The influence of variables on pseudo-Boolean functions with applications to game theory and multicriteria decision making

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Abstract

The power of players in a collective decision process is a central issue in game theory. For this reason, the concept of influence of players on a simple game has been introduced. More generally, the influence of variables on Boolean functions has been defined and studied.

We extend this concept to pseudo-Boolean functions, thus making it possible to appraise the degree of influence of any coalition of players in cooperative games. In the case of monotone games, we also point out the links with the concept of interaction among players. Although they do not have the same meaning at all, both influence and interaction functions coincide on singletons with the so-called Banzhaf power index.

We also define the influence of variables on continuous extensions of pseudo-Boolean functions. In particular, the Lovász extension, also called discrete Choquet integral, is used in multicriteria decision making problems as an aggregation operator. In such problems, the degree of influence of decision criteria on the aggregation process can then be quite relevant information. We give the explicit form of this influence for the Choquet integral and its classical particular cases.

Key words: pseudo-Boolean functions, game theory, power and interaction indices, multicriteria decision making.

1 Introduction

Let f be a Boolean function on n variables, and let S be a given subset of variables. The *influence of S over f*, denoted $I_f(S)$, is defined as follows [3, 11]. Assign values to the variables not in S at random, that is, variables are set independently of each other and the probability of a zero assignment is one half. This partial assignment may already suffice to set the value of f. The probability that f remains undetermined is defined as the influence of S over f.

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The motivation of this concept stems from the problem of searching for robust voting schemes in game theory. As an illustration, consider an n-person simple game G which proceeds according to the outcome of coin flips [2]. Every player flips an unbiased coin and announces the outcome: 0 or 1. The collective decision of G is then given by a consensus function $f : \{0,1\}^n \to \{0,1\}$, which characterizes the procedure of the game. We assume that the probability for G to end with one is equal to one half.

The simplest procedure is the dictatorial voting scheme, in which only one player flips his coin for all. However, such a procedure could be dangerous if not all players play the game fairly and that some of them announce outcomes according to their interest in the game and not by flipping their coin. Given a coalition of players S, the *influence of S on the game G* is then defined as the probability that the players of S may control the outcome of G when the rest of the players play fairly.

The dictatorial voting scheme is the most sensitive to the presence of an unfair player. Indeed, we clearly have

$$f(x_1,\ldots,x_n) = x_k \quad \Rightarrow \quad I_f(k) = 1 \quad \text{and} \quad I_f(i) = 0 \quad \forall i \neq k.$$

It is natural to search for voting schemes which are more robust, so that the influence of single players is as small as possible. For the majority voting, the influence of any player is $O(1/\sqrt{n})$. More precisely, one can easily show that (see Section 5)

$$f(x_1,\ldots,x_{2k-1}) = x_{(k)} \quad \Rightarrow \quad I_f(i) = \frac{1}{2^{2k-2}}\binom{2k-2}{k-1} \quad \forall i,$$

where $x_{(k)}$ is the median of the numbers $x_1,\ldots,x_{2k-1}$. Rather surprisingly, there are voting schemes significantly more robust than majority voting. Ben-Or and Linial [2] constructed a voting scheme that reduces the influence of each player to $O(\log n/n)$, which is asymptotically optimal.

In this paper we generalize the concept of influence to pseudo-Boolean functions (PBF), i.e., real-valued functions on Boolean variables. Back to the collective coin flipping game, we might assume that the global outcome is a real number, for example the sum of the individual outcomes, weighted by the importance of each player :

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} \omega_i\, x_i, \quad \text{with} \quad \sum_{i=1}^{n} \omega_i = 1 \quad \text{and} \quad \omega_i \geq 0 \ \forall i. \tag{1}$$

The influence over f of any coalition S of players, denoted $I_f(S)$, is then defined as the average amplitude of the range of f that S may control when the rest of the players play fairly. For the weighted mean (1), one can show that this influence is given by the sum of the individual weights :

$$I_f(S) = \sum_{i\in S} \omega_i\,.$$

More generally, let $G := (N, v)$ be a cooperative game, where $N := \{1,\ldots,n\}$ is the set of players and v is the characteristic function of G, that is a set function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Such a set function assigns to each coalition S of players a real number $v(S)$ representing the *worth* of S. Through the usual identification of coalitions $S \subseteq N$ with elements of $\{0,1\}^n$, one can regard the characteristic function v as a PBF, and the above definition of influence can be adapted to v (see Section 2 for a complete definition).

When v is monotone and such that $v(N) = 1$, this influence identifies with the arithmetic mean of the marginal contribution of S alone in all outer coalitions, that is,

$$I_v(S) = \frac{1}{2^{n-|S|}} \sum_{T \subseteq N \setminus S} [v(T \cup S) - v(T)], \qquad S \subseteq N.$$

Thus, we observe that the influence function coincides on singletons with the Banzhaf power index [1, 5].

We also apply the concept of influence to multicriteria decision making. In this context, N represents a set of decision criteria and v is a non-additive measure on N, that is, a monotone set function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$ and $v(N) = 1$. For any combination S of criteria, $v(S)$ is interpreted as the degree of importance of S, or its power to make the decision alone (without the remaining criteria). Now, from the satisfaction profile $x = (x_1, \ldots, x_n) \in [0,1]^n$ of a given alternative, one can compute a global evaluation $M_v(x)$ by means of an aggregation operator $M_v : [0,1]^n \to \mathbb{R}$, which takes into account the importance of the criteria. A suitable aggregation operator, whose use has been suggested by many authors [7], is the discrete Choquet integral, which is actually a continuous extension on the cube $[0,1]^n$ of the PBF that represents v.

Since the global evaluation depends on the importance of criteria, it would be interesting to appraise the degree of influence of any combination of criteria over the Choquet integral. In this paper we propose a definition of influence function for any continuous extension of a PBF. In the case of Choquet integral, this influence function coincides on singletons with the Shapley power index [18].

The outline of the paper is as follows. In Section 2 we introduce the concept of influence function on a PBF. We also give the explicit form of this function in terms of the PBF and its Möbius transform. In Section 3 we propose an analogous definition for continuous extensions. Two particular cases are investigated: the multilinear extension, whose influence coincides with that of the associated PBF, and the Lovász extension, which is nothing but the Choquet integral. In Section 4 we interpret the influence function in cooperative games and point out its connections with the *interaction* among players, a concept introduced axiomatically by Grabisch and Roubens [9]. In Section 5 we discuss the problem of robustness of voting schemes. On this issue, the concept of entropy of a non-additive measure proves to be helpful in determining robust collective decision rules. In Section 6 we study the influence function on the discrete Choquet integral, which is particularly relevant in multicriteria decision making. Finally, in Section 7 we propose an alternative definition of influence on PBFs from a specific probability distribution.

In order to avoid a heavy notation, cardinality of subsets $S, T, \ldots$ will be denoted whenever possible by the corresponding lower case letters $s, t, \ldots$, otherwise by the standard notation $|S|, |T|, \ldots$. Moreover, we will often omit braces for singletons, e.g. writing $a(i)$, $N \setminus i$ instead of $a(\{i\})$, $N \setminus \{i\}$. Also, for pairs, we will often write ij instead of $\{i, j\}$, as for example $a(ij)$.

For any subset $S \subseteq N$, e_S will denote the characteristic vector of S in $\{0,1\}^n$, i.e., the vector of $\{0,1\}^n$ whose i-th component is one if and only if $i \in S$.

The discrete cube $\{0,1\}^n$ can be assimilated with $\{0,1\}^N$, that is, the set of mappings $x : N \to \{0,1\}$. For any $x, y \in \{0,1\}^N$ we then define $x_S y_{-S} \in \{0,1\}^N$ as

$$x_S y_{-S} := \sum_{i \in S} x_i \, e_i + \sum_{i \in N \setminus S} y_i \, e_i$$

We proceed analogously with the entire cube $[0,1]^n$.

