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Claude Lamboray

PRUDENT RANKING RULES: THEORETICAL CONTRIBUTIONS AND APPLICATIONS

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Philippe Vincke, *directeur de thèse*
Université Libre de Bruxelles

Raymond Bisdorff, *co-directeur de thèse*
Université du Luxembourg

Thierry Marchant
Université de Gand

Bertrand Mareschal
Université Libre de Bruxelles

Marc Pirlot
Faculté Polytechnique de Mons

Philippe Van Ham
Université Libre de Bruxelles

Jean-Luc Marichal
Université du Luxembourg

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Contents

Conventions

- $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$: a set of n alternatives.
- $\mathcal{LO}$: the set of all linear orders on $\mathcal{A}$.
- $u = (O_1, O_2, \ldots, O_q)$: a profile of q linear orders on $\mathcal{A}$.
- B: the majority margin matrix.
- M: the strict majority relation.
- $\mathcal{PO}(u)$: the set of prudent orders of profile u.
- $\mathcal{XPO}(u)$: the set of extended prudent orders of profile u.
- $\mathcal{RP}(u)$: the set of linear orders obtained by the Ranked Pairs rule of profile u.
- $\mathcal{LPO}(u)$: the set of lexicographic prudent orders of profile u.
- $\mathcal{K}(u)$: the set of Kemeny orders of profile u.
- $\mathcal{E}(R)$: the set of linear extensions of the binary relation R.
- $t(R)$: the transitive closure of the binary relation R.
- $r(R)$: the reflexive closure of the binary relation R.
- $c(R)$: the complement of the binary relation R.
- $\delta(R_1, R_2)$: the symmetric difference distance between the binary relations R_1 and R_2.
- $\rho_O(a_i)$: the rank of alternative a_i in the linear order O. By convention, the alternative in the first position has rank 1, in the second position has rank 2, and so on.
- $|\{.\}|$: the cardinality of the set $\{.\}$.

Introduction

The subject of this PhD thesis is about ordinal ranking rules. An ordinal ranking rule is a procedure which combines several initial rankings into a global ranking.

There is nowadays a clear operational need for such tools. In Human Resource management for instance, it is common practice to rank candidates during a recruitment process or to rank employees during a performance appraisal. When more than one HR manager is involved, these individual preferences have to be combined in order to come up with a final ranking. In higher education, it becomes more and more common that universities are evaluated with respect to some indicators in order to measure the quality of the degrees they offer or the research they perform. Ordinal ranking rules are a very practical way of combining these indicators. In multicriteria decision aid, it is commonly accepted that criteria can be qualitative. But then you also need adequate tools to aggregate these qualitative criteria.

Despite these numerous fields of applications, ordinal ranking rules are often regarded with a certain part of suspiciousness. This can be due to the fact that engaging on the road of ordinal ranking rules will inevitably lead to the Borda-Condorcet debate and that Arrow's theorem seems always to be looming in the background. To some, this line of research seems rather vain. In our view however, there is a clear potential for ordinal ranking rules which address in a more appropriate way the particularities of a decision aid context.

From a cognitive point of view, it is very natural to compare alternatives two by two. Establishing a ranking that way is however a very difficult problem, since it is well known that the pairwise preference information can be contradicted on a global level. This is commonly referred to as Condorcet's paradox. Consequently, deriving a global ranking implicitly assumes forcing a transitivity, which, initially, was not necessarily present in the data.

That is why pairwise based ordinal ranking rule should be as transparent as possible in order to offer a useful insight into the decision problem. Given the difficulty and ambiguity of aggregating ordinal data, it is also worthwhile studying ranking rules which, depending on the initial data, do not lead to one, but to several global rankings, hence leaving some indeterminateness about the final result.

It appears that the concept of a prudent order, initially introduced by Arrow and Raynaud, is a possible answer from such a perspective. In this approach, pairwise majority margins are computed which count for any two alternatives the number of initial rankings that prefer the first over the second alternative. A prudent order is then defined as a linear order which maximizes the smallest pairwise majority margin.

Since the related literature lacks in solid theoretical foundations for this type of aggregation rule, it was our main objective in this thesis to thoroughly study and gain a better understanding of the family of prudent ranking rules. According to our definition, a prudent ranking rule is a rule which outputs only prudent orders.

To achieve this goal, we pursued different strategies: i) axiomatic characterizations which highlight the distinctive features of a ranking rule, ii) comparison of the properties or results of prudent ranking rules to those of other ranking rules, and iii) empirical simulations allowing for more quantitative conclusions.

Following the tradition of social choice theory, we decided to study prudent ranking rules from a very fundamental point of view. The theoretical working premises are that a set of linear orders must be aggregated into one or several final linear orders, under the condition of anonymity. The latter condition means that all the initial linear orders have the same importance. These working assumptions can be criticized, especially from an operational point of view. We think however that it is crucial to gain a solid understanding of prudent ranking rules in a more simplified framework before trying to extend them to more general situations, where more complex preference structures than linear orders are involved and importance coefficients may play a role.

The thesis is divided into 3 parts. **Part I** (Chapter 1 - Chapter 3) contains an introduction to the ordinal ranking problem. Our main theoretical contributions can be found in **Part II** (Chapter 4 - Chapter 8). This part is the most technical one. Finally, **Part III** (Chapter 9 - Chapter 10) illustrates the use of prudent ranking rules in two fields of applications.

More particularly, in **Chapter 1** the ordinal ranking problem is described in detail and its importance is motivated. In **Chapter 2**, the concept of prudent ranking rules is introduced. A review of the literature is made on prudent and non-prudent ranking rules. In **Chapter 3**, the interest of studying prudent ranking rules is motivated, especially from the perspective of using them as a decision support tool.

In Part II, we start in **Chapter 4** by studying some properties of the prudent order preference function which are commonly used to analyze ranking rules. **Chapter 5** presents an axiomatic characterization of the prudent order preference function. **Chapter 6** is devoted to Tideman's Ranked Pairs rule, which is characterized, using the same axiomatic framework as for the prudent order preference function. We define in **Chapter 7** a new prudent ranking rule based on an underlying leximin relation. Finally, in **Chapter 8**, we prove that we can construct profiles for which the result of a prudent ranking rule and a non-prudent ranking rule can be contradictory.

In Part III, we first discuss in **Chapter 9** the use of prudent ranking rules in the group ranking problem, while in **Chapter 10**, we address the problem of composite indicators. In both these chapters, we illustrate our models on some realistic data.

Of course, we were not able to study all the prudent ranking rules or to answer every question in depth. More importantly, our research has been able to open up the field of prudent ranking rules. In the **conclusion**, we then discuss the most interesting aspects which still deserve to be further investigated. Finally, the results of the simulations performed in Debord's PhD thesis on the number of prudent orders can be found in the **appendix**.

Publications and Conferences

The research presented in this PhD thesis has lead to several publications in peer-reviewed journals and proceedings.

C. Lamboray (2006) An axiomatic characterization of the prudent order preference function. Annales du Lamsade 6, 229-256.

> In this paper, we will axiomatize a preference function that associates to a profile of linear orders the set of its corresponding prudent orders. We will introduce axioms that will restrict the set of linear orders to the set of prudent orders. By strengthening these axioms, the prudent order preference function can be fully characterized. Finally, we will characterize the extended prudent order preference function by introducing an additional Condorcet-like criterion.

C. Lamboray (2007) A comparison between the prudent order and the ranking obtained with Borda's, Copeland's, Slater's and Kemeny's rules. Mathematical Social Sciences 54:1-16.

> Arrow and Raynaud suggested that the result of a ranking rule should be a prudent order. We prove that we can construct profiles of linear orders for which the unique prudent order is the exact opposite of the ranking obtained with Borda's rule or Copeland's rules. Furthermore, we show that we can construct profiles of linear orders such that the unique prudent order winner can be found at any position in the corresponding unique order found by Slater's or Kemeny's rules. Finally, we show that there exist profiles where the unique Slater or Kemeny order is the exact opposite of one prudent order.

C. Lamboray (2007) Supporting the search for a group ranking with robust conclusions on prudent orders. Annales du Lamsade 7:145-171.

We consider the problem where rankings, provided for instance by a group of evaluators, have to be combined into a common group ranking. In such a context, Arrow and Raynaud suggested that the compromise ranking should be a prudent order. In general, a prudent order is not unique. That is why, we propose to manage this possible multiplicity of compromise solutions by computing robust conclusions. This allows for a progressive refinement of the decision model and supports the group to eventually select one group ranking. The approach is illustrated on a problem where a group of junior researchers has to agree on a ranking of research domains.

C. Lamboray. A characterization of the prudent order preference function. Mathematical Social Sciences, to appear.

In this paper, we will characterize a preference function that associates to a profile of linear orders the set of its corresponding prudent orders. First, we will introduce axioms that will restrict the set of linear orders to the set of prudent orders. By strengthening these axioms, the prudent order preference function will then be fully characterized.

Besides, two papers are currently under review in *Group Decision and Negotiation* and *Social Choice and Welfare*.

Most of the research has also been presented in various conferences and seminars.

C. Lamboray (2007) A comparison between lexicographic prudent orders and Kemeny's rule. OSDA (Ordinal and Symbolic Data Analysis), Ghent, Belgium.

C. Lamboray (2007) Règles de rangement prudentes basées sur le Min, DiscriMin et LexiMin. Journée Mathématiques Discrètes et Sciences Sociales, ENST, Paris, France.

C. Lamboray (2007) Characterizations of some prudent ranking rules. 3^e cycle FNRS, ULB, Brussels, Belgium.

C. Lamboray (2006) An axiomatic characterization of the prudent order preference function. DIMACS-LAMSADE Workshop on Voting Theory and Preference Modelling, Paris, France.

C. Lamboray (2006) Comparison of the unique prudent order with the Borda, Copeland, Slater and Kemeny ranking. 8^{th} International Meeting of the Social Choice and Welfare Society, Istanbul, Turkey.

C. Lamboray (2006) An approach to support the search for a group ranking based on robust conclusions with prudent orders. 63^{rd} Meeting of the European Working Group on Multiple Criteria Decision Aiding, Porto, Portugal.

C. Lamboray (2006) Etablissement de quelques conclusions robustes sur les ordres prudents. ROADEF06, Lille, France.

C. Lamboray (2006) A comparison between the unique prudent order and the Borda, Copeland, Kemeny and Slater ranking. ORBEL 20, Ghent, Belgium.

Part I

Ordinal ranking rules

Chapter 1

Motivations for ordinal ranking rules

This introductory chapter aims at delimiting and motivating the topic of this thesis. In Section 1.1 we start by describing what we understand by an ordinal ranking problem and we introduce the main definitions and notation which will be necessary to formalize a ranking rule. In Section 1.2, we briefly discuss three particular fields of applications of ranking rules. Finally, in Section 1.3, we summarize the main reasons of using ordinal ranking rules in practice.

1.1 Definitions and notation

Let us suppose that a set of alternatives can be ranked from the best to the worst on various dimensions. In such a situation, we are naturally confronted with the problem of how to combine these various rankings into one global ranking. This is precisely the problem that we will be addressing in this thesis.

In fact, the ordinal ranking problem can be characterized by the following two aspects:

1. On the input side, we suppose that a set of alternatives can be ranked from the best to the worst on several dimensions.

2. On the output side, the aim will be to construct a global ranking which "best" combines the information contained in the initial rankings.

More particularly, we focus on mechanisms that aggregate or combine this multi-dimensional ordinal data into a new, so-called "compromise" or "consensus" or "group" or "global" ranking. Since on each dimension the

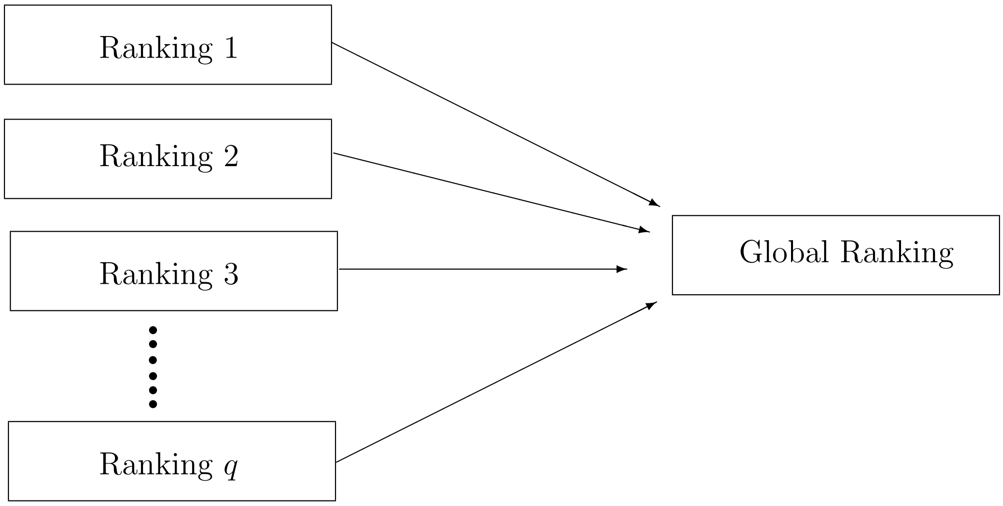


Figure 1.1: An ordinal ranking rule.

alternatives are evaluated on an ordinal scale, we speak of an "ordinal" ranking rule. This idea is represented in Figure 1.1.

Ordinal ranking rules will be at the core of this thesis. We provide now the definitions and notation needed to formalize such an ordinal ranking rule. This clarifies the assumptions on which we rely.

First of all, the construction of the set of alternatives is crucial and can be difficult and time consuming. For instance, in a decision process, Keeney [53] suggests to construct alternatives, both in a single decision maker or multiple decision maker context, by focusing on values. However, we will not concentrate on this step. Hence, we will assume that a set of n alternatives, denoted by $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, has been previously constructed and will remain stable throughout the decision process.

We are mainly concerned with ranking the alternatives from the best to the worst, which can conveniently be represented as an ordered list. For instance,

$$a_1 a_2 a_3 a_4$$

reads as follows: a_1 is preferred to a_2, which is preferred to a_3, which is preferred to a_4.

Such an ordered list can be modeled by means of a binary relation. A binary relation R on $\mathcal{A}$ is a subset of the Cartesian product $\mathcal{A}\times\mathcal{A}$. $\forall a_i, a_j \in \mathcal{A}$, $(a_i, a_j) \in R$ means that alternative a_i is as least as good as alternative a_j.

For instance, the binary relation corresponding to $a_1a_2a_3a_4$ is:

$$\{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_3), (a_2, a_4), (a_3, a_4)$$
$$(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4)\}.$$

We need the following properties of a binary relation R:

Definition 1 *Let R be a binary relation defined on $\mathcal{A}$.*

*R is **antisymmetric** if*
$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in R$ *and* $(a_j, a_i) \in R \Rightarrow a_i = a_j$.

*R is **complete** if*
$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in R$ *or* $(a_j, a_i) \in R$.

*R is **transitive** if*
$\forall a_i, a_j, a_k \in \mathcal{A}, \quad (a_i, a_j) \in R$ *and* $(a_j, a_k) \in R \Rightarrow (a_i, a_k) \in R$.

*R is **reflexive** if*
$\forall a_i \in \mathcal{A}, \quad (a_i, a_i) \in R$.

The following definitions of particular binary relations can for instance be found in [17].

Definition 2

*A **partial order** is a transitive and antisymmetric binary relation. We denote $\mathcal{PO}$ the set of all the partial orders on $\mathcal{A}$.*

*A **weak order** is a complete and transitive binary relation. We denote $\mathcal{WO}$ the set of all the weak orders on $\mathcal{A}$.*

*A **linear order** is a complete, transitive and antisymmetric binary relation. We denote $\mathcal{LO}$ the set of all the linear orders on $\mathcal{A}$.*

A profile $u = (O_1, O_2, O_3, \ldots, O_q)$ is a vector that contains the q rankings corresponding to the q dimensions which need to be combined. We will furthermore assume that in these rankings the alternatives are ordered from the best to the worst without allowing ties, which means that each ranking is a linear order.

An ordinal ranking rule can be formalized with a mapping f as follows:

$$\begin{array}{llcl} f: & \mathcal{LO}^q & \longrightarrow & \mathcal{D} \\ & u = (O_1, O_2, \ldots, O_q) & \mapsto & f(u) \end{array}$$

The solution space $\mathcal{D}$ can have various forms. As we will see in Section 2.2, the following two cases are generally encountered in the literature:

Case WO $f(u)$ is a weak order.

Case SLO $f(u)$ is a set of linear orders. In this case, we sometimes follow Young and Levenglick's terminology [109] and speak of f as a *preference function*. Vincke [106] calls ranking rules that lead to several solutions, for instance several linear orders, *preaggregation procedures*.

By ranking rule, we thus understand a well defined procedure or algorithm that takes as input a profile of linear orders and that computes an output that either consists of a weak order or several linear orders. Some authors call these solutions "consenus rankings" (see for instance [24]), others, who work in the arrowian tradition of social choice theory, rather call them "social orderings". We will in general adopt the terminology "compromise rankings" because, following the multicriteria decision aid paradigm, we are looking for a compromise of the rankings which belong to the initial profile and which may model some criteria.

We will be mostly manipulating binary relations. Several other concepts involving binary relations will be useful, namely the distance between two relations, the linear extensions of a relation, the transitive and reflexive closure of a relation and the complement of a relation.

Often, the distance between two binary relations R_1 and R_2 can be expressed using the symmetric difference between these two sets:

$$\delta(R_1, R_2) = \frac{|\{R_1 \setminus R_2 \cup R_2 \setminus R_1\}|}{2},$$

where $. \setminus .$ denotes the set difference and $|\{.\}|$ denotes the cardinality of the set $\{.\}$.

A linear extension of a relation R is simply a linear order that contains R, i.e. $R \subseteq O$. We denote $\mathcal{E}(R)$ the set of all the linear extensions of R:

$$\mathcal{E}(R) = \{O \in \mathcal{LO} : R \subseteq O\}.$$

Let us note that this set $\mathcal{E}(R)$ can possibly be empty. In fact, Szpilrajn [100] showed that if the relation R is acyclic, then it can be extended into at least one linear order. If however R contains a cycle, then the relation cannot be extended and the set of linear extensions is empty.

The transitive closure of a relation R, denoted by $t(R)$, is the smallest relation that is transitive and that contains R. The reflexive closure of a

relation R, denoted by $r(R)$ is the smallest relation that is reflexive and that contains R. The complement $c(R)$ of a relation is defined as follows:
$\forall a_i, a_j \in \mathcal{A}$,

$$(a_i, a_j) \in c(R) \iff (a_i, a_j) \notin R.$$

We refer to Fishburn [44] for a detailed discussion on the operators $t(.)$, $r(.)$ and $c(.)$.

1.2 Fields of applications

The ordinal ranking problem appears in a variety of situations. In this section, we briefly discuss three potential fields of applications:

1. The group ranking problem
2. The composite indicator problem
3. Multicriteria decision aid.

In Part III of this thesis, we will come back to the first two of these fields of applications. More particularly, we will discuss how prudent ranking rules can support a group in searching for a compromise ranking (see Chapter 9) and how they can be used to combine sub-indicators (see Chapter 10).

1.2.1 The group ranking problem

The group ranking problem can be characterized by a group of people belonging to the same organization or the same company who have to work together in order to come up with a common group ranking. The group ranking problem is not bounded to a particular industry, but can appear both in the private and public sector:

Example 1: R&D Project analysis

At NASA [101], the board has to decide on how to split the annual budget between different R&D projects. To do so, the research projects are evaluated by various departments (safety, systems engineering...). Here the group members correspond to the heads of the various departments. The management finally takes into account the preferences of the departments in order to rank the projects, which will help them decide on the budget spending.

Example 2: Recruitment

A group ranking problem in a company may also arise in recruitment situations (see for instance [102]). Candidates for a job position are evaluated and ranked by a group of human resource managers. These individual opinions have then to be combined in order to build a common ranking of the available candidates.

Example 3: Strategic decisions
The FNR, the scientific funding agency from Luxembourg, has asked a group of researchers to help them select research domains that should be developed in the medium and long term in Luxembourg. After discussion, the group selects a restricted set of the most pertinent research domains. The problem for the group will now be to agree on a common ranking of these research domains. This particular example will be further discussed in Section 9.7.

One way of addressing the group ranking problem is to assume that, first, each group member proposes his individual ranking. In a second step, these rankings have then to be combined in order to come up with a group ranking, using for instance an ordinal ranking rule.

1.2.2 The composite indicator problem

Socio-economic indicators have become a popular tool to evaluate countries, companies, universities etc. with respect to some particular issue. By trying to reflect a complex reality, indicators are usually a combination of different sub-indicators. That is why we call them *composite* indicators. These sub-indicators measure, though often indirectly, a certain aspect of the reality that we would like to represent. Let us look at two examples of composite indicators:

Example 1: Human Development Index
Historically, one of the first of such indicators is the so-called Human Development Index of the United Nation Development Program. This indicator tries to give a measure of the degree of development of a country. It is roughly based on three sub-indicators. The first sub-indicator takes into account life expectancy. The second one measures the level of education by looking, among other statistics, at the illiteracy rate. Finally, the third sub-indicator is based on the GDP. The precise definition of the indicator can for instance be found on page 394 of [88].

Example 2: Competitivity Indicator
The "Observatoire de la Compétivitité" of the Luxembourg Ministry of Economy has recently developed a so-called competitivity indicator

of the 25 EU countries [28]. This indicator should reflect the degree of advancement of the various countries with respect to the Lisbon strategy. To construct the indicator, 10 different sub-indicators have been selected. We will come back to this example in Section 10.3.

One approach to combine sub-indicators is to forget about the values of the evaluations of the objects on each sub-indicator. Instead we will only consider the underlying order of each sub-indicator. These rankings have then to be combined into a global ranking, using an ordinal ranking rule.

1.2.3 Multicriteria decision aid

Another field of applications of ordinal ranking rules is multicriteria decision aid (MCDA). This discipline aims at supporting a decision maker who is confronted with a set of possible alternatives that can be evaluated according to various, usually conflicting criteria by explicitly taken into account his or her preferences. This paradigm has given rise to many methodological, theoretical and practical developments [41]. The particular problem of aggregating ordinal data is also a major issue. In fact, when the criteria are ordinal, then we need techniques to combine them in order to propose a recommendation, e.g. a ranking, to the decision maker. This problem appears for instance in the well-known family of ELECTRE methods [91].

However, most of the models that we will develop in this thesis are not directly suitable for MCDA because of the following two reasons:

- We suppose that both the input and output preference structures are rather simple. More particularly, we assume that the rankings are linear orders. In MCDA however, often more complex preference structures, such as for instance preference structures that allow incomparability or non-transitive indifferences, are considered.
- We usually assume anonymity with respect to the input rankings. Intuitively, this means that all the rankings play the same role. In MCDA however, it is crucial to take into account differences of importance of the different criteria, since they contribute to model the preferences of the decision maker.

In fact, ordinal ranking rules can only be used in the particular case where all the criteria are linear orders and all the criteria have the same importance. This being said, many MCDA methods are inspired by ordinal ranking rules. For instance, the PROMETHEE method [76] can be seen as a generalization of Borda's rule (see Section 2.2.2 for a definition of this ranking rule). Following this idea, we will briefly discuss the use of weights in the framework of prudent ranking rules in Section 10.2.

1.3 Reasons for ordinality

In this section, we discuss the main reasons or benefits of using an ordinal ranking rule, for instance in one of the three situations described in the previous section.

The first feature of the ordinal ranking problem is that we solely rely on ordinal data as input. The main reason for this is that we do not want to make or that we are not able of making stronger assumptions:

- In some situations, evaluating alternatives on an ordinal scale is often easier than assigning a precise numerical value to an alternative on a particular dimension. For instance, in multicriteria decision aid, it can be realistic to tell that one alternative is "better" than another alternative on a more qualitative criterion such as "comfort", but it can be more difficult to tell by how much.

- Working with ordinal data as input also makes sense when the various dimensions which need to be combined are non-commensurable. Even if an alternative can be characterized by a precise numerical value on each dimension, the existence of different evaluation scales usually makes it impossible to properly combine these values. This is typically the case when the alternatives have been evaluated by independent sources. A good illustration of such a situation are composite indicators, where the different sub-indicators have sometimes been constructed by completely different organizations.

The second feature of the ordinal ranking problem is that the final solution should be a ranking. There are some reasons that motivate the need for constructing a global ranking:

- Working with a ranking is typically useful in prioritization problems, where we are interested in a priority list in order to decide how to spend the limited amount of funds or resources between the options. In fact, not only the project ranked first will get some funding, but also the subsequent ones receive still some, but probably less, fundings. For instance, in the strategic decision example of the group ranking problem, the problem rather consists in prioritizing research domains, and not selecting the "best" research domain.

- A ranking is also useful in the k-choice problem, which consists in choosing k best alternatives within the whole set of available alternatives. In such a situation, we could construct a ranking and select the

k first alternatives of that ranking (see Meyer and Bisdorff [78] for a more detailed discussion on k-choice problems). For instance, in the recruitment example of the group ranking problem, the company may not only seek one, but several new employees.

- A ranking is also interesting in situations where those who evaluate and rank the alternatives are not the decision makers themselves. In such a context, a ranking contains a richer information than solely a choice subset and leaves some amount of appreciation to the real decision maker. For instance, a particularity of composite indicators is usually that those who design them are different from those who use them in their decision process.

- A ranking can be of use in situations where the first alternatives can in a later stage disappear. In the recruitment example of the group ranking problem, it may happen that the candidate ranked first has already accepted a position in another company, and so the job will be offered to the candidate ranked second.

Finally, it is very important to acknowledge that in all these situations, the use of an ordinal ranking rule is usually embedded in a whole decision aid process. According to Roy, "Decision analysis consists in trying to answer questions raised by actors involved in a decision process using a model."[1] An ordinal ranking rule is thus only a tool which is used at a particular moment during a decision process. In the same line, Bouyssou & al. argue that the "... usefulness [of ranking rules] not only depends on their intrinsic formal qualities but also on the quality of their implementation (structuration of the problem, communication with actors involved in the process, transparency of the model, etc.)." [2]

[1] B. Roy (1996), *Multicriteria Methodology for Decision Analysis*, Kluwer Academic Publishers.

[2] D. Bouyssou, T. Marchant, M. Pirlot, A. Tsoukiàs and P. Vincke (2006), *Evaluation and Decision Models with Multiple Criteria*, Springer.

Chapter 2

Classification of ordinal ranking rules

In this chapter, we present several existing ordinal ranking rules. In Section 2.1, we first introduce the different informational levels on which these ranking rules are based. The definitions of some prudent and non-prudent ranking rules can be found in Section 2.2. Finally, in Section 2.3, we propose a classification of the ranking rules into different families. This will give a first hint on the particularities of prudent ranking rules.

2.1 Informational levels

A technical difference between the rules is that, although they all take as input a profile of linear orders, they work in reality with different types of information contained in that profile. We introduce in this section these different informational levels. These quantities will then be used in the definitions of the ranking rules presented in Section 2.2.

First of all, the information contained in a profile u can naturally be combined in a n times n dimensional concordance matrix C, where the value of row i and column j ($1 \leq i \leq n, 1 \leq j \leq n$), denoted by C_{ij}, simply counts the number of rankings in the profile u where alternative a_i is as least as good as alternative a_j. Let us note that since we work with linear orders orders, if $a_i \neq a_j$, then C_{ij} simply counts the number of rankings in the profile where a_i is preferred to a_j.

