

Alleviating the Mesh Burden in Computational Solid Mechanics

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Abstract

The goal of this chapter is to review recent avenues of investigation to alleviate meshing difficulties in computational mechanics and give a few exemplar applications.

Keywords: meshing; enrichment; meshfree methods; extended finite element methods; isogeometric analysis; advanced remeshing techniques.

1 Introduction

Computational modelling and simulation have strong economic implications by improving product quality and decreasing time to market and are now used in most fields of Engineering and Science. More importantly, virtual computer representations may be the only plausible route to study certain processes or phenomena where physical experiments are either extremely costly or outright impossible (e.g. nuclear applications). To support the development of those virtual platforms, computational methods have been intensively researched and have unarguably become extremely powerful tools for engineers and scientists.

We have entered the era of petascale computing where problem size and complexity will continue to increase faster than the yet mind-boggling computing power of the next generation supercomputers. Indeed, engineers are not content to predict the stress distribution in a turbine blade; they want to study the effects of temperature, moisture, turbulence, bird strikes, manufacturing techniques, variability in material properties/loading spectra, surface treatments, microstructural changes in the alloys, etc.

These more realistic models lead, in turn to more efficient and sustainable products. Because of obvious financial implications, engineers also expect the solutions to these increasingly complex problems to be provided quickly in order to reduce lead times and increase competitiveness. Similarly, scientists use computer modeling to study phenomena such as drug delivery, cell deformation, bacterial colonies, blood flow, or even to simulate the effects of surgical procedures where it is necessary that the computational software provide real-time interactive solutions with controlled accuracy. In other words, to meet these needs, simulations must be made faster, more accurate and should be obtained with a predefined accuracy level.

Much effort has been devoted from the scientific and engineering communities to reconcile these apparently incompatible and strenuous demands. This chapter addresses efforts directed towards reducing the mesh generation burden in numerical methods by focusing on three related advances made in the last decade:

- eXtended Finite Element Methods and Finite Element Methods in Ambient Space.;
- (Enriched) Meshfree Methods;

- Isogeometric Analysis;

Other important advances, which we will not review include:

- Advanced Meshing Technology;
- (Dual) Boundary Element Methods;
- Scaled Boundary Finite Element Methods.

2 Meshes

Most physical phenomena concerning solids can be described by considering the solid to be a continuum and to formulate appropriate partial differential equations (PDEs). Analytical solutions are in general not available for these PDE, but numerical methods can be used instead. The PDE can be discretized directly in which case the numerical method is said to be *strong form based* (finite differences, generalized finite differences, point collocation) or it can be multiplied by test functions and integrated over the domain (weighted residual method) in which case it is known as a *weak form based method*. Historically, the finite difference method (FDM), a strong form based method relying on a regular grid of points appeared first, followed by the finite element method (FEM), a weak form based method [92].

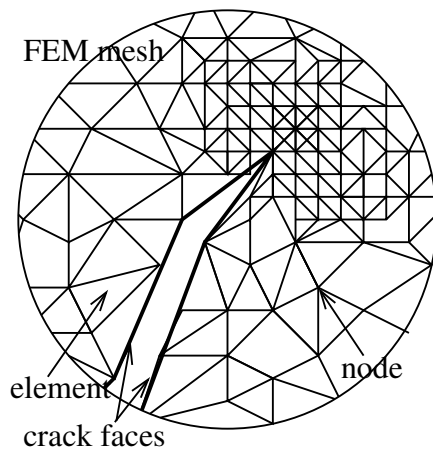


Figure 1: FEM for cracks.

The FEM is certainly the most widely employed numerical method today, for its robustness, accuracy, relative simplicity and ability to deal with complex geometries. In the FEM, the discretization points are known as *nodes* and are related to one another through a *mesh* (Figure 1), i.e. a finite set of geometrical entities (polygons in two dimensions, polyhedra in three dimensions) of finite size known as *elements*, which cover the solid. The vertices of those elements are located at the nodes, which builds a so-called *connectivity table*, i.e. a map providing the identities of the nodes for any given element. The union of these elements forms a mesh and the unknown fields are usually calculated at the nodes. To solve a given problem on a domain of interest, it is sufficient to generate a mesh which conforms with the geometry of the domain, and formulate the local behavior of each finite element according to the problem at hand which are then assembled to describe the global behavior. In FEM, the mesh of elements has essentially two roles:

- It allows the construction of a local polynomial *approximation*, based on the connectivity table;

- It is used for *numerical integration* of the weak form of the PDE.

There are several rather stringent requirements on the mesh which have been partially answered by recent research and which this chapter focuses upon:

Generating the mesh A first requirement is that this mesh conforms to the outer boundary of the domain and to any internal discontinuities such as holes, cracks or other interfaces. Consequently, if the geometry of the domain evolves, the mesh needs to be regenerated. Alleviating the stringency of this requirement has been the focus of two classes of methods: Partition of Unity Enriched Finite Elements (PUFEM) and meshless/meshfree methods (MMs) which will be the topic of Section 3 and Section 4 respectively.

Large gradients A second requirement is that the mesh be sufficiently fine to represent the behaviour of interest accurately (see Figure 1 for the case of a crack where the stress field increases significantly when approaching the crack front; this rapid spatial variation can only be captured by using very small elements). This is addressed by PUFEMs and enriched MMs.

Mesh quality A third requirement is that elements should be of “good quality”; for example, the shape of triangular elements should be maintained as close as possible to that of an equilateral triangle. In part, this has been addressed by the MMs as well as the smoothed finite element method (SFEM). The latter is reviewed in Section 6, although much work remains to be done in this area.

Iterations between CAD and FE A fourth requirement is the need to tie the mesh generation to the iterative design process. This has been addressed by an exciting new area of research known as isogeometric analysis, of which a brief overview is provided in Section 5.

Construction of approximations with higher-order continuity is cumbersome and not natural. MMs address this issue.

The mesh must conform to evolving discontinuities such as cracks, material interfaces, debonding zones, growing voids, etc. When large numbers of those features are present in a given model, regenerating the mesh is computationally expensive, decreases the robustness of the approach and increases human intervention. The PUFEM family addresses these issues.

Mesh adaptation where elements are split to capture large stress gradients and elements are enlarged where the solution is smooth is not a natural construct within the FEM as the nodal connectivity must be recomputed at each adaptive step. MMs also address this issue.