Finally, we introduce the notation $\tilde{x} := (x,\ldots,x) \in [0,1]^N$ for any $x \in [0,1]$.

2 The influence of variables on PBFs

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and let $S \subseteq N$. Define the *influence of S over f*, denoted by $I_f(S)$, as the probability that assigning values to the variables not in S at random, the value of f is undetermined, see [3, 11]. Formally, considering $\{0,1\}^N$ as a probability space with uniform distribution, we have

$$I_f(S) := \Pr\big(y \in \{0,1\}^{N\setminus S} \mid f(x_S y_{-S}) \text{ is not constant w.r.t. } x \in \{0,1\}^S\big).$$

The following immediate result, which seems to be previously unknown in the literature, gives the explicit form of $I_f(S)$ in terms of f.

Proposition 2.1 *For any function $f : \{0,1\}^n \to \{0,1\}$, we have*

$$I_f(S) = \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \Big[\max_{K\subseteq S} f(e_{T\cup K}) - \min_{K\subseteq S} f(e_{T\cup K})\Big], \qquad S \subseteq N.$$

Proof. We simply have

$$\begin{aligned} I_f(S) &= \Pr\Big(y \in \{0,1\}^{N\setminus S} \,\Big|\, \max_{x\in\{0,1\}^S} f(x_S y_{-S}) - \min_{x\in\{0,1\}^S} f(x_S y_{-S}) = 1\Big) \\ &= \frac{1}{2^{n-s}} \sum_{y\in\{0,1\}^{N\setminus S}} \Big[\max_{x\in\{0,1\}^S} f(x_S y_{-S}) - \min_{x\in\{0,1\}^S} f(x_S y_{-S})\Big], \end{aligned}$$

which is sufficient. ∎

One also defines the *influence of S towards zero* as follows [2, 11]. Let

$$p := \Pr(f = 0) = \Pr(x \in \{0,1\}^N \mid f(x) = 0)$$

and denote by p' the probability that assigning values to the variables not in S at random, it is possible to assign values to the variables in S so as to make f equal to zero. The difference $p' - p$ is defined to be $I^0_f(S)$, the *influence of S towards zero*. The influence towards one $I^1_f(S)$ is defined analogously.

Proposition 2.2 *For any function $f : \{0,1\}^n \to \{0,1\}$, we have*

$$\begin{aligned} I^0_f(S) &= \frac{1}{2^n} \sum_{T\subseteq N} f(e_T) - \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \min_{K\subseteq S} f(e_{T\cup K}), \qquad S \subseteq N, \\ I^1_f(S) &= \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \max_{K\subseteq S} f(e_{T\cup K}) - \frac{1}{2^n} \sum_{T\subseteq N} f(e_T), \qquad S \subseteq N, \end{aligned}$$

and hence $I_f = I^0_f + I^1_f$.

Proof. We have

$$\begin{aligned}
I_f^0(S) &= \Pr(y \in \{0,1\}^{N\setminus S} \,|\, \exists x \in \{0,1\}^S : f(x_S y_{-S}) = 0) - \Pr(f = 0)\\
&= \Pr\Big(y \in \{0,1\}^{N\setminus S} \,\Big|\, \min_{x\in\{0,1\}^S} f(x_S y_{-S}) = 0\Big) - \Pr(f = 0)\\
&= \Pr(f = 1) - \Pr\Big(y \in \{0,1\}^{N\setminus S} \,\Big|\, \min_{x\in\{0,1\}^S} f(x_S y_{-S}) = 1\Big)\\
&= \frac{1}{2^n}\sum_{T\subseteq N} f(e_T) - \frac{1}{2^{n-s}}\sum_{T\subseteq N\setminus S}\min_{K\subseteq S} f(e_{T\cup K})\,,
\end{aligned}$$

which proves the first equality. The second one can be established similarly. ∎

The definition of I_f can be extended in a natural way to PBFs as follows. Let $f : \{0,1\}^n \to \mathbb{R}$ and $S \subseteq N$. If f is constant everywhere then S has no influence on f. Otherwise, the influence of S on f is defined as the expected value of the highest relative variation of f when assigning values to the variables not in S at random. Formally, denoting by V_f the gap between the extremal values of f, that is,

$$V_f := \max_{x\in\{0,1\}^n} f(x) - \min_{x\in\{0,1\}^n} f(x)\,,$$

we define the influence function I_f as follows.

Definition 2.1 *Consider $\{0,1\}^N$ as a probability space with uniform distribution. The influence of $S \subseteq N$ on $f : \{0,1\}^n \to \mathbb{R}$ is defined by*

$$I_f(S) := \begin{cases} 0\,, & \text{if } f \text{ is constant,}\\[2mm] \dfrac{1}{V_f}\, E\Big[\max\limits_{x\in\{0,1\}^S} f(x_S y_{-S}) - \min\limits_{x\in\{0,1\}^S} f(x_S y_{-S})\Big]\,, & \text{otherwise,}\end{cases}$$

where the expectation is taken over all $y \in \{0,1\}^{N\setminus S}$.

One can readily see that $I_{rf} = I_f$ for all $r \in \mathbb{R}\setminus\{0\}$. Hence, replacing f by f/V_f, if necessary, we may assume that $V_f = 1$. The function f is then said to be *normalized.*

Thus, for any normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have

$$I_f(S) = \frac{1}{2^{n-s}} \sum_{y\in\{0,1\}^{N\setminus S}} \Big[\max_{x\in\{0,1\}^S} f(x_S y_{-S}) - \min_{x\in\{0,1\}^S} f(x_S y_{-S})\Big], \qquad S \subseteq N,$$

or equivalently,

$$I_f(S) = \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \Big[\max_{K\subseteq S} f(e_{T\cup K}) - \min_{K\subseteq S} f(e_{T\cup K})\Big], \qquad S \subseteq N, \tag{2}$$

which shows that this definition of influence is a generalization of that given for Boolean functions.

Now, for any normalized function $f : \{0,1\}^n \to \mathbb{R}$, we also have

$$I_f(S) = \frac{1}{2^{n}} \sum_{y\in\{0,1\}^{N}} \Big[\max_{x\in\{0,1\}^S} f(x_S y_{-S}) - \min_{x\in\{0,1\}^S} f(x_S y_{-S})\Big], \qquad S \subseteq N,$$

showing that the expectation in Definition 2.1 can also be taken over all $y \in \{0,1\}^N$. Indeed, this latter expression is written

$$\frac{1}{2^n}\sum_{T\subseteq N}\Big[\max_{K\subseteq S} f(e_{(T\cap(N\setminus S))\cup K}) - \min_{K\subseteq S} f(e_{(T\cap(N\setminus S))\cup K})\Big],$$

that is, partitioning $T \subseteq N$ into $T_1 \subseteq S$ and $T_2 \subseteq N \setminus S$,

$$\frac{1}{2^n}\sum_{T_1\subseteq S}\ \sum_{T_2\subseteq N\setminus S}\Big[\max_{K\subseteq S} f(e_{T_2\cup K}) - \min_{K\subseteq S} f(e_{T_2\cup K})\Big],$$

and we retrieve (2).

The case where f is monotone (i.e., non-decreasing in each variable) is particularly interesting. For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have

$$I_f(S) = E[f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})], \qquad S \subseteq N.$$

In this case the influence is linear with respect to f, as stated in the following result.