Definition 3 *The strict concordance matrix C is a n times n dimensional matrix defined as follows:*

$$\forall a_i, a_j \in \mathcal{A}, \quad C_{ij} = |\{k \in \{1, \ldots, q\} : (a_i, a_j) \in O_k\}|.$$

Since the profile consists of linear orders, it is easy to see that,

$$\forall i, j : 1 \le i \le n, 1 \le j \le n, i \ne j, \quad C_{ij} + C_{ji} = q.$$

Instead of working with C, many ranking rules are simply based on so-called majority margins. In fact, following the idea of balancing reasons (see Bisdorff [9]), a majority margin is defined for any two alternatives as the number of rankings in the profile where a_i is as least as good as a_j minus the number of rankings in the profile where a_i is not as least as good as a_j. In our particular case where the profile consists of linear orders, this is simply the number of rankings that prefer a_i over a_j minus the number of rankings that prefer a_j over a_i[1]:

Definition 4 *The majority margin matrix B is a n times n dimensional matrix defined as follows: :*
$\forall a_i, a_j \in \mathcal{A}$,

$$B_{ij} = |\{k \in \{1, \ldots, q\} : (a_i, a_j) \in O_k\}| - |\{k \in \{1, \ldots, q\} : (a_j, a_i) \in O_k\}|.$$

Again, because of the fact that the profile consists of linear orders, we have that

$$\forall i, j : 1 \le i \le n, 1 \le j \le n, \quad B_{ij} + B_{ji} = 0.$$

We call this the constant-sum property. If $B_{ij} > 0$, then there are more linear orders in the profile that prefer a_i over a_j than there are linear orders that prefer a_j over a_i. If $B_{ij} < 0$, then there are more linear orders that prefer a_j over a_i than there are linear orders that prefer a_i over a_j. In case, $B_{ij} = 0$, there is an equal number of linear orders preferring a_i over a_j and preferring a_j over a_i.

It is easy to see that, given C, we can directly obtain B, since $\forall a_i, a_j \in \mathcal{A}$, $B_{ij} = C_{ij} - C_{ji}$. However, the converse is not true, which means that, given B we cannot compute C without knowing the size q of the profile. In fact, it is easy to see that

$$\forall i, j : 1 \le i \le n, 1 \le j \le n, i \ne j, \quad C_{ij} = \frac{1}{2}(B_{ij} + q).$$

The reader may wonder at this stage, why such strong aggregation mechanisms are used, loosing that way the initial rankings contained in the profile. In that context, Debord [31] showed that, if a ranking rule verifies only two reasonable and rather weak assumptions (namely E-invariance and anonymity), then the result of that ranking rule must only depend on the

[1] Let us note that this definition does not apply if the profile contains more general preference structures than linear orders. We refer the reader to [9] for more details.

majority margins of that profile.

It is convenient to know that majority margins have been characterized by Debord [31]. Using the following result, we are able to tell if a given matrix B can be seen as a majority margin matrix of a profile of linear orders.

Proposition 1 *Let B be a n times n dimensional matrix such that*

$$\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n, \quad B_{ij} + B_{ji} = 0.$$

B can be seen as the majority margin matrix of a profile of linear orders if and only if one of the following two statements are valid

- $\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n$, *$B_{ij}$ is even.*
- $\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n, i \neq j$, *$B_{ij}$ is odd.*

It is important to note that we do not have such a characterization for concordance matrices. In fact, even under the constant-sum assumption, not every concordance matrix can be obtained from a profile of linear orders. It is impossible to obtain for instance a 3 times 3 matrix C, where $C_{12} = q$ (and consequently $C_{21} = 0$), $C_{23} = q$ (and consequently $C_{32} = 0$) and $C_{31} = q$ (and consequently $C_{13} = 0$). In fact, this would correspond to a profile where there is unanimity on a preference cycle, which is impossible, given the transitivity property of linear orders of the initial profile.

The information contained in B can be further weakened. In fact, a mathematical object that goes back to Condorcet is the so-called strict majority relation:

Definition 5 *The strict majority relation M is a binary relation defined as follows:*

$$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in M \iff B_{ij} > 0.$$

It is easy to see that, given B, we can obtain M but the converse is not true. At the core of the problem encountered in voting theory lies the fact that M is not necessarily transitive. Consider for instance the profile with three alternatives and three rankings $u = (abc, cab, bac)$. Then $(a, b) \in M$, $(b, c) \in M$ and $(c, a) \in M$. This is commonly referred to as Condorcet's paradox. There exists a huge literature on how often this paradox occurs and what are the domain conditions on the profile to avoid it (see for instance the recent book of Gehrlein [47] for an overview). We will however not insist on these two particular aspects in this thesis.

Finally, from a more cardinal or rank-oriented perspective,$\forall a_i \in \mathcal{A}, \forall O \in \mathcal{LO}$, let $\rho_O(a_i)$ denote the rank of alternative a_i in the linear order O. By

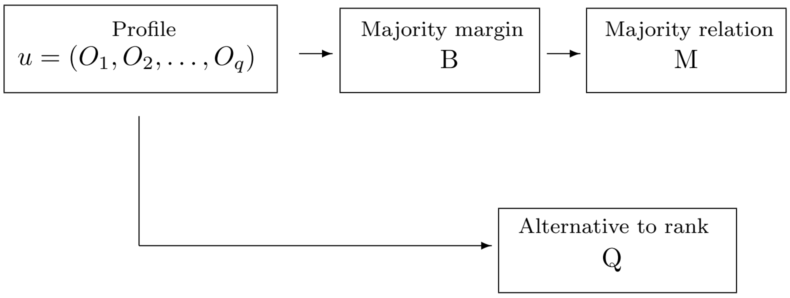


Figure 2.1: The relationships between the informational levels

convention, the alternative in the first position has rank 1, in the second position has rank 2, and so on. A profile can be combined into a matrix Q that counts the number of times that an alternative a_i occupies a rank j:

Definition 6 *The alternative to rank matrix Q is a n times n dimensional matrix defined as follows:*

$$\forall a_i \in \mathcal{A}, \forall j \in \{1, \ldots, q\} : \quad Q_{ij} = |\{k \in \{1, \ldots, q\} : \rho_{O_k}(a_i) = j\}|.$$

The relationships between the quantities introduced are schematically represented in Figure 2.1. Let us note that there is no explicit link between the matrix Q and the matrix B or the relation M.

2.2 Some ranking rules

In this section we present some popular ranking rules that have been introduced in the literature. Some of these rules have been developed on an ad-hoc basis, while others have stronger theoretical foundations. In Section 2.2.1, we first introduce prudent ranking rules, which will be at the core of this thesis, whereas other popular rules, which we refer to as non-prudent, are introduced in Section 2.2.2.

From a technical point of view, each rule is categorized according to:

- The type of solutions: a weak order **(WO)** or a set of linear orders **(SLO)**.
- The informational level on which the ranking rule is based: Majority relation **(M)**, majority margin matrix **(B)** or alternative to rank matrix **(Q)**.

Let us finally remark that some ranking rules rely on a heavy algorithmic resolution and many problems are NP-complete (see also Hudry [50] and

Bartholdi, Tovey and Trick [5]). This may cause problems from a practical point of view. Let us quote Bartholdi & al., who proved the NP-completeness of a voting rule proposed by Dodgson (aka Lewis Caroll): "We think Lewis Caroll would have appreciated the idea that a candidate's mandate might have been expired before it was ever recognized."[2] We will however less insist on the complexity of the rules. Instead, our main interest will rather be the definition of the rules.

2.2.1 Prudent ranking rules

Arrow and Raynaud [2] established a list of axioms which a ranking rule should verify. Among these, Axiom V' states that the compromise ranking should be a so-called prudent order. Let us now formally introduce this important concept.

Let $\lambda \in \{-q, \ldots, q\}$ and let us define the cut relations $R_{\geq\lambda}$ and $R_{>\lambda}$ as follows:
$\forall a_i, a_j \in \mathcal{A}$,

$$(a_i, a_j) \in R_{\geq\lambda} \text{ if } \begin{cases} B_{ij} \geq \lambda \text{ and } i \neq j \\ i = j \end{cases}$$

$$(a_i, a_j) \in R_{>\lambda} \text{ if } B_{ij} > \lambda \text{ and } i \neq j.$$

Hence, we always suppose that $\forall a_i \in \mathcal{A}$, the pairs (a_i, a_i) belong to the relation $R_{\geq\lambda}$, whereas $R_{>\lambda}$ is always irreflexive.

When $\lambda = -q$, then $R_{\geq\lambda}$ is a complete and symmetric relation, and consequently contains at least one linear order. By gradually increasing the cut value, ordered pairs will disappear from the corresponding cut relation. Let α be the largest value such that the corresponding cut relation still contains at least one linear order:

$$\alpha = \max\{\lambda \in \{-q, \ldots, q\} : R_{\geq\lambda} \text{ contains at least one linear order}\}.$$

We say that a relation R contains a cycle if there exists a subset of alternatives $\{a_{i_1}, a_{i_2}, \ldots, a_{i_p}\} \subseteq \mathcal{A}$ such that $(a_{i_1}, a_{i_2}) \in R$, $(a_{i_2}, a_{i_3}) \in R$, $\ldots$, $(a_{i_p}, a_{i_1}) \in R$. When $\lambda = q$, then $R_{>\lambda}$ is empty and consequently does not contain any cycle. By gradually decreasing the cut value, some ordered

[2] J.J. Bartholdi, C.A. Tovey and M.A. Trick (1989), *Voting schemes for which it can be difficult to tell who won the election*, Social Choice and Welfare 6, 157-165.

pairs will be added to the corresponding strict cut relation. Let β be the smallest value such that the corresponding strict cut relation is acyclic:

$$\beta = \min\{\lambda \in \{-q, \ldots, q\} : R_{>\lambda} \text{ is acyclic }\}.$$

Let us note that, consequently, $R_{\geq\beta}$ must contain at least one cycle involving at least two alternatives. Although working in a different context, Kramer [63] had already noticed the interest of the relation $R_{>\beta}$. However, he did not mention its close relationship with $R_{\geq\alpha}$. The following theorem of Arrow and Raynaud [2] establishes the link between these two relations.

Theorem 1 *If the constant sum property holds, then any linear order O containing $R_{>\beta}$ is contained in $R_{\geq\alpha}$ and any linear order contained in $R_{\geq\alpha}$ also contains $R_{>\beta}$.*

Arrow and Raynaud [2] thus proposed that the compromise ranking should be a prudent order.

Definition 7 *A prudent order O is a linear order that contains $R_{>\beta}$ and is contained in $R_{\geq\alpha}$:*

$$O \in \mathcal{LO} : \quad R_{>\beta} \subseteq O \subseteq R_{\geq\alpha}.$$

The authors justify such a ranking to be *prudent* by the fact that ordered pairs of alternatives that belong to the relation $R_{>\beta}$ are pairs with a high majority. If these pairs would not belong to the final compromise ranking, there would be a large majority against such a ranking. On the other hand, a group ranking which is not contained in $R_{\geq\alpha}$ has at least one ordered pair of alternatives (a_i, a_j) such that B_{ij} is strictly smaller than α. That is why such a group ranking should be discarded. Hence, a prudent order can be seen as the optimal trade-off between the absence of cycles and the existence of a linear order.

Let us note that in case the constant-sum property is verified, which will always be the case in our setting since we only work with profiles consisting of linear orders, then being contained in $R_{\geq\alpha}$ or containing $R_{>\beta}$ are two equivalent conditions according to Theorem 1. We will thus consider a preference function denoted by $\mathcal{PO}$, called prudent order preference function, that associates to every profile u the set of all linear extensions of $R_{>\beta}$:

$$\begin{aligned} \mathcal{PO}(u) &= \{O \in \mathcal{LO} : R_{>\beta} \subseteq O\} \\ &= \mathcal{E}(R_{>\beta}). \end{aligned}$$

Let us also stress that there always exists at least one prudent order, since any acyclic relation can be extended into at least one linear order (see

Szpilrajn [100]). From a technical point of view, the prudent order preference function is of type (SLO/B), which means it is based on the majority margin matrix B and the result is a set of linear orders.

Let us now illustrate this ranking rule on the following profile on 4 alternatives, where the number in front of the linear order specifies how often this linear order is repeated in the profile:

4	*abcd*	3	*bcad*
4	*dcab*	4	*dabc*
4	*cabd*	2	*cdab*
5	*dbca*	2	*bacd*
1	*cbda*	1	*acdb*

The following majority margins are obtained:

	a	*b*	*c*	*d*
a	.	8	–8	–2
b	–8	.	6	–2
c	8	–6	.	4
d	2	2	–4	.

In this example, $\beta = 6$, since:

- $R_{>6} = \{(c, a), (a, b)\}$ is an acyclic relation.
- $B(a, b) \geq 6, B(b, c) \geq 6$ and $B(c, a) \geq 6$, and so $R_{\geq 6}$ is not acyclic anymore.

The set of prudent orders corresponds to all the linear extensions of $R_{>6}$. There are in all 4 prudent orders:

$$\mathcal{PO}(u) = \{dcab, cabd, cdab, cadb\}.$$

More generally, we will be interested in ranking rules that always produce prudent orders. We call such rules prudent ranking rules.

Definition 8 *A preference function f is a prudent ranking rule if:*

$$\forall u, \quad f(u) \subseteq \mathcal{PO}(u).$$

The prudent order preference function is trivially a prudent ranking rule. Let us now present three other prudent ranking rules that have been proposed in the literature, namely the rule proposed by Kohler [2, 62], the rule proposed by Arrow and Raynaud [2] and the Ranked Pairs rule [104, 110].

- **Kohler's rule (SLO/B)**
  This rule can be seen as a sequential maximin rule[3]. At step r (where r goes from 1 to n):

  – Compute for each row i the smallest value $B_{ij} (j \neq i)$.
  – Select the alternative for which this minimum is maximal. If there are ties, select one alternative arbitrarily.
  – Put the selected alternative at position r in the final ranking.
  – Delete the row and the column corresponding to the selected alternative.

  It has been shown in [2, 62] that under the constant-sum property Kohler's rule is indeed a prudent ranking rule. The rule is illustrated in Figure 2.2. We first select alternative d which is put in the first position of the compromise ranking. This alternative is then removed from the matrix. In the second step, c is selected, put at the second position in the compromise ranking and removed from the matrix. In the third step a is selected and, finally b is put at the last position. We thus obtain the prudent order $dcab$. We denote $\mathcal{KO}(u)$ the set of all the linear orders that can be found using this ranking rule.

- **Arrow and Raynaud's rule (SLO/B)**
  This rule is very similar to Kohler's rule. At step r (where r goes from 1 to n):

  – Compute for each row i the largest value $B_{ij} (i \neq j)$.
  – Select the alternative for which this maximum is minimal. If there are ties, select one alternative arbitrarily.
  – Put the selected alternative at position $n - r + 1$ in the final ranking.
  – Delete the row and the column corresponding to the selected alternative.

  It has been shown in [2] that under the constant-sum property Arrow and Raynaud's rule is indeed a prudent ranking rule. The rule is illustrated in Figure 2.3, where the ranking $cabd$ is eventually obtained. We denote $\mathcal{AR}(u)$ the set of all the linear orders that can be found using this ranking rule.

[3] A maximin rule selects the alternative $a_i \in \mathcal{A}$ such that $\forall a_j \in \mathcal{A}$ we have that $\min_{k \neq i} B_{ik} \geq \min_{k \neq j} B_{jk}$

	a	b	c	d	min
a	.	8	–8	–2	–8
b	–8	.	6	–2	–8
c	8	–6	.	4	–6
d	2	2	–4	.	**–4**

	a	b	c	min
a	.	8	–8	–8
b	–8	.	6	–8
c	8	–6	.	**–6**

	a	b	min
a	.	8	**8**
b	–8	.	–8

Figure 2.2: An illustration of Kohler's ranking rule.

	a	b	c	d	max
a	.	8	–8	–2	8
b	–8	.	6	–2	6
c	8	–6	.	4	8
d	2	2	–4	.	**2**

	a	b	c	max
a	.	8	–8	8
b	–8	.	6	**6**
c	8	–6	.	8

	a	c	max
a	.	–8	**–8**
c	8	.	8

Figure 2.3: An illustration of Arrow and Raynaud's ranking rule.

- **Ranked Pairs rule (SLO/B)**
  - Rank the ordered pairs (a_i, a_j) according to their values B_{ij} from the largest to the smallest. Take any linear order compatible with this weak order.
  - Consider the pairs in that order and do the following:
    - If the ordered pair creates a cycle with the pairs already blocked, skip this ordered pair.
    - If the ordered pair does not create a cycle with the pairs already blocked, block this ordered pair.

By construction this is a prudent ranking rule. In fact, the ordered pairs of $R_{>\beta}$ are always blocked since no cycles can appear up to that point. Consequently, any linear order found by that ranking rule must contain $R_{>\beta}$. Let us illustrate this on the example. First, we block the two pairs (c, a) and (a, b) with a majority margin of 8. We then skip pair (b, c) with a majority margin of 6 since it creates a cycle with the pairs already blocked. We then block the pair (c, d) with a majority margin of 4. In fact, this pair does not create any cycle with the pairs already blocked. Finally, we block the pairs (d, a) and (d, b) with a majority margin of 2 and so the ranking *cdab* is eventually obtained. We denote $\mathcal{RP}(u)$ the set of all the linear orders that can be obtained with this rule. We will come back to this rule in Chapter 6.

$\mathcal{PO}(u)$	$\mathcal{KO}(u)$	$\mathcal{AR}(u)$	$\mathcal{RP}(u)$
dcab	X		
cabd		X	
cdab			X
cadb			

Table 2.1: The result of 4 prudent ranking rules.

All the results of these four prudent ranking rules are summarized in Table 2.1. Let us do the following remarks.

- It is clear that these are four different prudent ranking rules. In the example in Table 2.1 we even use a profile u such that

$$\mathcal{KO}(u) \cap \mathcal{AR}(u) \cap \mathcal{RP}(u) = \emptyset.$$

  Furthermore, all three ranking rules differ from the prudent order preference function.

- As $\mathcal{PO}$, the three ranking rules $\mathcal{KO}, \mathcal{AR}$ and $\mathcal{RP}$ can also lead to multiple solutions because of possible ex-aequos when selecting an alternative or when blocking an ordered pair.

- Kohler's rule is an illustration of a *ranking by choosing* procedure. First we apply a choice procedure to determine the alternative that is ranked first. This alternative is removed from the data set and the same choice procedure is reapplied on the not yet ranked set of alternatives. This scheme is sequentially repeated until all the alternatives have been ranked. For Kohler's rule, the choice procedure consists in choosing an alternative a_i such that the smallest B_{ij} is maximal (in case of ties we choose one alternative randomly). Arrow and Raynaud is a procedure that ranks upwards by eliminating always the worst choice.

- Apart from being prudent ranking rules, $\mathcal{KO}$ and $\mathcal{AR}$ are also sequentially prudent [2, 31]. This means that, at each step of the algorithm, the bi-partition of the alternatives already ranked and the alternatives that still need to be ranked verify a form of prudence.

- Unlike conjectured by Arrow and Raynaud in their book [2], Lansdowne [70] highlighted that Kohler's rule and Arrow and Raynaud's

Name	Type	Reference
Bernardo	SLO/Q	[8]
Cook & Seiford	SLO/Q	[25]
Dodgson	WO/u[1]	[11, 59]
MAH	WO/B	[7]
Robust Borda	WO/Q	[6]
Simpson	WO/B	[98]
Median Kendall	SLO/u[1]	[56]

Table 2.2: Some other ranking rules.

rule may not be sufficient to find the whole set of prudent orders, which means that there can exist a profile u such that $\mathcal{KO}(u) \cup \mathcal{AR}(u) \neq \mathcal{PO}(u)$ In fact, this can also be observed from the results of our introductory example in Table 2.1.

2.2.2 Non-prudent ranking rules

It is clear that not every ranking rule can be considered as prudent, according to Definition 8. In this section, we give the precise definitions of four well-known non-prudent ranking rules, namely the ones accredited to Borda, Copeland, Kemeny and Slater. Other ranking rules proposed in the literature can be found in Table 2.2 with the relevant references. We also refer to Cook [24] for a review on other distance based ad-hoc consensus models.

- **Borda's rule (WO/Q)**
  Borda's well-known rule [12] orders the alternatives according to their sums of ranks they occupy in the profile. This can be modeled by means of the matrix Q. The Borda score is thus defined as follows:

  $$\forall a_i \in \mathcal{A}, \quad b_i = \sum_{k=1}^{n} Q_{ik} k.$$

  The Borda ranking $\succeq_B$ is the weak order defined as follows:

  $$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in \succeq_B \iff b_i \leq b_j.$$

  Since Borda's rule seems very intuitive, it has received a large attention by the scientific community. A famous characterization of this rule is due to Young [108].

Finally, Borda's rule can also be seen as a special case of so-called scoring rules. Instead of assigning 1 point to the first position, 2 points to the second position, and so on, one may assign more generally s_1 points to the first position, s_2 points to the second position, and so on, as long as $s_1 \leq s_2 \leq \ldots \leq s_n$.

- **Copeland's rule (WO/M)**
Another "reasonable" ranking rule based on the majority relation has been proposed by Copeland [26].The idea is that the more a given alternative beats other alternatives at majority the better this alternative should be ranked. Similarly, the more other alternatives beat a given alternative at majority, the lower this alternative should be ranked. A score is attached to each alternative that translates these two objectives and the alternatives are then ranked according to these Copeland scores:
$\forall a_i \in \mathcal{A}$,

$$c_i = 2|\{a_k \in \mathcal{A} : (a_i, a_k) \in M\}| + |\{a_k \in \mathcal{A} : (a_k, a_i) \notin M \text{ and } (a_i, a_k) \notin M\}|.$$

The Copeland ranking $\succeq_C$ is the weak order defined as follows:

$$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in \succeq_C \iff c_i \geq c_j.$$

The Copeland choice function has been characterized by Henriet [49].

- **Kemeny's rule (SLO/B)**
Kemeny [54] and Kemeny and Snell [55] approached the problem of finding a compromise ranking by first showing that the symmetric difference distance δ is the only distance function that verifies a set of reasonable axioms. The final compromise rankings will then be defined as the linear orders which are closest, in average, to the linear orders of the profile according to the distance δ:

$$\begin{array}{ll} \min & \sum_{i=1}^{q} \delta(O, O_i) \\ s.t. & O \in \mathcal{LO}. \end{array}$$

One may show that the objective function of this optimization problem can also be replaced with $\sum_{(a_i,a_j)\in O} B_{ij}$ and consequently the model only depends on the majority margins. We refer the reader to Montjardet [79] for other formulations of the objective function of this

[1]This ranking rule relies upon the whole profile in the sense that the knowledge of M, B or Q is not sufficient to compute the result.

combinatorial optimization problem. In some contexts, a Kemeny order is also called a median order.

Kemeny orders have been widely studied in the literature and the properties they verify (e.g. strong consistency plus Young-Condorcet, see Young and Levenglick [109]) can be appealing. Unfortunately, finding Kemeny orders is an NP-complete problem [5, 50].

- **Slater's rule(SLO/M)**
  Slater [99] proposed to solve Condorcet's paradox by selecting the rankings that are closest, according to the symmetric difference distance δ, to the majority relation M. Formally, we have the following optimization problem that needs to be solved to find the so-called Slater orders:

$$\begin{array}{ll} \min & \delta(O, M) \\ s.t. & O \in \mathcal{LO}. \end{array}$$

  If O_S is a Slater order, then the distance $\delta(O_S, M)$ is sometimes called the Slater index.

2.3 Classification

In this section, we classify into ranking rule families the prudent and non-prudent ranking rules that we have encountered so far. To do so, we categorize the ranking rules on the basis of three aspects.

Condorcet ranking consistency (CRC)

We say that a ranking rule is *Condorcet ranking consistent* if the following holds: if the reflexive closure $r(M)$ of the strict majority relation of the profile is a linear order, then this linear order must be the result of the ranking rule.

From the perspective that we compare the alternatives pairwise, this property seems to be a minimal requirement. In fact, if there is a majority of rankings in the profile that prefer a_i over a_j and the majority relation is complete and transitive, then it will be difficult to justify why we would not put a_i before a_j in any compromise ranking. As explained by Arrow and Raynaud, we are interested in ranking rues "...that would be identical to the majority method, if applicable."[4]It is easy to see that Kemeny's rule, Slater's rule and Copeland's rule all verify CRC. Furthermore, any prudent ranking rule also verifies this property, as we will prove in Corollary 2 in Section 4.2. However, Borda's rule does not verify this basic requirement. Let us present an intriguing counterexample taken from [4], which shows that things are not so obvious:

1	$gabcfhde$
1	$fhgabcde$
1	$fdehgabc$

On the one hand, the Borda ranking of this profile is the weak order

$$fgh \sim abdce,$$

where h and a are in a tie. On the other hand, the majority relation leads to the linear order $fhgabcde$, which is thus different from Borda's ranking. Nevertheless, even such a transitive majority relation could be criticized with the following argument: g is placed once in the first, once in the third and once in the fifth position, whereas h is placed once in the second, once in the fourth and once in the sixth position. So it seems intuitive that g is preferred to h. This is the case in the Borda ranking but not in the transitive majority ranking! Even if we will pursue in this thesis the idea that CRC is essential in a decision aid context, we refer the interested reader to Saari (see for instance [96]), who insists that a property such as CRC is irrelevant.

B-ordinality (BO)

We say that a ranking rule is *B-ordinal* if the following holds: let u^1 be a profile with a majority margin matrix B^1 and let u^2 be a profile with a majority margin matrix B^2. If $\forall a_i, a_j \in \mathcal{A}$,

$$B^1_{ij} > 0 \iff B^2_{ij} > 0$$

$$B^1_{ij} = 0 \iff B^2_{ij} = 0$$

and $\forall a_i, a_j, a_k, a_l \in \mathcal{A}$,

$$B^1_{ij} > B^1_{kl} \iff B^2_{ij} > B^2_{kl}$$

$$B^1_{ij} = B^1_{kl} \iff B^2_{ij} = B^2_{kl},$$

then the result for u^1 and for u^2 must be identical.

[4]Arrow and Raynaud (1986), *Multicriterion Decision Making*, MIT Press, page 81.

B-ordinality means that the result of the ranking rule only depends on the order of the values of the majority margins, whereas the numeric values of the majority margins do not matter. Prudent ranking rules are usually BO. In fact, BO is closely related to Majority Profile Consistency and Weak Majority Profile Consistency, which are two properties that we will use in the axiomatic characterizations of some prudent ranking rules (see Chapters 5 and 6). A typical example for a non-BO ranking rule is Kemeny's rule. Let us consider the following counter example, where B^1 is the majority margin matrix of a profile u^1. According to Proposition 1, we know that a profile of linear orders corresponding to B^1 must exist.

B^1	a	b	c	d
a	.	1	1	–3
b	–1	.	3	3
c	–1	–3	.	3
d	3	–3	–3	.

$\rightarrow$ *bcda* is the unique Kemeny order.

Let us now consider another profile u^2 with a majority margin matrix B^2. Using Proposition 1, we know that such a profile of linear orders must exist.

B^2	a	b	c	d
a	.	3	3	–5
b	–3	.	5	5
c	–3	–5	.	5
d	5	–5	–5	.

$\rightarrow$ *abcd* is the unique Kemeny order.

The order of the majority margins for u^1 is the same as the the order of the majority margins for u^2:

$$B^1(a,b) = B^1(a,c) < B^1(b,c) = B^1(b,d) = B^1(c,d) = B^1(d,a).$$

$$B^2(a,b) = B^2(a,c) < B^2(b,c) = B^2(b,d) = B^2(c,d) = B^2(d,a).$$

However, the result for profile u^1 differs from the result for profile u^2.

A particular family of BO ranking rules are those rules that only depend on the strict majority relation.

M-invariance (MI)

We say that a ranking rule is *M-invariant* if the following holds: let u^1 be a profile with a majority margin matrix B^1 and let u^2 be a profile with a majority margin matrix B^2. If $\forall a_i, a_j \in \mathcal{A}$,

$$B^1_{ij} > 0 \iff B^2_{ij} > 0$$

$$B^1_{ij} = 0 \iff B^2_{ij} = 0,$$

then the result for u^1 and for u^2 must be identical.

MI ranking rules are those rules that only take into account the fact that there is a majority between two alternatives. We encountered two such rules: Slater's rule and Copeland's rule. BO ranking rules which do not verify MI, such as for instance most prudent ranking rules, work with a richer information because majority margins do count.

