in \mathcal{F}_N$.*

To prove this result, we need two technical lemmas, which give a characterization and a description of the Choquet integrals defined from 0-1 fuzzy measures, i.e. fuzzy measures $v \in \mathcal{F}_N$ such that $v(S) \in \{0,1\}$ for all $S \subseteq N$.

Lemma 4.1 *If v is a 0-1 fuzzy measure on N then*

$$\mathcal{C}_v(x) = \bigvee_{\substack{T\subseteq N\\ v(T)=1}} \bigwedge_{i\in T} x_i, \qquad x \in \mathbb{R}^n.$$

Proof. See Murofushi and Sugeno [23, Sect. 2]. ■

Lemma 4.2 *If $M : \mathbb{R}^n \to \mathbb{R}$ fulfils* **In**, **SPL**, *and is such that $M(e_S) \in \{0,1\}$ for all $S \subseteq N$, then there exists a 0-1 fuzzy measure v on N such that $M = \mathcal{C}_v$.*

Proof. One can easily see that the set function $v : 2^N \to \{0,1\}$ defined by $v(S) = M(e_S)$ for all $S \subseteq N$ is a 0-1 fuzzy measure on N.

Let $x \in [0,1]^n$. On the one hand, for all $T \subseteq N$, we have

$$M(x) \overset{\mathbf{In}}{\geq} M\big[(\bigwedge_{i\in T} x_i)\, e_T\big] \overset{\mathbf{SPL}}{=} v(T) \bigwedge_{i\in T} x_i$$

and thus

$$M(x) \geq \bigvee_{T\subseteq N} \Big[v(T) \bigwedge_{i\in T} x_i\Big].$$

On the other hand, let $T^* \subseteq N$ such that

$$v(T^*) \bigwedge_{i\in T^*} x_i = \bigvee_{T\subseteq N} \Big[v(T) \bigwedge_{i\in T} x_i\Big]$$

and set

$$J := \{j \in N \,|\, x_j \leq v(T^*) \bigwedge_{i\in T^*} x_i\}.$$

We should have $J \neq \emptyset$. Indeed, if $x_j > v(T^*) \bigwedge_{i\in T^*} x_i$ for all $j \in N$, then we have, since $v(N) = 1$,

$$v(N) \bigwedge_{i\in N} x_i > v(T^*) \bigwedge_{i\in T^*} x_i$$

which contradicts the definition of T^*. Moreover, we should have $v(N \setminus J) = 0$, for otherwise we would have, assuming $N \setminus J \neq \emptyset$,

$$v(N\setminus J) \bigwedge_{i\in N\setminus J} x_i = \bigwedge_{i\in N\setminus J} x_i > v(T^*) \bigwedge_{i\in T^*} x_i,$$

a contradiction. Finally, we have,

$$\begin{aligned}
M(x) &\overset{\mathbf{In}}{\leq} M\Big[\Big(v(T^*) \bigwedge_{i\in T^*} x_i\Big)\, e_J + e_{N\setminus J}\Big] \\
&\overset{\mathbf{SPL}}{=} v(T^*) \bigwedge_{i\in T^*} x_i + \Big(1 - v(T^*) \bigwedge_{i\in T^*} x_i\Big)\, v(N\setminus J) \\
&= v(T^*) \bigwedge_{i\in T^*} x_i \\
&= \bigvee_{T\subseteq N} \Big[v(T) \bigwedge_{i\in T} x_i\Big].
\end{aligned}$$

Finally, by Lemma 4.1, we then have $M = \mathcal{C}_v$ on $[0,1]^n$, and even on $\mathbb{R}^n$ by Proposition 3.1. ■

Proof of Theorem 4.1. (Sufficiency) Trivial.
(Necessity) Let $v \in \mathcal{F}_N$, with a its Möbius representation. By **LM** we have

$$M_v = \sum_{T\subseteq N} a(T)\, M_{v_T}.$$

Let us fix $T \subseteq N$. By **PW** we have $M_{v_T}(e_S) = v_T(S) \in \{0,1\}$ for all $S \subseteq N$. By Lemmas 4.1 and 4.2, we have

$$\begin{aligned} M_{v_T}(x) &= \mathcal{C}_{v_T}(x) = \bigvee_{\substack{K\subseteq N\\ v_T(K)=1}} \bigwedge_{i\in K} x_i \\ &= \bigvee_{K\supseteq T} \bigwedge_{i\in K} x_i = \bigwedge_{i\in T} x_i, \end{aligned}$$