Proposition 2.3 *For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have*

$$I_f(S) = \frac{1}{2^{n-s}}\sum_{T\subseteq N\setminus S}[f(e_{T\cup S}) - f(e_T)], \qquad S \subseteq N.$$

Hammer and Rudeanu [10] showed that any PBF has a unique expression as a multilinear polynomial in n variables :

$$f(x) = \sum_{T\subseteq N} a(T)\prod_{i\in T} x_i\,, \qquad x \in \{0,1\}^n, \tag{3}$$

with $a(T) \in \mathbb{R}$. In combinatorics, a viewed as a set function on N is called the *Möbius transform* of f (see e.g. Rota [17]), which is given by

$$a(S) = \sum_{T\subseteq S}(-1)^{s-t} f(e_T), \qquad S \subseteq N.$$

The transformation is invertible and we have

$$f(e_S) = \sum_{T\subseteq S} a(T), \qquad S \subseteq N. \tag{4}$$

The influence on any monotone normalized PBF can be expressed in terms of the Möbius representation of f as follows.

Proposition 2.4 *For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have*

$$I_f(S) = \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T)\,\frac{1}{2^{|T\setminus S|}}\,, \qquad S \subseteq N,$$

where a is the Möbius representation of f.

Proof. By using (4), we simply have

$$\begin{aligned} I_f(S) &= \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} [f(e_{T\cup S}) - f(e_T)] \\ &= \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} \sum_{\substack{K\subseteq T\cup S\\ K\cap S\neq\emptyset}} a(K) \\ &= \frac{1}{2^{n-s}} \sum_{\substack{K\subseteq N\\ K\cap S\neq\emptyset}} a(K) \sum_{T:K\setminus S\subseteq T\subseteq N\setminus S} 1 \\ &= \sum_{\substack{K\subseteq N\\ K\cap S\neq\emptyset}} a(K)\, \frac{1}{2^{|K\setminus S|}}\,, \end{aligned}$$

as expected. ∎

Example 2.1 *Let* $f : \{0,1\}^3 \to \mathbb{R}$ *be given by*

$$f(x_1, x_2, x_3) = \frac{1}{3}(x_1 + x_2 + x_2\, x_3).$$

Then the values of the influence function I_f *are*

$$\begin{array}{llll} I_f(\emptyset) = 0 & I_f(1) = 1/3 & I_f(12) = 5/6 & I_f(123) = 1 \\ & I_f(2) = 1/2 & I_f(13) = 1/2 & \\ & I_f(3) = 1/6 & I_f(23) = 2/3 & \end{array}$$

3 Case of continuous extensions

From any PBF $f : \{0,1\}^n \to \mathbb{R}$, we can define a variety of continuous extensions $\bar{f} : [0,1]^n \to \mathbb{R}$ which interpolate f at the 2^n vertices of $[0,1]^n$, that is, $\bar{f}(e_S) = f(e_S)$ for all $S \subseteq N$. In this section we propose a definition for the influence on such extensions. First, define $V_{\bar{f}}$ as the gap between the extremal values of $\bar{f}$, that is

$$V_{\bar{f}} := \sup_{x\in[0,1]^n} \bar{f}(x) - \inf_{x\in[0,1]^n} \bar{f}(x).$$

We then propose the following definition.

Definition 3.1 *Consider* $[0,1]^N$ *as a probability space with uniform distribution. The* in- fluence of $S \subseteq N$ on $\bar{f} : [0,1]^n \to \mathbb{R}$ *is defined by*

$$I_{\bar{f}}(S) := \begin{cases} 0\,, & \textit{if } \bar{f} \textit{ is constant,} \\ \dfrac{1}{V_{\bar{f}}}\, E\Big[\sup_{x\in[0,1]^S} \bar{f}(x_S y_{-S}) - \inf_{x\in[0,1]^S} \bar{f}(x_S y_{-S})\Big]\,, & \textit{otherwise,} \end{cases}$$

where the expectation is taken over all $y \in [0,1]^{N\setminus S}$.

Here again, we can assume without loss of generality that $\bar{f}$ is *normalized*, that is such that $V_{\bar{f}} = 1$. We then have trivially

$$I_{\bar{f}}(S) = \int_{[0,1]^{N\setminus S}} \Big[\sup_{x\in[0,1]^S} \bar{f}(x_S y_{-S}) - \inf_{x\in[0,1]^S} \bar{f}(x_S y_{-S}) \Big]\, dy, \qquad S \subseteq N.$$

Furthermore the expectation (that is, the integral) can also be taken over all $y \in [0,1]^N$.

If moreover $\bar{f}$ is monotone then

$$I_{\bar{f}}(S) = \int_{[0,1]^{N\setminus S}} \Big[\bar{f}(\tilde{1}_S y_{-S}) - \bar{f}(\tilde{0}_S y_{-S}) \Big]\, dy, \qquad S \subseteq N. \tag{5}$$

We now investigate two particular cases of continuous extensions of PBFs: the Owen multilinear extension and the Lovász extension.

3.1 Multilinear extension of PBFs

The polynomial expression (3) was used in game theory in 1972 by Owen [16] as the *multilinear extension* of a game.

Definition 3.2 *The multilinear extension of $f : \{0,1\}^n \to \mathrm{I\!R}$ is the function $\check{f} : [0,1]^n \to \mathrm{I\!R}$ defined by*

$$\check{f}(x) := \sum_{T\subseteq N} f(e_T) \prod_{i\in T} x_i \prod_{i\notin T} (1 - x_i) = \sum_{T\subseteq N} a(T) \prod_{i\in T} x_i\,, \qquad x \in [0,1]^n,$$

where a is the Möbius representation of f.

Proposition 3.1 *For any monotone normalized PBF $f : \{0,1\}^n \to \mathrm{I\!R}$, we have $I_{\check{f}} = I_f$.*

Proof. First, we observe that $\check{f}$ is monotone and normalized. Next, for any $y \in [0,1]^{N\setminus S}$, we have

$$\begin{aligned}
\check{f}(\tilde{1}_S y_{-S}) &= \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T) \prod_{i\in T\setminus S} y_i + \sum_{T\subseteq N\setminus S} a(T) \prod_{i\in T} y_i\,,\\
\check{f}(\tilde{0}_S y_{-S}) &= \sum_{T\subseteq N\setminus S} a(T) \prod_{i\in T} y_i\,.
\end{aligned}$$

Hence, by Eq. (5), we have

$$\begin{aligned}
I_{\check{f}}(S) &= \int_{[0,1]^{N\setminus S}} \Big[\sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T) \prod_{i\in T\setminus S} y_i \Big]\, dy\\
&= \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T)\, \frac{1}{2^{|T\setminus S|}}\,.
\end{aligned}$$

We then conclude by Proposition 2.4. ∎

3.2 Lovász extension of PBFs

The Lovász extension of a PBF is defined as follows (see [8] for more details).

Definition 3.3 *The Lovász extension of $f : \{0,1\}^n \to \mathbb{R}$ is the function $\hat{f} : [0,1]^n \to \mathbb{R}$ defined by*

$$\hat{f}(x) := \sum_{T \subseteq N} a(T) \min_{i \in T} x_i \,, \qquad x \in [0,1]^n,$$

where a is the Möbius representation of f.

Using the identity

$$\int_{[0,1]^n} \min_{i \in S} x_i \, dx = \frac{1}{s+1} \,, \qquad S \subseteq N,$$

(see [8]) and a proof analogous to that of Proposition 3.1, we can easily express the influence function associated to any monotone Lovász extension $\hat{f}$ in terms of the Möbius representation of f.

Proposition 3.2 *For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have*

$$I_{\hat{f}}(S) = \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} a(T) \, \frac{1}{|T \setminus S| + 1} \,, \qquad S \subseteq N, \tag{6}$$

where a is the Möbius representation of f.

From Eq. (6) we can derive the expression of $I_{\hat{f}}$ in terms of f. It is given in the next proposition.

