# A performance evaluation of weight-constrained conditioned portfolio optimisation using a new numerical scheme for multisignal problems

Marc BOISSAUX & Jang SCHILTZ

Luxembourg School of Finance
Faculty of Law, Economics and Finance
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# Outline

1. Portfolio optimisation with conditioning information

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2. General formulation of the problem
3. The case of several signals
   - Analysis
   - Empirical results

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# Problem context

- Discrete-time optimisation
- Minimise portfolio variance for a given expected portfolio mean
- Postulate that there exists some relationship $\mu(s)$ between a signal $s$ and each asset return $r$ observed at the end of the investment interval:

$$r_t = \mu(s_{t-1}) + \epsilon_t,$$

  with $E[\epsilon_t | s_{t-1}] = 0$.
- How do we optimally use this information in an otherwise classical (unconditional mean / unconditional variance) portfolio optimisation process?

# Problem history

- Hansen and Richard (1983): functional analysis argument suggesting that unconditional moments should enter the optimisation even when conditioning information is known
- Ferson and Siegel (2001): closed-form solution of unconstrained mean-variance problem using unconditional moments
- Chiang (2008): closed-form solutions to the benchmark tracking variant of the Ferson-Siegel problem
- Basu et al. (2006), Luo et al. (2008): empirical studies covering conditioned optima of portfolios of trading strategies

# Possible signals

Taken from a continuous scale ranging from purely macroeconomic indices to investor sentiment indicators. Indicators taking into account investor attitude may be based on some model or calculated in an ad-hoc fashion.

Examples include

- short-term treasury bill rates (Fama and Schwert 1977);
- CBOE Market Volatility Index (VIX) (Whaley 1993) or its European equivalents (VDAX etc.);
- risk aversion indices using averaging and normalisation (UBS Investor Sentiment Index 2003) or PCA reduction (Coudert and Gex 2007) of several macroeconomic indicators;

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# Possible signals (2)

- global risk aversion indices (GRAI) (Kumar and Persaud 2004) based on a measure of rank correlation between current returns and previous risks;
- option-based risk aversion indices (Tarashev et al. 2003);
- sentiment indicators directly obtained from surveys (e.g. University of Michigan Consumer Sentiment Index)

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# Unconditioned expected return and variance given conditioning information

These are obtained as expectation integrals over the signal domain. If a risk-free asset with return $r_t$ is available,

$$E(P) = E\left[u'(s)(\mu(s) - r_f 1)\right] = E\left[I_1(u, s)\right]$$

and

$$\sigma^2(P) = E\Big[u'(s)\big[(\mu(s) - r_t 1)(\mu(s) - r_t 1)' + \sigma_\epsilon^2\big]u(s)\Big] - \mu_P^2$$

$$= E\left[I_2(u, s)\right] - \mu_P^2$$

for an expected unconditional return of $\mu_P$ and a conditional covariance matrix $\sigma_\epsilon^2$.

# Optimal control formulation

$$
\begin{array}{ll}
\text{Minimise} & J_{[s^-,s^+]}(x,u) = \displaystyle\int_{s^-}^{s^+} l_2(u,s)p_s(s)ds \text{ as } s^- \to -\infty, s^+ \to +\infty \\
\text{subject to} & \dot{x}(s) = l_1(u,s)p_s(s)\ \forall s \in [s^-,s^+],\ \text{with} \\
 & \displaystyle\lim_{s\to-\infty} x(s) = x_-,\ \lim_{s\to+\infty} x(s) = x_+, \\
\text{and} & u(s) \in U,\ \forall s \in [s^-,s^+]
\end{array}
$$

where $U \subseteq \mathbb{R}^n$, $x(s) \in \mathbb{R}^m$ and $L$ as well as $f$ are continuous and differentiable in both $x$ and $u$.

Since the signal $s$ is not necessarily bounded, the resulting control problem involves expectation integrals with infinite boundaries in the general case.

# Necessity and sufficiency results generalised

- The Pontryagin Minimum Principle (PMP) and Mangasarian sufficiency theorem are shown to continue holding if the control problem domain corresponds to the full real axis: the corresponding optimal control problems are well-posed.
- The PMP is then used to show that the given optimal control formulation of the conditioned mean-variance problem generalises classical (Ferson and Siegel; Markowitz) problem expressions.