for all $x \in \mathbb{R}^n$, which completes the proof. ■

If $N = \{1,2\}$ then **LM** is a consequence of **In**, **SPL**, **PW**, see [16, Prop. 4.1]. Let us show that this is not true in general. Assume that $N = \{1,2,3\}$ and define $v^* \in \mathcal{F}_N$ by $v^*(12) = v^*(3) = 0$ and $v^*(13) = v^*(23) = 1/2$. Consider the class of operators $M_v : \mathbb{R}^3 \to \mathbb{R}$ defined by $M_v = \mathcal{C}_v$ for all $v \in \mathcal{F}_N \setminus \{v^*\}$ and

$$M_{v^*}(x) = \Big(\frac{x_1 + x_2}{2}\Big) \wedge x_3, \qquad x \in \mathbb{R}^3.$$

Of course, these operators satisfy **In**, **SPL**, and **PW**, but not **LM**.

We also see that the operators $M_v : \mathbb{R}^n \to \mathbb{R}$ ($v \in \mathcal{F}_n$) defined by

$$M_v(x) = \sum_{T\subseteq N} a(T) \prod_{i\in T} x_i, \qquad x \in \mathbb{R}^n,$$

fulfil **LM**, **In**, **PW**, but not **SPL**.

Finally, we will show in the next section that **PW** does not follow from **LM**, **In**, **SPL**.

5 Behavioral analysis of aggregation

Now that we have a tool for a suitable aggregation, an important question arises: how can we interpret the behavior of the Choquet integral or that of its associated fuzzy measure ? Of course the meaning of the values $v(T)$ is not always clear for the decision maker. These values do not give immediately the global importance of the criteria, nor the degree of interaction among them.

In fact, from a given fuzzy measure, it is possible to derive some indices or parameters that will enable to interprete the behavior of the fuzzy measure. In this final section, we present two types of indices: importance and interaction. Other indices, such as tolerance and dispersion, were proposed and studied by the author in [15].

5.1 Importance indices

The overall importance of a criterion $i \in N$ into a decision problem is not solely determined by the number $v(i)$, but also by all $v(T)$ such that $i \in T$. Indeed, we may have $v(i) = 0$, suggesting that element i is unimportant, but it may happen that for many subsets $T \subseteq N$, $v(T \cup i)$ is much greater than $v(T)$, suggesting that i is actually an important element in the decision.

Shapley [30] proposed in 1953 a definition of a coefficient of importance, based on a set of reasonable axioms. The *importance index* or *Shapley value* of criterion i with respect to v is defined by:

$$\phi(v,i) := \sum_{T\subseteq N\setminus i} \frac{(n-|T|-1)!\,|T|!}{n!}\,[v(T\cup i) - v(T)]. \quad (6)$$

The Shapley value is a fundamental concept in game theory expressing a power index. It can be interpreted as a weighted average value of the marginal contribution $v(T \cup i) - v(T)$ of element i alone in all combinations. To make this clearer, it is informative to rewrite the index as follows:

$$\phi(v,i) = \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T\subseteq N\setminus i\\ |T|=t}} [v(T\cup i) - v(T)].$$

Thus, the average value of $v(T \cup i) - v(T)$ is computed first over the subsets of same size t and then over all the possible sizes.

The use of the Shapley value in multicriteria decision making was proposed in 1992 by Murofushi [18]. It is worth noting that a basic property of the Shapley value is

$$\sum_{i=1}^{n} \phi(v,i) = 1.$$

Note also that, when v is additive, we clearly have

$$\phi(v,i) = v(i), \quad i \in N. \quad (7)$$

If v is non-additive then some criteria are dependent and (7) generally does not hold anymore. This shows that it is sensible to search for a coefficient of overall importance for each criterion.

In terms of the Möbius representation, the Shapley value takes a very simple form [30]:

$$\phi(v,i) = \sum_{T\ni i} \frac{1}{|T|}\, a(T). \quad (8)$$

Now, since the Shapley indices are non-negative and sum up to one, it would be interesting to consider the weighted arithmetic mean having these indices as weights. We call this operator the Shapley integral.