Proposition 3.3 *For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have*

$$I_{\hat{f}}(S) = \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \, [f(e_{T \cup S}) - f(e_T)], \qquad S \subseteq N. \tag{7}$$

Proof. By using (4), we simply have

$$\begin{aligned}
\sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \, [f(e_{T \cup S}) - f(e_T)] &= \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \sum_{\substack{K \subseteq T \cup S \\ K \cap S \neq \emptyset}} a(K) \\
&= \sum_{\substack{K \subseteq N \\ K \cap S \neq \emptyset}} a(K) \underbrace{\sum_{T : K \setminus S \subseteq T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!}}_{(*)}
\end{aligned}$$

Moreover, the expression $(*)$ is written

$$\begin{aligned}
&\sum_{t = |K \setminus S|}^{n-s} \binom{n-s-|K \setminus S|}{t - |K \setminus S|} \frac{(n-t-s)!\, t!}{(n-s+1)!} \\
&= \sum_{t=0}^{n-s-|K \setminus S|} \binom{n-s-|K \setminus S|}{t} \frac{(n-s-|K \setminus S|-t)!\, (t+|K \setminus S|)!}{(n-s+1)!}
\end{aligned}$$

$$= \frac{1}{(|K \setminus S|+1)\binom{n-s+1}{|K\setminus S|+1}} \sum_{t=0}^{n-s-|K\setminus S|} \binom{|K\setminus S|+t}{|K\setminus S|}$$

$$= \frac{\binom{n-s+1}{|K\setminus S|+1}}{(|K \setminus S|+1)\binom{n-s+1}{|K\setminus S|+1}}$$

$$= \frac{1}{|K\setminus S|+1}.$$

We then conclude by Proposition 3.2. ■

One can also see that, for any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have

$$I_{\hat{f}}(S) = \int_0^1 \Big[\check{f}(\tilde{1}_S \tilde{y}_{-S}) - \check{f}(\tilde{0}_S \tilde{y}_{-S})\Big]\, dy\,, \qquad S \subseteq N. \tag{8}$$

Indeed, from the expression of $\check{f}$ in terms of the Möbius representation of f, we immediately have

$$\check{f}(\tilde{1}_S \tilde{y}_{-S}) - \check{f}(\tilde{0}_S \tilde{y}_{-S}) = \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T)\, y^{|T\setminus S|},$$

which is sufficient.

As shown in the next example, Eq. (8) allows a rather quick computation of $I_{\hat{f}}$.

Example 3.1 *For the PBF given in Example 2.1, one can see that $I_{\hat{f}} = I_f$. The computation can be done as follows. Suppose $S = \{1,2\}$, then we have*

$$\check{f}(\tilde{1}_S \tilde{y}_{-S}) = \check{f}(1,1,y) = \frac{2+y}{3} \quad \text{and} \quad \check{f}(\tilde{0}_S \tilde{y}_{-S}) = \check{f}(0,0,y) = 0\,,$$

so that

$$I_{\hat{f}}(12) = \int_0^1 \Big[\check{f}(\tilde{1}_S \tilde{y}_{-S}) - \check{f}(\tilde{0}_S \tilde{y}_{-S})\Big]\, dy = \int_0^1 \frac{2+y}{3}\, dy = 5/6\,.$$

4 Influence of coalitions in games

Let $G = (N, v)$ be a cooperative game, where $N = \{1,\ldots,n\}$ is the set of players and v is the characteristic function of G. When there is no fear of ambiguity, the game will be simply denoted by v. In this section, we assume that v is monotone and normalized.

By identifying v with its corresponding PBF on $\{0,1\}^N$, we see by Proposition 2.3 that the influence of any coalition $S \subseteq N$ on v is the expectation of the marginal contribution of S when joining a coalition picked at random from among the 2^{n-s} outer coalitions :

$$I_v(S) = \frac{1}{2^{n-s}} \sum_{T\subseteq N\setminus S} [v(T\cup S) - v(T)], \qquad S \subseteq N. \tag{9}$$

Such an expression is in accordance with the idea of an influence. The influence of any coalition S on v should not be solely determined by the number $v(S)$, but also by all $v(S \cup T)$ such that $T \subseteq N \setminus S$. Indeed, the worth $v(S)$ may be very low, suggesting that S has a rather weak importance, while $v(S \cup T)$ may be much greater than $v(S)$ for many coalitions $T \subseteq N \setminus S$, suggesting that S actually has an important influence.

We also note that the influence function I_v coincides on singletons with the Banzhaf power index [1, 5]:

$$I_v(i) = B_v(i) = \frac{1}{2^{n-1}} \sum_{T \subseteq N \setminus i} [v(T \cup i) - v(T)], \qquad i \in N.$$

We thus see that the influence function is a generalization of the Banzhaf power index. Indeed, it enables to express the global importance (or influence) not only of each player, but also of any coalition of players.

The concept of influence presents some links with that of *interaction* among players, which has been introduced axiomatically by Grabisch and Roubens [9] as extensions of the Banzhaf and Shapley power indices :

- The *Banzhaf interaction index* related to v is defined by

$$I_v^{\mathrm{B}}(S) := \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} \sum_{L \subseteq S} (-1)^{s-l}\, v(L \cup T), \qquad S \subseteq N.$$

- The *Shapley interaction index* related to v is defined by

$$I_v^{\mathrm{Sh}}(S) := \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \sum_{L \subseteq S} (-1)^{s-l}\, v(L \cup T), \qquad S \subseteq N.$$

Grabisch *et al.* [8] proved that both Banzhaf and Shapley interaction indices, viewed as set functions on N, are equivalent representations of v. The conversion formulas involving the Möbius representation are given by :

$$I_v^{\mathrm{B}}(S) = \sum_{T \supseteq S} (\tfrac{1}{2})^{t-s}\, a(T), \qquad S \subseteq N, \tag{10}$$

$$a(S) = \sum_{T \supseteq S} (-\tfrac{1}{2})^{t-s}\, I_v^{\mathrm{B}}(T), \qquad S \subseteq N, \tag{11}$$

$$I_v^{\mathrm{Sh}}(S) = \sum_{T \supseteq S} \frac{1}{t-s+1}\, a(T), \qquad S \subseteq N, \tag{12}$$

$$a(S) = \sum_{T \supseteq S} B_{t-s}\, I_v^{\mathrm{Sh}}(T), \qquad S \subseteq N, \tag{13}$$

where B_n is the n-th Bernoulli number, that is the n-th element of the numerical sequence $(B_n)_{n \in \mathbb{N}}$ defined recursively by

$$\begin{cases} B_0 = 1, \\ \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \qquad n \in \mathbb{N} \setminus \{0\}. \end{cases}$$

Since the influence function I_v has clearly a Banzhaf-like form, it would be interesting to express it in terms of the Banzhaf interaction index. The following proposition gives the conversion formula. Note that one can easily show that I_v is not an equivalent representation of v, that is, there is no conversion formula from I_v to v. In particular, no conversion formula from I_v to I_v^{B} can be found.

