A Finite Mixture Model with Trajectories Depending on Covariates

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joint work with
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Outline

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General description of Nagin's model

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- mixture : population composed of a mixture of unobserved groups
- finite : sums across a finite number of groups

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where $P^j(Y_i)$ is probability of Y_i if subject i belongs to group j.

The Likelihood Function (2)

Aim of the analysis: Find r groups of trajectories of a given kind (for instance polynomials of degree 4, $P(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4$.

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We try to estimate a set of parameters $\Omega = \left\{ \beta_0^j, \beta_1^j, \beta_2^j, \beta_3^j, \beta_4^j, \pi_j \right\}$ which allow to maximize the probability of the measured data.

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- censored data $\Rightarrow$ Censored normal distribution

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$$L = \frac{1}{\sigma} \prod_{i=1}^{N} \sum_{j=1}^{r} \pi_j \prod_{t=1}^{T} \phi\left(\frac{y_{i_t} - \beta^j t_{i_t}}{\sigma} \right). \tag{2}$$

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It is too complicated to get closed-forms equations.

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Just implements a zero-inflation Poission model.

Model Selection (1)

Bayesian Information Criterion:

$$\text{BIC} = \log(L) - 0,5k\log(N), \tag{3}$$

where k denotes the number of parameters in the model.

Rule:

The bigger the BIC, the better the model!

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Model Selection (2)

Leave-one-out Cross-Validation Apporach:

$$CVE = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\left|y_{i_t} - \hat{y}_{i_t}^{[-i]}\right|. \tag{4}$$

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Posterior Group-Membership Probabilities

Posterior probability of individual i's membership in group j : $P(j/Y_i)$.

Bayes's theorem

$$\Rightarrow P(j/Y_i) = \frac{P(Y_i/j)\hat{\pi}_j}{\sum_{j=1}^{r} P(Y_i/j)\hat{\pi}_j}. \tag{5}$$

Bigger groups have on average larger probability estimates.

To be classified into a small group, an individual really needs to be strongly consistent with it.

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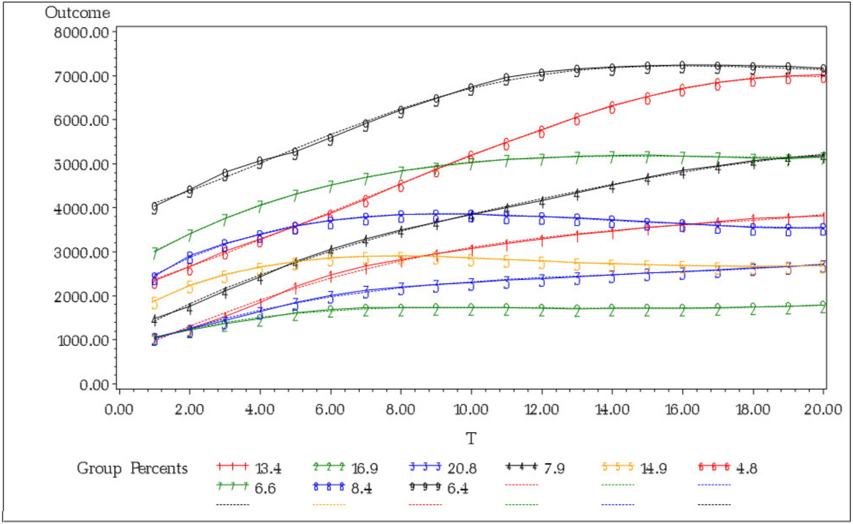

Results for 9 groups (dataset 1)

Maximum Likelihood Estimates
Model: Censored Normal (CNORM)

Group	Parameter	Estimate	Standard Error	T for H0: Parameter=0	Prob > |T|
1	Intercept	589.03067	18.46813	31.894	0.0000
	Linear	387.72145	11.31617	34.263	0.0000
	Quadratic	-14.36621	2.12997	-6.745	0.0000
	Cubic	-0.01563	0.15109	-0.103	0.9176
	Quartic	0.00856	0.00358	2.395	0.0166
2	Intercept	784.79156	15.75939	49.798	0.0000
	Linear	277.63602	9.78078	28.386	0.0000
	Quadratic	-28.36731	1.83236	-15.481	0.0000
	Cubic	1.17739	0.12972	9.076	0.0000
	Quartic	-0.01635	0.00307	-5.330	0.0000
3	Intercept	709.28728	15.90545	44.594	0.0000
	Linear	318.88029	8.97949	35.512	0.0000
	Quadratic	-21.54540	1.69611	-12.703	0.0000
	Cubic	0.62010	0.12002	5.167	0.0000
	Quartic	-0.00440	0.00284	-1.554	0.1203

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Multinomial logit model:

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$$L = \frac{1}{\sigma}\prod_{i=1}^{N}\sum_{j=1}^{r}\frac{e^{x_i\theta_j}}{\sum_{k=1}^{r} e^{x_i\theta_k}}\prod_{t=1}^{T}\phi\left(\frac{y_{i_t} - \beta^j t_{i_t}}{\sigma}\right). \tag{7}$$

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where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma)$, σ being a constant standard deviation and z_l are covariates that may depend or not upon time t.