The overall situation of the ranking rules that we have presented in the previous section with respect to the three properties Condorcet ranking consistency, B-ordinality and M-invariance are summarized in a tree in Figure 2.4. We already included in the classification the extended prudent order and the lexicographic prudent order preference functions, which are two prudent ranking rules that will be introduced in Chapter 5 and in Chapter 7. In this thesis we will be mainly concerned with ranking rules that are Condorcet ranking consistent, B-ordinal but not M-invariant.

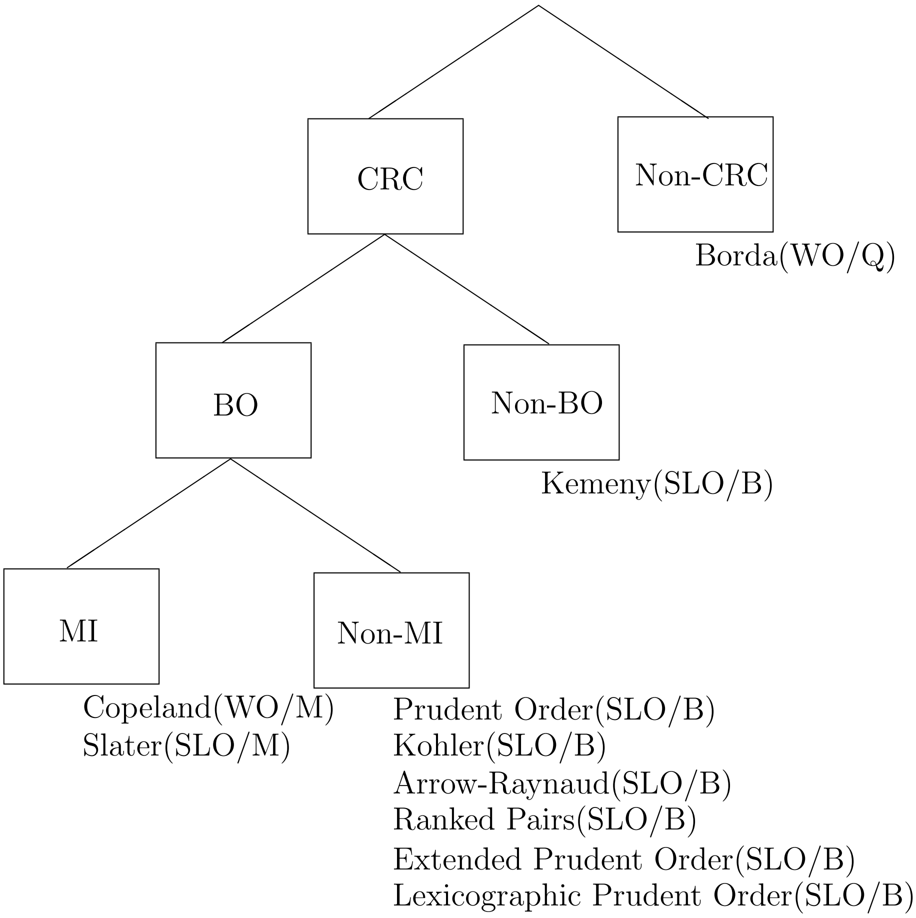


Figure 2.4: Summary of the ranking rules with respect to Condorcet ranking consistency, B-Ordinality and M-invariance.

Chapter 3

Ordinal ranking rules as a decision support tool

In this chapter we postulate that prudent ranking rules can be an appropriate tool in practice. First, in Section 3.1, we introduce social choice theory and we highlight some differences between an election and the type of problems we are considering. We focus our discussion on prudent ranking rules in Section 3.2.

3.1 Social choice theory

According to Riker, "Social choice theory is the description and analysis of the way that the preferences of individual members of a group are amalgamated into a decision of the group as a whole."[1]. Scholars of social choice theory study aggregation procedures, and in particular ordinal ranking rules, from a very abstract perspective. Usually the nature of the group or the nature of the decision at stake is not specified. This approach has lead to the introduction of rather general concepts and has favored the development of very deep results. Probably the most disturbing and most influential result was Arrow's famous impossibility theorem [1].

Despite being a rather theoretical discipline, social choice theory eventually aims at designing voting rules that could be applied in an election. It is however crucial to clearly distinguish between, on the one hand, the problem of an election and, on the other hand, the group ranking problem or the MCDA problem described in Section 1.2.

First of all, we use a different vocabulary, depending on the fact if we

[1]W. Riker (1986), *The art of political manipulation*, Yale University Press.

consider an election, a group ranking problem or a MCDA problem:

	Election	**Group**	**MCDA**
Who ranks?	Voters	Group members	Criteria
What is ranked?	Candidates	Alternatives	Alternatives

As we have seen, in all three situations, we need algorithms that combine multi-dimensional ordinal data. Despite this technical similarity, let us more particularly insist on the following conceptual differences, which have been partly put forward by Bouyssou, Marchant and Perny [14] and by Marchant [75].

- *Definition of the problem*
  A candidate is a person that accepts to run in an election. Since each candidate is personally responsible to manifest his interest, the set of candidates is objectively well-defined and stable before the start of any election. In the same way, it is usually clear who are the voters, which means the people that are allowed to cast a ballot. They can for instance be the members of a society, the adult population of a country, etc.

  In a group ranking problem, it is usually easy to determine the group members. These are the people of the same organization or the same company working on a particular problem. In a MCDA problem, establishing a family of criteria can be a more tedious process. In both contexts, the definition of the set of alternatives is usually far from obvious. Usually this set is not an a priori of the problem, but has to be constructed by the decision makers. During the decision process, some alternatives might dynamically change, others will disappear and new ones are likely to appear.

- *Indeterminateness of the result*
  In an election, the rules are clearly set a priori. The ballots are sufficient to unambiguously determine the result. This result has to be accepted by the voters and candidates and it is not open to further discussion or negotiation. Consequently, the result cannot be indeterminate.

  In a group ranking problem or in a MCDA problem, the recommendation proposed by any decision aid method can be questioned and discussed and eventually rejected. Hence, it makes totally sense to highlight the indeterminateness of the result. Pointing out the more problematic parts of the solution will hopefully stimulate the discussion on the problem. Furthermore, there can be a possibility that the result is progressively refined.

Let us summarize our discussion:

	Election	**Group**	**MCDA**
Def. of cand./alt.	easy	difficult	difficult
Def. of voters/group members/criteria	easy	easy	difficult
Indeterminateness of the result	not useful	useful	useful

The main point that we would like to make here is that underlining the indeterminateness of the the result does not make sense in an election but can be useful in more operational decision aid situations. Unfortunately, most of the ranking rules are not suitable from this perspective. This can be explained by the fact that most of the ordinal ranking rules have been developed and studied by the social choice community who are less interested by ambiguous results. Our thesis is that there is a need for ordinal ranking rules that explicitly take into account the difficulty and ambiguity of aggregating ordinal data. We show in the next section why prudent orders can be a possible answer to this concern.

3.2 Motivations for prudent ranking rules

The choice of a particular, not necessarily prudent, ranking rule is a fundamental problem. Roy and Bouyssou [93] (page 359 - 360) bring up this issue when discussing the use of ordinal ranking rules in a decision aid context. The problem is complicated by the fact that, although the rules that we could use all take as input the same ordinal data and they all look, a priori, very reasonable, they can possibly lead to very different results. Consequently, the result rather depends on the choice of a particular ranking rule than on the input data itself. This may cause problems since it will be difficult to justify, in practice, the use of a particular ranking rule.

The difficulty of a choosing a ranking rule can apparently be overcome by adopting the following ad-hoc solution. In fact, one could consider all the solutions obtained by a set of different popular ranking rules (see for instance Colson [23]). In a way, a new rule is defined by combining existing rules. If the different rules agree, then we are tempted to interpret this as a confirmation of the results since each ranking rule has its own logic and consequently treats the data from its own perspective. On the contrary, this approach can cause confusion, misunderstandings and misinterpretations in case the different results are contradictory. Even if the results of various ranking rules do agree, the interpretation is not straightforward.

We will however not pursue this more pragmatic approach. Instead, we give some hints why prudent ranking rules can be interesting to consider.

3.2.1 Interpretation of a prudent order

Intuitively, a prudent order is a compromise ranking such that the weakest pairwise preference link is maximal. Equivalently, a prudent order is a compromise ranking such that the strongest opposition against this solution is minimal. According to Arrow and Raynaud, this makes especially sense when "... working in an industrial or business-like context."[2] Let us illustrate this idea of "maximizing the strongest opposition" in the context of the group ranking problem.

Consider the following example with three alternatives and 9 group members.

GM_1	*abc*	GM_6	*bca*
GM_2	*abc*	GM_7	*abc*
GM_3	*cab*	GM_8	*abc*
GM_4	*bca*	GM_9	*cab*
GM_5	*cab*		

Given this situation, a compromise ranking has to be constructed. Since there are 3 alternatives, there are in total 3!=6 linear orders that can be considered. Let us now analyze these 6 solutions one by one. Each solution consists of 3 ordered pairs of alternatives and we focus on the opposition that any such ordered pair gets from the group members.

- Solution *abc*.

Pair of altern.	(a,b)	(a,c)	(b,c)	**Strong. Coal.**
Oppos. from *GM*	4,6	3,4,5,6,9	3,5,9	3,4,5,6,9

- Solution *acb*.

Pair of altern.	(a,c)	(a,b)	(c,b)	**Strong. Coal.**
Oppos. from *GM*	3,4,5,6,9	4,6	1,2,4,6,7,8	1,2,4,6,7,8

[2] Arrow and Raynaud (1986), *Multicriterion Decision Making*, MIT Press.

- Solution *bac*.

Pair of altern.	(b,a)	(b,c)	(a,c)	**Strong. Coal.**
Oppos. from *GM*	1,2,3,5,7,8,9	3,5,9	3,4,5,6,9	1,2,3,5,7,8,9

- Solution *bca*.

Pair of altern.	(b,c)	(b,a)	(c,a)	**Strong. Coal.**
Oppos. from *GM*	3,5,9	1,2,3,5,7,8,9	1,2,7,8	1,2,3,5,7,8,9

- Solution *cab*.

Pair of altern.	(c,a)	(c,b)	(a,b)	**Strong. Coal.**
Oppos. from *GM*	1,2,7,8	1,2,4,6,7,8	4,6	1,2,4,6,7,8

- Solution *cba*.

Pair of altern.	(c,b)	(c,a)	(b,a)	**Strong. Coal.**
Oppos. from *GM*	1,2,4,6,7,8	3,4,5,6,9	1,2,3,5,7,8,9	1,2,3,5,7,8,9

Take a closer look at the first solution *abc*: there are 2 group members that are opposed to the fact that a is preferred to b (namely GM_4 and GM_6), there are 5 group members that are opposed to the fact that a is preferred to c and there are 3 group members that are opposed to the fact that b is preferred to c. Hence, the strongest coalition against this ranking consists of the group members 3, 4, 5, 6 and 9, who all agree on the fact that c is preferred to a. Let us now imagine that this ranking will be the final group solution. Then, the group members belonging to the strongest coalition could join their forces and veto against this solution. Put in another way, by adopting this ranking, we go against the common will of the group members of the strongest coalition.

Since we have to chose at least one group ranking, following the argumentation of the previous paragraph, we are going to select the ranking for which the strongest coalition against this ranking is smallest possible. After analyzing the six solutions, we conclude that the first ranking *abc* could be a potential group ranking, since the strongest coalition against this ranking only consists of 5 group members, whereas the strongest coalition of all the other solutions consists of at least 6 group members.

To our point of view, the concept of a prudent order is thus an interesting, and above all, easy and transparent interpretation. An analyst or a decision maker is surely more comfortable with using a ranking rule based on a principle which he can easily grasp than having to use a ranking rule as a black-box tool.

3.2.2 Properties of prudent ranking rules

In theory, the choice of a particular ranking rule can only be justified by looking at its properties. In Bouyssou & al. [16], some guidelines are given for popular social choice rules or multicriteria decision aid methods. Since no perfect aggregation mechanism exists, we select the one that performs well on those aspects relevant to our particular context. Let us come back to the properties introduced in the classification in Section 2.3.

- *Condorcet ranking consistency*
  We think that it is interesting from a cognitive point of view to compare alternatives pairwisely. A decision maker can always compare two alternatives and ask himself which of the two alternatives she or he prefers. Once you accept to compare alternatives pairwisely, it will be difficult not to require Condorcet ranking consistency.
- *B-ordinality*
  B-ordinality implies that a very high majority margin cannot compensate for a very low majority margin. This makes especially sense when the pairwise preference intensities are not on a cardinal, but rather on an ordinal scale. For instance, a consequence of B-ordinality will be that the use of importance coefficients will become more transparent (see Chapter 10).
- *Absence of M-invariance*
  We think that the ranking rule should not solely depend on the strict majority relation. There is a difference between a majority margin of +1 and a majority margin of +100. This needs to be taken into account somehow.

Many ranking rules can be imagined which follow the prudence principle. Those prudent ranking rules which we will consider in this thesis all satisfy the three properties that we have just mentioned.

3.2.3 Multiplicity of prudent orders

We will especially focus on the largest (with respect to inclusion) prudent ranking rule, namely the prudent order preference function which outputs the whole set of prudent orders.

Let us directly acknowledge that the number of prudent orders can be rather large. Following some simulations on the cardinality of the set of prudent orders for small profiles, Debord concluded that: "La simulation [...] semblerait indiquer que, statistiquement, le cardinal de la [...] procédure prudente reste élevé et donc que l'ambiguïté quant au choix final demeure."[3] The result of Debord's simulations can be found in the appendix.

At first sight, enlarging the set of solutions seems counter-intuitive. In fact, a ranking rule that possibly leads to a large amount of rankings is usually considered as a bad aggregation mechanism. Consider for instance a rule that associates to every profile all the linear orders. Such a rule is a very poor decision support. Furthermore, the large amount of different solutions can be very contradictory (see also Perny [86]). This may cause problems from a conceptual point of view, since we do not know which compromise ranking, among all the "optimal" compromise rankings, we should eventually choose.

Although the high number of prudent orders is perceived by Debord (and many others) as an inconvenience, it is precisely this feature that makes them attractive to us. We claim that the multiplicity of results should not be perceived as an inconvenience, but rather as a consequence of the difficulty of aggregating ordinal data. As discussed broadly in Section 3.1, leaving open some indeterminateness can make sense in some situations.

How are we going to manage this set of compromise rankings in practice? In this thesis, we will suggest to analyze the set of potentially interesting compromise rankings using the idea of robust information. We will be looking for conclusions that remain valid for all the compromise rankings (see Roy [92]). That way, more solid information can be obtained. Moreover, this approach also allows to highlight problems and conflicts, since the quality of the robust conclusions obtained is inversely proportional to the degree of contradictions contained in the results. Hence, robustness is used as an ex-post exploitation tool (see also Dias [33] for a discussion on the use of robustness in decision aid). If feasible, an interactive approach can finally help to manage the possible diversity of the results and support the exploration and refinement of the set of solutions.

3.2.4 Variety of prudent ranking rules

Instead of using the prudent order preference function, we could also consider a prudent ranking rule which outputs only one (or a few) "good" prudent orders. This is useful in situations where one final ranking is immediately requested and there is not the possibility of interactively refining the set of prudent orders. In a way, the refinement of the set of prudent orders needs to be done in an automatic way. That is why we will discuss more in detail

[3] B. Debord (1987), *Axiomatisation de procédures d'agrégation de préférences*, PhD Thesis, Université Scientifique et Médicale de Grenoble.

in this thesis three prudent ranking rules different from the prudent order preference function:

- The extended prudent order preference function $\mathcal{XPO}(u)$ (see Chapter 5).
- The Ranked Pairs rule $\mathcal{RP}(u)$ (see Chapter 6).
- The lexicographic prudent order preference function $\mathcal{LPO}(u)$ (see Chapter 7).

In fact, these ranking rules offer an increasingly sharper refinement of the set of prudent orders:

$$\forall u, \quad \mathcal{PO}(u) \supseteq \mathcal{XPO}(u) \supseteq \mathcal{RP}(u) \supseteq \mathcal{LPO}(u).$$

The choice of a particular ranking rule is highly dependent on the context. On the one hand, a prudent ranking rule such as for instance $\mathcal{LPO}$ are more suitable from a "prescriptive" perspective, since it unveils the best possible compromise ranking from a prudent point of view, given the preference information contained in the profile. On the other hand, a prudent ranking rule such as for instance $\mathcal{PO}$ is more suitable from a "constructive" perspective, since it leaves some indeterminateness on the result and allows to interactively discover the best compromise ranking. Of course, the frontier between constructive and prescriptive approaches remains fuzzy. We refer the reader to Chapter 2 in [16] for a discussion on these concepts, although these authors rather discuss different approaches than methods.

Since in practice it can be difficult to correctly assess the degree of constructiveness or prescriptiveness of a situation, we believe that the most interesting prudent ranking rules are the two extremes: $\mathcal{PO}$ and $\mathcal{LPO}$. Let us finally note that these two prudent ranking rules could also be used together. For instance, we compute the best prudent orders using the lexicographic prudent order preference function. These rankings can then possibly be enriched with the information available about the whole set of prudent orders.

Part II

Prudent ranking rules: theoretical contributions

Chapter 4

The prudent order preference function

In this chapter, we focus on technical properties that are commonly used to analyze ordinal ranking rules. Such an insight into the behavior of the prudent order preference function can help to indicate the reasonableness of the prudence principle as an aggregation mechanism.

More particularly, we come back in Section 4.1 to the definition of the prudent order preference function. We highlight in Section 4.2 the link of prudent orders with the strict majority relation. In Section 4.3, we address the issue of consistency. In Section 4.4, we analyze the impact on the result when removing alternatives, especially in view of Arrow's independence of irrelevant alternatives axiom. Finally, in Section 4.5, we make some comments on the prudent choice problem.

4.1 Definition

As we have seen in Section 2.2.1, a prudent order is a linear extension of $R_{>\beta}$, where β is the smallest value such that the corresponding strict cut relation is acyclic. Equivalently, a prudent order is a linear order contained in $R_{\geq\alpha}$, where α is the largest possible value such that the corresponding cut relation contains at least one linear order.

For "problematic" profiles, β is large and α is small. As the two relations $R_{>\beta}$ and $R_{\geq\alpha}$ drift further apart, the number of prudent orders increases, until the trivial case where every linear order on $\mathcal{A}$ is a prudent order (see for instance Lansdowne [70] for such an example). The following proposition points out some particularities of these coefficients α and β.

Proposition 2 *Let u be a profile with q linear orders. Let α and β be the two optimal cut values for this profile.*

1. $\alpha + \beta = 0$.
2. *If $\beta < 0$, then the reflexive closure of the strict majority relation of u must be a linear order.*
3. $\beta < q$.
4. *If $\beta = -q$, then u must be a profile consisting of q times a same linear order.*

Proof:

1. This equality has been proved in Arrow and Raynaud [2]. We recall that $c(R)$ denotes the complement and $t(R)$ denotes the transitive closure of a binary relation R.

2. By definition, the strict majority relation M is equal to the strict cut relation $R_{>0}$. Since $\beta < 0$, we have $R_{>0} \subseteq R_{>\beta}$. Consequently, we have $M \subseteq R_{>\beta}$. Since $R_{>\beta}$ is acyclic, so must be M. We are now going to show that $r(M)$ is also complete, where $r(.)$ denotes the reflexive closure. This will imply that $r(M)$ is a linear order. Let us suppose by contradiction that $r(M)$ is not complete, i.e. there exists two different alternatives a_i and a_j with $(a_i, a_j) \notin M$ and $(a_j, a_i) \notin M$. Because of the constant-sum property ($\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n$, $B_{ij} + B_{ji} = 0$), this implies that $B_{ij} = B_{ji} = 0$. Since we suppose that $\beta < 0$, this means that $(a_i, a_j) \in R_{>\beta}$ and $(a_j, a_i) \in R_{>\beta}$. This is a contradiction since $R_{>\beta}$ is by construction acyclic.

3. Let us suppose by contradiction that $\beta = q$. This means that $R_{\geq q}$ contains a cycle: $B_{i_1 j_1} = q, B_{i_2 j_2} = q, \ldots, B_{i_{p-1} j_p} = q, B_{j_p i_1} = q$. Having an unanimous cycle is not possible since the profile consists of linear orders, which are by definition acyclic.

4. Since $\beta = -q$, Point 1 of this proposition implies that $\alpha = q$. Consequently, the relation $R_{\geq q}$ must contain a linear order O. In other words, $\forall (a_i, a_j) \in O$, we must have that $B_{ij} = q$. This is only possible with a profile containing q times the linear order O.

$\square$

The next proposition gives some further equivalent definitions of a prudent order.

Proposition 3 *Let O_P be a linear order. The following statements are equivalent:*

1. *O_P is a prudent order.*

2. $O_P \subseteq ctc(R_{\geq\alpha})$.

3. *O_P is an optimal solution of* $\max_{O\in\mathcal{LO}} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij}$.

4. $t(R_{>\beta}) \subseteq O_P$.

5. *O_P is an optimal solution of* $\min_{O\in\mathcal{LO}} \max_{(a_i,a_j)\notin O} B_{ij}$.

Proof:

- Equivalence between 1 and 2
  O_P is a prudent order $\iff O_P \subseteq R_{\geq\alpha} \iff c(R_{\geq\alpha}) \subseteq c(O_P)$
  $\iff tc(R_{\geq\alpha}) \subseteq tc(O_P) = c(O_P)$ (since $c(O_P)$ is transitive)
  $\iff cc(O_P) = O_P \subseteq ctc(R_{\geq\alpha})$.

- Equivalence between 1 and 3
  Let us first show that

  $$\alpha = \max_{O\in\mathcal{LO}} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij}.$$

  Let us suppose by contradiction that there exists $O' \in \mathcal{LO}$ such that

  $$\min_{(a_i,a_j)\in O', a_i\neq a_j} B_{ij} = \alpha' > \alpha.$$

  This implies that:

  $$\forall (a_i,a_j) \in O', \quad B_{ij} \geq \alpha'.$$

  Consequently $O' \subseteq R'_{\geq\alpha'}$. Hence there exists $\alpha' > \alpha$ such that $R_{\geq\alpha'}$ still contains at least one linear order. This contradicts the definition of α.

  O_P is a prudent order if and only if $O_P \subseteq R_{\geq\alpha}$. This is equivalent to stating that

  $$\forall (a_i,a_j) \in O_P,\ B_{ij} \geq \alpha.$$

  This is equivalent to stating that

  $$O_P \in arg \max_{O\in\mathcal{LO}} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij},$$

  since $\alpha = \max_{O\in\mathcal{LO}} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij}$.

- Equivalence between 1 and 4
  O_P is a prudent order $\iff R_{>\beta} \subseteq O_P \iff t(R_{>\beta}) \subseteq t(O_P) = O_P$ (since O_P is transitive).

- Equivalence between 3 and 5 follows directly from the constant-sum assumption ($\forall i, j : 1 \le i \le n, 1 \le j \le n,\ B_{ij} + B_{ji} = 0$) and the fact that linear orders are antisymmetric.

□

The interpretation of Point 3 of this proposition is that prudent orders are those linear orders which maximize the weakest link, which means the smallest pairwise majority margin. The interpretation of Point 5 is that prudent orders are those linear orders which minimize the strongest opposition. Consequently, $\forall u$, the set $\mathcal{PO}(u)$ is never empty, since $\mathcal{LO}$ is finite and by exhaustive enumeration at least one linear order is optimal under each of these two problems.

The formulation of prudent orders as an optimization problem allows us also to put prudent orders into perspective with the so-called Kemeny orders [54]. We have seen in Section 2.2.2 that O_K is a Kemeny order if and only if $\forall O \in \mathcal{LO}, \sum_{(a_i,a_j)\in O_k} B_{ij} \ge \sum_{(a_i,a_j)\in O} B_{ij}$. Hence, Kemeny orders are linear orders optimal under a *sum* operator, whereas, according to Point 3 of Proposition 3, prudent orders are optimal using a *min* operator.

There is a straightforward approach to enumerate all the prudent orders (see also Debord [31], page 102). First, use Kohler's algorithm (see Section 2.2.1) to find one prudent order denoted by O_P. Find the strongest opposition against this ranking O_P, which means the largest B_{ij} such that $(a_i, a_j) \notin O_P$. This value corresponds to β. Hence, we can easily compute $R_{>\beta}$ and $t(R_{>\beta})$.

Enumerating all the prudent orders then boils down to enumerating all the linear extensions of the partial order $t(R_{>\beta})$. A constant amortized time algorithm for linear extension enumeration, that is an algorithm that runs in $O(|\mathcal{PO}(u)|)$, is presented in Pruesse and Ruskey [89].

Debord [31] performed some simulations on the number of prudent orders for profiles up to 7 alternatives and up to 17 rankings. His results can be found in the appendix. Unfortunately, the number of prudent orders increases dramatically with the number of alternatives. That is why the simulations of Debord were restricted to small profiles since, by relying on complete enumeration, the computational limits of this approach were quickly reached. In fact, counting linear extensions of a partial order is shown by Brightwell and Winkler [18] to be a #P-complete problem. This means that it is as difficult as finding the number of assignments of a 3-SAT instance.

Any set of linear extensions of a partial order can be represented with a so-called linear extension graph. A vertex of this graph corresponds to one linear extension and there is an edge between two vertices if the corresponding two linear extensions only differ by exactly one transposition of adjacent alternatives. It has been shown that this graph is connected for every partial order (see for instance the paper of Pruesse and Ruskey [89]).

Since the set of prudent orders is the set of linear extensions of the partial order $t(R_{>\beta})$, it can be represented with a linear extension graph. Since the linear extension graph is connected, we can tell that any two prudent orders are linked by a chain of prudent orders and two consecutive prudent orders of this chain only differ by a transposition of two adjacent alternatives. A similar result has been highlighted in Debord's PhD thesis (see corollary 7.2 in [30]), however without referring to the linear extension graph.

Let us illustrate the linear extension graph on a profile with 4 alternatives a, b, c and d. Let $u = (abcd, bcda, cdab, dabc, dcba)$. This profile has the following majority margins:

	a	b	c	d
a	.	1	–1	–3
b	–1	.	1	–1
c	1	–1	.	1
d	3	1	–1	.

By applying Kohler's rule, the linear order $dabc$ can be obtained. The oppositions against this solution are as follows:

$$B(c, b) = -1 \quad B(c, a) = 1 \quad B(c, d) = 1$$

$$B(b, a) = -1 \quad B(b, d) = -1 \quad B(a, d) = -3$$

Hence, the strongest opposition against this ranking is 1, and so $\beta = 1$. The set of prudent orders then corresponds to all the linear extensions of the relation $R_{>1} = \{(d, a)\}$. The linear extension graph of this partial order is given in Figure 4.1. In this graph, the twelve vertices correspond to the twelve prudent orders.

4.2 Majority

The set of prudent orders is closely linked with the strict majority relation M. In general, the reflexive closure of the strict majority relation M is not a linear order. However, if $r(M)$ is a linear order, then Lansdowne [69] showed that $r(M)$ must be a prudent order. In fact, this is the only prudent order, as will be shown by the following more general result:

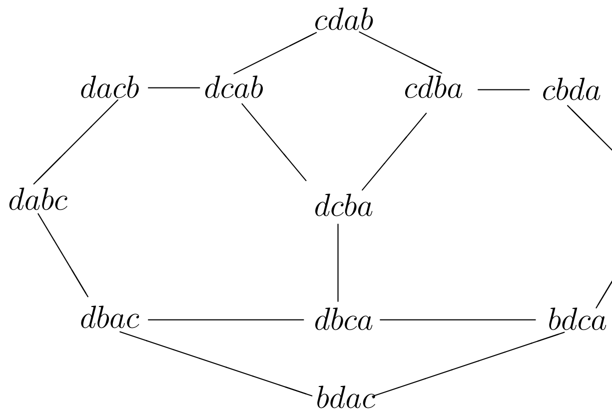


Figure 4.1: The linear extension graph.

Proposition 4 *Let M be the strict majority relation of profile u. If M is acyclic, then $\mathcal{PO}(u) = \mathcal{E}(M)$.*

Proof: Let us suppose M acyclic. Either $r(M)$ is complete or not.