Proposition 4.1 *For any monotone normalized game v on N, we have*

$$I_v(S) = \sum_{\substack{T\subseteq S\\ t \text{ odd}}} (\frac{1}{2})^{t-1}\, I_v^{\mathrm{B}}(T), \qquad S\subseteq N.$$

Proof. Combining Eq. (11) with Proposition 2.4 provides, for a given $S\subseteq N$,

$$\begin{aligned} I_v(S) &= \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} (\frac{1}{2})^{|T\setminus S|} \sum_{K\supseteq T} (-\frac{1}{2})^{k-t}\, I_v^{\mathrm{B}}(K) \\ &= \sum_{\substack{K\subseteq N\\ K\cap S\neq\emptyset}} \sum_{\substack{T\subseteq K\\ T\cap S\neq\emptyset}} (\frac{1}{2})^{|T\setminus S|}(-\frac{1}{2})^{k-t}\, I_v^{\mathrm{B}}(K). \end{aligned}$$

Partitioning $T\subseteq K$ into $T_1\subseteq S\cap K$ and $T_2\subseteq K\setminus S$, we have

$$\begin{aligned} I_v(S) &= \sum_{\substack{K\subseteq N\\ K\cap S\neq\emptyset}} (-\frac{1}{2})^{k}\, I_v^{\mathrm{B}}(K) \sum_{\substack{T_1\subseteq S\cap K\\ T_1\neq\emptyset}} (-2)^{t_1} \underbrace{\sum_{T_2\subseteq K\setminus S} (-1)^{t_2}}_{(1-1)^{|K\setminus S|}} \\ &= \sum_{\substack{K\subseteq S\\ K\neq\emptyset}} (-\frac{1}{2})^{k}\, I_v^{\mathrm{B}}(K) \underbrace{\sum_{\substack{T_1\subseteq K\\ T_1\neq\emptyset}} (-2)^{t_1}}_{(-1)^k-1} \\ &= \sum_{\substack{K\subseteq S\\ k \text{ odd}}} (-\frac{1}{2})^{k-1}\, I_v^{\mathrm{B}}(K), \end{aligned}$$

which proves the result. ∎

In Example 2.1, we can observe that I_v is an additive set function. Of course, this is not the case in general. The question then arises of determining conditions on v that assure additivity of I_v. It is easy to see that I_v is additive whenever v is of order ≤ 2, that is such that $a(S)=0$ for all $S\subseteq N$, with $s\geq 3$. Rather interestingly, one can readily see by (10) and (11) that v is of order ≤ 2 if and only if $I_v^{\mathrm{B}}(S)=0$ for all $S\subseteq N$, with $s\geq 3$. However, this condition is not necessary for I_v to be additive, as the following result shows.

Proposition 4.2 *Let v be a monotone normalized game on N. Then I_v is additive if and only if $I_v^{\mathrm{B}}(S)=0$ for all $S\subseteq N$ such that s is odd and ≥ 3.*

Proof. The set function I_v is additive if and only if

$$I_v(S) = \sum_{i\in S} I_v(i), \qquad S\subseteq N,\ S\neq\emptyset.$$

Since $I_v(i)=B_v(i)=I_v^{\mathrm{B}}(i)$ for all $i\in N$, by Proposition 4.1, this condition is equivalent to

$$\sum_{\substack{T\subseteq S\\ t \text{ odd},\geq 3}} (\frac{1}{2})^{t-1}\, I_v^{\mathrm{B}}(T) = 0, \qquad S\subseteq N,\ S\neq\emptyset. \tag{14}$$

Now, for any $S\subseteq N$, with $s=3$, the identity in (14) leads to $I_v^{\mathrm{B}}(S)=0$. Going on with $s=5,7,9,\ldots$, we obtain

$$I_v^{\mathrm{B}}(S)=0 \quad \text{for all } S\subseteq N \text{ with } s \text{ odd and } \geq 3.$$

Conversely, this latter condition implies trivially (14). ∎

Example 4.1 *Let v be the monotone normalized game on $N = \{1, 2, 3, 4\}$ defined by $a(i) = 1/2$, $a(ij) = 0$, $a(ijk) = -1/2$ for all $i, j, k \in N$, and $a(N) = 1$. It is easily verified that $I_v(S) = s/4$ and hence I_v is additive.*

Before closing this section, we point out an interesting link between the influence of players and the Banzhaf power index. We also present a recursive formula that characterizes the influence on monotone simple games from the Banzhaf power index.

Let v be any monotone normalized game on N. The *reduced game* with respect to a coalition $S \subseteq N$ [15] is a game denoted $v_{[S]}$ defined on the set $(N \setminus S) \cup [S]$ of $n - s + 1$ players, where $[S]$ indicates a single hypothetical player, which is the representative of the players in S. This game is defined by

$$v_{[S]}(T) = \begin{cases} v(T), & \text{if } [S] \notin T, \\ v(T \cup S), & \text{if } [S] \in T. \end{cases}$$

Now, it is easily verified that the influence of the coalition S on v is nothing but the Banzhaf power index of the representative $[S]$ in the reduced game $v_{[S]}$:

$$I_v(S) = I_{v_{[S]}}([S]) = B_{v_{[S]}}([S]), \qquad S \subseteq N. \tag{15}$$

Indeed, by (9), we have

$$\begin{aligned} B_{v_{[S]}}([S]) &= \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} [v_{[S]}(T \cup [S]) - v_{[S]}(T)] \\ &= \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} [v(T \cup S) - v(T)], \end{aligned}$$

which is sufficient.

Let v be any monotone simple game on N, that is a monotone game such that $v(S) \in \{0, 1\}$ for all $S \subseteq N$. Given $T \subseteq N$, we denote by v_{-T} the game on $N \setminus T$ defined by $v_{-T}(S) = v(S)$ for all $S \subseteq N \setminus T$. This is equivalent to consider only coalitions in $N \setminus T$. We also denote by $v_{\cup T}$ the game on $N \setminus T$ defined by

$$v_{\cup T}(S) = v(S \cup T) - v(T), \qquad S \subseteq N \setminus T.$$

This game consists in considering only coalitions containing T. Subtraction of $v(T)$ is introduced simply to satisfy the condition $v_{\cup T}(\emptyset) = 0$.

Proposition 4.3 *For any monotone simple game v on N, the influence function I_v obeys the following recurrence formula :*

$$I_v(S \cup i) = I_{v_{-i}}(S) + I_{v_{\cup S}}(i) \qquad \forall i \in N \ \ \forall S \subseteq N \setminus i. \tag{16}$$

Proof. By (9), we have

$$\begin{aligned} I_{v_{-i}}(S) &= \frac{1}{2^{n-s-1}} \sum_{T \subseteq N \setminus (S \cup i)} [v(T \cup S) - v(T)], \\ I_{v_{\cup S}}(i) &= \frac{1}{2^{n-s-1}} \sum_{T \subseteq N \setminus (S \cup i)} [v(T \cup S \cup i) - v(T \cup S)], \end{aligned}$$

and

$$I_v(S \cup i) = \frac{1}{2^{n-s-1}} \sum_{T \subseteq N \setminus (S \cup i)} [v(T \cup S \cup i) - v(T)],$$

which proves the result. ■

The recursive formula (16) has an interesting interpretation. It says that the influence of the union $S \cup i$ on the simple game v is equal to the influence of S in the absence of i plus the influence of i in the presence of S. A symmetric version of this formula is written

$$I_v(S \cup i) = I_{v_{\cup i}}(S) + I_{v_{-S}}(i) \qquad \forall i \in N \;\; \forall S \subseteq N \setminus i.$$

Notice that when v is a non-simple game, the games v_{-i} and $v_{\cup S}$ are not necessarily normalized and the recursive formula (16) does not hold in general. However, it always holds when v is simple, even if v_{-i} and $v_{\cup S}$ are identically zero.

We also observe that Eq. (16) characterizes uniquely the influence function from its values on singletons, namely the Banzhaf power index. A similar characterization was proposed by Grabisch and Roubens [9] for the concept of interaction, namely

$$I_v^{\mathrm{B}}(S \cup i) = I_{v_{\cup i}}^{\mathrm{B}}(S) - I_{v_{-i}}^{\mathrm{B}}(S) \qquad \forall i \in N \;\; \forall S \subseteq N \setminus i.$$

Both concepts of influence and interaction can be combined in an interesting way. For any monotone simple game v on N and any pair of players $i, j \in N$, we have

$$I_v(ij) = I_{v_{-j}}(i) + I_{v_{-i}}(j) + I_v^{\mathrm{B}}(ij).$$

Thus, the influence of the pair $\{i, j\}$ is the influence of i in the absence of j, plus the influence of j in the absence of i, plus the interaction (positive or negative) between i and j.