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Unfortunately the influence of the covariates in this model is limited to the intercept of the trajectory.

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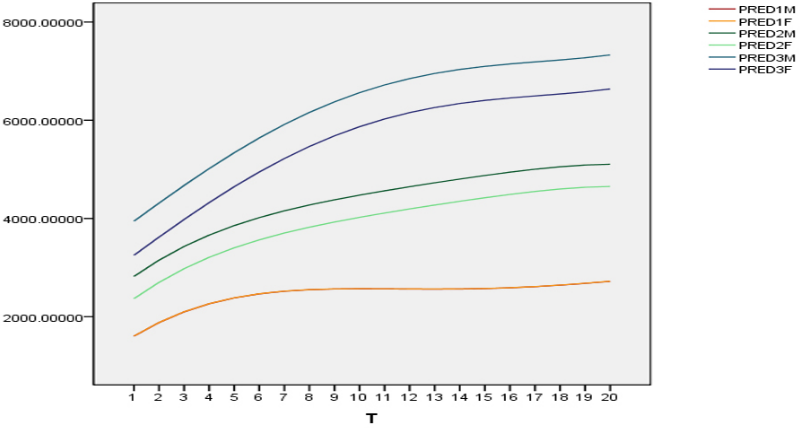

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We propose the following model:

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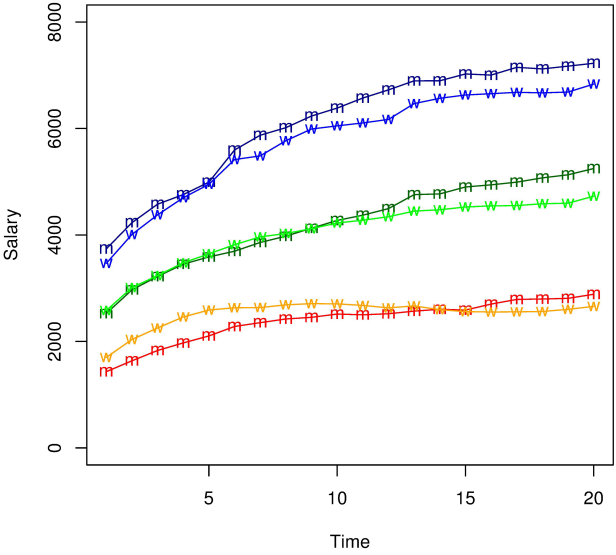
$$+ \left(\beta_2^j + \sum_{l=1}^{L} \alpha_{2l}^j x_l + \gamma_2^j z_{i_t}\right) Age_{i_t}^2 + \left(\beta_3^j + \sum_{l=1}^{L} \alpha_{3l}^j x_l + \gamma_3^j z_{i_t}\right) Age_{i_t}^3$$
$$+ \left(\beta_4^j + \sum_{l=1}^{L} \alpha_{4l}^j x_l + \gamma_4^j z_{i_t}\right) Age_{i_t}^4 + \varepsilon_{i_t},$$

where $\varepsilon_{i_t} \sim \mathcal{N}(0, \sigma)$, σ being a constant standard deviation.

Men versus women



Statistical Properties

Both Nagin's and our model can be written as

$$L = \frac{1}{\sigma} \prod_{i=1}^{N} \sum_{j=1}^{r} \pi_j \prod_{t=1}^{T} \phi \left(\frac{\text{observed data - modelled data}}{\sigma} \right). \tag{9}$$

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Our model:

$$\begin{aligned} y_{i_t} = {} & \left(\beta_0^j + \sum_{l=1}^{L} \alpha_{0l}^j x_l + \gamma_0^j z_{i_t} \right) + \left(\beta_1^j + \sum_{l=1}^{L} \alpha_{1l}^j x_l + \gamma_1^j z_{i_t} \right) Age_{i_t} \\ & + \left(\beta_2^j + \sum_{l=1}^{L} \alpha_{2l}^j x_l + \gamma_2^j z_{i_t} \right) Age_{i_t}^2 + \left(\beta_3^j + \sum_{l=1}^{L} \alpha_{3l}^j x_l + \gamma_3^j z_{i_t} \right) Age_{i_t}^3 \\ & + \left(\beta_4^j + \sum_{l=1}^{L} \alpha_{4l}^j x_l + \gamma_4^j z_{i_t} \right) Age_{i_t}^4 + \varepsilon_{i_t}, \end{aligned}$$

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- Apply proc.traj to the data without covariates do the clustering and obtain the number of groups and the constitution of the groups.
- Use your favorite regression model software to get the trajectories separately for each group.

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