When dealing with moving interfaces, these four requirements combine to demand of the analyst the generation of possibly very fine meshes at each step of the simulation. Despite much work and some formidable advances, some of the most impressive

by SimpleWare [simpleware, Franc3d, Bouchard], automatic mesh generators are not sufficiently robust so that today still, 80% of a typical analysis is spent generating the mesh. For moving interfaces in 3D such as cracks, automatic mesh generation often simply fails and analyses must be restarted from the previous converged solution. When more than one crack or evolving discontinuity is present, this remeshing step becomes intractable.

Several alternatives have appeared to combat this strenuous need for meshing and remeshing boundaries and evolving discontinuities, some of which we will review in the remainder of this chapter. Others, which we do not cover here include the Boundary Element Method, Scaled Boundary Finite Element Methods, and others. For example, Boundary Element Methods (BEM), as stand-alone packages or coupled to FEM are successful techniques to reduce the difficulties associated with remeshing for crack propagation simulations, mainly in the context of linear elastic fracture mechanics (LEFM) [41, 42] and gradient elasticity [47]. Enriched versions of the BEM have also recently been proposed by Robert Simpson in his PhD thesis [89]. Several commercial packages are available to this end such as BEASY and Zencrack. See [24] for a text book account. However, BEM is not yet well-developed for non-linear materials and multiple cracks.

3 Partition of Unity/Generalized/Extended Finite Element Methods

3.1 Motivation and Historical Perspectives

Enrichment methods based on the decomposition of the solution in a form $\mathbf{u} + \tilde{\mathbf{u}}$ where \mathbf{u} is typically a finite element solution and $\tilde{\mathbf{u}}$ provides information about the behaviour of the solution in the form of a *global* analytical function, are attributed to [?] and are known as *global-local methods*. Those were later revisited by [30], [76], [59, 95] and more recently [60]. Global-local methods have been applied to a variety of problems including shear bands and cracks [5, 72]. However, these global enrichment methods decrease the sparsity of the resulting matrix. Another technique allowing enrichment of the finite element method is known as the *s*-version by Fish [39].

The development of local enrichment, condensed out at the element level, followed the early global enrichment techniques. Some of the pioneering work on this is due to [6], among others.

Enriched methods in general are particularly attractive to model crack propagation since they do not require the mesh to conform to the evolving discontinuities.

3.1.1 Embedded Discontinuities: Enrichment at the Element Level

Embedded elements introduced in [36], and improved in subsequent work by Oliver among others [77, 78], [79, 80], can handle arbitrary crack paths without remeshing.

The embedded element method (EFEM) is based on an enrichment at the element level such that the appropriate crack kinematics is obtained. Crack opening is assumed to be piecewise constant though recent studies have shown that it is also possible to incorporate linear crack opening into EFEM, [53], later extended to finite deformation [1] and branching [54].

One advantage of the EFEM is that the enrichment can be condensed at the element level. Hence, EFEM can be incorporated into commercial software with relatively small effort. However, though the EFEM is capable of handling arbitrary crack paths, crack path continuity is still required in order to remove spurious mesh-dependence, [71]. A very good review of EFEM is presented by [46].

3.1.2 Partition of Unity Enrichment: Enrichment at the Node Level

The extended finite element method (XFEM) [69] is also based on a $\mathbf{u} + \tilde{\mathbf{u}}$ decomposition, where the local Partition of Unity (PU) concept [63], [64] and [62] is being applied to industrial problems, especially in the aerospace industry: [10, 13] and [98, 99]. In contrast to the EFEM, the additional degrees of freedom inherent to the nodes are introduced in the variational formulation and have to be solved for. This makes the method very flexible, e.g. with respect to the application spectrum and in the sense that higher order crack openings can be incorporated quite easily (Figure 2). However, it also makes the method more cumbersome to implement into commercial finite element packages.

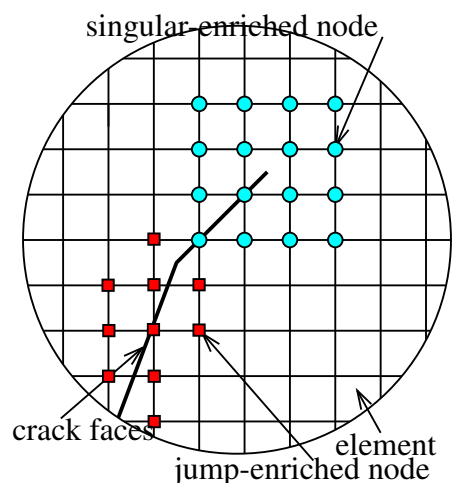


Figure 2: XFEM for cracks.

Recent re-interpretations of the original XFEM might be an alternative for the incorporation into commercial software [90, 91], which was first performed in the context of standard XFEM in [13]. The major difference between these versions and original XFEM is that the crack kinematics is captured by overlapping elements rather than by adding additional degrees of freedom. This way, the existing structure of a given code can be better exploited.

A very interesting comparative study between element deletion, interelement-separation methods and XFEM was recently presented by [90].

A close cousin of the XFEM is the Generalized Finite Element Method (GFEM) [93], which was also successful at modelling 3D dynamic crack propagation in [32] and coupled to a global/local method to introduce cracks of subelement length in [48].

3.2 Introduction to Ambient Space Methods

3.2.1 Definition

The mesh generation task in finite element analysis (around 80% of the processing time and still heavy in user intervention) can be simplified by embedding the complex domain to be analyzed into another, simpler and easier to mesh domain whose mesh is independent of the geometry of the component. Different names are used for this class of approaches: Fixed Grid FEM (FGFEM) [26,40], Fictitious Domain [19,43,83], Implicit Meshing [9], Immersed FEM [101], Immersed Boundary method [101]. In the following we will put those methods under the large umbrella of “*Finite Elements in Ambient Space*.” These techniques have been used to solve several types of problems, both in the context of the FEM and in that of the finite volume method. During the last years, advanced FEM formulations such as the extended finite element method (XFEM) and the generalized finite element method (GFEM) have also facilitated alleviating the mesh burden by allowing meshes to be independent of the geometry.

Partition of unity methods such as XFEM are not only useful for problems involving moving boundaries such as crack propagation. They have also been used to solve finite element problems where the boundary of the domain itself is not meshed explicitly.

3.2.2 Motivation and background

Before discussing recent developments where Partition of Unity Methods have been used to this effect, it is useful to review recent advances in implicit boundary definition for FEA in ambient space e.g. [88]. In a series of papers, [49, 50], Kumar and colleagues proposed an approach based on constructive solid geometry (CSG) [21], where the geometry is defined implicitly by a regularized Heaviside step function, in a way similar to level set methods. These Heaviside functions are regularized and represented by high-order polynomials, which requires high-order approximations. The disadvantages of this approach is that (i) a regularization parameter has to be defined, and tailored, locally, to the geometry to be described; (ii) high-order shape functions must be used to suitably represent the large spatial variation of the regularized step function. Similar approaches based on Heaviside functions were proposed in the context of the extended finite element method (XFEM): see [94] and [14] for details, and a review and implementation aspects, respectively.