The previous formula can be extended to any coalition of players. For any monotone simple game v on N and any $S \subseteq N$, we have (combine (15) and Recursive axiom 1 in [9]):

$$I_v(S) = \sum_{T \subsetneq S} I_{v_{-T}}^{\mathrm{B}}(S \setminus T).$$

5 Robust voting schemes

Let us come back to the simple game presented in the introduction. Each player $i \in N$ proposes its vote $x_i \in \{0, 1\}$, and the collective decision is given by a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ such that $\Pr(f = 1) = 1/2$, that is,

$$\frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x) = \frac{1}{2}\,.$$

Assume that n is odd, $n = 2k - 1$, and consider the majority voting scheme, whose corresponding simple game v is defined by $v(T) = 1$ if and only if $t \geq k$ and 0 otherwise. In that case, the influence of any coalition $S \subseteq N$ is given by

$$\begin{aligned} I_v(S) &= \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} [v(T \cup S) - v(T)] \\ &= \frac{1}{2^{n-s}} \sum_{\substack{T \subseteq N \setminus S \\ k-s \leq t \leq k-1}} 1\,, \end{aligned}$$

that is,

$$\begin{aligned} I_v(S) &= \frac{1}{2^{2k-s-1}} \sum_{t=\max(k-s,0)}^{k-1+\min(k-s,0)} \binom{2k-s-1}{t} \\ &= \begin{cases} 1\,, & \text{if } s \geq k\,, \\ \frac{1}{2^{2k-s-1}} \sum_{t=1}^{s} \binom{2k-s-1}{k-t}\,, & \text{if } s < k\,. \end{cases} \end{aligned}$$

In particular, the influence of single players in the majority voting is written

$$I_v(i) = \frac{1}{2^{2k-2}} \binom{2k-2}{k-1}, \qquad i \in N.$$

that is, $I_v(i) = O(1/\sqrt{n})$ for all $i \in N$.

Ben-Or and Linial [2] proposed another voting scheme that reduces the influence of single players to $O(\log n/n)$, which is much smaller than $O(1/\sqrt{n})$. The corresponding simple game v is constructed as follows. Partition N into subsets $S_1, \ldots, S_\ell$ of size $\log n - \log\log n + c$ (c is an appropriate constant) and define $v(T) = 1$ if and only if T contains S_j for some j.

It has been proved that this voting scheme is asymptotically optimal. More precisely, Kahn *et al.* [11] proved the following result.

Theorem 5.1 *There exists an absolute constant C so that for every function $f : \{0,1\}^n \to \{0,1\}$ with $\Pr(f = 1) = p \leq 1/2$, there is at least one $j \in N$ such that*

$$I_f(j) \geq C\,p\,\frac{\log n}{n}\,.$$

Now, as the influence function I_v is not additive in general, it might happen that the single players have a very small influence while some medium-sized coalitions have a very large influence. Back to the majority voting procedure, we have $I_v(S) = 1$ for all $S \subseteq N$ such that $s \geq k$. Indeed, all these coalitions, being in the majority, are decisive. In that case, half of the coalitions have influence equal to one.

Therefore, a proper definition of the robustness of voting schemes should take into consideration the influence of all coalitions of players. The following result [3, 11] shows that we can confine ourselves to monotone games.

Proposition 5.1 *Given a function $f : \{0,1\}^n \to \{0,1\}$, there is a monotone function $g : \{0,1\}^n \to \{0,1\}$ such that $\Pr(f = 1) = \Pr(g = 1)$ and*

$$I_f(S) \geq I_g(S), \qquad S \subseteq N.$$

In order to avoid as much as possible coalitions that have a large influence, we could search for monotone simple games whose influence function increases from 0 to 1 as uniformly as possible. However, there are several ways of defining such a "uniformity". For example, one might search for games that minimize the objective function

$$M(v) = \max_{i \in N} \max_{T \subseteq N \setminus i} [I_v(T \cup i) - I_v(T)].$$

We propose here another objective function, based on the concept of *entropy of a non-additive measure*. Let w be a non-additive measure on N, that is a monotone set function $w : 2^N \to \mathbb{R}$ such that $w(\emptyset) = 0$ and $w(N) = 1$. When w is additive (probability measure), it is defined solely from the values $w(i)$ $(i \in N)$ whose evenness can be measured by means of the so-called Shannon entropy of w, that is,

$$H(w) = -\sum_{i\in N} w(i)\, \log_n w(i),$$

with the convention that $0 \log_n 0 := 0$.

For non-additive measures, an entropy-like measure of uniformity has been proposed recently by the author [12, §6.2.4] in the framework of aggregation. Its expression, derived from the Shapley power index, is written :

$$H(w) = -\sum_{i\in N}\sum_{T\subseteq N\setminus i} \frac{(n-t-1)!\,t!}{n!}\,[w(T\cup i) - w(T)]\, \log_n [w(T\cup i) - w(T)].$$

Such an index measures somehow the regularity of the monotonicity of w from 0 to 1. The more regular the monotonicity of w, the higher its entropy $H(w)$. Some properties of $H(w)$ are shown in [12, 14]. For example, $H(w)$ is maximum $(= 1)$ only when $w(S) = s/n$ for all $S \subseteq N$, and minimum $(= 0)$ only when $w(S) \in \{0, 1\}$ for all $S \subseteq N$.

Given a monotone simple game v on N, its influence function is clearly a non-additive measure and the regularity of its monotonicity can be measured by $H(I_v)$. Thus, searching for robust voting schemes amounts to solving the following convex programming problem in Boolean variables :

$$\text{maximize} \quad H(I_v)$$

subject to

$$\begin{cases} \frac{1}{2^n} \sum_{T\subseteq N} v(T) = \frac{1}{2} \\ v(T\cup i) - v(T) \geq 0 \qquad \forall i \in N \quad \forall T \subseteq N \setminus i \\ v(\emptyset) = 0,\; v(N) = 1 \\ v(T) \in \{0, 1\} \qquad \forall T \subseteq N. \end{cases}$$

This problem has been solved for $n \leq 5$ by an exhaustive enumeration. The solutions are the following :

- *Case* $n = 2$. There are two optimal solutions (uninteresting), which are the two dictatorial voting schemes. $H(I_v) = 0$.
- *Case* $n = 3$. There is only one optimal solution, which is the majority voting. $H(I_v) = \log_3 2 \approx 0.63$.
- *Case* $n = 4$. There are 12 symmetric optimal solutions, which can be built as follows:
  1. If $s \geq 3$ then $v(S) = 1$.
  2. If $s \leq 1$ then $v(S) = 0$.
  3. Take $a, b, c, d \in N$ and set $v(S) = 0$ whenever $s = 2$, except $v(ab) = v(cd) = v(ac) = 1$ (12 possible choices).

The optimal value is $H(I_v) = 17/24 \approx 0.71$.

- *Case* $n = 5$. There are 30 symmetric optimal solutions, which can be built as follows:

  1. If $s \geq 4$ then $v(S) = 1$.
  2. If $s \leq 1$ then $v(S) = 0$.
  3. For $s = 2$ and $s = 3$, v is constructed like this: Choose $a, b, c, d \in N$, and set $v(S) = 0$ whenever $s = 2$, except $v(ab) = v(cd) = 1$ (15 possible choices). Then, set $v(S) = 1$ whenever $s = 3$, except $v(ace) = v(bde) = 0$, where e is the remaining element in N (2 possible choices).

  The optimal value is $H(I_v) = (155 \log_5 2 - 36 \log_5 3)/60 \approx 0.70$. Notice that the majority voting is not optimal ($H(I_v) \approx 0.67$).