Definition 5.1 *Let $v \in \mathcal{F}_N$. The* Shapley integral *of $x \in \mathbb{R}^n$ with respect to v is defined by*

$$\mathrm{Sh}_v(x) = \sum_{i\in N} \phi(v,i)\, x_i.$$

In terms of the Möbius representation, the Shapley integral has an interesting form.

Proposition 5.1 *Any Shapley integral $\mathrm{Sh}_v : \mathbb{R}^n \to \mathbb{R}$ can be written*

$$\mathrm{Sh}_v(x) = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} a(T) \Big(\frac{1}{|T|} \sum_{i\in T} x_i\Big), \qquad x \in \mathbb{R}^n,$$

where a is the Möbius representation of v.

Proof. By (8), we simply have

$$\mathrm{Sh}_v(x) = \sum_{i \in N} \Big(\sum_{T \ni i} \frac{1}{|T|}\, a(T) \Big)\, x_i.$$

Permuting the sums leads immediately to the result. ■

Although the Shapley integral takes into account the dependence among criteria expressed in the underlying fuzzy measure, it remains an unattractive aggregation operator. Indeed, since the Shapley integral is nothing less than a weighted arithmetic mean, it is not suitable to aggregate criteria when mutual preferential independence is violated.

We also note that the class of Shapley integrals satisfies **LM**, **In**, **SPL**, but not **PW**.

5.2 Interaction indices

Another interesting concept is that of *interaction* among criteria. We have seen that when the fuzzy measure is not additive then some criteria interact. Of course, it would be interesting to appraise the degree of interaction among any subset of criteria.

Consider first a pair $\{i, j\} \subseteq N$ of criteria. It may happen that $v(i)$ and $v(j)$ are small and at the same time $v(ij)$ is large. Clearly, the number $\phi(v, i)$ merely measures the average contribution that criterion i brings to all possible combinations, but it does not explain why criterion i may have a large importance. In other words, it gives no information on the interaction phenomena existing among criteria.

We have seen in Section 2.1 that depending on whether the correlation between i and j is ≤ 0 or ≥ 0, the expression

$$v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$$

is ≥ 0 or ≤ 0 for all $T \subseteq N \setminus ij$, respectively. We call this expression the marginal interaction between i and j, conditioned to the presence of elements of the combination $T \subseteq N \setminus ij$. Now, an interaction index for $\{i, j\}$ is given by an average value of this marginal interaction. Murofushi and Soneda [19] proposed in 1993 to calculate this average value as for the Shapley value. Setting

$$(\Delta_{ij}\, v)(T) := v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T),$$

the *interaction index* of criteria i and j related to v is then defined by

$$I(v, ij) = \sum_{T \subseteq N \setminus ij} \frac{(n - |T| - 2)!\, |T|!}{(n-1)!}\, (\Delta_{ij}\, v)(T). \qquad (9)$$

We immediately see that this index is negative as soon as i and j are positively correlated or substitutive. Likewise, it is positive when i and j are negatively correlated or complementary. Moreover, it has been shown in [8] that $I(v, ij) \in [-1, 1]$ for all $i, j \in N$.

It should be mentioned that, historically, the interaction index (9) was first introduced in 1972 by Owen (see Eq. (28) in [25]) in game theory to express a degree of complementarity or competitiveness between elements i and j.

The interaction index among a combination S of criteria was introduced by Grabisch [8] as a natural extension of the case $|S| = 2$. The *interaction index* of S ($|S| \geq 2$) related to v, is defined by

$$I(v, S) := \sum_{T \subseteq N \setminus S} \frac{(n - |T| - |S|)!\, |T|!}{(n - |S| + 1)!}\, (\Delta_S\, v)(T),$$

where we have set

$$(\Delta_S\, v)(T) := \sum_{L \subseteq S} (-1)^{|S| - |L|} v(L \cup T).$$

In terms of the Möbius representation, this index is written (see [8, 11])

$$I(v, S) = \sum_{T \supseteq S} \frac{1}{|T| - |S| + 1}\, a(T), \qquad S \subseteq N.$$

Viewed as a set function, it coincides on singletons with the Shapley value (6).

6 Concluding remarks

We have shown that the Choquet integral is an appropriate substitute to the weighted arithmetic mean to aggregate dependent decision criteria. The motivation is based mainly on an axiomatic characterization of the class of Choquet integrals, but also on an intuitive interpretation of its expression.

We hope that this will encourage people of the multicriteria decision making community to use this innovative technique of aggregation, which seems very promising.

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