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# Backtesting data set for all empirical results

- 11 years of daily data, from January 1999 to February 2010 (2891 samples)
- Risky assets: 10 different EUR-based funds commercialised in Luxembourg chosen across asset categories (equity, fixed income) and across Morningstar style criteria
- Money market fund (KBC) included to represent a near risk-free asset
- Signals: VDAX, volatility of bond index, PCA-based indices built using both 2 and 4 factors and estimation window sizes of 50, 100 and 200 points, Kumar and Persaud currency-based GRAI obtained using 1 month and 3 month forward rates

# Individual backtest for all empirical results

- Rebalance Markowitz-optimal portfolio alongside conditioned optimal portfolio over the 11-year period
- Assume lagged relationship $\mu(s)$ between signal and return can be represented by a linear regression
- Use kernel density estimates for signal densities
- Estimate the above using a given rolling window size (15 to 120 points)
- Use direct collocation discretisation method for numerical problem solutions

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# Empirical study for conditioned mean-variance problem involving constrained portfolio weights

- Mean-variance (MV) optimisation problem with and without risk-free asset using discretised efficient frontiers
- Various signals tested, best performance seen for VDAX
- For VDAX, robust improvements typically of the order of 25% for most metrics (returns, SR...), both ex ante and ex post, for different problem parameter settings

# Empirical study for conditioned problems involving higher moments of returns and constrained portfolio weights

- Mean-kurtosis (MK) optimisation problem using discretised efficient frontiers; mean-variance-kurtosis (MVK) and mean-variance-skewness-kurtosis (MVSK) problems using quartic polynomial utility functions
- While MVSK objective function is nonconvex, the results obtained for that case seem consistent with the other variants
- Improvements seen with respect to classical (unconditioned) portfolio optimisation are of the same order of magnitude as seen for the mean-variance problem

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# Optimal control translation

Two signals $s^{(1)}$ and $s^{(2)}$ with $s = (s^{(1)} s^{(2)})$, investor utility function $U(x) = a_1 x + a_2 x^2$, joint signal density $p_s$ give

$$
\begin{aligned}
\text{minimise} \quad & J_{I_S}(x(s), u(s)) = \int_{I_S} \left( a_1 \frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} + a_2 \frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} \right) ds \\
\text{subject to} \quad & \frac{\partial^2 x_1}{\partial s^{(1)} \partial s^{(2)}} = u'(s)\mu(s) p_s(s), \\
& \frac{\partial^2 x_2}{\partial s^{(1)} \partial s^{(2)}} = \left( \Big( u'(s)\mu(s) \Big)^2 + u'(s) \Sigma_\epsilon^2 u(s) \right) p_s(s), \\
& x_1(s^-) = x_2(s^-) = 0 \\
\text{and} \quad & u(s) \in U \ \forall s \in I_S
\end{aligned}
$$

as the resulting mean-variance equivalent optimisation problem formulation.

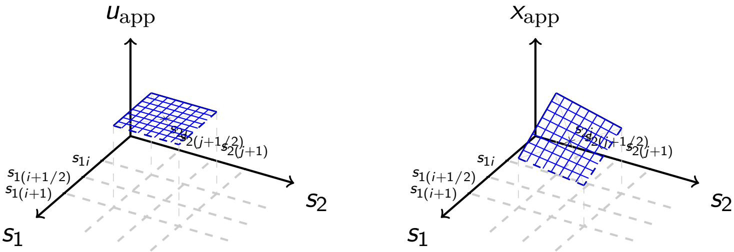
# Multidimensional results

- Optimal control problems involving a higher-dimensional objective function integration variable and first-order state PDEs are called *Dieudonné-Rashevsky* problems
- Multidimensional analogues of PMP have been established (Cesari 1969) for problems of the Dieudonné-Rashevsky type
- The problem with cross-derivatives just given represents a form equivalent to Dieudonné-Rashevsky (Udriste 2010)

# 2-D discretisation scheme

- Use a 2-D direct collocation scheme: *direct* means both control and state variables are discretised, *collocation* means PDE and other constraints have to be met exactly at prespecfied (collocation) points on the grid
- Use control values constant on each surface element and state values on vertices to which bilinear interpolation is applied
- Provide analytical expressions for the (sparse) gradient and Hessian matrices to the numerical solver so convergence rate and computational cost remain manageable

# 2-D discretisation scheme (2)




(a) Control discretisation constant over surface elements.