- $r(M)$ is complete.
  We can note that, since $r(M)$ is complete, $\forall (a_i, a_j)$ with $(a_i, a_j) \notin r(M)$, we must have $B_{ij} < 0$. Let $\gamma = \max\{B_{ij} : (a_i, a_j) \notin r(M)\}$. By construction we have that $R_{>\gamma} = M$ and so $R_{>\gamma}$ is acyclic. Furthermore, there exists $(a_i, a_j) \notin r(M)$ such that $B_{ij} = \gamma$. Since $r(M)$ is complete and $(a_i, a_j) \notin r(M)$, we must have that $(a_j, a_i) \in r(M)$. Consequently, $(a_i, a_j) \in R_{\geq\gamma}$ and $(a_j, a_i) \in R_{\geq\gamma}$. This means that $R_{\geq\gamma}$ contains a cycle. Hence, $\beta = \gamma$ and $\mathcal{PO}(u) = \mathcal{E}(R_{>\beta}) = \mathcal{E}(R_{>\gamma}) = \mathcal{E}(M)$.

- $r(M)$ is not complete.
  By definition, $M = R_{>0}$ and since M is acyclic, so must be $R_{>0}$. Since $r(M)$ is not complete, there must exist two different alternatives a_i and a_j such that $(a_i, a_j) \notin R_{>0}$ and $(a_j, a_i) \notin R_{>0}$. Since $B_{ij} \leq 0, B_{ji} \leq 0$ and $B_{ij} + B_{ji} = 0$, we must have $B_{ij} = B_{ji} = 0$. Consequently $R_{\geq 0}$ contains a cycle. Hence $\beta = 0$, and so $R_{>\beta} = R_{>0} = M$. Consequently, $\mathcal{PO}(u) = \mathcal{E}(M)$.

□

Corollary 1 *If $r(M)$ is a partial order, then $\mathcal{PO}(u) = \mathcal{E}(r(M))$.*

Corollary 2 *If $r(M)$ is a linear order, then $\mathcal{PO}(u) = \{r(M)\}$.*

Another interpretation of prudent orders worth mentioning here has been highlighted by Debord [30]. Let us suppose that the profile u is such that the reflexive closure of the strict majority relation is not a linear order. We denote $\forall \lambda > 0$ and $\forall O \in \mathcal{LO}$ by $u + \lambda O$ a profile consisting of the linear orders of profile u and of λ times the linear order O.

For any linear order $O \in \mathcal{LO}$ we denote by μ_O the minimal number of times that one has to add O to u such that the reflexive closure of the strict majority relation of the profile $u + \mu_O O$ corresponds exactly to the linear order O. In other words, μ_O corresponds to the necessary strength of the linear order O to impose itself as the majority solution. We define

$$\mu_{min} = \min_{O \in \mathcal{LO}} \mu_O.$$

Debord [30] then proved the following theorem.

Theorem 2 *Let u be a profile such that the reflexive closure of the strict majority relation is not a linear order. O is a prudent order if and only if the reflexive closure of the strict majority relation of the profile $u + \mu_{min} O$ is equal to O.*

Hence, a prudent order can be interpreted as a linear order that one has to add the smallest number of times to the profile so that the reflexive closure of the strict majority relation of the new profile corresponds exactly to this linear order.

We illustrate this theorem on the profile introduced in Section 2.2.1. In this case $\mu_{min} = 7$. We know that *dcab* is a prudent order. Let us now add this linear order 7 times to the initial profile u:

$$u^* = (u, dcab, dcab, dcab, dcab, dcab, dcab, dcab).$$

The reflexive closure of the strict majority relation of this new profile u^* is exactly the linear order *dcab*. This can be checked by looking at the majority margins of u^*:

	a	b	c	d
a	.	15	–15	–9
b	–15	.	–1	–9
c	15	1	.	–3
d	9	9	3	.

Consider now the linear order *abcd*, which is not a prudent order. The following majority margins will be obtained when adding *abcd* 7 times to the initial profile u:

Property	Name	Mentioned by
$f(u^1) \cap f(u^2) \neq \emptyset$ $\Rightarrow f(u^1 + u^2) = f(u^1) \cap f(u^2)$	Strong consistency	Young and Levenglick [109]
$f(u^1) = f(u^2)$ $\Rightarrow f(u^1 + u^2) = f(u^1) = f(u^2)$	Weak consistency	Saari [94]
$f(u^1) \cap f(u^2) \cap f(u^1 + u^2) \neq \emptyset$	Very weak consistency	Durand [39]

Table 4.1: Some Consistency properties of a preference function f.

	a	b	c	d
a	.	15	–1	5
b	–15	.	13	5
c	1	–13	.	11
d	–5	–5	–11	.

The reflexive strict majority relation of this new profile is not equal to $abcd$ (for instance (a, c) belongs to this linear order but does not belong to the strict majority relation). In fact, $abcd$ has to be added 9 times in order to impose itself as the majority solution.

4.3 Consistency

In this section, we study different formulations of what can be called a consistency property. Given two profiles

$$u^1 = (O^1_1, O^1_2, \ldots, O^1_q) \text{ and } u^2 = (O^2_1, O^2_2, \ldots, O^2_{q'}),$$

we construct a new profile denoted by

$$u^1 + u^2 = (O^1_1, O^1_2, \ldots, O^1_q, O^2_1, O^2_2, \ldots, O^2_{q'})$$

which consists in concatenating the two initial profiles. The relationships between $\mathcal{PO}(u^1), \mathcal{PO}(u^2)$ and $\mathcal{PO}(u^1 + u^2)$ are then studied. In Table 4.1, we give some examples of consistency properties which have been mentioned in the literature.

Durand [39] was studying in his PhD thesis the consistency of prudent choice rules. He noticed already that the prudence principle is hardly compatible with the idea of consistency. In fact, in his models, no positive result about consistency has been mentioned. More particularly, Durand's work consists in performing simulations in order to estimate the frequency of profiles where some kind of consistency is nevertheless verified.

	u^1
4	cab
3	abc
1	bca
3	cba
3	acb

$\rightarrow$

B^1	a	b	c
a	.	6	–2
b	–6	.	–6
c	2	6	.

$\rightarrow \quad \mathcal{PO}(u^1) = \{cab\}$

	u^2
4	bac
5	acb
5	cba

$\rightarrow$

B^2	a	b	c
a	.	–4	4
b	4	.	–6
c	–4	6	.

$\rightarrow \quad \mathcal{PO}(u^2) = \mathcal{E}(\{(c,b)\})$

$u^1 + u^2 \quad \rightarrow$

S	a	b	c
a	.	2	2
b	–2	.	–12
c	–2	12	.

$\rightarrow \quad \mathcal{PO}(u^1 + u^2) = \{acb\}$

Figure 4.2: Reference example for different consistency properties.

Let us however remark that Durand was working with prudent choice and not prudent ranking functions. Unfortunately, the picture seems to be not more encouraging when working with rankings instead of choice subsets. In the example in Figure 4.2, $\mathcal{PO}(u^1) = \{cab\}$, $\mathcal{PO}(u^2)$ consists of all the linear extensions of $\{(c,b)\}$ and $\mathcal{PO}(u^1 + u^2) = \{acb\}$. So, in this case, $\mathcal{PO}(u^1) \cap \mathcal{PO}(u^2) \cap \mathcal{PO}(u^1 + u^2) = \emptyset$! Using Durand's terminology (see Table 4.1), very weak consistency is not verified. This implies that strong consistency, used by Young and Levenglick [109] in their characterization of Kemeny's rule, cannot be verified for prudent orders: in the example in Figure 4.2, although $\mathcal{PO}(u^1) \cap \mathcal{PO}(u^2) \neq \emptyset$, it is not true that $\mathcal{PO}(u^1 + u^2) = \mathcal{PO}(u^1) \cap \mathcal{PO}(u^2)$.

However, a weaker form of consistency can nevertheless be stated for prudent orders.

Theorem 3 *Let $R^s_{>\beta_s}$ be the optimal strict cut relation of profile u^s ($s = 1,2$).If*

$$\mathcal{PO}(u^1) \cap \mathcal{PO}(u^2) \neq \emptyset,$$

then

$$\mathcal{PO}(u^1 + u^2) \subseteq \mathcal{E}(R^1_{>\beta_1} \cap R^2_{>\beta_2}).$$

Proof: Let B be the majority margin matrix of profile $u = u^1 + u^2$. Let β be the optimal cut-value of profile u. We need the following lemma:

Lemma 1 *$R^1_{>\beta_1} \cup R^2_{>\beta_2}$ is acyclic if and only if $\mathcal{PO}(u^1) \cap \mathcal{PO}(u^2) \neq \emptyset$, i.e. there exists a linear order O such that $R^1_{>\beta_1} \subseteq O$ and $R^2_{>\beta_2} \subseteq O$.*

Proof of the lemma:

- $\Rightarrow$
  Since $R^1_{>\beta_1} \cup R^2_{>\beta_2}$ is acyclic, there must exist a linear order O (see Szpilrajn [100]) such that $R^1_{>\beta_1} \cup R^2_{>\beta_2} \subseteq O$. This implies that $R^1_{>\beta_1} \subseteq O$ and $R^2_{>\beta_2} \subseteq O$.

- $\Leftarrow$
  There exists a linear order O such that $R^1_{>\beta_1} \subseteq O$ and $R^2_{>\beta_2} \subseteq O$. This implies that $R^1_{>\beta_1} \cup R^2_{>\beta_2} \subseteq O$. Let us suppose by contradiction that $R^1_{>\beta_1} \cup R^2_{>\beta_2}$ contains a cycle. Hence there exist two different alternatives a_i and a_j such that $(a_i, a_j) \in t(R^1_{>\beta_1} \cup R^2_{>\beta_2})$ and $(a_j, a_i) \in t(R^1_{>\beta_1} \cup R^2_{>\beta_2})$, which implies that $(a_i, a_j) \in O$ and $(a_j, a_i) \in O$. This is a contradiction since O is antisymmetric.

Proof of the theorem:
Let us cut the majority margins B of profile $u = u^1 + u^2$ at level $\beta_1 + \beta_2$. Then :

$$R^1_{>\beta_1} \cap R^2_{>\beta_2} \subseteq R_{>\beta_1+\beta_2} \subseteq R^1_{>\beta_1} \cup R^2_{>\beta_2}$$

On the one hand, if $(a_i, a_j) \in R^1_{>\beta_1} \cap R^2_{>\beta_2}$, then we have $B^1_{ij} > \beta_1$ and $B^2_{ij} > \beta_2$, which implies that $B_{ij} > \beta_1+\beta_2$, which implies that $(a_i, a_j) \in R_{>\beta_1+\beta_2}$. On the other hand, if $(a_i, a_j) \notin R^1_{>\beta_1}$ and $(a_i, a_j) \notin R^2_{>\beta_2}$, then we have $B^1_{ij} \leq \beta_1$ and $B^2_{ij} \leq \beta_2$, which implies that $B_{ij} \leq \beta_1 + \beta_2$, which implies that $(a_i, a_j) \notin R_{>\beta_1+\beta_2}$.

Since $\mathcal{PO}(u^1) \cap \mathcal{PO}(u^2) \neq \emptyset$, Lemma 1 tells us that $R^1_{>\beta_1} \cup R^2_{>\beta_2}$ is acyclic. That is why $R_{>\beta_1+\beta_2}$ is also acyclic. Consequently $\beta \leq \beta_1 + \beta_2$ and so $R_{>\beta_1+\beta_2} \subseteq R_{>\beta}$. Consequently, $R^1_{>\beta_1} \cap R^2_{>\beta_2} \subseteq R_{>\beta}$, which implies that $\mathcal{PO}(u^1 + u^2) \subseteq \mathcal{E}(R^1_{>\beta_1} \cap R^2_{>\beta_2})$.

□

The example in Figure 4.2 illustrates this kind of consistency property. The intersection of $\mathcal{PO}(u^1)$ with $\mathcal{PO}(u^2)$ is non empty. Since

$$R^1_{>\beta_1} = \{(c, a), (c, b), (a, b)\}$$

and $R^2_{>\beta_2} = \{(c, b)\}$, we have that $R^1_{>\beta_1} \cap R^2_{>\beta_2} = \{(c, b)\}$. Indeed, the reader can check that $\mathcal{PO}(u^1 + u^2) \subseteq \mathcal{E}(\{(c, b)\})$.

A consequence of Theorem 3 is the following corollary.

Corollary 3 *If $R^1_{>\beta_1} = R^2_{>\beta_2}$, then $\mathcal{PO}(u^1 + u^2) \subseteq \mathcal{PO}(u^1)$.*

	u^1
1	*abcd*
1	*bcda*
1	*cdab*
1	*dabc*

→

B^1	a	b	c	d
a	.	2	0	–2
b	–2	.	2	0
c	0	–2	.	2
d	2	0	–2	.

→ $\mathcal{PO}(u^1) = \mathcal{LO}$

	u^2
2	*abdc*
2	*bdca*
2	*cabd*
2	*dacb*

→

B^2	a	b	c	d
a	.	4	–4	0
b	–4	.	0	4
c	4	0	.	–4
d	0	–4	4	.

→ $\mathcal{PO}(u^2) = \mathcal{LO}$

$u^1 + u^2$ →

B	a	b	c	d
a	.	6	–4	–2
b	–6	.	2	4
c	4	–2	.	–2
d	2	–4	2	.

→ $\mathcal{PO}(u^1 + u^2) = \{cabd\}$

Figure 4.3: Although profiles u^1 and u^2 have the same set of prudent orders, $u^1 + u^2$ is smaller.

Even in this particular formulation, we cannot replace the set inclusion by an equality. In Figure 4.3, for both profiles the optimal strict cut relations are empty and consequently for both profiles every linear order is a prudent order. However, when considering the two profiles together, the set of prudent orders has shrunk to one linear order. Hence, following Saari's terminology (see Table 4.1), prudent orders do not verify weak consistency.

Let us now suppose that $O \in \mathcal{PO}(u)$. When adding the prudent order O to the initial profile, we can now state the following proposition, where the second inclusion is a consequence of Theorem 3.

Proposition 5 *For every linear order O with $O \in \mathcal{PO}(u)$, we have that*

$$\{O\} \subseteq \mathcal{PO}(u + O) \subseteq \mathcal{PO}(u).$$

Proof:

- Let us first show that $\mathcal{PO}(u + O) \subseteq \mathcal{PO}(u)$. In order to use the same notation as in Theorem 3, let us denote $u^1 = u$ and u^2 the profile consisting solely of the linear order O. It is easy to see that $\mathcal{PO}(u^2) = \{O\}$ and that $r(R^2_{>\beta_2}) = O$ (where $r(.)$ denotes the reflexive closure). Since $O \in \mathcal{PO}(u^1)$, we can also tell that $R^1_{>\beta_1} \subseteq O$. We can conclude that $R^1_{>\beta_1} \cap R^2_{>\beta_2} = R^1_{>\beta_1}$. Applying Theorem 3, we finally get that:

$$\mathcal{PO}(u{+}O) = \mathcal{PO}(u^1{+}u^2) \subseteq \mathcal{E}(R^1_{>\beta_1} {\cap} R^2_{>\beta_2}) = \mathcal{E}(R^1_{>\beta_1}) = \mathcal{PO}(u^1) = \mathcal{PO}(u).$$

- Let us now show that $\{O\} \subseteq \mathcal{PO}(u+O)$. We denote by B the majority margin matrix of profile u and by B' the majority margin matrix of profile $u+O$.

  Since $O \in \mathcal{PO}(u)$, we have (see Proposition 3):

  $$\forall \tilde{O} \in \mathcal{LO}, \quad \min_{\substack{(a_i,a_j)\in O,\\ a_i \neq a_j}} B_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O},\\ a_i \neq a_j}} B_{ij}.$$

  This implies that:

  $$\forall \tilde{O} \in \mathcal{LO}, \quad (\min_{\substack{(a_i,a_j)\in O,\\ a_i \neq a_j}} B_{ij}) + 1 \geq (\min_{\substack{(a_i,a_j)\in \tilde{O},\\ a_i \neq a_j}} B_{ij}) + 1.$$

  Let us note that adding O to the profile u will change the majority margins as follows. For every i, j with $1 \leq i \leq n, 1 \leq j \leq n$, $i \neq j$, we have $B'_{ij} = B_{ij} + 1$, if $(a_i, a_j) \in O$ and $B'_{ij} = B_{ij} - 1$, if $(a_i, a_j) \notin O$. Hence, on the left hand side of the last inequality we have:

  $$(\min_{\substack{(a_i,a_j)\in O,\\ a_i \neq a_j}} B_{ij}) + 1 = \min_{\substack{(a_i,a_j)\in O,\\ a_i \neq a_j}} B'_{ij}.$$

  On the right hand side we have:

  $$(\min_{\substack{(a_i,a_j)\in \tilde{O},\\ a_i \neq a_j}} B_{ij}) + 1 \geq \min_{\substack{(a_i,a_j)\in \tilde{O},\\ a_i \neq a_j}} B'_{ij}.$$

  Consequently:

  $$\forall \tilde{O} \in \mathcal{LO}, \quad \min_{\substack{(a_i,a_j)\in O,\\ a_i \neq a_j}} B'_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O},\\ a_i \neq a_j}} B'_{ij}.$$

  Hence $O \in \mathcal{PO}(u + O)$.

□

Combining Proposition 5 with Theorem 2, we can iteratively deduce the following Corollary:

Corollary 4 *Let u be a profile such that the reflexive closure of the strict majority relation is not a linear order. Let $O \in \mathcal{PO}(u)$. Then:*

$$\{O\} = \mathcal{PO}(u{+}\mu_{min}O) \subseteq \mathcal{PO}(u{+}(\mu_{min}{-}1)O) \subseteq \ldots \subseteq \mathcal{PO}(u{+}O) \subseteq \mathcal{PO}(u)$$

Hence, by progressively adding a prudent order O to a profile, the set of prudent orders progressively refines until the situation where O corresponds to the majority relation.

Let us mention that other, more particular, types of consistency properties are verified by prudent orders. We refer the reader to Weak Majority Profile Consistency, Majority Profile Consistency, Weak Homogeneity and Homogeneity, which can all be seen as particular forms of consistency and which are introduced in Chapter 5.

4.4 Removal of alternatives

Not surprisingly, prudent orders do not verify Arrow's axiom of independence with respect to irrelevant alternatives. This has already been noticed by Kohler [62]. Let us present his counterexample. There are 3 alternatives a, b and c. The profile consists of 42 orders abc, 25 orders bca and 33 orders cab. The following majority margins are obtained:

B	a	b	c
a	.	50	–16
b	–50	.	34
c	16	–34	.

The unique prudent order is abc. If one removes alternative b, then the unique prudent order is ca.

Nevertheless, some kind of independence with respect to *irrelevant* alternatives is verified for prudent orders. In fact, it all depends on defining what is an "irrelevant" pair in a prudent order context. To do so, we suppose that we restrict our analysis to a particular subset of alternatives $\mathcal{A}' \subseteq \mathcal{A}$.

We denote $(R)_{\mathcal{A}'}$ the binary relation R restricted to $\mathcal{A}' \subseteq \mathcal{A}$:

$$(R)_{\mathcal{A}'} = \{(x, y) \in R : x \in \mathcal{A}' \text{ and } y \in \mathcal{A}'\}.$$

We denote $u_{\mathcal{A}'}$ the profile $u = (O_1, \dots, O_q)$ restricted to the alternatives of $\mathcal{A}'$:

$$u_{\mathcal{A}'} = ((O_1)_{\mathcal{A}'}, \dots, (O_q)_{\mathcal{A}'}).$$

Let $\mathcal{PO}_{\mathcal{A}'}(u)$ be the set of prudent orders restricted to the alternatives of $\mathcal{A}'$.

$$\mathcal{PO}_{\mathcal{A}'}(u) = \{(O)_{\mathcal{A}'} : O \in \mathcal{PO}(u)\}.$$

Let $P = t(R_{>\beta})$ be the transitive closure of the relation $R_{>\beta}$ corresponding to a profile u.

Definition 9 *Let $\mathcal{A}' \subseteq \mathcal{A}$ be a subset of alternatives. We say that $\mathcal{A}'$ is P-compatible if:*

$$\forall a_i, a_j \in \mathcal{A}',\ \forall a_k \in \mathcal{A}, \quad (a_i, a_k) \in P \text{ and } (a_k, a_j) \in P \Rightarrow a_k \in \mathcal{A}'.$$

We then have the following result when restricting the computations to a subset of alternatives which are P-compatible:

Proposition 6 *Let $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathcal{A}'$ is P-compatible. Then:*

$$\mathcal{PO}(u_{\mathcal{A}'}) \subseteq \mathcal{PO}_{\mathcal{A}'}(u).$$

Proof: Let us recall that:

- β is the optimal cut value of profile u.
- $P = t(R_{>\beta})$.

We will further use the following notation:

- $R'_{>\lambda}$ is the strict λ-cut relation of the majority margins of profile $u_{\mathcal{A}'}$.
- β' is the optimal cut value of profile $u_{\mathcal{A}'}$.
- $P' = t(R'_{>\beta'})$.

We have:

$$(R_{>\beta})_{\mathcal{A}'} = R'_{>\beta}. \tag{4.1}$$

Since $R_{>\beta}$ is an acyclic relation, $(R_{>\beta})_{\mathcal{A}'}$ is an acyclic relation and so is $R'_{>\beta}$. Hence $\beta' \leq \beta$, which implies that:

$$R'_{>\beta} \subseteq R'_{>\beta'}. \tag{4.2}$$

Combining (4.1) and (4.2), we get:

$$t((R_{>\beta})_{\mathcal{A}'}) \subseteq t(R'_{>\beta'}) = P'. \tag{4.3}$$

Let $O \in \mathcal{PO}(u_{\mathcal{A}'})$. We want to show that $O \in \mathcal{PO}_{\mathcal{A}'}(u)$. Since $O \in \mathcal{PO}(u_{\mathcal{A}'})$, we have by definition that $P' \subseteq O$. Combining this with (4.3), we get that:

$$t((R_{>\beta})_{\mathcal{A}'}) \subseteq O. \tag{4.4}$$

We are now going to show the following equality:

$$t((R_{>\beta})_{\mathcal{A}'}) = (t(R_{>\beta}))_{\mathcal{A}'}. \tag{4.5}$$

$$
\begin{array}{cl}
& (x,y) \in t((R_{>\beta})_{\mathcal{A}'}) \\
\iff & \exists x, z_1, \ldots, z_p, y \in \mathcal{A}' : (x, z_1) \in (R_{>\beta})_{\mathcal{A}'} \ldots (z_p, y) \in (R_{>\beta})_{\mathcal{A}'} \\
\iff & \exists x, z_1, \ldots, z_p, y \in \mathcal{A}' : (x, z_1) \in R_{>\beta} \ldots (z_p, y) \in R_{>\beta} \\
\iff & \exists x, z_1, \ldots, z_p, y \in \mathcal{A}, x, y \in \mathcal{A}' : (x, z_1) \in R_{>\beta} \ldots (z_p, y) \in R_{>\beta} \\
\iff & (x,y) \in t(R_{>\beta}), x, y \in \mathcal{A}' \\
\iff & (x,y) \in (t(R_{>\beta}))_{\mathcal{A}'}
\end{array}
$$

Let us consider the third equivalence. $\Rightarrow$ is obvious. $\Leftarrow$ comes from the following observation: Since $(x, z_1) \in R_{>\beta} \ldots (z_p, y) \in R_{>\beta}$, it follows that $(x, z_1) \in P$ and $(z_1, y) \in P$. Since $x, y \in \mathcal{A}'$ and $\mathcal{A}'$ is P-compatible, $z_1 \in \mathcal{A}'$. In a similar way, we get that, $\forall i \in \{1, \ldots, p\}$, $z_i \in \mathcal{A}'$.

Combining (4.4) and (4.5), we have:

$$(t(R_{>\beta}))_{\mathcal{A}'} = (P)_{\mathcal{A}'} \subseteq O.$$

Let us consider the following two sets:

$$\mathcal{A}^+ = \{a_i \notin \mathcal{A}' : \forall a_j \in \mathcal{A}', (a_i, a_j) \in P \text{ or } ((a_i, a_j) \notin P \text{ and } (a_j, a_i) \notin P)\}.$$

$$\mathcal{A}^- = \{a_i \notin \mathcal{A}' : \forall a_j \in \mathcal{A}', (a_j, a_i) \in P \text{ or } ((a_i, a_j) \notin P \text{ and } (a_j, a_i) \notin P)\}.$$

Let us note that, since $\mathcal{A}'$ is P-compatible, we must have that

$$\mathcal{A}^+ \cup \mathcal{A}^- = \mathcal{A} \setminus \mathcal{A}'.$$

A linear extension of a partial order can be constructed sequentially as follows: remove any maximal element from the partial order and rank it below the already ranked alternatives in the linear extension; stop the procedure when all the alternatives have been ranked. In our case, let us apply this scheme to P by considering first the alternatives of $\mathcal{A}^+$, then the alternatives of $\mathcal{A}'$ and then the alternatives of $\mathcal{A}^- \setminus \mathcal{A}^+$. Let $\tilde{O}$ be a linear extension obtained that way. Since by construction we have that $(\tilde{O})_{\mathcal{A}'} = O$ and that $P \subseteq \tilde{O}$, we can conclude that $O \in \mathcal{PO}_{\mathcal{A}'}(u)$, which terminates the proof.

□

Hence, by restricting to a specific set of alternatives, some kind of consistency is verified. Although prudent orders may disappear, since the removal of some particular alternatives can resolve some problems, no new prudent orders can appear. We will come back to this proposition in Section 9.6, when we will analyze the convergence of a process which interactively applies the prudent order preference function to support a group of decision

$\mathcal{PO}(u)$	$\mathcal{PO}_{\mathcal{A}'}(u)$	$\mathcal{PO}(u_{\mathcal{A}'})$
dcba	*dca*	
dcab	*dca*	
dbca	*dca*	
dacb	*dac*	
dbac	*dac*	
dabc	*dac*	
cdba	*cda*	X
cdab	*cda*	X
bdca	*dca*	
bdac	*dac*	
cbda	*cda*	X
bcda	*cda*	X

Table 4.2: Restricting the set of alternatives to $\mathcal{A}' = \{a, c, d\}$.

makers in determining a compromise ranking.

In order to illustrate Proposition 6, let us come back to the profile $u = (abcd, bcda, cdab, dabc, dcba)$ introduced in Section 4.1. We know already that $R_{>\beta} = \{(d, a)\} = t(R_{>\beta}) = P$ and that there are 12 prudent orders, which are listed in the first column of table 4.2. We now assume that $\mathcal{A}' = \{a, c, d\}$. It is clear that $\mathcal{A}'$ is P-compatible. In the second column of table 4.2, the prudent orders of profile u restricted to $\mathcal{A}'$ are listed. In the third column of this table, a 'X' indicates that the linear order also belongs to $\mathcal{PO}(u_{\mathcal{A}'})$. In fact, $\mathcal{PO}_{\mathcal{A}'}(u) = \{dca, cda, dac\}$ and $\mathcal{PO}(u_{\mathcal{A}'}) = \{cda\}$, which illustrates the fact that $\mathcal{PO}(u_{\mathcal{A}'}) \subseteq \mathcal{PO}_{\mathcal{A}'}(u)$.

4.5 Prudent choice

In this PhD thesis, we are studying preference functions that combine a profile of linear orders into one or several linear orders. However, in some situations, the aim is not to rank the alternatives from the best to the worst, but to chose the "best" alternatives available, given the preferences contained in the profile. This can for instance happen in an election where a president has to be elected but we are not necessarily interested in ranking all the candidates. That is why a lot of effort has been spent on social *choice* rules, which are rules that select one or several good alternatives based on the linear orders of the profile.