It is worth mentioning that the optimal solutions for $n = 4$ and $n = 5$ do not privilege any coalition. These are symmetric solutions. One can cover all of them by first permuting the players at random, and then applying one of the solutions.

Note also that depending upon what is expected from the game v, other constraints can be added to the optimization problem. For example, $I_v^0 = I_v^1$.

We now introduce yet another objective function. Let us consider the lattice $\mathcal{L}(N)$ related to the power set of N. We can represent $\mathcal{L}(N)$ as a graph called the Hasse Diagram $H(N)$, whose nodes correspond to the coalitions $S \subseteq N$ and the edges represent adding a player to the bottom coalition to get the top coalition. A *maximal chain* of $H(N)$ is an ordered collection of $n + 1$ nested and distinct coalitions, that is

$$\mathcal{M} = (\emptyset = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = N).$$

It is then natural to search for games v on N that minimize

$$Z(v) = \max_{\mathcal{M} \in C(N)} \frac{1}{n} \sum_{i=0}^{n} I_v(M_i),$$

where $C(N)$ denotes the set of maximal chains of $H(N)$. Since, for each maximal chain $\mathcal{M}$ there exists a unique permutation π on N such that $M_i = \{\pi(1), \ldots, \pi(i)\}$ for all $i \in N$, this objective function can be written

$$Z(v) = \max_{\pi \in \Pi(N)} \frac{1}{n} \sum_{i=1}^{n} I_v(\pi(1), \ldots, \pi(i)),$$

where $\Pi(N)$ is the set of permutations on N.

For the dictatorial voting, we obtain the maximum value: $Z(v) = 1$. For the majority voting, we obtain

$$Z(v) = \frac{1}{2k-1} \Big[\sum_{s=1}^{k-1} \frac{1}{2^{2k-s-1}} \sum_{t=1}^{s} \binom{2k-s-1}{k-t} + k \Big], \tag{17}$$

that is, $Z(v) = 0.83, 0.82, 0.83, 0.84, 0.95, 0.96$ for $k = 2, 3, 4, 5, 100, 200$, respectively. An exhaustive search showed that the value (17) is optimal for $k = 2$ and $k = 3$.

We thus observe that more than one objective function can be considered. However, we must be very cautious when choosing the objective function. For example, the average value of I_v over all the coalitions, that is

$$\frac{1}{2^n}\sum_{T\subseteq N} I_v(T) \quad \text{or} \quad \sum_{T\subseteq N}\frac{(n-t)!\,t!}{(n+1)!}\,I_v(T),$$

is not an appropriate function to minimize. It is smaller for the dictatorial voting than for the majority voting.

We can also observe that, under the condition $\Pr(f=1)=1/2$, there will always be small dominant coalitions. This is actually a result established in [3, 11]:

Theorem 5.2 *For every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ so that for every function $f:\{0,1\}^n \to \{0,1\}$ with $\Pr(f=1)=1/2$, there is $S\subseteq N$, with $s = C(\varepsilon)\,n/\log n$ so that*

$$I_f(S) \geq 1-\varepsilon.$$

6 Influence of criteria in multicriteria decision making problems

As mentioned in the introduction, the so-called discrete Choquet integral can be used to aggregate criteria in many multicriteria decision making problems. In this section, we investigate the influence function associated to this aggregation operator.

The concept of Choquet integral has been first introduced in capacity theory [4]. Its use as an aggregation operator in decision making has been proposed by several authors (see [7] and the references therein). Moreover, an axiomatic characterization was proposed by the author in [12, 13].

Recall that a non-additive measure on N is a monotone set function $v:2^N\to[0,1]$ such that $v(\emptyset)=0$ and $v(N)=1$. For any combination $S\subseteq N$ of criteria, $v(S)$ is then interpreted as the weight or the degree of importance of S.

Definition 6.1 *Let v be a non-additive measure on N. The Choquet integral of $x\in[0,1]^n$ with respect to v is defined by*

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)}\,[v(A_{(i)})-v(A_{(i+1)})],$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)}\leq\ldots\leq x_{(n)}$. Also $A_{(i)}=\{(i),\ldots,(n)\}$, and $A_{(n+1)}=\emptyset$.

For instance, if $x_3\leq x_1\leq x_2$, we have

$$\mathcal{C}_v(x_1,x_2,x_3) = x_3\,[v(3,1,2)-v(1,2)] + x_1\,[v(1,2)-v(2)] + x_2\,v(2).$$

The Choquet integral is closely related to the Lebesgue integral (weighted arithmetic mean), since both coincide when the measure v is additive :

$$\mathcal{C}_v(x) = \sum_{i=1}^{n} v(i)\,x_i, \qquad x\in[0,1]^n.$$

It was proved [12, 13] that the Choquet integral $\mathcal{C}_v$ is nothing but the Lovász extension of the PBF which represents v. By Propositions 3.2 and 3.3, we then have

$$I_{\mathcal{C}_v}(S) = \sum_{T\subseteq N\setminus S} \frac{(n-t-s)!\,t!}{(n-s+1)!}\,[v(T\cup S)-v(T)]\,, \tag{18}$$

$$I_{\mathcal{C}_v}(S) = \sum_{\substack{T\subseteq N\\ T\cap S\neq\emptyset}} a(T)\,\frac{1}{|T\setminus S|+1}\,, \tag{19}$$

for all $S\subseteq N$. It follows that this influence function $I_{\mathcal{C}_v}$ coincides on singletons with the Shapley power index [18]:

$$I_{\mathcal{C}_v}(i) = Sh_v(i) = \sum_{T\subseteq N\setminus i} \frac{(n-t-1)!\,t!}{n!}\,[v(T\cup i)-v(T)], \qquad i\in N.$$

We now calculate the influence function with respect to two particular Choquet integrals, namely the weighted arithmetic mean and the ordered weighted averaging.

Any vector $\omega\in[0,1]^n$ such that $\sum_i \omega_i = 1$ will be called a *weight vector* as we continue.

6.1 The weighted arithmetic mean

Definition 6.2 *For any weight vector $\omega\in[0,1]^n$, the weighted arithmetic mean operator* WAM_ω *associated to ω is defined by*

$$\mathrm{WAM}_\omega(x) = \sum_{i=1}^n \omega_i\,x_i\,.$$

We have seen that WAM_ω is a Choquet integral $\mathcal{C}_v$ with respect to an additive measure :

$$v(S) = \sum_{i\in S}\omega_i\,, \qquad S\subseteq N.$$

Moreover, it is clear that the influence of $S\subseteq N$ over WAM_ω is given by the sum of the weights related to S :

$$I_{\mathrm{WAM}_\omega}(S) = \sum_{i\in S}\omega_i\,, \qquad S\subseteq N.$$

6.2 The ordered weighted averaging

Yager [19] has defined in 1988 the ordered weighted averaging operators (OWA) as follows.

Definition 6.3 *For any weight vector $\omega\in[0,1]^n$, the ordered weighted averaging operator* OWA_ω *associated to ω is defined by*

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^n \omega_i\,x_{(i)}$$

with the convention that $x_{(1)}\leq\cdots\leq x_{(n)}$.

The following result, due to Grabisch [6], shows that any OWA operator is a Choquet integral with respect to a non-additive measure that depends only on the cardinality of subsets, also called *cardinality-based non-additive measure.*

Proposition 6.1 *Let v be a non-additive measure on N. Then the following assertions are equivalent.*

i) For any $S, S' \subseteq N$ such that $|S| = |S'|$, we have $v(S) = v(S')$.

ii) There exists a weight vector $\omega \in [0,1]^n$ such that $\mathcal{C}_v = \mathrm{OWA}_\omega$.

iii) $\mathcal{C}_v$ is a symmetric function.