(b) Bilinear state discretisation.

# 2-D discretisation scheme convergence result

**Theorem**

*At the collocation points $s_{i+1/2,j+1/2}$, the Pontryagin costate equations are verified to order the chosen grid mesh $h$:*

$$\nabla_s \cdot \lambda = -\sum_{\alpha=1}^{2} \lambda^{(\alpha)}_{i+1/2,j+1/2} \frac{\partial f^{(\alpha)}_{i+1/2,j+1/2}}{\partial x} + O(h).$$

*Also, for any optimal control interior to the admissible set $U$, the proposed scheme is consistent with the first-order condition on the Hamiltonian $\mathcal{H}$*

$$\frac{\partial \mathcal{H}}{\partial u(s)} = 0 \ \forall s \in I_S.$$

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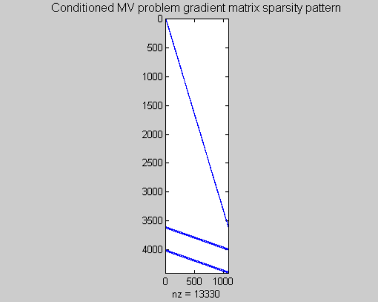
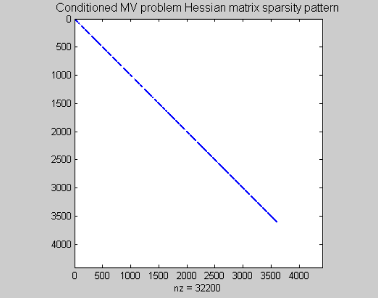
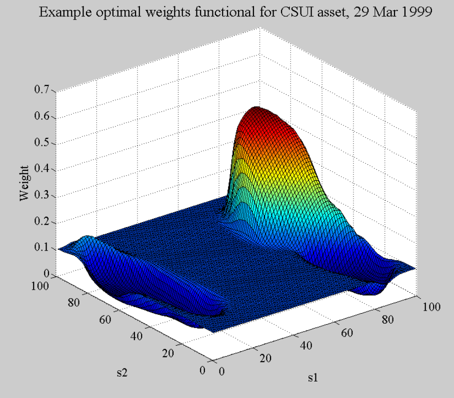
# 2-D discretisation gradient and Hessian matrix sparsity patterns




- Gradient dimensions for $N \times N$-point grid and $n$ assets are $\left[(N-1)^2 n + 2N^2\right] \times \left[3(N-1)(N-2) + 3(N-2) + 5\right]$
- Hessian dimensions in that case are $\left[(N-1)^2 n + 2N^2\right] \times \left[(N-1)^2 n + 2N^2\right]$

# Typical optimal weight functional




Optimal weights are found as vector functions of the two signals

# 2-signal backtest

- Simultaneously use VDAX (pure equity risk) and BONDIDX (volatility of Barclays Aggregate Euro Bond Index, pure interest rate risk) as signals
- Obtain optimal portfolio weights for daily rebalancing by optimising unconditional expected utilities for quadratic investor utility functions $U(x) = a_1 x + a_2 x^2$ and three different levels of risk aversion: $a_2 = -0.2$, $a_2 = -0.5$ and $a_2 = -0.7$.
- Compare utilities and Sharpe ratios (ex ante and ex post), maximum drawdowns / drawdown durations (MD/MDD) and observed returns time paths for Markowitz, 1 signal and 2 signal strategies