The problem of ranking and choosing are related. On the one hand, a choosing procedure can trivially be derived from a ranking procedure by selecting as winners the alternatives ranked first. On the other hand, some authors ([2, 39, 13]) have been studying procedures which compute a ranking by reapplying iteratively a choosing procedure on the not yet ranked alternatives. These are called "ranking by choosing" procedures (for a prudent ranking by choosing procedure see for instance Kohler's rule presented in Section 2.2.1 and for a non-prudent ranking by choosing procedure see for instance the ranking rule MAH proposed by Beck [7]).

Moulin however pleads [80] to clearly distinguish the question of ranking alternatives from the one of selecting winners. Similarly, in the field of multicriteria decision aid, Bernard Roy [93] explicitly distinguishes between the problem of choosing (the so-called problematic α) and the problem of ranking (the so-called problematic γ).

Although in our research we have concentrated only on prudent ranking rules, this section should point out some ways of defining prudent choice rules. A very natural way of defining a prudent choice rule is to select all the alternatives which are ranked first in at least one prudent order. More formally, we are going to define the prudent choice function, denoted by $\mathcal{PCF}$, as follows:

$$\mathcal{PCF}(u) = \{a_i \in \mathcal{A} : \exists O \in \mathcal{PO}(u) : \rho_O(a_i) = 1\}.$$

To our knowledge, this social choice function has not been seriously mentioned in the literature. Hence, it would be surely relevant to analyze the properties of such a choice rule. We have not done this yet. Let us nevertheless point out a possible inadequacy of $\mathcal{PCF}$ in a choice context.

To formulate our observation, we will need the notion of a top-cycle. The top-cycle $\mathcal{TC}(u)$ of a profile u is the smallest possible subset (in the sense of the inclusion) of $\mathcal{A}$ such that for all $a_i \in TC(u)$ and for all $a_j \in \mathcal{A} \setminus TC(u)$ we have $(a_i, a_j) \in M$, where M denotes the strict majority relation of profile u.

The top-cycle is often considered as the weakest of all Condorcet social choice rules (see for instance Laslier [64]). This set can be very large and consequently every other Condorcet social choice rule should be smaller than the top-cycle. Unfortunately, it may happen that $\mathcal{TC}(u) \subset \mathcal{PCF}(u)$. Consider the following majority margins:

	a	b	c	d
a	0	4	−4	2
b	−4	0	4	2
c	4	−4	0	2
d	−2	−2	−2	0

For a profile u with such majority margins, it is easy to see that

$$\mathcal{TC}(u) = \{a, b, c\},$$

whereas every linear order is a prudent order and so

$$\mathcal{PCF}(u) = \{a, b, c, d\}.$$

Hence, $\mathcal{TC}(u) \subset \mathcal{PCF}(u)$. Let us note that there also exist profiles where the inclusion is the other way around. After slightly changing the majority margins, we obtain:

	a	b	c	d
a	0	4	–2	2
b	–4	0	4	2
c	2	–4	0	4
d	–2	–2	–4	0

In this case, we still have that $\mathcal{TC}(u) = \{a, b, c\}$, but now $abcd$ will be the unique prudent order and so $\mathcal{PCF}(u) = \{a\}$. Hence, we have that $\mathcal{PCF}(u) \subset \mathcal{TC}(u)$.

We stop here our discussion on $\mathcal{PCF}$ and refer the reader to Section 5.7, where we will introduce a new preference function $\mathcal{XPO}$ which will reconcile the top-cycle principle with the set of prudent orders.

Apart from $\mathcal{PCF}$, other "prudent" choice functions could be defined. To do so, we suggest to distinguish between two approaches:

- In a first approach, we consider any prudent ranking rule. The choice set then consists of all the alternatives that are ranked first in at least one of the compromise rankings. $\mathcal{PCF}$ is an illustration of this strategy. We have also seen that Kohler's rule is a prudent ranking rule. The associated choice rule then consists in selecting those alternatives which are ranked first in one of the orders found by this ranking rule. Pérez analyzed this choice rule theoretically in [84] and empirically in [85]. In the same manner, the Ranked Pairs rule, which we will further discuss in Chapter 6, has initially been introduced by Tideman [104] as a choice rule, and not as a ranking rule.

- Another approach consists in redefining the prudence principle in a choice context without referring to rankings. For instance, an approach based on so called α-elites has been proposed in Debord's thesis [30]. More generally, the concept of a prudent k choice is discussed in [32], where a rule is defined which takes as output all the prudent choice subsets of a fixed size k.

On the one hand, the literature on choice rules is very rich. On the other hand, the field of prudent choice rules seems rather unexploited. That is why some further work should be spent on defining and analyzing prudent choice rules, especially in view of applying them in a decision aid context.

Chapter 5

A characterization of the prudent order preference function

In this chapter, we propose a characterization of the prudent order preference function. This work has been presented at the LAMSADE-DIMACS Workshop on Voting Theory and Preference Modeling in Paris during October 2006 and has been published in [65] and in [68]. I would like to thank especially Thierry Marchant and Marc Pirlot for their help and their comments concerning this part of the thesis.

The chapter is organized as follows. First, in Section 5.1, we give an introduction to axiomatic characterizations. In Section 5.2, we analyze which sets of linear orders can be considered as a set of prudent orders. We introduce in Section 5.3 the axioms used in our characterization results and discuss these axioms in Section 5.4. In Section 5.5, we present our main results, whereas in Section 5.6 we check the independence of the axioms. Finally, in Section 5.7, we refine the set of prudent orders by taking into account an additional Condorcet-like criterion.

5.1 Introduction to axiomatic characterizations

An axiomatic characterization of a preference function consists in establishing a list of independent axioms and proving that this particular preference function is the only preference function that verifies all the axioms of that list simultaneously. Such an approach has been followed by many decision theorists. For instance, we can mention Young's characterization of Borda's method [108] and his seminal characterization of Kemeny orders [109]. The Copeland choice function has for instance been characterized by Henriet [49].

Saari [95] (Section 8.2) has pointed out that in most of these results it is more appropriate to use the term "property" instead of "axiom". In fact, he argues that "axiom" refers to something basic. When two axioms are combined to form a new axiom, then we do not consider the latter to be an axiom anymore, since it can be split into two more atomic axioms. Moreover, he criticizes that most "axiomatic" characterizations consist usually of an artificial collection of properties that are only brought up together in order to prove the desired characterization result. He claims ambitiously that "...for a price, give me any decision/election procedure you wish. Tell me whether you want it promoted or attacked. I will design an appropriate *axiomatic* characterization that will do the job."[1]

Although there is surely some truth in Saari's arguments, we feel that a characterization with axioms (or with properties) is a very powerful and enlightening approach to understand the intrinsic features of a preference function, even if this can be due to a less intuitive axiom (or property). Since the question of characterizing the prudent order preference function has not been addressed yet, this will be the topic of this chapter. This will help to highlight the main properties implied by the prudence principle. More generally, we are going to build an axiomatic framework which can be used to characterize other prudent ranking rules, namely the extended prudent order preference function (see Section 5.7) and the Ranked Pairs rule (see Chapter 6).

Let us emphasize that, in our setting, the type of solution which we will characterize is neither a ranking, nor a choice subset, but a *set* of rankings. This has also been the case in Young's [109] axiomatization of the set of Kemeny orders. A major difference however with the Kemeny model is that the prudent order model is B-ordinal (see Section 2.3). In the literature, we can find characterizations of ordinal ranking models by Barberà [3], Pirlot [87] and Fortemps and Pirlot [45], although these authors were working in very different contexts.

Throughout the next two chapters, we adopt the following notation. For all $a_i, a_j \in \mathcal{A}, a_i \neq a_j$, we denote by $a_i a_j x_{-ij}$ a linear order in which a_i is followed by a_j and then by the alternatives x_{-ij} with x_{-ij} being an arbitrary permutation of the alternatives $\mathcal{A} \setminus \{a_i, a_j\}$. Furthermore, we denote by $-x_{-ij}$ the reverse permutation of x_{-ij}.

Before presenting the characterization of the prudent order preference function, we first analyze in the next section the structure of the set of pru-

[1] D.G. Saari (2001), *Decisions and Elections*, Cambridge University Press.

dent orders.

5.2 The set of the sets of prudent orders

In this section, we answer the question whether a given set of linear orders can be (or cannot be) considered as a set of prudent orders corresponding to a profile of linear orders.

We know that the set of prudent orders is by definition the set of all the linear extensions of the relation $R_{>\beta}$. The reader may wonder if $R_{>\beta}$ can be any possible acyclic relation on the set of alternatives $\mathcal{A}$. The answer will be given by the following proposition.

Proposition 7 *For every acyclic relation H on the set of alternatives $\mathcal{A}$, there exists a profile u of linear orders such that the optimal strict cut relation of that profile $R_{>\beta}$ is equal to H.*

Proof: If $r(H)$ is a linear order, then u trivially consists of this one linear order. If $r(H)$ is not a linear order, then we construct the profile u as follows. For every ordered pair (a_i, a_j) such that $(a_i, a_j) \in H$, we consider the two linear orders V^1_{ij} and V^2_{ij}:

$$V^1_{ij} = a_i a_j x_{-ij} \quad V^2_{ij} = -x_{-ij} a_i a_j.$$

The profile u then consists of all the linear orders V^1_{ij} and V^2_{ij} such that $(a_i, a_j) \in H$:

$$u = \sum_{(a_i,a_j)\in H} V^1_{ij} + V^2_{ij}.$$

This leads to the following preference margin matrix:
$\forall a_i, a_j \in \mathcal{A}$,

$$B_{ij} = \begin{cases} 2 & \text{if } (a_i, a_j) \in H \\ -2 & \text{if } (a_j, a_i) \in H \\ 0 & \text{otherwise} \end{cases}$$

We show that in this case $\beta = 0$ and consequently $R_{>\beta} = H$. On the one hand, $R_{>0}$ is acyclic since H is acyclic. On the other hand, we show that $R_{\geq 0}$ contains a cycle. H, being an acyclic relation such that $r(H)$ is not a linear order, is not complete. Hence there must exist a_i and a_j such that $(a_i, a_j) \notin H$ and $(a_j, a_i) \notin H$. Hence $(a_i, a_j) \in R_{\geq 0}$ and $(a_j, a_i) \in R_{\geq 0}$. This proves that $R_{\geq 0}$ contains a cycle. □

A consequence of this proposition is that, apart from being acyclic, we cannot make any additional assumptions on the relation $R_{>\beta}$. At the same time this proposition allows us to characterize the set of all the sets of prudent orders of a profile with linear orders on $\mathcal{A}$. In fact, this set, denoted by $\Pi\Omega_{\mathcal{A}}$, simply consists of all the sets of linear extensions of all the acyclic binary relations on $\mathcal{A}$:

$$\Pi\Omega_{\mathcal{A}} = \{\mathcal{E}(R) : \ R \text{ is an acyclic relation on } \mathcal{A}\}.$$

5.3 Axioms

In this section, we introduce the axioms that we will need to characterize the prudent order preference function. We first concentrate on giving the formal description of the axioms. For a more detailed discussion on the interpretation of these axioms, we refer the reader to Section 5.4.

The object which we would like to characterize is a preference function. A preference function f is a procedure that combines a profile of linear orders u into a non-empty set of linear orders $f(u)$.

$$\begin{array}{rlcl} f: & \mathcal{LO}^q & \mapsto & P(\mathcal{LO}) \setminus \emptyset \\ & u & \rightarrow & f(u). \end{array}$$

In general, the strict majority relation M contains cycles, which is commonly referred to as Condorcet's paradox. However, in case M is acyclic, then the first axiom says that this information must be included in the set of solutions.

Axiom 1 *Weak Condorcet Extension (WCE):*
If M is acyclic, then:

$$f(u) \subseteq \mathcal{E}(M).$$

In other words, this means that, if M is acyclic and if $(a_i, a_j) \in M$, then a_i must be preferred to a_j in all the linear orders of $f(u)$. A stronger version of axiom WCE says that, if M is acyclic, then $f(u)$ corresponds exactly to all the linear extensions of this relation M.

Axiom 2 *Condorcet Extension (CE):*
If M is acyclic, then:

$$f(u) = \mathcal{E}(M).$$

It is easy to see that CE implies WCE.

Let us recall that (see Section 4.3) if $u = (O_1, O_2, \ldots, O_q)$ is a first profile and $u' = (O'_1, O'_2, \ldots, O'_{q'})$ is a second profile, then we denote $u + u'$ the profile $(O_1, O_2, \ldots, O_q, O'_1, O'_2, \ldots, O'_{q'})$.

Let u_E be a profile such that $\forall a_i, a_j \in \mathcal{A}$, we have that $B_{ij} = 0$. Adding such a balanced profile to a given profile can enlarge the set of solutions.

Axiom 3 *Weak E-Invariance (WEI)*

$$f(u) \subseteq f(u + u_E).$$

A stronger version of this axiom says that adding a balanced profile u_E to a given profile does not alter the set of solutions.

Axiom 4 *E-Invariance (EI)*

$$f(u) = f(u + u_E).$$

EI implies WEI.

The next axiom says that if the size of the profile is odd and we create a new profile by taking twice the initial profile, then the set of solutions may only increase.

Axiom 5 *Weak Homogeneity for Odd Profiles (WHOP):*
If q is odd, then:

$$f(u) \subseteq f(u + u).$$

A stronger version of this axiom simply says that if we double an odd profile, then the result does not change at all.

Axiom 6 *Homogeneity for Odd Profiles (HOP):*
If q is odd, then:

$$f(u) = f(u + u).$$

HOP implies WHOP.

Before presenting our main axiom, we have to introduce the definition of a majority profile. Let u be a profile where M is the corresponding strict majority relation. We analyze now each pair $\{a_i, a_j\}$ once:

- If $(a_i, a_j) \in M$, construct the two linear orders

$$V_{ij}^1 = a_i a_j x_{-ij} \quad V_{ij}^2 = -x_{-ij} a_i a_j.$$

- If $(a_i, a_j) \notin M$ and $(a_j, a_i) \notin M$, consider one of the following two exclusive possibilities:

– Skip this pair.

– Construct two linear orders V^1_{ij} and V^2_{ij} such that:

$$V^1_{ij} = a_i a_j x_{-ij} \quad V^2_{ij} = -x_{-ij} a_i a_j.$$

Definition 10 *Let M be a strict majority relation of profile u. We say that $u(M)$ is a majority profile of profile u if $u(M)$ can be written as follows:*

$$u(M) = \sum_{(a_i,a_j)\in M} (V^1_{ij} + V^2_{ij}) + \sum_{(a_i,a_j)\in \zeta} (V^1_{ij} + V^2_{ij}),$$

where $\zeta \subseteq \{(a_i, a_j) : (a_i, a_j) \notin M$ and $(a_j, a_i) \notin M\}$.

It is clear that for a given profile u, different majority profiles $u(M)$ can be constructed. First of all, the sequence x_{-ij} can be arbitrarily chosen when we construct a linear order $a_i a_j x_{-ij}$. Furthermore, for pairs $\{a_i, a_j\}$ such that $B_{ij} = 0$, breaking the indifference between a_i and a_j in a certain direction or leaving the indifference untouched could lead to different majority profiles.

The next axiom says that if we add to a profile u a majority profile $u(M)$, and the new profile $u + u(M)$ contains cycles (either existing cycles of profile u or new cycles created by adding $u(M)$), then the set of compromise solutions either stays the same or shrinks.

Axiom 7 *Weak Majority Profile Consistency (WMPC):*
Let u be a profile and let $u(M)$ be a majority profile of u. If the strict majority relation of $u + u(M)$ contains at least one cycle, then:

$$f(u + u(M)) \subseteq f(u).$$

Different majority profiles can pull the set of compromise solutions $f(u+u(M))$ in possibly different directions. Whatever choice will be made, the new set $f(u + u(M))$ must always be contained in the set $f(u)$.

We will also use a stronger version of the WMPC axiom, namely Majority Profile Consistency, which says that adding a majority profile does not alter the result at all.

Axiom 8 *Majority Profile Consistency (MPC):*
Let u be a profile and let $u(M)$ be a majority profile of u. If the strict majority relation of $u + u(M)$ contains at least one cycle, then:

$$f(u + u(M)) = f(u).$$

MPC means that if we add a majority profile $u(M)$ to any given profile u, and the new profile $u+u(M)$ contains cycles, then the set of solution rankings must stay the same. Removing the cyclicity condition of profile $u + u(M)$ from this axiom leads to a contradiction with axiom CE. In fact, if the strict majority relation of profile $u + u(M)$, denoted by M', is acyclic, then the strict majority relation of profile u must also be acyclic, since one can show that $M \subseteq M'$. According to CE, $f(u) = \mathcal{E}(M)$ and $f(u + u(M)) = \mathcal{E}(M')$. If we suppose that $M \subset M'$, then it can happen that $f(u + u(M)) \subset f(u)$.

Axiom MPC implies axiom WMPC, since the inclusion is simply replaced by an equality.

5.4 Discussion

Weak Condorcet Extension and **Condorcet Extension** are axioms which indicate that, whenever possible, the solution should comply with the Condorcet principle. In our framework, a profile can reasonably be considered as easy if the strict majority relation does not contain any cycles. In such a case, if there is a path from alternative a_i to alternative a_j in that acyclic strict majority relation, then a_i should be preferred to a_j in all the solution rankings. In fact, from a pairwise comparison perspective, it would be difficult to justify or to explain a solution where a_j is preferred to a_i.

Any preference function satisfying these two axioms must also satisfy what we called in Section 2.3 Condorcet Ranking Consistency. If the reflexive closure of the majority relation is a linear order, then this linear order is the unique solution obtained by the preference function.

WCE and CE are generally verified by those procedures which are based on pairwise comparisons and which produce as a solution a set of linear orders. For instance, Slater's rule or Kemeny's rule both verify this type of axioms (see Section 2.2.2 for a definition of these rules). In the literature other formulations of axioms can be found which translate the idea of complying, whenever possible, with the Condorcet principle (see for instance Truchon [105] or Young and Levenglick [109]).

E-Invariance, which has been used by Debord [31], and **Weak E-Invariance** both deal with adding a completely balanced profile to any given profile. It seems rather natural to assume that adding a balanced profile does not change the result, or, at least, keeps the solution rankings of the initial profile. Interestingly, Debord [31, 30] showed that a preference function only depends on the pairwise majority margins if and only if it is anonymous and E-invariant. Consequently preference functions which do not depend on the majority margins but are anonymous, such as for instance the model presented by Cook and Seiford [25], do not verify E-Invariance.

Weak Homogeneity for Odd Profiles and **Homogeneity for Odd Profiles** require that doubling a profile of odd size does not change the result, or, at least, keeps the solution rankings of the initial profile. These are two axioms which weaken the more usual homogeneity condition that asks for result-invariance when doubling a profile of any size (see for instance Debord [31] or Saari [94]). Restricting to profiles of odd size can be explained by the fact that we are only interested in linking profiles of odd size with similar profiles of even size.

Weak Majority Profile Consistency and **Majority Profile Consistency** deal with adding a so-called majority profile to an existing profile. Let us take again a closer look at such a majority profile. The profile $u(M)$ translates in fact the information contained in the strict majority relation M of profile u:

- If $(a_i, a_j) \in M$, then there is a strict majority of rankings in the initial profile u that prefer a_i over a_j. Adding the two linear orders $a_i a_j x_{-ij}$ and $-x_{-ij} a_i a_j$ only confirms this fact. The majority margins resulting from these two linear orders will be +2 for (a_i, a_j) and –2 for (a_j, a_i), whereas the majority margins for all the remaining ordered pairs will be 0.

- If $(a_j, a_i) \notin M$ and $(a_i, a_j) \notin M$, then there are as many rankings in the profile u that prefer a_i over a_j than there are rankings that prefer a_j over a_i. For such a pair, one of the two mutually exclusive possibilities has to be considered:
  - We skip this pair since we do not want to discriminate between a_i and a_j.
  - We add the two linear orders $a_i a_j x_{-ij}$ and $-x_{-ij} a_i a_j$, which breaks the indifference by improving the situation of a_i with respect to a_j.

In fact, a majority profile $u(M)$ is, in terms of majority margins, equivalent (up to the pairs with a zero majority margin) to a profile consisting

of twice the strict majority relation. More formally, let B^M denote the majority margins of profile $u(M)$. We then have:
$\forall a_i, a_j \in \mathcal{A}$,

$$B^M_{ij} = \begin{cases} 2 & \text{if } B_{ij} > 0 \\ -2 & \text{if } B_{ij} < 0 \end{cases} \tag{5.1}$$

If $B_{ij} = B_{ji} = 0$, then either $B^M_{ij} = 2$ and $B^M_{ji} = -2$, or $B^M_{ij} = -2$ and $B^M_{ji} = 2$, or $B^M_{ij} = 0$ and $B^M_{ji} = 0$.

A majority profile is not a totally new concept. In fact, Mc Garvey [77] showed that any complete binary relation can be seen as the strict majority relation of a profile of linear orders. In his proof, he precisely used a majority profile. Mc Garvey's result has in turn been generalized by Debord (see Proposition 1 in Section 2.1).

Axiom WMPC and MPC establish a relationship between the result of profile u and the result of profile $u + u(M)$. These axioms suggest that "confirming the majority" of a profile should not lead to creating new solutions. This is in line with the idea that reinforcing the majority does not fundamentally change the aggregation problem.

Whereas under axiom MPC we assume that the result of profile u and of profile $u + u(M)$ is exactly the same, axiom WMPC allows the set of solutions of profile $u+u(M)$ to possibly refine. This can be explained by the fact that a majority profile can break the indifference between two alternatives. A majority margin between a_i and a_j of zero in profile u can become a strict preference of a_i over a_j in profile $u + u(M)$. In such a situation, depending on the preference function, the set of solutions of profile $u + u(M)$ may lose some linear orders which have been obtained for profile u and where a_j is preferred to a_i.

From a technical perspective, performing a translation of the majority margins (+2 for the positive majority margins of profile u and –2 for the negative majority margins of profile u) is linked to what we called in Section 2.3 B-ordinality. In fact, if we ignore all the pairs with a majority margin of 0 ($\zeta = \emptyset$), then the order of the pairs of profile u according to their majority margins and the order of the pairs of profile $u + u(M)$ according to their majority margins is the same.

5.5 Characterization

In the proofs of this section, we will need the following lemma:

Lemma 2 *The strict majority relation of a profile is acyclic if and only if the optimal cut value β for that profile is non-positive.*

Proof: If M is acyclic, then $R_{>0} = M$ is acyclic. Consequently $\beta \leq 0$. Reciprocally, let $\beta \leq 0$. Then $M = R_{>0} \subseteq R_{>\beta}$. Since $R_{>\beta}$ is acyclic, so must be M. □

First, we are going to show that the prudent order preference function verifies the axioms introduced so far.

Proposition 8 *The prudent order preference function verifies Condorcet Extension, E-Invariance, Homogeneity for Odd Profiles and Majority Profile Consistency.*

Proof: It is easy to see that the prudent order preference function verifies EI and HOP. CE has been proved in Proposition 4 in Section 4.2.

We finally show that the prudent order preference function also verifies MPC. Let us suppose that the strict majority relation of profile $u + u(M)$ contains at least one cycle. We denote by $R_{>\lambda}$ and $R_{\geq\lambda}$ the cut relations based on the majority margins of profile u and by $R'_{>\lambda}$ and $R'_{\geq\lambda}$ the cut relations based on the majority margins of profile $u + u(M)$. Let β be the optimal cut value of profile u and β' be the optimal cut value of profile $u + u(M)$.

If $\beta < 0$, then this means that the strict majority relation of profile u is a linear order (see Proposition 2 in Section 4.1). Consequently, the strict majority relation of profile $u+u(M)$ is also a linear order. We are not interested in this case since we suppose that the strict majority relation of profile $u+u(M)$ contains at least one cycle. Let us from now on suppose that $\beta \geq 0$.

The majority margins of profile u, denoted by B, and the majority margins of profile $u + u(M)$, denoted by B', are linked as follows:

$$B'_{ij} = \begin{cases} B_{ij} + 2 & \text{if } B_{ij} > 0 \text{ or } (a_i, a_j) \in \zeta \\ B_{ij} - 2 & \text{if } B_{ij} < 0 \text{ or } (a_j, a_i) \in \zeta \\ B_{ij} & \text{otherwise} \end{cases} \tag{5.2}$$

That is why we have that:

$$R'_{>\beta+2} \quad = \quad R_{>\beta}. \tag{5.3}$$

Since $R_{>\beta}$ is acyclic, this implies that

$$\beta' \leq \beta + 2. \tag{5.4}$$

We distinguish between two cases: either $\beta = 0$ or $\beta > 0$.

- $\beta = 0$
  We know that $\beta' > 0$, since we suppose that the strict majority relation of profile $u + u(M)$ contains at least one cycle (see Lemma 2). We also know that $\beta' \leq \beta + 2 = 2$ (see inequality 5.4). Since $\beta = 0$, the profile u must be even. Consequently, the profile $u + u(M)$ must also be even and the majority margins and the optimal cut values of $u+u(M)$ take only even values. Hence, $\beta' = 2$. Following equation 5.3, we have that $R'_{>\beta'} = R_{>\beta}$, which means that $\mathcal{PO}(u) = \mathcal{PO}(u+u(M))$.

- $\beta > 0$
  In that case, we have that $R'_{\geq\beta+2} = R_{\geq\beta}$. Since $R_{\geq\beta}$ contains at least one cycle, then this means that $R'_{\geq\beta+2}$ also contains at least one cycle and consequently $\beta' \geq \beta + 2$. We also know that $\beta' \leq \beta + 2$ (see inequality 5.4). Consequently, $\beta' = \beta+2$. Following equation 5.3, we have that $R'_{>\beta'} = R_{>\beta}$, which means that $\mathcal{PO}(u) = \mathcal{PO}(u+u(M))$.

We thus showed that, if the the strict majority relation of profile $u + u(M)$ contains at least one cycle, then $\mathcal{PO}(u) = \mathcal{PO}(u + u(M))$. This proves MPC.

□

Corollary 5 *The prudent order preference function verifies Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles and Weak Majority Profile Consistency.*

Let us now present our first result. In fact, we show that if i) the preference function should satisfy Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles and Weak Majority Profile Consistency and ii) the set of compromise solutions should be as large as possible, then we must use the prudent order preference function. In a way, since it is pointless to consider all the linear orders, the axioms restrict the set of possible solutions to all the prudent orders.

Theorem 4 *The prudent order preference function is the largest preference function (with respect to inclusion) that verifies Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles and Weak Majority Profile Consistency.*

Proof: We are going to show that any preference function f that verifies the above mentioned axioms is such such that

$$f(u) \subseteq \mathcal{PO}(u).$$

Since, by Corollary 5, the prudent order preference function verifies these axioms, the proof will be complete.

Let us suppose that the size of profile u is even. Let B be the majority margin matrix of this profile and let β be the optimal cut value of this profile. Hence, $\mathcal{PO}(u) = \mathcal{E}(R_{>\beta})$. If $\beta \leq 0$, then this implies (see Lemma 2) that the strict majority relation is acyclic and consequently axiom WCE tells us that $f(u) \subseteq \mathcal{E}(M) = \mathcal{PO}(u)$. Let us from now on suppose that the strict majority relation of profile u contains at least one cycle and consequently $\beta > 0$.