Numerical examples shown in [50] include calculation of volumes for various shapes and objects, based on boolean operations on volumes, which show that their method is able to represent sharp corners. Yet, this is done by tailoring an additional regularization parameter, ϵ . As this regularization parameter goes to zero, the solid representation becomes exact. For non-zero ϵ , this representation is inexact, but the error can be made negligible compared to other numerical sources of error. The authors also show in [51] how their technique can be used to solve two and three-dimensional finite element problems in ambient space, i.e. with a fixed background grid of quadri-

lateral elements. The imposition of essential boundary conditions, however, depends on parameter ε and requires modifying the problem formulation. The method was recently applied to compute effective properties of composite structures in [18].

An alternative was proposed in [70] where Lagrange multipliers avoid the need for this boundary regularization approach. Finally, in [17], it is shown how a higher order approximation (relying on NURBS) based on a regular background grid, can improve the accuracy of the geometrical description.

An elegant technique based on oct-tree mesh refinement and a discontinuous Galerkin method was recently proposed in [37]. In this paper, the authors use a regular mesh and local oct-tree refinement to enrich the approximation space. Doing so, they successfully solve a potential flow problem around multiple obstacles.

In a very recent paper [61], an interface handling technique for 3d higher order extended finite element computations in fluid structure interaction is proposed. In this paper, the authors do not use level sets, but place points along the interface, which requires information about the analytical form of the interface geometry, at least initially. In this work, the surfaces are explicitly defined, and the authors state that an implicit level set description of the interface is part of their plans for future work.

A conjecture is that one can remedy the above difficulties by employing multiple level set functions to describe the boundary of the domain and resolve vertices as well as sharp edges. To capture the geometry accurately, an integration technique based on the recursive subdivision of suitably chosen integration cells could prove sufficient.

3.2.3 Partition of Unity Methods for Finite Elements in Ambient Space

Partition of unity methods (PUM) such as XFEM lend themselves naturally to performing ambient space finite element simulations where the domain's boundary is defined independently of the mesh. The use of PUM for ambient space FEA pursues a similar goal to that of isogeometric analysis (Section 5), i.e. tying CAD to FEA, but approaches the problem from a different angle:

- The underlying shape functions are generally the usual piecewise linear polynomials, hence, the usual arsenal of methods available to FEA can be directly used. This facilitates implementation and numerical integration as well as the imposition of boundary conditions. The approximation space can, if needed, be enriched through partition of unity;
- The geometry is defined implicitly, through level set functions;
- Contrary to isogeometric analysis, the geometrical description is *not exact*, but the geometrical error vanishes as the mesh is refined and the number of subgrid integration cells increases.

To our knowledge, the first paper on this approach is due to Belytschko and colleagues in [8], where the boundary of solids is described implicitly using radial basis

functions (RBF). In this case, the underlying approximation is, however, not polynomial. This concept was further developed by Moës' group in the paper [68] where it was shown that complex microstructures can be represented without requiring a conformal mesh.

Similar ideas have also been used by Duysinx and colleagues [67] to simplify shape optimization. In the following part of this section, we will briefly provide the reader with the necessary tools required to understand the literature in this area. For details, refer to, the papers referred to above.

3.3 Implicit Geometry and Finite Elements in Ambient Space

3.3.1 Definition

A domain Ω is divided into 2 domains Ω_A and Ω_B . The interface (or surface of discontinuity) between the 2 domains is denoted by Γ (Figure 3).

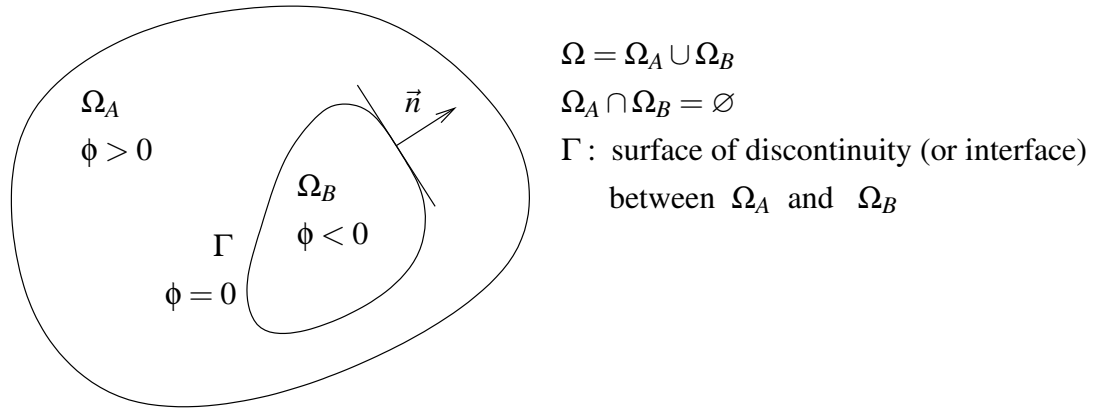


Figure 3: Description of a two-domain problem by using level sets.

The level set function $\phi(\mathbf{x})$ is such that:

$$\phi(\mathbf{x}) > 0 \text{ if } \mathbf{x} \in \Omega_A,$$

$$\phi(\mathbf{x}) < 0 \text{ if } \mathbf{x} \in \Omega_B,$$

$$\phi(\mathbf{x}) = 0 \text{ if } \mathbf{x} \in \Gamma.$$

In other terms

$$\Omega_A = \{\mathbf{x} \in \Omega / \phi(\mathbf{x}) > 0\},$$

$$\Omega_B = \{\mathbf{x} \in \Omega / \phi(\mathbf{x}) < 0\},$$

$$\Gamma = \{\mathbf{x} \in \Omega / \phi(\mathbf{x}) = 0\}.$$

The interface Γ is the zero iso contour of $\phi(\mathbf{x})$. Note that this function can be time-dependant, the zero iso-contour of $\phi(\mathbf{x}, t)$ is then the position of the interface at time t [81] [87] [38].