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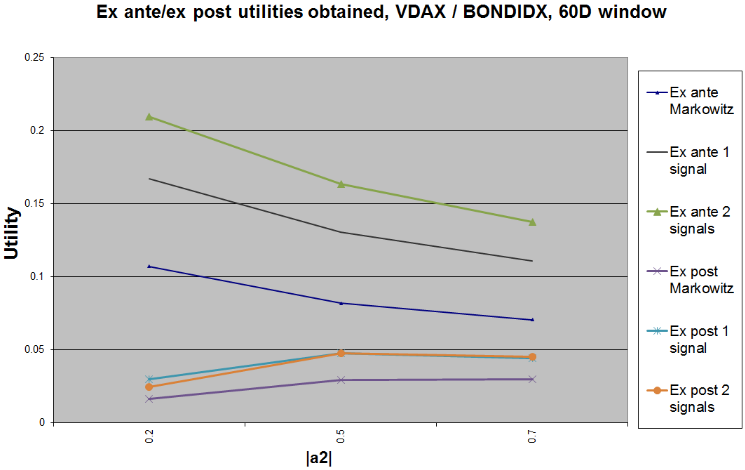
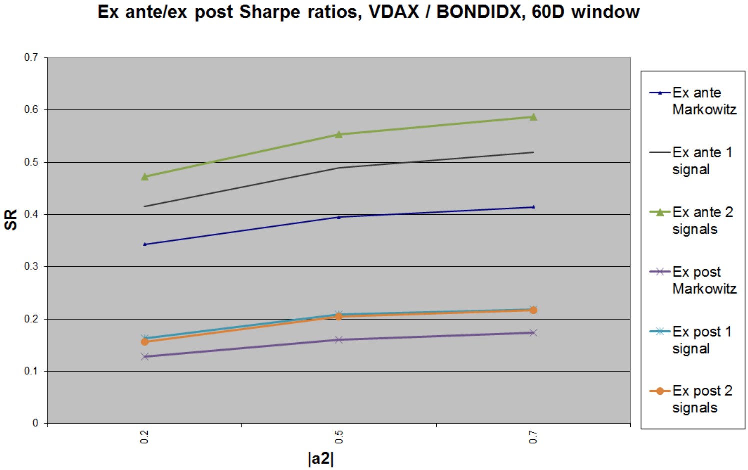
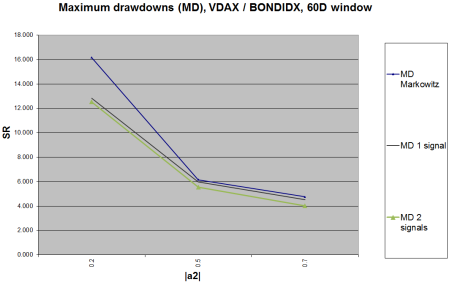
# Backtest average utility values

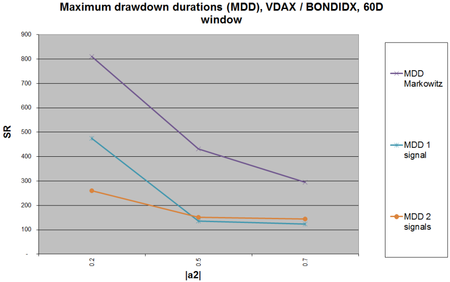


# Backtest average Sharpe ratios

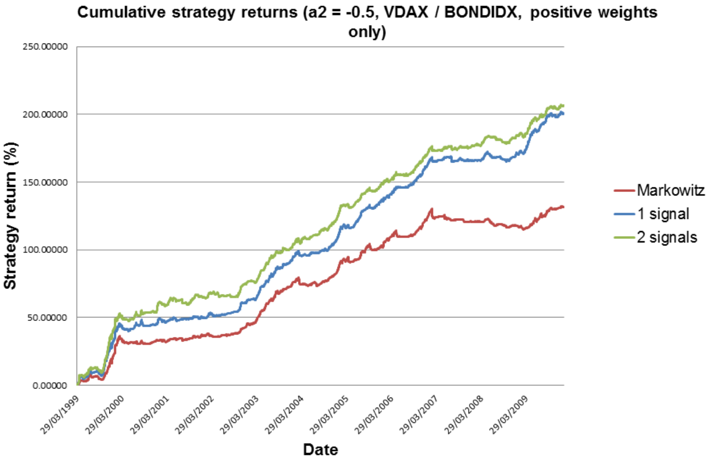


# Backtest average maximum drawdown (durations)



# Backtest cumulative return time paths, $a_2 = -0.5$



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# Summary

- Improvement with a second signal is substantial ex ante, but very marginal ex post: estimation risk larger than for a single signal
- The suggested numerical solution scheme can be generalised to even more signals, but a curse of dimensionality applies:
  - computational cost: will diminish in impact over time
  - statistical (kernel density estimate): fundamentally prevents the use of more than three signals unless simplifications are made.
- Marginal ex post improvements, however, suggest an averaging effect (as seen for single PCA indices in earlier single signal study) takes place for more signals, such that this limitation is not seen as that restrictive

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