Consider the following $p = \frac{\beta}{2}$ binary relations:

$$\Lambda_1 = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A} : B_{ij} = 2\}.$$

$$\Lambda_2 = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A} : B_{ij} = 4\}.$$

$$\Lambda_3 = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A} : B_{ij} = 6\}.$$

$$\vdots$$

$$\Lambda_p = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A} : B_{ij} = \beta\}.$$

For every $(a_i, a_j) \in \Lambda_s$ ($s \in \{1, \ldots, p\}$) and for every $(a_i, a_j) \in R_{>\beta}$, we are going to consider the following two linear orders:

$$V^1_{ji} = a_j a_i x_{-ji} \quad V^2_{ji} = -x_{-ji} a_j a_i.$$

We are going to define a new profile u^0 as follows:

$$u^0 = u + \sum_{(a_i,a_j)\in\Lambda_1} (V^1_{ji} + V^2_{ji}) + 2 \sum_{(a_i,a_j)\in\Lambda_2} (V^1_{ji} + V^2_{ji}) + \ldots$$

$$+p \sum_{(a_i,a_j)\in\Lambda_p} (V^1_{ji} + V^2_{ji}) + p \sum_{(a_i,a_j)\in R_{>\beta}} (V^1_{ji} + V^2_{ji}).$$

Let us denote B^0 the majority margin matrix of profile u^0. In fact, B^0 is linked to B in the following way:
$\forall a_i, a_j \in \mathcal{A}$,

$$B^0_{ij} = \begin{cases} B_{ij} - \beta & \text{if } B_{ij} > \beta \\ B_{ij} + \beta & \text{if } B_{ij} < -\beta \\ 0 & \text{otherwise} \end{cases}$$

Let M^0 be the strict majority relation of profile u^0. In fact, M^0 is equal to $R_{>\beta}$, and so M^0 is acyclic. By applying axiom WCE, we can tell that:

$$f(u^0) \subseteq \mathcal{E}(M^0) = \mathcal{E}(R_{>\beta}) = \mathcal{PO}(u).$$

Given profile u^0, construct a majority profile $u^0(M^0)$ in the following way.

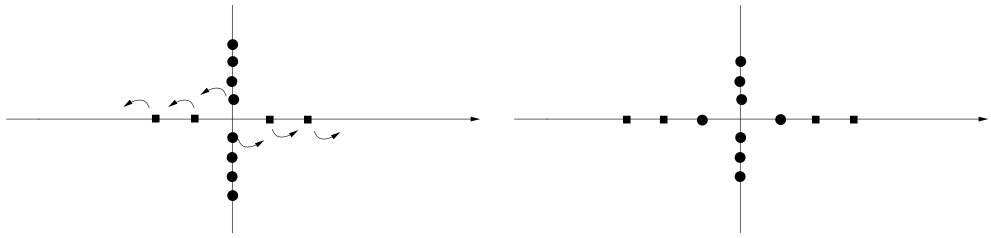


Figure 5.1: The transformation from B^0 into B^1.

- If $(a_i, a_j) \in M^0$ (and consequently $(a_i, a_j) \in R_{>\beta}$), then consider the following two linear orders:

$$V_{ij}^1 = a_i a_j x_{-ji} \quad V_{ij}^2 = -x_{-ji} a_i a_j.$$

If $(a_i, a_j) \notin M^0$ and $(a_j, a_i) \notin M^0$ and $(a_i, a_j) \in \Lambda_p$, then consider the following two linear orders:

$$V_{ij}^1 = a_i a_j x_{-ji} \quad V_{ij}^2 = -x_{-ji} a_i a_j.$$

We thus have:

$$u^0(M^0) = \sum_{(a_i,a_j)\in M^0} (V_{ij}^1 + V_{ij}^2) + \sum_{(a_i,a_j)\in \Lambda_p} (V_{ij}^1 + V_{ij}^2).$$

We define $u^1 = u^0 + u^0(M)$. Let B^1 be the corresponding preference margins of profile u^1. B^1 can be obtained from the majority margins B^0 of the profile u^0 by shifting to the right the ordered pairs with positive majority margins ($B_{ij}^0 > 0$), and, consequently, to the left the ordered pairs with negative majority margins ($B_{ij}^0 < 0$). Furthermore, the pairs such that $B_{ij}^0 = 0$ and $(a_i, a_j) \in \Lambda_p$ are shifted to the right whereas the pairs such that $B_{ij}^0 = 0$ and $(a_j, a_i) \in \Lambda_p$ are shifted to the left. The remaining pairs such that $B_{ij}^0 = 0$ and $(a_i, a_j) \notin \Lambda_p$ and $(a_j, a_i) \notin \Lambda_p$ simply do not move. The transformation from B^0 into B^1 is schematically illustrated in Figure 5.1.

Furthermore, the strict majority relation of profile u^1, denoted by M^1, must contain at least one cycle since $M^1 = R_{>\beta} \cup \Lambda_p = R_{\geq\beta}$ and by definition of the cut value, $R_{\geq\beta}$ is not acyclic. By applying axiom WMPC, we know that:

$$f(u^1) = f(u^0 + u^0(M)) \subseteq f(u^0) \subseteq \mathcal{PO}(u).$$

Construct a majority profile $u^1(M^1)$ in the following way.

- If $(a_i, a_j) \in M^1$ (and consequently $(a_i, a_j) \in R_{>\beta} \cup \Lambda_p$), then consider the following two linear orders:

$$V_{ij}^1 = a_i a_j x_{-ji} \quad V_{ij}^2 = -x_{-ji} a_i a_j.$$

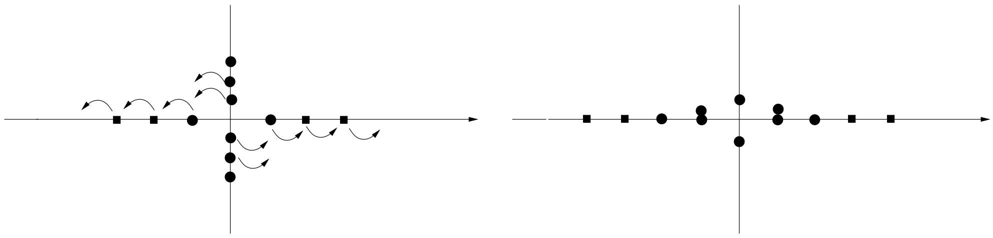


Figure 5.2: The transformation from B^1 into B^2.

- If $(a_i, a_j) \notin M^1$ and $(a_j, a_i) \notin M^1$ and $(a_i, a_j) \in \Lambda_{p-1}$, then consider the following two linear orders:

$$V_{ij}^1 = a_i a_j x_{-ji} \quad V_{ij}^2 = -x_{-ji} a_i a_j.$$

We thus have:

$$u^1(M^1) = \sum_{(a_i,a_j)\in M^1} (V_{ij}^1 + V_{ij}^2) + \sum_{(a_i,a_j)\in \Lambda_{p-1}} (V_{ij}^1 + V_{ij}^2).$$

We define $u^2 = u^1 + u^1(M^1)$. Let B^2 be the preference margin matrix corresponding to profile u^2. The transformation from B^1 into B^2 is schematically represented in Figure 5.2.

The strict majority relation of profile u^2 is denoted by M^2. In fact, $M^1 \subseteq M^2$ and since M^1 is not acyclic, M^2 is not acyclic. Consequently, we can reapply axiom WMPC:

$$f(u^2) = f(u^1 + u^1(M)) \subseteq f(u^1) \subseteq f(u^0) \subseteq \mathcal{PO}(u).$$

By reapplying this construction p times, we finally get:

$$f(u^p) \subseteq f(u^{p-1}) \subseteq \ldots \subseteq f(u^2) \subseteq f(u^1) \subseteq f(u^0) \subseteq \mathcal{PO}(u).$$

In fact, the linear orders of the profile u^p can be reordered as follows:

$$u^p = u + \sum_{(a_i,a_j)\in\Lambda_1} (V_{ij}^1 + V_{ji}^2 + V_{ji}^1 + V_{ij}^2) + 2 \sum_{(a_i,a_j)\in\Lambda_2} (V_{ij}^1 + V_{ji}^2 + V_{ji}^1 + V_{ij}^2) + \ldots$$

$$+ p \sum_{(a_i,a_j)\in\Lambda_p} (V_{ij}^1 + V_{ji}^2 + V_{ji}^1 + V_{ij}^2) + p \sum_{(a_i,a_j)\in R_{>\beta}} (V_{ij}^1 + V_{ji}^2 + V_{ji}^1 + V_{ij}^2).$$

Since $\forall (a_i, a_j) \in \cup_{i=1}^{p} \Lambda_i \cup R_{>\beta}$,

$$V_{ij}^1 = a_i a_j x_{-ji} \quad V_{ji}^2 = -x_{-ji} a_j a_i \quad V_{ji}^1 = a_j a_i x_{-ji} \quad V_{ij}^2 = -x_{-ji} a_i a_j,$$

u^p can be rewritten as $u + u_E$, where u_E is a profile where all the majority margins are zero. Using axiom WEI, we thus have that $f(u) \subseteq f(u+u_E) = f(u^p)$. Consequently:

$$f(u) \subseteq \mathcal{PO}(u).$$

This completes the proof for even profiles. Let us now suppose that u has an odd size. We then create an even profile by taking the profile u twice. Applying the previous result to the even profile $u + u$ and using axiom WHOP, we get:

$$f(u) \subseteq f(u + u) \subseteq \mathcal{PO}(u + u) = \mathcal{PO}(u).$$

This completes the proof for odd profiles. □

Using stronger axioms, the following theorem fully characterizes the prudent order preference function.

Theorem 5 *The prudent order preference function is the only preference function that verifies Condorcet Extension, E-Invariance, Homogeneity for Odd Profiles and Majority Profile Consistency.*

Proof: We know from Proposition 8 that the prudent order preference function verifies CE, EI, HOP and MPC.

We suppose that the size of profile u is even. Let B be the majority margins of this profile. The optimal cut value for this profile is denoted by β. Hence, $\mathcal{PO}(u) = \mathcal{E}(R_{>\beta})$. If $\beta \leq 0$, this implies that (see Lemma 2) the strict majority relation is acyclic and consequently axiom CE tells us that $f(u) = \mathcal{E}(M) = \mathcal{E}(R_{>\beta}) = \mathcal{PO}(u)$. Let us from now on suppose that the majority relation contains at least one cycle and consequently $\beta > 0$.

As in the proof of Theorem 4, define a profile u^0 with majority margins B^0 and an acyclic strict majority relation M^0. Applying axiom CE, we have:

$$f(u^0) = \mathcal{E}(M^0) = \mathcal{E}(R_{>\beta}) = \mathcal{PO}(u).$$

As in the proof of Theorem 4, define a profile u^1 by adding a majority profile $u^0(M^0)$ to u^0: $u^1 = u^0 + u^0(M^0)$. Let us denote by M^1 the strict majority relation of profile u^1. In fact, $M^1 = R_{>\beta} \cup \Lambda_p = R_{\geq\beta}$, which must contain at least one cycle. Hence M^1 is not acyclic and we can apply axiom MPC:

$$f(u^1) = f(u^0 + u^0(M^0)) = f(u^0) = \mathcal{PO}(u).$$

As in the proof of Theorem 4, define a profile u^2 by adding a majority profile $u^1(M^1)$ to u^1:

$$f(u^2) = f(u^1 + u^1(M^1)) = f(u^1) = \mathcal{PO}(u).$$

By reapplying the same construction p times (as in the proof of Theorem 4), and by using axiom EI, we finally get:

$$f(u) = f(u + u_E) = f(u^p) = \ldots = f(u^2) = f(u^1) = f(u^0) = \mathcal{PO}(u).$$

This completes the proof for even profiles. In case the profile u is odd, we apply the previous result to the even profile $u + u$ and using axiom HOP we have:

$$f(u) = f(u + u) = \mathcal{PO}(u + u) = \mathcal{PO}(u).$$

This completes the proof for odd profiles. □

In comparison to Theorem 4, we strengthened Weak Condorcet Extension by Condorcet Extension, Weak Homogeneity for Odd Profiles by Homogeneity for Odd Profiles and finally Weak Majority Profile Consistency by Majority Profile Consistency. In fact, all the inclusions have simply been replaced by an equality.

5.6 Independence of the axioms

Let us emphasize the independence of the five axioms used in Theorem 5.

1. *Condorcet Extension*

   The preference function that associates to every profile the whole set of linear orders trivially verifies MPC, EI and HOP but clearly does not verify CE.

2. *E-Invariance*

   We consider the following six linear orders:

O_1	*abcd*	O_3	*cdab*	O_5	*dbca*
O_2	*dabc*	O_4	*bcda*	O_6	*cadb*

   We denote $u^* = (O_1, O_2, O_3, O_4, O_5, O_6)$ the profile that consists of these six linear orders. The following majority margin matrix B^* is associated with this profile u^*:

B^*	a	b	c	d
a	.	2	–2	–2
b	–2	.	2	–2
c	2	–2	.	2
d	2	2	–2	.

We now consider any majority profile $u(M)$ relative to this profile u^*. It is easy to see that the majority margins of any majority profile $u(M)$ must be the same as the majority margins of profile u^*. Furthermore, $u(M)$ must contain 12 linear orders corresponding to the 6 ordered pairs with positive majority margins. We denote by $\mathcal{M}$ the set of all the majority profiles of profile u^*.

We are going to construct a particular class of profiles, denoted by $\mathcal{U}$. If $u \in \mathcal{U}$, then this means that u can be decomposed as follows:

- The first 6 linear orders of profile u corresponds to the first six linear orders of profile u^*
- These 6 linear orders can possibly be followed by a finite sequence of profiles taken out of $\mathcal{M}$.

More formally, we have:

$$\mathcal{U} = \{u : u = u^* \text{ or } \exists u_1, u_2, \ldots, u_t \in \mathcal{M} : u = u^* + \sum_{i=1}^{t} u_i\}.$$

Although each profile of $\mathcal{U}$ is finite, the size of the set $\mathcal{U}$ is infinite.

Below is an example of a profile which belongs to $\mathcal{U}$:

$$u = (O_1, O_2, O_3, O_4, O_5, O_6,$$

$$abcd, dcab, bcda, adbc, cdab, bacd, cabd, dbca, dabc, cbda, dbac, cadb).$$

We are now going to define a new preference function g_2 as follows:

$$g_2(u) = \begin{cases} abcd & \text{if } u \in \mathcal{U} \\ \mathcal{PO}(u) & \text{otherwise} \end{cases}$$

Hence g_2 corresponds to the prudent order preference function except for profiles belonging to $\mathcal{U}$. It is easy to see that g_2 verifies CE. In fact, the strict majority relation of all the profiles belonging to $\mathcal{U}$ contains at least one cycle. Hence, if u is a profile with an acyclic majority relation M, then $u \notin \mathcal{U}$, and, consequently, $g_2(u) = \mathcal{PO}(u) = \mathcal{E}(M)$.

The function g_2 also verifies HOP. If u is an odd profile, then we know that $u \notin \mathcal{U}$ since $\mathcal{U}$ only contains even profiles. This follows from the observation that the majority margins of an odd profile must be all odd (see Section 2.1). We thus have that $g_2(u) = \mathcal{PO}(u)$. Furthermore we will show that $u+u \notin \mathcal{U}$, which will imply that $g_2(u+u) = \mathcal{PO}(u+u)$. If $u+u$ is a profile of $\mathcal{U}$, then this would mean that:

$$u + u = u^* \text{ or } \exists u_i \in \mathcal{M} : \quad u + u = u^* + \sum_{i=1}^{t} u_i$$

It is clearly not possible that $u+u = u^*$, since, for instance, the linear order *abcd* only appears once in u^*. If $u+u = u^* + \sum_{i=1}^{t} u_i$, then the size of profile $u+u$ will be $6+12t$, since the size of profile u^* is 6 and the size of any majority profile u_i is 12. We know that $O_1 = O_{4+6t} = abcd$ and $O_2 = O_{5+6t} = dabc$. Since O_{4+6t} and O_{5+6t} are not linked in the way that $a_i a_j x_{-ij}$ is linked with $-x_{-ij} a_i a_j$, O_{5+6t} must be linked with O_{6+6t}. Since $O_{5+6t} = O_2 = dabc$, we must have that $O_{6+6t} = cbda$. This is a contradiction, since $O_3 = O_{6+6t} = cdab$.

g_2 also verifies MPC. One may check that if $u \in \mathcal{U}$, then the set of majority profiles corresponding to u is equal to the set of majority profiles $\mathcal{M}$ corresponding to profile u^*. Let $u(M)$ be a majority profile of profile u. If $u \in \mathcal{U}$, then $u + u(M) \in \mathcal{U}$, and so $g_2(u) = g_2(u + u(M)) = abcd$.

We will now show that, if $u \notin \mathcal{U}$, then $u + u(M) \notin \mathcal{U}$. In fact, we will show that if $u + u(M) \in \mathcal{U}$, then $u \in \mathcal{U}$.

If $u + u(M) \in \mathcal{U}$, then this implies that:

$$u + u(M) = u^* \text{ or } \exists u_i \in \mathcal{M} : \quad u + u(M) = u^* + \sum_{i=1}^{t} u_i,$$

We first show that $u + u(M) = u^*$ is impossible. In fact, this would mean that O_6, the last linear order of profile u^*, must belong to a majority profile $u(M)$. If $O_6 = cadb$ belongs to a majority profile, then O_5 must be *dbac*, which is not the case.

If $u + u(M) = u^* + \sum_{i=1}^{t} u_i$ and if $t > 1$, then:

$$u = u^* + \sum_{i=1}^{t-1} u_i$$

If $u + u(M) = u^* + \sum_{i=1}^{t} u_i$ and if $t = 1$, then:

$$u = u^*.$$

In both these last two cases, we have that $u \in \mathcal{U}$.

However, g_2 does not verify EI: add the two linear orders *abcd* and *dcba* to profile u^*. We then have that $g_2(u^*) = abcd$ but $g_2(u^* + abcd + dcba) = \mathcal{LO}$.

3. *Homogeneity for Odd Profiles*

We consider the preference margin matrix B^* and the preference margin matrix B' defined as follows:

B^*	a	b	c
a	.	3	–1
b	–3	.	3
c	1	–3	.

B'	a	b	c
a	.	2	–2
b	–2	.	2
c	2	–2	.

We are going to define a new preference function g_3 as follows, where B denotes the majority margin matrix of profile u.

$$g_3(u) = \begin{cases} \mathcal{LO} & \text{if } \exists\lambda \in \{0, 1, 2, \ldots\} : B = B^* + \lambda B' \\ \mathcal{PO}(u) & \text{otherwise} \end{cases}$$

Hence g_3 corresponds to the prudent order preference function except for profiles with majority margins $B^* + \lambda B'$.

Such a procedure g_3 verifies CE and EI. It also verifies MPC. However, g_3 does not verify HOP. Let u be a profile with majority margins equal to B^*. Consequently, $g_3(u) = \mathcal{LO}$. However, $g_3(u + u) = \{abc\}$. Although the size of profile u is odd, $g_3(u) \neq g_3(u + u)$.

4. *Majority Profile Consistency*

Kemeny orders (see Section 2.2) can be defined as follows:

$$g_4(u) = \{O_K \in \mathcal{LO} : \forall O \in \mathcal{LO}, \sum_{(a_i,a_j)\in O_K} B_{ij} \geq \sum_{(a_i,a_j)\in O} B_{ij}\}.$$

Kemeny orders verify CE, EI, HOP but not MPC. Consider a profile with the following majority margin matrix B.

B	a	b	c	d
a	.	1	1	–3
b	–1	.	3	3
c	–1	–3	.	3
d	3	–3	–3	.

A profile with such a majority margin matrix yields one unique Kemeny order: $g_4(u) = \{bcda\}$. Construct a majority profile $u(M)$. Let us then compute the Kemeny orders of the profile $u + u(M)$ with preference margin matrix B':

B'	a	b	c	d
a	.	3	3	–5
b	–3	.	5	5
c	–3	–5	.	5
d	5	–5	–5	.

We have that $g_4(u + u(M)) = \{abcd\}$. Hence $g_4(u + u(M)) \neq g_4(u)$.

5.7 The extended prudent order preference function

Before introducing a new prudent ranking rule, let us consider the following example that can be found in Taylor [103]. There are five alternatives a, b, c, d and e and the profile consists of 7 linear orders:

O_1	*abcde*	O_5	*cdbae*
O_2	*adbec*	O_6	*bcdae*
O_3	*adbec*	O_7	*ecdba*
O_4	*cdbea*		

We have the following majority margin matrix:

	a	b	c	d	e
a	.	–1	–1	–1	3
b	1	.	1	–1	5
c	1	–1	.	3	1
d	1	1	–3	.	5
e	–3	–5	–1	–5	.

In this case, $\beta = 1$, since

- $R_{>1}$ is acyclic,

- $R_{\geq 1}$ contains a cycle: $B(c,d) \geq 1$, $B(d,b) \geq 1$ and $B(b,c) \geq 1$.

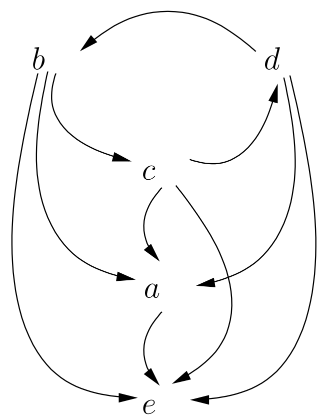
Consequently, the relation

$$R_{>\beta} = R_{>1} = \{(a,e), (b,e), (c,d), (d,e)\}.$$

The set of prudent orders corresponds to all the linear extensions of this relation. These 12 prudent orders are listed below:

PO_1	*acbde*	PO_7	*cadbe*
PO_2	*abcde*	PO_8	*bcade*
PO_3	*cabde*	PO_9	*cdabe*
PO_4	*acdbe*	PO_{10}	*cbdae*
PO_5	*cbade*	PO_{11}	*cdbae*
PO_6	*bacde*	PO_{12}	*bcdae*

Let us now take a look at the strict majority relation of this profile. This relation can be graphically represented as follows:



This strict majority graph seems to indicate that b, c and d could be put before a in a compromise ranking. However, the prudent orders $PO_1 - PO_9$ do not follow this logic. One reason for this is that the prudent order preference function does not verify what Truchon [105] calls the extended Condorcet criterion. Let us introduce this additional criterion.

Given a strict majority relation M, we say that $TC(M)$ is the top-cycle of M if it is the smallest possible subset of $\mathcal{A}$ such that for all $a_i \in TC(M)$ and for all $a_j \in \mathcal{A} \setminus TC(M)$ we have $(a_i, a_j) \in M$ (see also Section 4.5 where we first introduced the top-cycle). We partition $\mathcal{A}$ into ordered subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ with $\mathcal{A}_i = TC(M|\mathcal{A} \setminus \cup_{j<i}\mathcal{A}_j)$, where $M|\mathcal{A} \setminus \cup_{j<i}\mathcal{A}_j$ denotes the strict majority relation of profile u restricted to the set of alternatives $\mathcal{A} \setminus \cup_{j<i}\mathcal{A}_j$. We call this ordered partition the top-cycle partition

(see for instance Truchon [105] or Klamler [59] for further comments on this partition). Saari [96] calls the different classes of the top-cycle partition "layers". In Taylor's example, the top-cycle partition consists of three layers: $\mathcal{A}_1 = \{b, c, d\}, \mathcal{A}_2 = \{a\}$ and $\mathcal{A}_3 = \{e\}$.

Given a top-cycle partition, we can very naturally define the following partial order T:

$$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in T \text{ if } a_i \in \mathcal{A}_k \text{ and } a_j \in \mathcal{A}_l \text{ and } k < l$$

In order to reconcile the prudent order preference function with the top-cycle principle, we define the following new preference function $\mathcal{XPO}$, called the extended prudent order preference function.

Definition 11 *The extended prudent order preference function associates to a profile u the set of linear extensions of the binary relation $R_\beta \cup T$, where $R_{>\beta}$ is the optimal cut relation used in the prudent order preference function and T is the partial order induced by the top-cycle partition.*

$$\forall u, \quad \mathcal{XPO}(u) = \mathcal{E}(R_{>\beta} \cup T).$$

The set of compromise rankings thus corresponds to all the linear extensions of the relation $R_{>\beta} \cup T$. We will show that $R_{>\beta} \cup T$ is acyclic. Since we can always extend an acyclic relation into a linear order (see Szpilrajn [100]), the set $\mathcal{E}(R_{>\beta} \cup T)$ is never empty and consequently $\mathcal{XPO}$ is a true preference function.

Proposition 9 *The relation $R_{>\beta} \cup T$ is acyclic.*

Proof: If $\beta < 0$, then the strict majority relation M is a linear order and $M = R_{>\beta} = T$. Consequently, $R_{>\beta} \cup T = R_{>\beta}$ is acyclic. Let us suppose $\beta \geq 0$. By contradiction we assume that $R_{>\beta} \cup T$ contains a cycle. Since this cycle cannot appear inside a layer of the top-cycle-partition (by definition T is empty inside a layer and $R_{>\beta}$ is acyclic), there must exist a_i and a_j such that $a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l, k < l$ and $(a_j, a_i) \in R_{>\beta}$. Hence, $B_{ji} > \beta \geq 0 \Rightarrow (a_j, a_i) \in M \Rightarrow (a_i, a_j) \notin M$. This is impossible since we supposed that a_i belongs to a higher layer in the top-cycle partition than a_j.

□

It is clear that for any profile u, we have that $\mathcal{XPO}(u) \subseteq \mathcal{PO}(u)$. Hence $\mathcal{XPO}$ is a prudent ranking rule since it refines the set of prudent orders. In Taylor's example, the set of extended prudent orders consists of the prudent orders 10 to 12.

If the strict majority relation is acyclic, and if a preference function f verifies WCE, then this implies that $\forall a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l$ such that $k < l$, we have that:

$$\forall O \in f(u), \quad (a_i, a_j) \in O.$$

The following axiom says that, also for profiles with a non-acyclic strict majority relation, the top-cycle partition should not be contradicted by any solution belonging to the set of compromise rankings.

Axiom 9 *Truchon Condorcet(TC):*
Let us suppose that the strict majority relation contains at least one cycle. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ be the top-cycle-partition. We say that a preference function f verifies Truchon Condorcet if $\forall a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l$ such that $k < l$, we have that

$$\forall O \in f(u), \quad (a_i, a_j) \in O.$$

Axiom TC simply says that for profiles with a cyclic strict majority relation, we must have that $f(u) \subseteq \mathcal{E}(T)$. It is obvious that the extended prudent order preference function verifies TC, but, interestingly, it also verifies the axioms used in Theorem 4.

Proposition 10 *The extended prudent order preference function verifies Weak Condorcet Extension, Weak Majority Profile Consistency, Weak E-Invariance, Weak Homogeneity for Odd Profiles and Truchon Condorcet.*

Proof: WEI, WHOP and TC are easy to check.

Let us show that WCE is verified. If M is acyclic, then $R_{>\beta} = M$. Furthermore, $T \subseteq M$, which implies that $T \subseteq R_{>\beta}$, which implies that $R_{>\beta} \cup T = R_{>\beta} = M$. Consequently, $\mathcal{XPO}(u) = \mathcal{E}(R_{>\beta} \cup T) = \mathcal{E}(M)$.

Let us show that WMPC is verified. We denote M the strict majority relation and T the top-cycle relation of profile u. We denote M^* the strict majority relation and T^* the top-cycle relation of profile $u + u(M)$. Furthermore, we denote $R_{>\lambda}$ a cut relation of profile u and $R^*_{>\lambda}$ a cut relation of profile $u + u(M)$. Let β be the optimal cut value for profile u and β^* be the optimal cut value for profile $u + u(M)$.

If $\beta < 0$, then this means that the strict majority relation of profile u is a linear order (see Proposition 2 in Section 4.1). Consequently, the strict majority relation of profile $u + u(M)$ is also a linear order. We are not interested in this case since we suppose that the strict majority relation of profile $u + u(M)$ contains at least one cycle. Let us from now on suppose that $\beta \geq 0$.

We know from the inclusion 5.3 and the inequality 5.4 that

$$R_{>\beta} = R'_{\beta+2} \subseteq R^*_{>\beta^*}.$$

We are now going to show that $T \subseteq T^*$, which will prove WMPC since then

$$\mathcal{PO}(u + u(M)) = \mathcal{E}(T^* \cup R^*_{>\beta^*}) \subseteq \mathcal{E}(T \cup R_{>\beta}) = \mathcal{XPO}(u).$$

Let us show that if $(a_i, a_j) \in T$, then $(a_i, a_j) \in T^*$. Since $(a_i, a_j) \in T$, there exists two layers in the top-cycle partition $\mathcal{A}_k$ and $\mathcal{A}_l$ such that $a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l$ and such that $k < l$. Let $\mathcal{A}^*_{k'}$ be the layer of the top-cycle partition of M^* to which a_i belongs and let $\mathcal{A}^*_{l'}$ be the layer of the top-cycle partition of M^* to which a_j belongs. We want to show that $k' < l'$, which will prove that $(a_i, a_j) \in T^*$. Let us suppose by contradiction that $k' \geq l'$.