3.3.2 Normal and curvature

Normal The normal \vec{n} to the interface Γ at the point $\mathbf{x} \in \Gamma$ is defined by

$$\vec{n} = \frac{\vec{grad}\phi}{\|\vec{grad}\phi\|}.$$

If $\|\vec{grad}\phi\| = 1$ then

$$\begin{aligned}\vec{n} &= \vec{grad}\phi, \\ n_i &= \phi_{,i}.\end{aligned}$$

The normal \vec{n} is oriented from Ω_B to Ω_A if Ω_B is the ϕ -negative domain and Ω_A is the ϕ -positive domain.

Curvature The curvature of Γ at a point $\mathbf{x} \in \Gamma$ is defined by

$$\mathcal{K} = \text{div } \vec{n} = \text{div} \left(\frac{\vec{grad}\phi}{\|\vec{grad}\phi\|} \right) = n_{i,i}$$

If $\|\vec{grad}\phi\| = 1$ then

$$\mathcal{K} = n_{i,i} = \phi_{,ii}$$

3.3.3 Signed distance function

A usual level set function is the signed distance function (Figure 4). This function reaches the property of being zero on the surface of discontinuity.

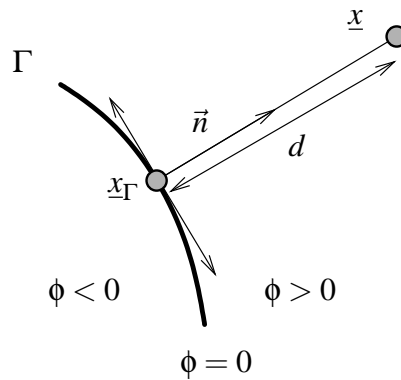


Figure 4: Signed distance function.

The distance d from a point \mathbf{x} to the interface Γ is

$$d = \|\mathbf{x} - \mathbf{x}_\Gamma\|$$

where \mathbf{x}_Γ is the normal projection of \mathbf{x} on Γ . The level-set function $\phi(\mathbf{x})$ is set to

$$\phi(\mathbf{x}) = d \text{ in } \Omega_A,$$

$$\phi(\mathbf{x}) = -d \text{ in } \Omega_B.$$

This can be written

$$\phi(\mathbf{x}) = \underbrace{\min}_{\tilde{\mathbf{x}} \in \Gamma} \|\mathbf{x} - \tilde{\mathbf{x}}\| \operatorname{sign}(\underline{n} \cdot (\mathbf{x} - \tilde{\mathbf{x}})).$$

Note that the signed distance function reaches the following property:

$$\|\vec{\operatorname{grad}}\phi\| = 1.$$

3.3.4 Discretization of the level set

In the general case, the signed distance to the interface is not known analytically. The level set is usually discretized. The discretization is based on a finite element mesh where $N_I(\mathbf{x})$ is the shape function associated to the node I . The set of nodes which belong to the mesh is denoted \mathcal{S} . The discretized level set is

$$\phi(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) \phi_I$$

where ϕ_I is the value of the level set on node I . This discretization becomes useful when the value of the level set is needed at the element level: it can be evaluated by interpolation. Moreover the derivative of the level set,

$$\phi(\mathbf{x})_{,i} = \sum_{I \in \mathcal{S}} N_{I,i}(\mathbf{x}) \phi_I,$$

involves only the well known derivatives of the shape functions.

3.3.5 Updating the level set

When the discontinuity evolves in time, the level set has to be updated. Different methods can be used to do so.

Signed distance computation If the position of the discontinuity is known at each time step, the signed distance function can be re-compute in order to build the level-set.

Time integration using conservation equation In general, the position of the discontinuity is not known *a priori*. The level set has to be zero on the interface. The material time derivative of ϕ has to be zero:

$$\frac{D\phi(\mathbf{x}, t)}{Dt} = 0.$$

By assuming that the discontinuity has a \vec{v} velocity field, the previous equation can be written

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} + \vec{grad} \phi(\mathbf{x}, t) \cdot \vec{v}(\mathbf{x}, t) = 0,$$

or

$$\dot{\phi} + \phi_{,i} v_i = 0.$$

Using a first-order time scheme leads to

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = -\phi_{,i}^n v_i^n$$

and to

$$\phi^{n+1} = \phi^n - \Delta t \phi_{,i}^n v_i^n$$

where Δt is the time step. The Courant-Friedreichs-Lewis (CFL) condition is

$$\Delta t < \frac{\Delta x}{\max(v_i)}.$$

The \vec{v} velocity field can be given by the flow for a bi-phasic fluid [22] or by the physics of the phenomena: e.g. solidification front velocity [23] , biofilm velocity extension [33, 34].

3.3.6 Solid shape description

This level set approach can be used to model the shape of a solid [8] [94] or of a fluid structure interface [52]. The solid, defined by domain Ω_A is included in the Ω domain called computational domain (Figure 5). The shape of the solid is defined by the zero iso-contour of the level set $\phi(\mathbf{x})$.

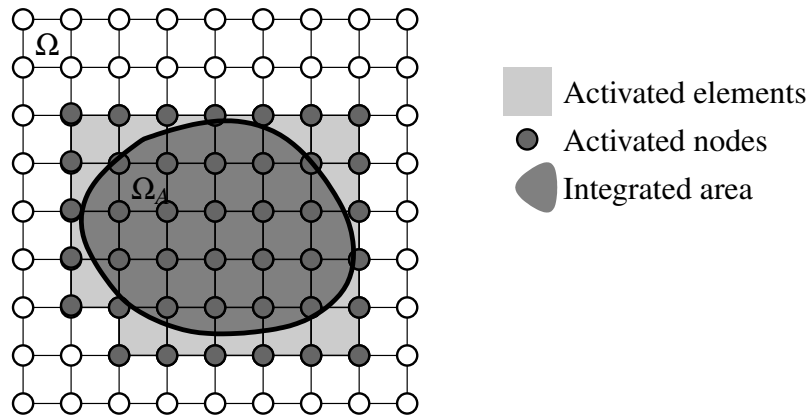


Figure 5: Solid shape defined by a level set.

The elements which have at least one ϕ -positive node are activated for the computation. For instance, a node which is ϕ -negative and which belongs to an element cut

by the zero level set is taken into account in the computation even if it is outside of the solid.

Remarks:

- In order to impose boundary conditions, it is better to have a part of the imposed displacement and the imposed stress vector boundaries compatible to the mesh [70].
- The elements cut by the zero level set have to be cut in 2 parts for the integration (Section 3.4.6). The part which is outside of the solid is not integrated since there is no material.

The advantages of this method are that the shape of the solid can have an analytical expression and that there is no need to remesh when a shape optimization procedure is performed.