The non-additive measure v associated to OWA_ω is given by

$$v(S) = \sum_{i=n-s+1}^{n} \omega_i\,, \qquad S \subseteq N,\ \ S \neq \emptyset. \tag{20}$$

Finally, the following proposition gives the influence function associated to an OWA operator.

Proposition 6.2 *For any weight vector $\omega \in [0,1]^n$, we have*

$$I_{\mathrm{OWA}_\omega}(S) = \frac{1}{n-s+1} \sum_{i=1}^{n} \omega_i \, \min(i, s, n-i+1, n-s+1), \qquad S \subseteq N.$$

Proof. Fix $k \in N$ and consider the operator OWA_ω defined by $\omega_i = 1$ iff $i = k$, and 0 otherwise. This operator is actually the k-th order statistic

$$\mathrm{OS}_k(x) = x_{(k)}, \qquad x \in [0,1]^n.$$

By Proposition 6.1, we can set $v_t := v(T)$ for all $T \subseteq N$. Fixing $S \subseteq N$, we have, by (18),

$$\begin{aligned} I_{\mathrm{OS}_k}(S) &= \frac{1}{n-s+1} \sum_{T \subseteq N \setminus S} \frac{1}{\binom{n-s}{t}} (v_{t+s} - v_t) \\ &= \frac{1}{n-s+1} \sum_{t=0}^{n-s} (v_{t+s} - v_t). \end{aligned}$$

By (20), we have

$$v_t = \begin{cases} 1, & \text{if } t \geq n-k+1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence,

$$v_{t+s} - v_t = \begin{cases} 1, & \text{if } n-k-s+1 \leq t \leq n-k, \\ 0, & \text{otherwise.} \end{cases}$$

We then have

$$I_{\mathrm{OS}_k}(S) = \frac{1}{n-s+1} \sum_{t=\max(0,n-k-s+1)}^{\min(n-k,n-s)} 1$$

and two cases can be considered :

- If $k \leq s$ then

$$I_{\mathrm{OS}_k}(S) = \frac{(n-s+1) - \max(0, n-k-s+1)}{n-s+1} = \frac{\min(n-s+1, k)}{n-s+1}$$

- If $k \geq s$ then

$$I_{\mathrm{OS}_k}(S) = \frac{(n-k+1) - \max(0, n-k-s+1)}{n-s+1} = \frac{\min(n-k+1, s)}{n-s+1}$$

Summing up, we obtain

$$I_{\mathrm{OS}_k}(S) = \frac{\min(k, s, n-k+1, n-s+1)}{n-s+1}.$$

Finally, for any weight vector $\omega \in [0,1]^n$, we have

$$\mathrm{OWA}_\omega = \sum_{i=1}^{n} \omega_i \, \mathrm{OS}_i$$

and the result follows from the linearity of the influence with respect to the non-additive measure. ■

In particular, for $\min = \mathrm{OS}_1$ and $\max = \mathrm{OS}_n$, we have

$$I_{\min}(S) = I_{\max}(S) = \frac{1}{n-s+1}\,, \qquad S \subseteq N.$$

For the median function ($\mathrm{median} = \mathrm{OS}_k$, with $n = 2k-1$), we have, for any $S \subseteq N$,

$$\begin{aligned} I_{\mathrm{median}}(S) &= \frac{\min(k, s, 2k-s)}{2k-s} \\ &= \begin{cases} 1\,, & \text{if } s \geq k\,, \\ \frac{s}{2k-s}\,, & \text{if } s < k\,. \end{cases} \end{aligned}$$

Notice that the underlying non-additive measure of the median function corresponds to the majority voting scheme.

7 An alternative definition of influence on PBFs

In this final section, we show that, for any monotone normalized PBF f, the influence function $I_{\hat{f}}$ can also be viewed as an influence function associated to f and defined from a specific probability distribution.

We have seen in Section 2 that, for any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have

$$I_f(S) = E[f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})], \qquad S \subseteq N,$$

where the expectation if defined from the uniform distribution. However, the uniform distribution does not take into account the fact that there are elements inside each subset, and that a single element is involved several times in different subsets, especially with subsets of around $n/2$ elements, which are the most numerous. This means that a distribution taking into account this combinatorial aspect should be used to avoid this effect. For instance, consider $\{0,1\}^N$ as a probability space with the following distribution :

$$p(y) := \frac{1}{n+1} \binom{n}{\sum_i y_i}^{-1}, \qquad y \in \{0,1\}^N. \tag{21}$$

This is a well-defined distribution since $p(y) \geq 0$ for any $y \in \{0,1\}^N$, and

$$\sum_{y \in \{0,1\}^N} p(y) = \frac{1}{n+1} \sum_{T \subseteq N} \binom{n}{t}^{-1} = \frac{1}{n+1} \sum_{t=0}^{n} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} 1 = 1\,.$$

Moreover, with this distribution, the expectation of any function is calculated first over the subsets of the same size $t \in \{0, \ldots, n\}$ and then over all the possible sizes.

Now, for any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have, by Eq. (7),

$$I_{\hat{f}}(S) = \frac{1}{n-s+1} \sum_{T \subseteq N \setminus S} \binom{n-s}{t}^{-1} [f(e_{T \cup S}) - f(e_T)], \qquad S \subseteq N,$$

or equivalently,

$$I_{\hat{f}}(S) = \frac{1}{n-s+1} \sum_{y \in \{0,1\}^{N \setminus S}} \binom{n-s}{\sum_i y_i}^{-1} [f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})], \qquad S \subseteq N.$$

Therefore, we have

$$I_{\hat{f}}(S) = E[f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})], \qquad S \subseteq N,$$

where the expectation if defined from the distribution (21). Actually, this expectation can be taken over all $y \in [0,1]^N$, as the following result shows.

Proposition 7.1 *For any monotone normalized function $f : \{0,1\}^n \to \mathbb{R}$, we have*

$$I_{\hat{f}}(S) = \frac{1}{n+1} \sum_{y \in \{0,1\}^N} \binom{n}{\sum_i y_i}^{-1} [f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})], \qquad S \subseteq N.$$

Proof. For any $S \subseteq N$, we have

$$\begin{aligned}
& \frac{1}{n+1} \sum_{y \in \{0,1\}^N} \binom{n}{\sum_i y_i}^{-1} [f(\tilde{1}_S y_{-S}) - f(\tilde{0}_S y_{-S})] \\
= \; & \frac{1}{n+1} \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} a(T) \sum_{y \in \{0,1\}^N} \binom{n}{\sum_i y_i}^{-1} \prod_{i \in T \setminus S} y_i \\
= \; & \frac{1}{n+1} \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} a(T) \sum_{K \subseteq N} \binom{n}{k}^{-1} \prod_{i \in T \setminus S} (e_K)_i \\
= \; & \frac{1}{n+1} \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} a(T) \sum_{K \supseteq T \setminus S} \binom{n}{k}^{-1}
\end{aligned}$$

Moreover, for any $T \subseteq N$, we have

$$\begin{aligned}
\sum_{K \supseteq T \setminus S} \binom{n}{k}^{-1} &= \sum_{k=|T \setminus S|}^{n} \binom{n - |T \setminus S|}{k - |T \setminus S|} \binom{n}{k}^{-1} \\
&= \binom{n}{|T \setminus S|}^{-1} \sum_{k=|T \setminus S|}^{n} \binom{k}{|T \setminus S|} = \frac{n+1}{|T \setminus S| + 1}.
\end{aligned}$$

We then conclude by Proposition 3.2. ∎

Consequently, when we reason on elements rather than subsets, it seems that the influence function on f should be given by $I_{\hat{f}}$ instead of I_f. A similar analysis has been done for Banzhaf and Shapley interaction indices, see [8, §2].

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