If $k' > l'$, then this means that $(a_j, a_i) \in M^*$, since a_i is in a lower layer than a_j. This is impossible: the fact that $M \subseteq M^*$ and the fact that $(a_i, a_j) \in M$ imply that $(a_i, a_j) \in M^*$, which implies that $(a_j, a_i) \notin M^*$. Hence a_i cannot be in a lower layer than a_j.

If $k' = l'$, then this means that a_i and a_j belong to the same layer in the top-cycle partition of M^*. We denote this layer by $\mathcal{A}^* = \mathcal{A}^*_{k'} = \mathcal{A}^*_{l'}$. We consider the following subsets of alternatives: $D^*_1 = \mathcal{A}^* \cap \mathcal{A}_1$, $D^*_2 = \mathcal{A}^* \cap \mathcal{A}_2, \ldots,$ $D^*_p = \mathcal{A}^* \cap \mathcal{A}_p$. Let $D^+ = \cup_{r=1}^{k} D^*_r$ and let $D^- = \mathcal{A} \setminus D^+$. We know that both D^+ and D^- are non-empty since $\{a_i\} \subseteq D^*_k \subseteq D^+$ and $\{a_j\} \subseteq D^*_l \subseteq D^-$. We know that $\forall x \in D^+$ and $\forall y \in D^-$, $(x, y) \in M$, which implies that $(x, y) \in M^*$, since $M \subseteq M^*$. Hence $\mathcal{A}'$ cannot be a layer of the top-cycle partition of M^* since D^+ is dominating D^-.

This proves that $T \subseteq T^*$. □

It will now be easy to show that if i) we want to use the axioms Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles, Weak Majority Profile Consistency and Truchon Condorcet and ii) we want to have a set of compromise solutions as large as possible, then we must use the extended prudent order preference function.

Theorem 6 *The extended prudent order preference function is the largest preference function (with respect to inclusion) that verifies Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles, Weak Majority Profile Consistency and Truchon Condorcet.*

Proof: Either the majority relation is acyclic, or not. In the first case, we have that $M = R_{>\beta}$ and $T \subseteq M$. Consequently,

$$\mathcal{XPO}(u) = \mathcal{E}(R_{>\beta} \cup T) = \mathcal{E}(M \cup T) = \mathcal{E}(M).$$

Axiom WCE implies that

$$f(u) \subseteq \mathcal{E}(M) = \mathcal{XPO}(u).$$

This concludes the proof when the strict majority relation is acyclic. If this relation is not acyclic, then we know from Theorem 4 that axioms WCE, WEI, WHOP and WMPC imply that

$$f(u) \subseteq \mathcal{E}(R_{>\beta}).$$

Axiom TC implies that

$$f(u) \subseteq \mathcal{E}(T).$$

Combining these two inclusions, we get:

$$f(u) \subseteq \mathcal{E}(T \cup R_{>\beta}) = \mathcal{XPO}(u).$$

Since the extended prudent order preference function verifies the 5 axioms (see Proposition 10), it is consequently the largest preference function that verifies the 5 axioms. This completes the proof. □

We have seen that the prudent order preference function is the largest preference function with respect to inclusion which satisfies WCE, WEI, WHOP and WMPC. Although the extended prudent order preference function also verifies these four axioms, it is not the largest such preference function. In fact, we need to add axiom TC in order to come up with a similar characterization result.

Although the proof of Theorem 6 has been rather straightforward, let us insist again on the fact that it is non-trivial that $\mathcal{XPO}$ verifies the first four axioms. A prudent ranking rule, which by definition always constructs a subset of the set of prudent orders, does not automatically inherit all the properties of the prudent order preference function.

Following the correspondence between the results of Theorem 4 and Theorem 5, the reader probably expects at this point a result which states that the extended prudent order preference function is the "only" preference function verifying a particular list of axioms which are preferably similar to the ones used in Theorem 6. However, we have not been able to come up with such a result. The following comments can nevertheless be made:

- It does not make sense to simply replace the four first axioms by their stronger counterparts CE, EI, HOP and MPC. In fact, we have shown that these four axioms completely characterize the prudent order preference function. Since $\mathcal{PO}$ does not verify TC, there does not exist any ranking rule satisfying simultaneously CE, EI, HOP, MPC and TC.

- Another strategy could be to somehow strengthen axiom TC. This axiom basically says that, if the strict majority relation contains cycles, then $f(u) \subseteq \mathcal{E}(T)$. Hence, we may be tempted to replace the inclusion between $f(u)$ and $\mathcal{E}(T)$ by an equality. This means that for every profile u where the strict majority relation is not acyclic, we have that $f(u) = \mathcal{E}(T)$. Unfortunately, since such an axiom completely discards any impact of $R_{>\beta}$ on the result, it is not verified by the extended prudent order preference function.

- In order to characterize the extended prudent order preference function, we probably have to add another new axiom. However, the task is not easy, since $\mathcal{XPO}$ works on the one hand with T, which only depends on the strict majority relation, and, on the other hand, with $R_{>\beta}$, which takes into account the values of the majority margins. Similar to axiom WMPC, we need to find an axiom which reconciles both types of relations.

Chapter 6

A characterization of the Ranked Pairs rule

In this chapter, we characterize the Ranked Pairs rule using the axiomatic framework built in the previous chapter. This work has been presented at the FNRS Seminar "Modélisation des préférences" in Brussels in May 2007 and at the Seminar "Mathématiques discrètes et Sciences Sociales" at ENST, Paris, in June 2007. I would like to thank Marc Pirlot for his valuable comments concerning this chapter.

In Section 6.1, we first come back to a particular discrimin-like relation. In Section 6.2, we then show that the Ranked Pairs rule is equivalent to finding a set of linear orders which are optimal according to that discrimin relation. The axioms are introduced in Section 6.3 and the characterizations are presented in Section 6.4. In Section 6.5, we check the independence of the axioms. Finally, in Section 6.6, all the results are summarized.

6.1 Discrimin relation

Given a profile u with a majority margin matrix B, two linear orders O and $\tilde{O}$ are compared on the basis of their weakest link. More formally, we build a binary relation $\succeq^u_{min}$ defined on $\mathcal{LO}$ as follows:

$$\forall O, \tilde{O} \in \mathcal{LO}, \quad O \succeq^u_{min} \tilde{O} \text{ if } \min_{\substack{(a_i,a_j)\in O \\ a_i \neq a_j}} B_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O} \\ a_i \neq a_j}} B_{ij}.$$

The relation $\succeq^u_{min}$ is transitive and complete, hence a weak order. We recall that a prudent order is a linear order which maximizes the weakest link (see Proposition 3 in Section 4.1). That is why the set of maximal elements of the weak order $\succeq^u_{min}$ corresponds exactly to the set of prudent orders:

$$\mathcal{PO}(u) = \{O \in \mathcal{LO} : \forall \tilde{O} \in \mathcal{LO}, \quad O \succeq^u_{min} \tilde{O}\}.$$

We have already stressed that the set of prudent orders can be rather large. Intuitively, this can be explained by the "drowning effect" of the min relation [37, 38]: by maximizing only the weakest link, the majority margins of the other ordered pairs belonging to the linear order are neglected. This may lead to large sets of ex aequo's in the weak order $\succeq^u_{min}$ and consequently to a large first equivalence class.

A possible refinement of the min relation is the so-called discrimin relation, which we denote by $\succeq^u_{disc}$. Given a profile u, two linear orders O and $\tilde{O}$ are compared on the basis of their weakest link, but only considering those ordered pairs on which they differ:

$$\forall O, \tilde{O} \in \mathcal{LO} \quad O \succeq^u_{disc} \tilde{O} \text{ if } \begin{cases} O = \tilde{O} \text{ or} \\ \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} \end{cases}$$

We refer the reader to Fortemps and Pirlot [45] for other names and equivalent definitions of the discrimin relation. We denote by $\sim^u_{disc}$ the symmetric and by $\succ^u_{disc}$ the asymmetric parts of the discrimin relation. It is clear that the relation $\succeq^u_{disc}$ is complete. However $\succeq^u_{disc}$ is not transitive. Consider a profile u in which $B(b,c) = B(b,d) = B(d,a) = B(d,c) = 2$. We know that a profile u with such majority margins must exist (see Section 2.1). Then

$$dabc \sim^u_{disc} abcd$$

since the smallest majority margin of the pairs belonging to $dabc$ and not to $abcd$ is –2 (consider the minimal value of $B(d,a) = 2, B(d,b) = -2, B(d,c) = 2$) and the smallest majority margin of the pairs belonging to $abcd$ and not to $dabc$ is also –2. In the same way, we can check that

$$abcd \sim^u_{disc} adcb.$$

However we also have that

$$dabc \succ^u_{disc} adcb,$$

which shows that $\succeq^u_{disc}$ is not transitive. Nevertheless, $\succ^u_{disc}$, the asymmetric part of $\succeq^u_{disc}$, is transitive indeed. Hence we can define the set of maximal linear orders according to the relation $\succeq^u_{disc}$ denoted by $\mathcal{D}(u)$:

$$\mathcal{D}(u) = \{O \in \mathcal{LO} : \forall \tilde{O} \in \mathcal{LO}, \quad O \succeq^u_{disc} \tilde{O}\}.$$

One may show that $\forall O, \tilde{O} \in \mathcal{LO}$, if $O \succ^u_{min} \tilde{O}$, then $O \succ^u_{disc} \tilde{O}$. It follows from this observation that

$$\forall u, \quad \mathcal{D}(u) \subseteq \mathcal{PO}(u).$$

Hence $\mathcal{D}$, being a refinement of the set of prudent orders, is a prudent ranking rule.

We illustrate the sets $\mathcal{PO}(u)$ and $\mathcal{D}(u)$ on the following profile with 5 alternatives $\{x, y, z, v, w\}$. The example can also be found in Tideman [104]. The number in front of each linear order counts how often this linear order is repeated in the profile.

7	$vwxyz$
3	$zyvwx$
6	$yzwxv$
3	$wxvzy$
5	$zxvwy$
3	$yxvwz$

The majority margin matrix of this profile is as follows:

	v	w	x	y	z
v	0	9	–7	3	–1
w	–9	0	11	3	–1
x	7	–11	0	3	–1
y	–3	–3	–3	0	5
z	1	1	1	–5	0

The reader may check that $\mathcal{PO}(u) = \mathcal{E}(R_{>7})$, where $R_{>7} = \{(v, w), (w, x)\}$. There are in total twenty prudent orders which are enumerated in Table 6.1. For each prudent order, we listed all the ordered pairs belonging to that prudent order and the majority margin corresponding to each ordered pair. The smallest majority margin of any prudent order is -7 corresponding to the pair (v, x). That is why all the linear orders listed in Table 6.1 are equivalent according to the relation $\succeq^u_{min}$.

We now analyze the solutions according to the relation $\succeq^u_{disc}$. We show that $PO_{20} = vwxyz$ is the unique discrimin optimal solution, which means that $\mathcal{D}(u) = \{vwxyz\}$.

Let us compare for instance $PO_{20} = vwxyz$ and $PO_1 = zyvwx$. These two linear orders differ by the following pairs:

$$\{(v, y), (v, z), (w, y), (w, z), (x, y), (x, z), (y, z)\} \subset PO_{20}$$

while

$$\{(y, v), (z, v), (y, w), (z, w), (y, x), (z, x), (z, y)\} \subset PO_1.$$

When considering these pairs only, the smallest majority margin for PO_{20} is –1, whereas the smallest majority margin for PO_1 is –3. Consequently $PO_{20} \succ^u_{disc} PO_1$. In a similar way, the reader may check that

$$\forall i \in \{1, \ldots, 19\}, \quad PO_{20} \succ^u_{disc} PO_i.$$

Consequently, PO_{20} is the only discrimin optimal solution.

6.2 Ranked Pairs rule

Tideman and Zavist [104, 110] proposed the so-called Ranked Pairs ranking rule. We have introduced this rule already in Section 2.2.1. First, rank the ordered pairs (a_i, a_j) according to their values B_{ij} from the largest to the smallest $(1 \leq i \leq n, 1 \leq j \leq n, i \neq j)$. We denote $E_1, E_2, \ldots, E_r$ the equivalence classes of this weak order. Take any linear order compatible with this weak order. Consider the pairs in that order and do the following: if the selected pair creates a (directed) cycle with the pairs already blocked, skip this pair. If the selected pair does not create a cycle with the pairs already blocked, block this pair. This ranking rule leads to possibly more than one solution, since for each equivalence class E_i $(i \in \{1, \ldots, r\})$ we must consider all the possible orderings (or permutations) of the pairs belonging to that equivalence class. We recall that $\mathcal{RP}(u)$ denotes the set of linear orders found by the Ranked Pairs ranking rule.

We apply this rule to the profile introduced in the previous section. In Table 6.2, we have described the weak order on the ordered pairs. In this example there are 12 equivalence classes. First, we select (w, x) with a majority margin of 11 and block it. Then we select (v, w) with a majority margin of 9 and block it. Since (x, v) with a majority margin of 7 would create a cycle, we skip it and go on to (y, z) with a majority margin of 5. Since no cycle is created with this pair, we block it. We then add the pairs $(v, y), (w, y)$ and (x, y), all with majority margins of 3. At this stage, we have blocked a chain which goes from v to w to x to y to z and consequently the only final ranking can be $vwxyz$. In fact, after equivalence class E_5, we are only allowed to block the pairs (v, z), (w, z), (x, z) (all three with a majority margin of –1) and pair (v, x) (with a majority margin of –7).

It appears that for this profile the solution obtained by the Ranked Pairs rule is the discrimin optimal solution computed in the previous section. In fact, we show that both ranking rules always give identical results.

Proposition 11

$$\forall u, \quad \mathcal{RP}(u) = \mathcal{D}(u).$$

PO_1	(v,w)	(v,x)	(y,v)	(z,v)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(z,y)
$zyvwx$	9	–7	–3	1	11	–3	1	–3	1	–5
PO_2	(v,w)	(v,x)	(y,v)	(z,v)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(y,z)
$yzvwx$	9	–7	–3	1	11	–3	1	–3	1	5
PO_3	(v,w)	(v,x)	(y,v)	(v,z)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(y,z)
$yvzwx$	9	–7	–3	–1	11	–3	1	–3	1	5
PO_4	(v,w)	(v,x)	(y,v)	(v,z)	(w,x)	(y,w)	(w,z)	(y,x)	(z,x)	(y,z)
$yvwzx$	9	–7	–3	–1	11	–3	–1	–3	1	5
PO_5	(v,w)	(v,x)	(y,v)	(v,z)	(w,x)	(y,w)	(w,z)	(y,x)	(x,z)	(y,z)
$yvwxz$	9	–7	–3	–1	11	–3	–1	–3	–1	5
PO_6	(v,w)	(v,x)	(v,y)	(z,v)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(z,y)
$zvywx$	9	–7	3	1	11	–3	1	–3	1	–5
PO_7	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(z,y)
$vzywx$	9	–7	3	–1	11	–3	1	–3	1	–5
PO_8	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(y,w)	(z,w)	(y,x)	(z,x)	(y,z)
$vyzwx$	9	–7	3	–1	11	–3	1	–3	1	5
PO_9	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(y,w)	(w,z)	(y,x)	(z,x)	(y,z)
$vywzx$	9	–7	3	–1	11	–3	–1	–3	1	5
PO_{10}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(y,w)	(w,z)	(y,x)	(x,z)	(y,z)
$vywxz$	9	–7	3	–1	11	–3	–1	–3	–1	5
PO_{11}	(v,w)	(v,x)	(v,y)	(z,v)	(w,x)	(w,y)	(z,w)	(y,x)	(z,x)	(z,y)
$zvwyx$	9	–7	3	1	11	3	1	–3	1	–5
PO_{12}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(z,w)	(y,x)	(z,x)	(z,y)
$vzwyx$	9	–7	3	–1	11	3	1	–3	1	–5
PO_{13}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(y,x)	(z,x)	(z,y)
$vwzyx$	9	–7	3	–1	11	3	–1	–3	1	–5
PO_{14}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(y,x)	(z,x)	(y,z)
$vwyzx$	9	–7	3	–1	11	3	–1	–3	1	5
PO_{15}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(y,x)	(x,z)	(y,z)
$vwyxz$	9	–7	3	–1	11	3	–1	–3	–1	5
PO_{16}	(v,w)	(v,x)	(v,y)	(z,v)	(w,x)	(w,y)	(z,w)	(x,y)	(z,x)	(z,y)
$zvwxy$	9	–7	3	1	11	3	1	3	1	–5
PO_{17}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(z,w)	(x,y)	(z,x)	(z,y)
$vzwxy$	9	–7	3	–1	11	3	1	3	1	–5
PO_{18}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(x,y)	(z,x)	(z,y)
$vwzxy$	9	–7	3	–1	11	3	–1	3	1	–5
PO_{19}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(x,y)	(x,z)	(z,y)
$vwxzy$	9	–7	3	–1	11	3	–1	3	–1	–5
PO_{20}	(v,w)	(v,x)	(v,y)	(v,z)	(w,x)	(w,y)	(w,z)	(x,y)	(x,z)	(y,z)
$vwxyz$	9	–7	3	–1	11	3	–1	3	–1	5

Table 6.1: The 20 prudent orders.

Equiv. class	Pair	Maj. Margin
E_1	(w, x)	11
E_2	(v, w)	9
E_3	(x, v)	7
E_4	(y, z)	5
E_5	$(v, y), (w, y), (x, y)$	3
E_6	$(z, v)(z, w), (z, x)$	1
E_7	$(v, z)(w, z), (x, z)$	-1
E_8	$(y, v), (y, w), (y, x)$	-3
E_9	(z, y)	-5
E_{10}	(v, x)	-7
E_{11}	(w, v)	-9
E_{12}	(x, w)	-11

Table 6.2: The ranking of the equivalence classes.

For any linear order O, we denote $E_t(O) = (E_1 \cup \ldots \cup E_t) \cap O$. In the proof of Proposition 11 we need the following lemma.

Lemma 3 $O \in \mathcal{RP}(u) \iff \forall t,\ E_t(O)$ *is maximal with respect to cyclicity:*

$$\forall (x, y) \in E_t \setminus O, \quad E_t(O) \cup \{(x, y)\} \text{ is not acyclic.}$$

Proof of the lemma.

Proof:

- $\Rightarrow$
  Consider a linear order $O \in \mathcal{RP}(u)$. Assume by contradiction that at step t there exists $(x, y) \in E_t \setminus O$ such that $E_t(O) \cup \{(x, y)\}$ is acyclic. When the Ranked Pairs algorithm arrives at E_t, it has already selected $E_{t-1}(O)$. When it comes to consider (x, y), it may have already selected a part A of $E_t \cap O$. In any case, (x, y) can be added to $E_{t-1}(O) \cup A$ without creating cycles, since

  $$E_{t-1}(O) \cup A \cup \{(x, y)\} \subseteq E_t(O) \cup \{(x, y)\}$$

  and we assumed that $E_t(O) \cup \{(x, y)\}$ is acyclic. Hence it is not possible that O does not contain (x, y).

- $\Leftarrow$
  Let $\succ^*$ be any linear order of the pairs that satisfies the following constraints:

  – all pairs in E_i are ranked before all pairs in $E_{i+1}, \forall i$.

– all pairs in $E_i \cap O$ are ranked before all pairs in $E_i \setminus O, \forall i$.

If $\forall t$, $E_t(O)$ is maximal with respect to acyclicity, applying the Ranked Pairs algorithm to $\succ^*$ yields O.

□

Proof of the proposition.

Proof:

- We first prove that if $O \in \mathcal{D}(u)$, then $O \in \mathcal{RP}(u)$. Let us suppose by contradiction that $O \notin \mathcal{RP}(u)$. Then there exists t and $(x, y) \in E_t \setminus O$ such that $E_t(O) \cup \{(x, y)\}$ is acyclic (Lemma 3). Consider an order $\succ^*$ on the pairs such that all pairs in E_i are ranked before all pairs in $E_{i+1}, \forall i$ and such that in E_t, (x, y) is just ranked after $E_t \cap O$. Applying the Ranked Pairs algorithm to such an order $\succ^*$ will yield a linear order O' such that $E_t(O') \supseteq E_t(O) \cup \{(x, y)\}$. It is easy to see that $O' \succ^u_{disc} O$, which is a contradiction, since we assumed that $O \in \mathcal{D}(u)$.

- We now prove that if $O \in \mathcal{RP}(u)$ then $O \in \mathcal{D}(u)$. Let O' be any linear order different from O such that $O' \succ^u_{disc} O$. We then have that:

$$\min_{\substack{(a_i,a_j)\in O' \\ (a_i,a_j)\notin O}} B_{ij} > \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin O'}} B_{ij}. \tag{6.1}$$

  Let
  $$t = \min\{s : E_s \cap (O \setminus O') \neq \emptyset\}$$
  and
  $$t' = \min\{s : E_s \cap (O' \setminus O) \neq \emptyset\}.$$

  According to (6.1), we must have that $t > t'$. That is why $E_{t'}(O') \setminus E_{t'}(O) \neq \emptyset$. Take $(x, y) \in E_{t'}(O') \setminus E_{t'}(O)$. $E_{t'}(O) \cup \{(x, y)\}$ has no cycle since
  $$E_{t'}(O) \cup \{(x, y)\} \subseteq E_{t'}(O')$$
  and $E_{t'}(O')$ is acyclic because O' is a linear order. Consequently $E_{t'}(O)$ is not maximal with respect to cyclicity. According to Lemma 3, this means that $O \notin \mathcal{RP}(u)$, which is a contradiction. Hence there cannot exist any linear order O' such that $O' \succ^u_{disc} O$.

□

A related result is presented in Tideman [104] (page 201), where the author claims that the linear orders found by the Ranked Pairs ranking rule are the maximal elements of a particular relation. Let us present this relation, denoted by $\succ^u_t$, which is however slightly different from $\succ^u_{disc}$.

$$\forall O, \tilde{O} \in \mathcal{LO}, \quad O \succ_t^u \tilde{O} \iff$$

$$\begin{cases} \max_{(a_i,a_j)\notin O} B_{ij} < \max_{(a_i,a_j)\notin \tilde{O}} B_{ij} \text{ or} \\ \begin{cases} \max_{(a_i,a_j)\notin O} B_{ij} = \max_{(a_i,a_j)\in \tilde{O}} B_{ij} = a^* \text{ and} \\ \{(a_i,a_j) : (a_i,a_j) \notin O \text{ and } B_{ij} = a^*\} \subset \{(a_i,a_j) : (a_i,a_j) \notin \tilde{O} \text{ and } B_{ij} = a^*\} \end{cases} \end{cases}$$

Because of the constant-sum property ($\forall i, j,\ B_{ij} + B_{ji} = 0$) and the fact that a linear order is asymmetric, this relation $\succ_t^u$ can be equivalently defined as follows:

$$\forall O, \tilde{O} \in \mathcal{LO}, \quad O \succ_t^u \tilde{O} \iff$$

$$\begin{cases} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij} > \min_{(a_i,a_j)\in \tilde{O}, a_i\neq a_j} B_{ij} \text{ or} \\ \begin{cases} \min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij} = \min_{(a_i,a_j)\in \tilde{O}, a_i\neq a_j} B_{ij} = b^* \text{ and} \\ \{(a_i,a_j) : (a_i,a_j) \in O \text{ and } B_{ij} = b^*\} \subset \{(a_i,a_j) : (a_i,a_j) \in \tilde{O} \text{ and } B_{ij} = b^*\} \end{cases} \end{cases}$$

The relation $\succ_{disc}^u$ which we have introduced in the first section of this chapter is in fact a refinement of the relation $\succ_t^u$ used in the paper of Tideman. For any two linear orders O and $\tilde{O}$, if

$$\min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij} \neq \min_{(a_i,a_j)\in \tilde{O}, a_i\neq a_j} B_{ij},$$

then both relations are equivalent: $O \succ_t^u \tilde{O} \iff O \succ_{disc}^u \tilde{O}$. If however

$$\min_{(a_i,a_j)\in O, a_i\neq a_j} B_{ij} = \min_{(a_i,a_j)\in \tilde{O}, a_i\neq a_j} B_{ij},$$

then it is easy to see that if $O \succ_t^u \tilde{O}$, then $O \succ_{disc}^u \tilde{O}$. However, the converse is not true, which means it may happen that $O \succ_{disc}^u \tilde{O}$, but $O \not\succ_t^u \tilde{O}$. Consider for instance the example presented in Figure 6.1. The smallest majority margin of any prudent order is equal to –7 corresponding to the pair (v, x). Furthermore, this is the only pair with a majority margin of –7. Consequently, no prudent order dominates, according to the relation $\succ_t^u$ another prudent order. However, we have seen that O^{20} dominates, according to the relation $\succ_{disc}^u$, the remaining 19 prudent orders.

The claim that the Ranked Pairs ranking rule is equivalent to the maximal linear orders of the relation $\succ_t^u$ is thus false. Instead, we have shown that the Ranked Pairs ranking rule is equivalent to the maximal linear orders of the relation $\succ_{disc}^u$. In the example presented in Section 6.1, only one linear order has been obtained by the Ranked Pairs ranking rule. At the same time, this linear order also corresponds to the unique non-dominated linear order according to the relation $\succ_{disc}^u$. However, according to the relation $\succ_t^u$, every prudent order is non-dominated.

Equiv. class	Pair	Maj. Margin
E_1	(d, a)	3
E_2	$(a, b), (b, c), (c, d), (c, a), (d, b)$	1
E_3	$(b, a), (c, b), (d, c), (a, c), (b, d)$	-1
E_4	(a, d)	-3

Table 6.3: The ranking of the equivalence classes.

E_1	E_2	E_3	Solution
(d, a)	$(b, c), (c, a), (d, b)$	$(b, a), (d, c)$	*dbca*
(d, a)	$(a, b), (b, c), (d, b)$	$(d, c), (a, c)$	*dabc*
(d, a)	$(a, b), (c, d), (c, a), (d, b)$	(c, b)	*cdab*
(d, a)	$(b, c), (c, d), (c, a)$	$(b, a), (b, d)$	*bcda*

Table 6.4: The solutions obtained by the Ranked Pairs algorithm.

Following Lemma 3, the Ranked Pairs ranking rule constructs a linear order O such that for any t, $E_t(O)$ is maximal with respect to cyclicity. This does not mean that if two different linear orders O and O' are obtained, then the number of pairs selected in $E_t(O)$ and the number of pairs selected in $E_t(O')$ are the same for every t. Consider the example which has already been introduced in Section 4.1. There are 4 alternatives and the profile is given by $u = (abcd, bcda, cdab, dabc, dcba)$. The ranking of the equivalence classes is presented in Table 6.3.

Given the large equivalence classes E_2 and E_3, the Ranked Pairs algorithm constructs for this profile four different solutions. They are given in Table 6.4, as well as the pairs blocked in each equivalence class that yield these four solutions. It appears that in order to obtain *cdab*, the Ranked Pairs algorithm blocks 4 pairs in E_2, whereas in order to obtain *dbca*, the Ranked Pairs algorithm only blocks 3 pairs in E_2. We will come back to this example in Section 7.1, where we will introduce a new ranking rule which further refines the Ranked Pairs rule by lexicographically maximizing the number of pairs blocked in each equivalence class.

6.3 Axioms

In this section, we detail the axioms which we need in the characterization of the Ranked Pairs ranking rule. In fact, we build on the axiomatic framework which has been presented in Chapter 5. We recall that we are interested in characterizing a preference function denoted by f which associates to any profile of linear orders u a non-empty set of linear orders $f(u)$ (see Section 4). More particularly, we refer to Section 5.3 for a precise formalization of the following axioms:

- Weak Condorcet Extension (WCE)
- Weak E-Invariance (WEI)
- Weak Homogeneity for Odd Profiles (WHOP)
- Weak Majority Profile Consistency (WMPC)

Recall that Theorem 4 in Section 5.5 states that the prudent order preference function is the largest preference function with respect to inclusion which verifies WCE, WEI, WHOP and WMPC. As we will check in Proposition 12, the Ranked Pairs rule also verifies these four axioms. However, unlike the prudent order preference function, the Ranked Pairs rule is not the largest such preference function.