3.4 Partition of Unity and Enriched Approximations

3.4.1 Partition of unity

A partition of unity is a set of function $f_i(\mathbf{x})$ defined on Ω^{pu} such that

$$\forall \mathbf{x} \in \Omega^{\text{pu}}, \sum_i f_i(\mathbf{x}) = 1$$

By choosing arbitrary a function $\psi(\mathbf{x})$ defined on Ω^{pu} , the following property is obvious,

$$\sum_i f_i(\mathbf{x})\psi(\mathbf{x}) = \psi(\mathbf{x}).$$

In a finite element approach, the collection of shape functions is usually a partition of unity.

3.4.2 Enrichment

Principle A partition of unity allows to introduce an arbitrary function $\psi(\mathbf{x})$ in the approximation field of $u_i(\mathbf{x})$ [63] [64] [62].

- The standard part of the approximation is

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii}$$

where

- \mathcal{S} is the set of nodes of the mesh,

- $N_I(\mathbf{x})$ is the shape function associated to node I , note that $N_I(\mathbf{x}_J) = \delta_{IJ}$,
 - \mathbf{x}_I are the node I coordinates,
 - U_{Ii} is the nodal unknown of node I for the i^{th} component.
- The enriched approximation field is

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii} + \sum_{J \in \mathcal{P}.\mathcal{U}} f_J(\mathbf{x}) \psi(\mathbf{x}) A_{Ji}$$

where

- $\mathcal{P}.\mathcal{U}$ is the set of functions building the partition of unity,
- $f_J(\mathbf{x})$ is the J^{th} function of the partition of unity,
- $\psi(\mathbf{x})$ is the enriched or additional function,
- A_{Ji} is the additional unknown associated to the function $f_J(\mathbf{x})$.

If one takes all the additional unknowns A_{Ji} equal to 1 and all the standard nodal unknowns u_{Ii} equal to zero, the approximation can recover the additional function $\psi(\mathbf{x})$.

The support of the partition of unity is called Ω^{pu} . This support is usually located around the discontinuity but it can be extended if one needs to model a high gradient as for a crack tip.

3.4.3 Shape functions as a partition of unity

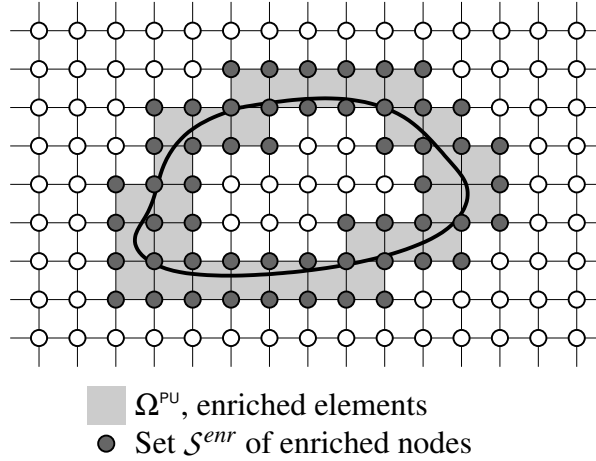


Figure 6: Set of enriched nodes \mathcal{S}^{enr} , enriched elements and support of the partition of unity.

Since the collection of the shape functions is a partition of unity, they can be used as a support for the additional function. The new approximation is then:

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii} + \sum_{J \in \mathcal{S}^{enr}} N_J(\mathbf{x}) \psi(\mathbf{x}) A_{Ji}$$

where \mathcal{S}^{enr} is the set of the so-called enriched nodes. The set of enriched nodes \mathcal{S}^{enr} is included in the set of nodes \mathcal{S} .

The set of enriched nodes \mathcal{S}^{enr} are the nodes that are connected to the elements cut by the discontinuity (Figure 6).

Numerically, these enriched nodes can be easily found by looking at the signs of the nodal level set values in an element: if at least 2 nodes have different signs, the element is enriched.

From the enriched approximation

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii} + \sum_{J \in \mathcal{S}^{enr}} N_J(\mathbf{x}) \psi(\mathbf{x}) A_{Ji}$$

one can see that the value of $u_i(\mathbf{x})$ on an enriched node K in \mathcal{S}^{enr} is

$$u_i(\mathbf{x}_K) = U_{Ki} + \psi(\mathbf{x}_K) A_{Ki}.$$

This last expression is not the standard expected relation where the nodal unknown U_{Ki} is the real displacement on the node.

In order to satisfy this relation, the following approximation is often used:

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii} + \sum_{J \in \mathcal{S}^{enr}} N_J(\mathbf{x}) \left(\psi(\mathbf{x}) - \psi(\mathbf{x}_J) \right) A_{Ji}$$

Using this last approximation, the expected property is reached:

$$u_i(\mathbf{x}_K) = U_{Ki}.$$

3.4.4 General expression for several enriched functions

The method can be extended to several enriched functions $\psi^k(\mathbf{x})$. The enriched approximation field is

$$u_i(\mathbf{x}) = \sum_{I \in \mathcal{S}} N_I(\mathbf{x}) U_{Ii} + \sum_k \sum_{J \in \mathcal{P}, \mathcal{U}^k} f_J^k(\mathbf{x}) \psi^k(\mathbf{x}) A_{Ji}^k.$$

In this last expression, each additional function $\psi^k(\mathbf{x})$ uses its own partition of unity $\mathcal{P}, \mathcal{U}^k$ and has its own additional unknowns A_{Ji}^k .

3.4.5 Choosing the enrichment

The choice of the enriched functions depends on the *a priori* solution of the problem. The enrichment is usually given in terms of the level set. Table 1 shows a few typical enrichment.

kind of problem	displacement	strain	enrichment
bi-material	continuous	discontinuous	ramp: $\Psi(\mathbf{x}) = \phi(\mathbf{x}) $
crack	discontinuous	-	Heaviside: $\Psi(\mathbf{x}) = \text{sign}(\phi(\mathbf{x}))$
crack tip (local (θ, r) coordinates to the crack tip)	discontinuous for $\theta = \pm\pi$, \sqrt{r} order	high gradient, $\frac{1}{\sqrt{r}}$ order	$\sqrt{r} \sin \frac{\theta}{2}$, $\sqrt{r} \cos \frac{\theta}{2}$, $\sqrt{r} \sin \frac{\theta}{2} \sin \theta$, $\sqrt{r} \cos \frac{\theta}{2} \sin \theta$.

Tableau 1: Examples of choice of enriched functions.

3.4.6 Numerical integration

Since the enriched part of the discretized gradient operator is discontinuous through the interface, the element stiffness matrix of an enriched element has to be integrated separately in each domain. Moreover, the material properties can be different in the 2 domains.

Firstly, the element is divided into 2 sub-domains:

where $\phi > 0$ (Ω_A^e),

and where $\phi < 0$ (Ω_B^e).

By interpolating the nodal values of ϕ , one can find the points on the edges of the element where $\phi = 0$.

Secondly, each sub-domain is sub-divided into simple shape elements in which the integration is possible. For a 2D element, a triangle sub-division is usually used (Figure 7). For a 3D element, one can use a tetrahedra sub-division (Figure 8).

Recently, an analytical method has been proposed to integrate the enriched elements [96]. Other techniques aiming at removing integration subcells include work by Natarajan and colleagues [73, 74]

3.5 Imposition of Boundary Conditions on Implicit Boundaries

Because boundaries are defined implicitly, i.e no node is present on the boundaries, imposition of essential boundary conditions requires special attention. We do not review this point here, but refer the interested reader to the very nice recent article by [29] and references therein.

3.6 Challenges and Future Work

There are a number of challenges ahead for methods such as the extended FEM. Many of the intrinsic challenges which appear as the flip side of its advantages have already been heavily researched, namely:

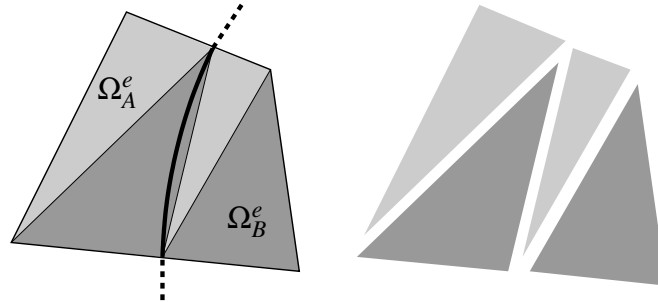


Figure 7: Sub-division of an enriched 2D element into triangles. Each triangle is integrated with a standard Gauss point quadrature.

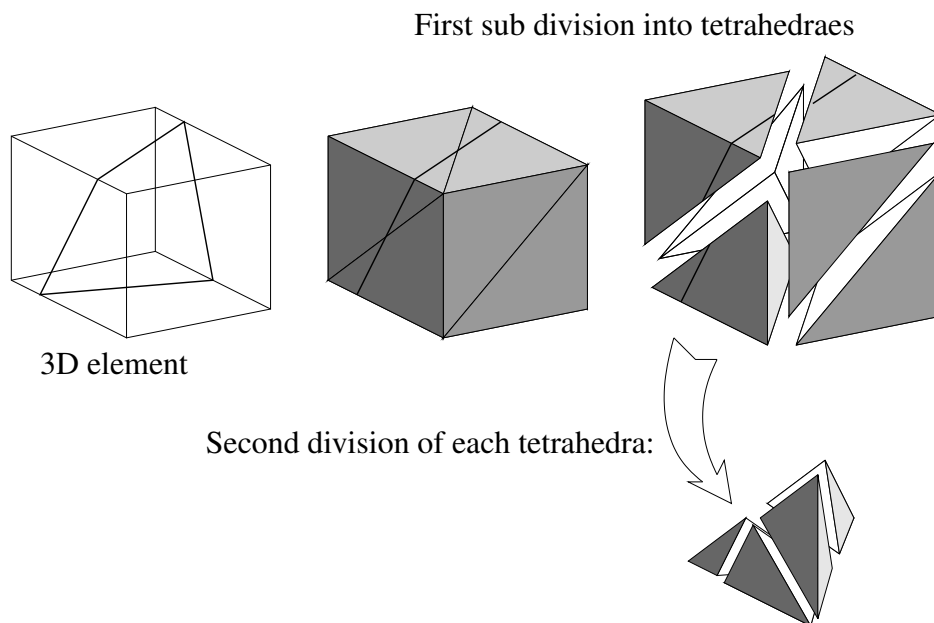


Figure 8: Sub-division of an enriched 3D element into tetrahedra. Each tetrahedron is integrated with a standard Gauss point quadrature.

- Numerical Integration of non-polynomial and non-smooth functions;
- Imposition of essential boundary conditions;
- Absence of analytical information to be used for enrichment, resulting in the need for numerical or/and parametric enrichment;
- Preconditioning of global system matrix to overcome special cases where boundaries pass close to nodes and ill-conditioning issues related to enriched approximations with specific functions;
- Lack of smoothness of the approximation resulting in lack of accuracy of the gradient fields;
- High performance computing and parallelization/domain decomposition methods for enriched approximations;
- Generic, robust and automatic implicit boundary definition of multiple growing open surfaces such as cracks.

Concerning implicit boundary definition for ambient space finite elements still suffer from difficulties, including:

- In existing approaches, the mesh must not be too coarse to represent the boundary [68] and in areas of large curvature and around sharp corners or vertices, high order polynomial functions are usually required to obtain a geometrically converged model. Representing solids using level sets does not, at the moment allow to represent the geometry of the volume to machine precision and regularization parameters make methods which depend on them sensitive to the choice of these parameters.
- In the current literature, the geometry description uses basic level sets, i.e. analytical functions for ellipses, circles, planes, etc. but is not capable of representing arbitrary parametric surfaces provided, e.g. by CAD or micro-structure scans;
- It would be beneficial to obtain the level set functions directly from the parametric description provided by CAD data. This would increase the tightness of CAD and FEA integration. A means to generate the level set data from a parametric description of surfaces (as provided by CAD) is required;¹
- Sharp edges and corners cannot be represented exactly because of the first order finite elements used to approximate the single level set function that describes

¹If this were achieved, it would be straightforward to represent the “inverse” of the volume by inverting the direction of the level set propagation, which can be useful in fluid flow analysis around microstructures

the solid combined with an insufficiently accurate numerical integration (improvements were very recently proposed in [103]). This representation of the solid boundary with a single level set can either overestimate or underestimate the volume. For example, the calculated surface area of a disk moving over a background disk will depend on the location of the disk. This leads to difficulties in shape optimization, for example, see [67]. One possibility to make this possible would be to associate one level set function to each face of the volume's boundary. Currently, it is not possible to represent sharp corners, large curvatures or vertices without refining the background mesh.

- The need for recovery techniques is especially important in these versions of the finite element method as the accuracy can be significantly reduced in the elements located along the boundary of the domain [40]. On the other hand, in order to improve results, we should keep in mind that the error estimator in energy norm developed by Zienkiewicz and Zhu (ZZ) [102], requires the use of a recovered stress field to define the approximation error committed by the approximate solution. Some recent advances in this direction are reported in other contributions made to this Conference. Automatic enrichment based on error estimation and adaptivity has been proposed in [11] [12] [35] [86] [97] and [82]. Additionally automatic numerical calculation of enrichment functions [20] for anisotropic fracture have been developed in [66].
- Preconditioners are required to deal with cases where the boundaries pass close to nodes and to avoid linear dependencies created by large enrichment zones and multiple enrichment functions. Preliminary work was presented in [4] and generalized in [65]. Furthermore, the cut-off XFEM is a good way to decrease the conditioning number of the enriched global stiffness matrix [20].

4 Meshfree Methods

4.1 General Principles

To alleviate the difficulties mentioned in Introduction encountered by the FEM, so-called *meshfree* or *meshless* methods have appeared. In contrast to finite element methods, they do not employ elements in the construction of the interpolants. Instead, a set of nodes is accompanied by a domain of influence for each node, whereby the overlap of domains of influence accounts for the interconnectivity between nodes.

Despite their name, most meshfree methods do not completely suppress the need for a mesh. More precisely, if meshfree methods usually do not require elements to construct the approximation, most weak-form based meshfree methods require some type of mesh for the numerical integration.

The basic idea behind meshfree methods is to lift the strict connectivity require-

ments posed by the FEM. The solid is represented by an “arbitrary” set of nodes² related to one another by *influence domains*. Based on these nodes and the size of their domain of influence, an approximation is constructed (see e.g. the recent review [75] for details and implementation aspects) in a much more flexible way than in the case of the FEM:

- Mesh generation is vastly simplified;
- For most meshfree methods, the influence of the node placement is not as severe as in FEM, which makes them suitable for very large deformation problems where the FEM has difficulties;
- Arbitrary continuity and completeness can be achieved which can be useful for higher order continua or some plate/shell formulations;
- Discontinuities can be treated more easily than with the FEM (see Section 4.3);
- Nodes can be added easily to locally adapt the approximation [84].

Most meshfree methods may be recast as a particular case of the *weighted residual method*, with particular choices of the *trial* and *test* spaces. Depending on these choices and whether local or global weak forms are employed, a variety of methods can be constructed, with their respective advantages and drawbacks in terms of stability, accuracy, computational cost, convergence and robustness.

4.2 Stability of Meshfree Methods

An important issue regarding the stability of meshless methods is the kernel function (i.e. the weighting function used to construct the approximation). This kernel function can be expressed in terms of material (original configuration) or spatial (deformed configuration) coordinates. These kernels are known as Lagrangian and Eulerian, respectively. Early meshless methods such as SPH used an Eulerian kernel, but those methods have consequently suffered from so-called tensile instabilities (the method becomes unstable, i.e. spurious oscillations/modes appear where tensile stresses occur). Meshless methods based on a Lagrangian kernel do not suffer from these instabilities [7]. However, for very large deformations, methods based on Lagrangian kernels also become unstable because the domain of influence in the current configuration can become extremely distorted. Recently, Rabczuk and Belytschko have come up with a method where a switch from Lagrangian (for small deformations) to Eulerian (once deformations become large) is performed. This enables the study of very large deformations occurring, for example during fragmentation or some fluid-structure interaction problems [85].

²this arbitrariness is quite relative since the quality of the approximation is known to be dependent on the geometrical location of the nodes

4.3 Enrichment of Meshfree Methods for Capturing Evolving Discontinuities

As mentioned in the introduction, discontinuous fields occur in many applications in solid mechanics, e.g. cracks or between solid-fluid interfaces (strong discontinuities) or solid-solid interfaces (weak discontinuities). We will focus on strong discontinuities, i.e. cracks since modeling cracks was one main focus of meshfree methods.

Chapter “*Meshfree Methods for Dynamic Fracture*” by Rabczuk, Bordas and Askes summarizes the state of the art in the area with a special focus on modelling discontinuities, including:

- The visibility, transparency, see-through and diffraction criteria to model cracks including pitfalls of these methods;
- Extrinsic enrichment through partition of unity (see also Section 3 in the present chapter);
- Intrinsic enrichment of Moving Least Squares approximations.

Because these are treated in detail in the aforementioned chapter of this book, we will only, in the following section, present some ideas for future directions.

4.4 Challenges and Future Work

We believe this is a particularly important topic for meshless methods, as they are not fully mastered from a mathematical point of view, hence very few if any theoretical error bounds are available. Using suitable error estimators (which could be devised for specific quantities of interest, e.g. drag) would decrease computational time significantly and provide a way for the engineers to control the error. As such, this would answer the concerns of computational resource limits, for example as in the case of the FIA regulations and also resonates with one of the findings of the NSF Report of the National Science Foundation Blue Ribbon Panel on Simulation-Based Engineering Science:

While verification and validation and uncertainty quantification have been subjects of concern for many years, their further development will have a profound impact on the reliability and utility of simulation methods in the future. New theory and methods are needed for handling stochastic models and for developing meaningful and efficient approaches to the quantification of uncertainties. As they stand now, verification, validation, and uncertainty quantification are challenging and necessary research areas that must be actively pursued.

Meshless methods have huge potential to help Industry solve increasingly complex problems. Arguably, one of the main reasons why these methods are not widely adopted is their lack of robustness. To alleviate this difficulty, and for meshless methods to

deliver on their full potential, a group consortium of mathematicians, computer scientists and engineers should be involved in the future development of techniques. This would improve the understanding of meshless methods and answer some of the important questions related to the following:

Accuracy and Stability Validation and Verification On this front, a few questions are worth mentioning:

- What is the influence of point placement on accuracy and stability?
- How can we develop a priori/(goal oriented) a posteriori error measures to verify and validate emerging numerical techniques?
- What is the accuracy and robustness versus computational cost of meshless in the context of industrially relevant problems? Studying point collocation methods rigourously from a combined mathematical and engineering point of view would be beneficial.
- How can we better solve coupled problems and track the propagation of uncertainties/errors?

High Performance Computing How can we utilise the next generation of high performance computing in a transparent way?

Hybrid Methods How best can we develop hybrid methods, using meshless methods where they are most effective, coupled to other methods in regions of the problem domain where those are sufficiently accurate?

Training Future Engineers What is the most effective way to train the next generation of engineers to be capable of making informed decisions in a constantly evolving computational mechanics market?

More detailed analysis of the state of the art in meshless methods can be found in the DTI Report “*CFMS Whitepaper: An Industrial and Academic Perspective on Meshless Methods in CFD*” [58].

5 Isogeometric Analysis

5.1 Motivation

The main purpose of Isogeometric Analysis [44] is to increase the ties between Computer Aided Design (CAD) data and Finite Element Analysis (FEA). For clarity, let us take the example of linear elastic structural mechanics problems. In engineering practice, the geometry is described by Computer Aided Design (CAD) packages which allow the construction of very complex three-dimensional (3D) geometries, usually described by NURBS (Non-Uniform Rational B-Splines). This geometry then needs

to be meshed³ and analyzed (e.g. by the FEM) to obtain an approximate stress distribution within the structure. This stress analysis is used by the engineer to assess the design. If alterations to this design are required, they must then be communicated to the drafting team, and the geometry of the component suitably modified. A new mesh must then be regenerated by the analysis team to perform a second stress analysis, and the process is repeated as many times as is required for a suitable design to be obtained.

Considering this design cycle, it becomes clear that tying the geometrical CAD data to the mesh in an automatic and direct way, where the mesh would be defined automatically from the CAD data is highly desirable. This desirable integration is evident from the consolidation trends observed in the CAD/Analysis industry where CAD companies join forces with leaders in analysis, with the goal to streamline the design cycle and decrease lead time in computer aided design.

In Isogeometric Analysis, [44], the geometry can be represented *exactly* and directly from the CAD data, through the use of NURBS-based approximations. This approach is being pursued widely both in the engineering community by improving the basis functions and seeking adaptive methods [2, 31, 55], applying the ideas to fields outwith mechanics, such as electromagnetics [16]. It is also being taken up by the applied mathematics community through the derivation of error estimates [25]. This engineering and mathematics research should allow rapid development of the method through application-oriented investigations and sound mathematical analysis, respectively.

5.2 T-Splines and PHT-Splines

Recently, T-splines, a generalization of NURBS enabling local refinement, have been applied to isogeometric analysis. A posteriori error estimation techniques for local h-refinement with T-splines have been very recently introduced by Dorfel and colleagues [31]. Bazilevs [2] have tested T-splines on some elementary two-dimensional and three-dimensional fluid and structural analysis problems.

PHT-splines proposed in [27] (see also [28]) are constructed on the base of T-splines and thus inherit their advantageous properties. Moreover, unlike T-splines, PHT-splines are polynomial instead of rational; and the local refinement algorithm of PHT-splines is local and simple. The conversion between NURBS and PHT-splines is very fast, while conversion between NURBS and T-splines is a bottleneck of T-splines in practical applications. Compared with T-splines and hierarchical B-splines, PHT-splines are only C^1 continuous. However, PHT splines contain a close set (linearly independent) of basis functions, which is important for combining them to FEM.

³This meshing operation is inexact, and only as the mesh size goes to zero is the geometry exactly represented. This geometry approximation error is palliated by the recently developed Isogeometric Analysis [44].

5.3 Challenges and Future Work

Isogeometric Analysis is likely to generate quite a revolution by facilitating the coupling of computer aided design and analysis. There are a number of interesting points to address in order to ease this transition:

Integration Efficient numerical integration of the underlying approximation in isogeometric analysis is difficult since most of those are non-polynomial. This difficulty is common to this method, enriched FEM and most meshless methods. The most recent advance known to the authors is [45];

Mesh distortion The behaviour of isogeometric approximations for distorted meshes has been studied very recently in [55];

Boundary Conditions The lack of Kronecker Delta property of many isogeometric approximations (again similar to many meshfree methods) leads to complications in the imposition of essential boundary conditions. A recent paper on this topic is [3];

Error Estimation and Adaptivity As in all the methods discussed in this chapter, error estimation and adaptivity are of prime importance for reliable isogeometric analyses. A method based on h -refinement was proposed very recently by [31]. However, p - and k - (see [44] for a definition) refinements are also possible, which will offer a great flexibility in the way the approximation space can be tailored to the unknown solution.

The reader will have noticed the similarities with meshfree methods. This emanates from the common features of the approximation used in meshfree and isogeometric analysis, which are usually non-polynomial, smooth and do not possess the Kronecker Delta Property. This implies that the existing and extensive meshfree literature can be utilized to inform the developments of isogeometric analysis.

6 Smoothed Finite Element Methods

6.1 Motivation

Recently, novel finite element methods were born from the coupling of stabilized conforming nodal integration with the standard finite element method [57]. An overarching theory has been developed in the recent paper [56]. The main premise of this theory is the wish to achieve reliable results using simplicial meshes (triangles, tetrahedra) for which automatic mesh generators are quite robust and automatic. If this could be achieved, the need for human-intervention-heavy hexahedral meshes would be suppressed. This class of methods can thus be seen as yet another means of alleviating the mesh burden. In this case, the focus is mainly on

- Reducing the sensitivity of the finite element method to mesh distortion;
- Alleviating the over stiff behavior of simplex elements (triangular and tetrahedral) thereby allowing the use of unstructured meshes, much easier to generate automatically than structured HEX based meshes;
- Obtaining upper bounds of the solution both compared to the displacement-based FEM solution and to the exact solution.

The basic ideas of strain smoothing in FEM and XFEM have been reviewed recently in [15] and the reader is referred to this paper for more details.

Examples of gradient smoothing in FEM include the node-based smoothed point interpolation method (NS-PIM), smoothed (extended) finite element method (S(X)FEM), node-based smoothed finite element method (N-SFEM), edge-based smoothed finite element method (E-SFEM), cell-based smoothed point interpolation method (CS-PIM), etc.

6.2 Challenges and Future Work

The main challenges for smoothed finite element methods are related to the following points:

- Construction of higher-order approximations;
- Stabilization of the modes present in the low-subcell versions;
- Behavior of the method when partition of unity enrichment is used;
- Non-linear problems including severe mesh distortion and large deformations;
- Complete mathematical analysis and a priori error estimates. Such upper and lower bounds for the smoothed FEM in linear elasticity are now available from [100].

7 Conclusion

The topic of this chapter was a brief review of various methods that have appeared over the last decade or so and aim at alleviating the mesh burden in computational solid mechanics. Future research directions include

- Robust partition of unity methods for ambient space finite element simulations. Special attention must in particular be brought to the imposition of essential boundary conditions on implicit interfaces;
- A posteriori error estimates for ambient space simulation;

- Robust point collocation meshless methods for moving boundaries. Point collocations offer great potential in terms of computational cost savings but are not well understood mathematically and rarely used in the engineering community because of their lack of robustness. Systematic mathematical studies are required.
- Isogeometric analysis offers great promise. It will be particularly interesting to develop adaptive procedures and a posteriori error estimates for such techniques. T-Splines, for example, offer such possibilities.

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