In order to characterize the Ranked Pairs rule as being the largest preference function verifying a particular set of axioms, we will need an additional axiom, which we present now. It says that improving the strength of an ordered pair that belongs to a compromise ranking will not discard that particular compromise ranking.

Axiom 10 *Monotone Consistency (MC)*
Let $O \in f(u)$. Then for all (a_i, a_j) such that $a_i \neq a_j$ and $(a_i, a_j) \in O$, we have:

$$O \in f(u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j).$$

This axiom is related to the monotonicity issue, since it imposes a logical impact on the result when improving the pairwise majority margins between two alternatives. Unlike most of the monotonicity conditions which have been studied and proposed in the literature (see for instance Bouyssou [13]), axiom MC is rather particular in the sense that it does not apply to any ordered pairs (a_i, a_j), but only to those which belong already to at least one compromise ranking in $f(u)$.

To further stress the particular degree of monotonicity implied by axiom MC, we now present a weaker and a stronger monotonicity condition. Let us suppose, as in axiom MC, that $O \in f(u)$ and that a_i is preferred to a_j in the linear order O. A weaker monotonicity condition imposes that there must exist at least one linear order $\tilde{O}$ such that $\tilde{O} \in f(u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j)$ and such that $(a_i, a_j) \in \tilde{O}$. A stronger monotonicity condition requests that for all $\tilde{O} \in f(u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j)$ we have that $(a_i, a_j) \in \tilde{O}$. The strong monotonicity condition implies Monotone Consistency, which implies the weak monotonicity condition.

Both this weak and strong monotonicity conditions are of no use in our setting. On the one hand, the weaker condition is verified by the prudent order preference function, and consequently does not allow us to distinguish the Ranked Pairs rule from the prudent order preference function. On the other hand, the stronger condition is not verified by the Ranked Pairs rule, and consequently it is irrelevant for any characterization purposes.

Finally, in order to completely characterize the Ranked Pairs rule, we will need to strengthen axiom WMPC. In fact, WMPC deals with so-called majority profiles. We generalize this idea and work with qualified majority profiles. Let us now formally define this notion.

A qualified majority at level γ ($\gamma \geq 0$) is a binary relation defined as follows:

$$\forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in M_\gamma \text{ if } B_{ij} > \gamma.$$

We now consider each pair $\{a_i, a_j\}$ in turn:

- If $(a_i, a_j) \in M_\gamma$, construct the two linear orders

$$V_{ij}^1 = a_i a_j x_{-ij} \quad V_{ij}^2 = -x_{-ij} a_i a_j.$$

- If $B_{ij} = \gamma$, consider one of the following two exclusive possibilities:
  - Skip this pair.
  - Construct two linear orders V_{ij}^1 and V_{ij}^2 such that:

$$V_{ij}^1 = a_i a_j x_{-ij} \quad V_{ij}^2 = -x_{-ij} a_i a_j.$$

We then construct a new profile, denoted by $u(M_\lambda)$, which contains all the linear orders obtained that way.

Definition 12 *For every $\lambda \geq 0$, let M_λ be the qualified majority relation of profile u. We say that $u(M_\lambda)$ is a qualified majority profile of profile u if $u(M_\lambda)$ can be written as follows:*

$$u(M_\gamma) = \sum_{(a_i,a_j) \in M_\gamma} (V_{ij}^1 + V_{ij}^2) + \sum_{(a_i,a_j) \in \zeta} (V_{ij}^1 + V_{ij}^2),$$

where $\zeta \subseteq \{(a_i, a_j) : B_{ij} = \gamma\}$.

The information contained in $u(M_\gamma)$ confirms the information contained in M_γ. Furthermore, a qualified majority profile is indeed a generalization of a majority profile (see Section 5.2 for the definition of a majority profile). It suffices to set $\gamma = 0$ to obtain again the definition of a majority profile. The next axiom then says that adding a qualified majority profile cannot create any new compromise rankings.

Axiom 11 *Weak Qualified Majority Profile Consistency (WQMPC)*
Let u be a profile and let $u(M_\gamma)$ be a qualified majority profile of u at level γ, with $\gamma \geq 0$. Then:

$$f(u + u(M_\gamma)) \subseteq f(u).$$

Axiom WQMPC implies axiom WMPC. In fact, by fixing $\lambda = 0$, axiom WQMPC says that if u is a profile and $u(M)$ a majority profile of u, then

$$f(u + u(M)) \subseteq f(u).$$

Axiom WMPC exactly states this inclusion, under the condition that profile $u + u(M)$ contains at least one cycle.

The interpretation of axiom WQMPC is similar to the interpretation of axiom WMPC. In fact, it suggests that "confirming any qualified majority" of a profile should not lead to creating new solutions.

6.4 Characterization

In this section we present two main results. Before coming to these results, we first check that the Ranked Pairs rule verifies the axioms introduced in the preceding section.

Proposition 12 *The Ranked Pairs ranking rule verifies Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles, Weak Majority Profile Consistency, Weak Qualified Majority Profile Consistency and Monotone Consistency.*

Proof: WEI, WHOP are obvious. We now prove WCE. Let M be acyclic. Since $\mathcal{PO}$ verifies WCE (see Corollary 5 in Section 5.5), we have that:

$$\mathcal{RP}(u) \subseteq \mathcal{PO}(u) \subseteq \mathcal{E}(M).$$

This shows WCE.

We now prove WQMPC. Let B denote the majority margins of profile u and B' the majority margins of profile $u + u(M_\gamma)$, where

$$u(M_\gamma) = \sum_{(a_i,a_j)\in M_\gamma} (V_{ij}^1 + V_{ij}^2) + \sum_{(a_i,a_j)\in\zeta} (V_{ij}^1 + V_{ij}^2),$$

with $\zeta \subseteq \{(a_i, a_j) : B_{ij} = \gamma\}$. Let $E_1, E_2, \ldots, E_s, \ldots, E_t, \ldots, E_r$ be the equivalence classes of the weak order of the ordered pairs relative to profile u, where $E_s = \{(a_i, a_j) : B_{ij} = \gamma\}$ and $E_t = \{(a_i, a_j) : B_{ij} = -\gamma\}$. We recall that, in each equivalence class, the Ranked Pairs rule considers all the permutations of the ordered pairs belonging to that equivalence class. For each of these permutations, the ordered pairs are considered one by one and blocked, if the solutions so far constructed remains acyclic, or skipped, in case a cycle appears with pairs already blocked. We show that adding $u(M_\gamma)$ to u has the effect of not considering all permutations in the equivalence classes $E_1, E_2, \ldots, E_s, \ldots, E_t \ldots E_r$, but only a subset of these permutations. Consequently, a solution which can be found for $u + u(M_\gamma)$ can also be found for u, or, in other words, $\mathcal{RP}(u + u(M_\gamma)) \subseteq \mathcal{RP}(u)$.

Note that B and B' are related as follows:

$$B'_{ij} = \begin{cases} B_{ij} + 2 & \text{if } B_{ij} > \gamma \text{ or } (a_i, a_j) \in \zeta \\ B_{ij} - 2 & \text{if } B_{ij} < -\gamma \text{ or } (a_j, a_i) \in \zeta \\ B_{ij} & \text{otherwise} \end{cases}$$

That is why the weak order of the ordered pairs relative to profile $u+u(M_\gamma)$ can be represented as follows:

$$E_1, E_2, \ldots, E_{s-1}, \zeta, E_s \setminus \zeta, \ldots, E_t \setminus c(\zeta), c(\zeta), \ldots, E_r,$$

where $c(\zeta)$ denotes the converse of the relation ζ:

$$c(\zeta) = \{(a_i, a_j) \in \mathcal{A} \times \mathcal{A} : (a_j, a_i) \in \zeta\}.$$

This illustrated in Figure 6.1. Hence, at equivalence class E_s, we only consider those permutations that start with ordered pairs belonging to ζ, and at equivalence class E_t we only consider those permutations that end with ordered pairs belonging to $c(\zeta)$. This proves WQMPC.

The Ranked Pairs rule also satisfies WMPC since WQMPC implies WMPC.

We finally prove MC. Let $O \in \mathcal{RP}(u)$. According to Proposition 11, this means that

$$\forall \tilde{O} \in \mathcal{LO}, \quad O \succeq_{disc}^{u} \tilde{O}.$$

Suppose that $(a_k, a_l) \in O$ and denote

$$u^* = u + a_k a_l x_{-kl} + -x_{-kl} a_k a_l.$$

We need to show that $O \in \mathcal{RP}(u^*)$, which means that, according to Proposition 11:

$$\forall \tilde{O} \in \mathcal{LO}, \quad O \succeq^{u^*}_{disc} \tilde{O}.$$

Let B denote the majority margins of profile u and B^* the majority margins of profile u^*. Let us note that B and B^* are linked as follows:

$$\begin{cases} B^*_{kl} = B_{kl} + 2 \\ B^*_{lk} = B_{lk} - 2 \\ B^*_{ij} = B_{ij} & \forall (i,j) \notin \{(k,l),(l,k)\} \end{cases}$$

We supposed that $(a_k, a_l) \in O$. Let $\tilde{O}$ be any linear order. We distinguish the two cases whether $(a_k, a_l) \in \tilde{O}$ or $(a_l, a_k) \in \tilde{O}$.

- $(a_k, a_l) \in \tilde{O}$
  In that case, we have that

  $$\min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B_{ij} = \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B^*_{ij}$$

  and

  $$\min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} = \min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij}.$$

  Consequently $O \succeq^{u}_{disc} \tilde{O}$ implies that $O \succeq^{u^*}_{disc} \tilde{O}$.

- $(a_l, a_k) \in \tilde{O}$
  In that case, we have that

  $$\min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B_{ij} \leq \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B^*_{ij}$$

  and

  $$\min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij}.$$

  Consequently $O \succeq^{u}_{disc} \tilde{O}$ implies that $O \succeq^{u^*}_{disc} \tilde{O}$.

We can conclude that $\forall \tilde{O} \in \mathcal{LO}$, we have that $O \succeq^{u^*}_{disc} \tilde{O}$. Hence $O \in \mathcal{RP}(u^*)$, which proves MC.

□

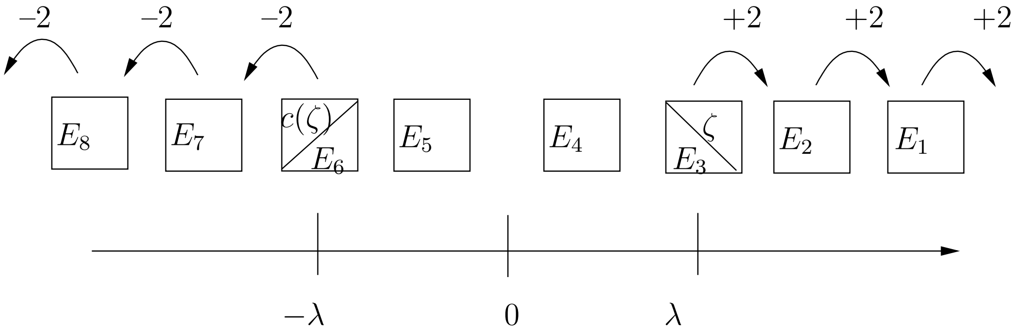


Figure 6.1: The transformation from profile u to profile $u + u(M_\lambda)$.

The first result says that i) if the preference function should satisfy axioms WCE, WEI, WHOP, WMPC and MC and ii) the set of compromise rankings should be as large as possible, then we must use the Ranked Pairs ranking rule.

Theorem 7 *The Ranked Pairs rule is the largest preference function (with respect to inclusion) that verifies Weak Condorcet Extension, Weak E-Invariance, Weak Homogeneity for Odd Profiles, Weak Majority Profile Consistency, and Monotone Consistency.*

Proof: We are going to show that any preference function f that verifies the above mentioned axioms is such such that

$$f(u) \subseteq \mathcal{RP}(u).$$

Since, by Proposition 12, the Ranked Pairs ranking rule verifies these axioms, the proof will be complete.

More particularly, we are going to show that, $\forall \tilde{O} \in \mathcal{LO}$, if there exists a linear order $O \in \mathcal{LO}$ such that $O \succ^u_{disc} \tilde{O}$, then $\tilde{O} \notin f(u)$. In view of Proposition 11, this will imply that $f(u) \subseteq \mathcal{RP}(u)$.

Let B denote the majority margin matrix of profile u. Consider the following set of ordered pairs

$$\zeta = \{(a_k, a_l) : a_k \neq a_l \text{ and } (a_k, a_l) \in O \cap \tilde{O} \text{ and } B_{kl} \leq \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij}\}.$$

In fact, ζ consists of those pairs that belong to both O and $\tilde{O}$, but whose majority margins are less than or equal to $\min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij}$. If $\zeta = \emptyset$, then this means that if $(a_k, a_l) \in O$ and $(a_k, a_l) \in \tilde{O}$, then

$$B_{kl} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} \geq \min_{(a_i,a_j)\in\tilde{O}} B_{ij}.$$

Since we assume that $O \succ^u_{disc} \tilde{O}$, then we moreover have that if $(a_k, a_l) \in O$ and $(a_k, a_l) \notin \tilde{O}$, then:

$$B_{kl} \geq \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin\tilde{O}}} B_{ij} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} \geq \min_{(a_i,a_j)\in\tilde{O}} B_{ij}.$$

We learn from the last two lines of inequalities that:

- If $(a_k, a_l) \in O$ and $(a_k, a_l) \in \tilde{O}$, then $B_{kl} > \min_{(a_i,a_j)\in\tilde{O}} B_{ij}$.
- If $(a_k, a_l) \in O$ and $(a_k, a_l) \notin \tilde{O}$, then $B_{kl} > \min_{(a_i,a_j)\in\tilde{O}} B_{ij}$.

Consequently, we have that

$$\min_{(a_i,a_j)\in O} B_{ij} > \min_{(a_i,a_j)\in\tilde{O}} B_{ij}.$$

In other words, this means that $O \succ^u_{min} \tilde{O}$, which implies that $\tilde{O} \notin \mathcal{PO}(u)$. Theorem 4 in Section 5.5 says that axioms WCE, WEI, WHOP, and WMPC imply that $f(u) \subseteq \mathcal{PO}(u)$. Since $\tilde{O} \notin \mathcal{PO}(u)$, we must have that $\tilde{O} \notin f(u)$. This completes the proof if $\zeta = \emptyset$.

We suppose from now on that $\zeta \neq \emptyset$.

Assume that $\tilde{O} \in f(u)$. We compute γ_{kl} for each $(a_k, a_l) \in \zeta$:

$$\gamma_{kl} = \frac{\min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} + 2 - B_{kl}}{2}.$$

γ_{kl} is an integer since B_{kl} and $\min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij}$ have the same parity. It is a positive integer since $B_{kl} \leq \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij}$.

Let us construct the profile u^*, which consists of adding γ_{kl} times for each pair $(a_k, a_l) \in \zeta$ the two linear orders $a_k a_l x_{-kl}$ and $-x_{-kl} a_k a_l$ to profile u:

$$u^* = u + \sum_{(a_k,a_l)\in\zeta} \gamma_{kl}(a_k a_l x_{-kl} + -x_{-kl} a_k a_l).$$

Since we supposed that $\tilde{O} \in f(u)$ and since $(a_k, a_l) \in \tilde{O}$, we conclude that $\tilde{O} \in f(u^*)$ by applying γ_{kl} times axiom MC for each pair $(a_k, a_l) \in \zeta$. Consequently, we have shown that

$$\tilde{O} \in f(u) \Rightarrow \tilde{O} \in f(u^*). \tag{6.2}$$

B^* denotes the majority margin matrix of profile u^*. In fact, B^* and B are linked as follows:
$\forall a_i, a_j \in \mathcal{A}$,

$$B^*_{ij} = \begin{cases} B_{ij} + 2\gamma_{ij} & \text{if } (a_i, a_j) \in \zeta \\ B_{ij} - 2\gamma_{ij} & \text{if } (a_j, a_i) \in \zeta \\ B_{ij} & \text{otherwise} \end{cases} \tag{6.3}$$

Since ζ only contains ordered pairs belonging to both O and $\tilde{O}$, we have:
$\forall a_i, a_j \in \mathcal{A}$,

$$\left.\begin{array}{l} (a_i, a_j) \in O \text{ and} (a_i, a_j) \notin \tilde{O} \\ \text{ or} \\ (a_i, a_j) \notin O \text{ and} (a_i, a_j) \in \tilde{O} \end{array}\right\} \Rightarrow B_{ij} = B^*_{ij}. \tag{6.4}$$

We are now going to check that:

$$\forall (a_k, a_l) \in O, a_k \neq a_l : \quad B^*_{kl} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij} \tag{6.5}$$

Let $(a_k, a_l) \in O$. We distinguish two cases: $(a_k, a_l) \in \tilde{O}$ and $(a_k, a_l) \notin \tilde{O}$.

- If $(a_k, a_l) \in \tilde{O}$, then we distinguish between $(a_k, a_l) \in \zeta$ and $(a_k, a_l) \notin \zeta$.

  - If $(a_k, a_l) \in \zeta$, then this means that:

    $$B^*_{kl} = B_{kl} + 2\gamma_{kl} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} = \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij}.$$

    The first equality comes from (6.3), the second strict inequality is a consequence of the definition of γ_{kl} and the third equality results from (6.4).
  - If $(a_k, a_l) \notin \zeta$, then this means that:

    $$B^*_{kl} = B_{kl} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} = \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij}.$$

    The first equality comes from (6.3), the second strict inequality is a consequence of the definition of ζ and the third equality results from (6.4).

- If $(a_k, a_l) \notin \tilde{O}$, then we have:

$$B^*_{kl} \geq \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin\tilde{O}}} B^*_{ij} = \min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin\tilde{O}}} B_{ij} > \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B_{ij} = \min_{\substack{(a_i,a_j)\in\tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij}.$$

The first inequality results from the fact that we assume that $(a_k, a_l) \in O$ and $(a_k, a_l) \notin \tilde{O}$. The second equality comes from (6.4), the third strict inequality comes from the fact that $O \succ^u_{disc} \tilde{O}$ and the fourth equality comes again from (6.4).

The last two bullets prove (6.5), which implies that:

$$\min_{\substack{(a_i,a_j)\in O \\ a_i \neq a_j}} B^*_{ij} > \min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B^*_{ij} \geq \min_{\substack{(a_i,a_j)\in \tilde{O} \\ a_i \neq a_j}} B^*_{ij}.$$

This shows that $O \succ^{u^*}_{min} \tilde{O}$ and consequently $\tilde{O} \notin \mathcal{PO}(u^*)$. Theorem 4 in Section 5.5 says that axioms WCE, WEI, WHOP, and WMPC imply that $f(u^*) \subseteq \mathcal{PO}(u^*)$. Since $\tilde{O} \notin \mathcal{PO}(u^*)$, we thus also have that $\tilde{O} \notin f(u^*)$. Given 6.2, we can conclude that $\tilde{O} \notin f(u)$. □

We knew already that the largest preference function verifying axioms WCE, WEI, WHOP and WMPC is the prudent order preference function. The Ranked Pairs ranking rule also verifies these four axioms, but it is not the largest such preference function. If we additionally require axiom MC, then the largest possible preference function is the Ranked Pairs ranking rule. A consequence of these results is that MC cannot be verified by the prudent order preference function (see however a very similar property described in Proposition 5 (Section 4.3), which is indeed verified by the prudent order preference function).

The next corollary follows naturally from Theorem 7 and from the convention that the image of a preference function never is the empty set.

Corollary 6 *Let f be a preference function which verifies Weak Condorcet Extension, Weak E-invariance, Weak Homogeneity for Odd Profiles, Weak Majority Profile Consistency and Monotone Consistency. For a profile u such that $|\mathcal{RP}(u)| = 1$ we have:*

$$f(u) = \mathcal{RP}(u).$$

We will also need the following lemma:

Lemma 4 *Let O and $\tilde{O}$ be two linear orders. Let u be a profile of linear orders with a majority margin matrix B. If*

$$\min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B_{ij} = \min_{\substack{(a_i,a_j)\in \tilde{O} \\ (a_i,a_j)\notin O}} B_{ij},$$

then

$$\min_{\substack{(a_i,a_j)\in O \\ (a_i,a_j)\notin \tilde{O}}} B_{ij} \leq 0.$$

Proof: Suppose by contradiction that

$$\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} > 0.$$

Hence there exists (a_k, a_l) such that $(a_k, a_l) \in O$ and $(a_k, a_l) \notin \tilde{O}$ and $B_{kl} > 0$. Consequently there exists (a_l, a_k) such that $(a_l, a_k) \in \tilde{O}$ and $(a_l, a_k) \notin O$ and $B_{lk} < 0$. Hence

$$\min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij} < 0.$$

Since $\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} = \min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij}$, this implies that:

$$\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} < 0.$$

This is a contradiction since we initially supposed that $\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} > 0$.
□

By strengthening the Weak Majority Profile Consistency axiom, the Ranked Pairs ranking rule can be completely characterized.

Theorem 8 *The Ranked Pairs ranking rule is the only preference function that verifies Weak Condorcet Extension, Weak E-invariance, Weak Homogeneity for Odd profiles, Weak Qualified Majority Profile Consistency and Monotone Consistency.*

Proof: We have shown in Proposition 12 that the Ranked Pairs ranking rule verifies the axioms stated in the theorem. Let f be a preference function verifying these axioms.

We know from Theorem 7 that the Ranked Pairs rule is the largest preference function verifying WCE, WEI, WHOP, WMPC and MC. Since WQMPC implies WMPC, we already know that:

$$\forall u, \quad f(u) \subseteq \mathcal{RP}(u).$$

We still need to prove that if $O \in \mathcal{RP}(u)$, then $O \in f(u)$. Let us assume that $O \in \mathcal{RP}(u)$. Consider the remaining linear orders of $\mathcal{RP}(u)$:

$$\Omega(u) = \mathcal{RP}(u) \setminus \{O\}.$$

If $\Omega(u) = \emptyset$, then this means that O is the unique solution found by the Ranked Pairs rule for profile u. We can thus apply Corollary 6 and conclude that $f(u) = \mathcal{RP}(u)$. Otherwise, let $\tilde{O} \in \Omega(u)$ be any linear order of that set. We must have that, according to Proposition 11, $O \sim^u_{disc} \tilde{O}$, and, consequently:

$$\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} = \min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij} = \gamma,$$

where γ denotes the common value of the two terms. We know from Lemma 4 that $\gamma \leq 0$. We are going to construct a qualified majority relation at level $-\gamma$:

$$(a_i, a_j) \in M_{-\gamma} \iff B_{ij} > -\gamma.$$

A qualified majority profile will be constructed as follows:

$$u(M_{-\gamma}) = \sum_{(a_i,a_j)\in M_{-\gamma}} V^1_{ij} + V^2_{ij} + \sum_{(a_i,a_j)\in \zeta} V^1_{ij} + V^2_{ij},$$

where $\zeta = \{(a_i, a_j) : B_{ij} = -\gamma \text{ and } (a_i, a_j) \in O\}$. Let us denote

$$u^1 = u + u(M_{-\gamma}).$$

The following three observations can be made:

1. $f(u^1) \subseteq f(u)$.
2. $O \in \mathcal{RP}(u^1)$.
3. $\mathcal{RP}(u^1) \subset \mathcal{RP}(u)$.

These three observations can be explained as follows:

1. We assumed that f verifies WQMPC.
2. We assumed that $O \in \mathcal{RP}(u)$. In order to transform profile u into profile $u + \sum_{(a_i,a_j)\in M_{-\gamma}} V^1_{ij} + V^2_{ij}$, we add +2 to the majority margins B_{ij} such that $B_{ij} > \gamma$ and we substract –2 to the the majority margins B_{ij} such that $B_{ij} < -\gamma$. The order of the equivalence classes of the ordered pairs is the same in both profiles and that is why we have that:

   $$\mathcal{RP}(u) = \mathcal{RP}(u + \sum_{(a_i,a_j)\in M_{-\gamma}} V^1_{ij} + V^2_{ij}).$$

   Consequently we have that

   $$O \in \mathcal{RP}(u + \sum_{(a_i,a_j)\in M_{-\gamma}} V^1_{ij} + V^2_{ij}).$$

   Since the Ranked Pairs rule verifies MC (see Proposition 12) and since $\zeta \subseteq O$, this last observation implies that:

   $$O \in \mathcal{RP}(u + \sum_{(a_i,a_j)\in M_{-\gamma}} V^1_{ij} + V^2_{ij} + \sum_{(a_i,a_j)\in \zeta} V^1_{ij} + V^2_{ij}) = \mathcal{RP}(u^1).$$

3. We showed in Proposition 12 that the Ranked Pairs ranking rule verifies WQMPC. Hence $\mathcal{RP}(u^1) \subseteq \mathcal{RP}(u)$. We now explain the strict inclusion. More particularly, we show that $O \succ_{disc}^{u^1} \tilde{O}$, which implies that on the one hand $\tilde{O} \in \mathcal{RP}(u)$ and on the other hand $\tilde{O} \notin \mathcal{RP}(u^1)$. Recall that, under profile u, we have that:

$$\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} = \min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij} = \gamma.$$

The majority margin matrix of profile u^1 is denoted by B^1. Adding the linear orders

$$\sum_{(a_i,a_j)\in M_{-\gamma}} V_{ij}^1 + V_{ij}^2 + \sum_{(a_i,a_j)\in \zeta} V_{ij}^1 + V_{ij}^2$$

to profile u where

$$\zeta = \{(a_i, a_j) : B_{ji} = \lambda \text{ and } (a_j, a_i) \notin O\}$$

has the following consequences. If $B_{ij} = \lambda$ and $(a_i, a_j) \in O$, then $B_{ij}^1 = B_{ij}$. If $B_{ij} = \lambda$ and $(a_i, a_j) \notin O$, then $B_{ij}^1 = B_{ij} - 2$. This implies that:

$$\min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij}^1 = \min_{\substack{(a_i,a_j)\in O\\(a_i,a_j)\notin \tilde{O}}} B_{ij} = \gamma.$$

$$\min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij}^1 = (\min_{\substack{(a_i,a_j)\in \tilde{O}\\(a_i,a_j)\notin O}} B_{ij}) - 2 = \gamma - 2.$$

It follows from the last two lines that $O \succ_{disc}^{u^1} \tilde{O}$.

We can now compute the new set $\Omega(u^1) = \mathcal{RP}(u^1) \setminus \{O\}$. We will iteratively construct that way a sequence of t profiles $u^1, u^2, \ldots, u^t$, until profile u^t is such that $\Omega(u_t) = \emptyset$, which means that $\mathcal{RP}(u^t) = \{O\}$. We know that such a t exists since observation 2 tells us that the set $\mathcal{RP}(u^i)$ $(i = 1, \ldots, t)$ always contains the linear order O and observation 3 tells us that at each step the cardinality of this (finite) set strictly decreases. We can furthermore conclude from observation 1 that:

$$f(u^t) \subseteq ... \subseteq f(u).$$

Since $\mathcal{RP}(u_t) = \{O\}$, we can apply Corollary 6. This means that $f(u^t) = \{O\}$, and consequently $O \in f(u)$, which proves the theorem. □

6.5 Independence of the axioms

Let us now check the independence of the axioms used in Theorem 8.

- *Weak Condorcet Extension*

  The preference function that associates to every profile the whole set of linear orders verifies WEI, WHOP, WQMPC, MC but not WCE.

- *Weak E-invariance*
  We consider a profile u^* which contains 5 times the linear order $O_1 = abc$, 4 times the linear order $O_2 = cab$ and 3 times the linear order $O_3 = bca$.

  $$u^* = (O_1, O_1, O_1, O_1, O_1, O_2, O_2, O_2, O_2, O_3, O_3, O_3).$$

  This profile yields the following majority margin matrix:

	a	b	c
a	0	6	–2
b	–6	0	4
c	2	–4	0

  Let us consider the following set of profiles:

  $$\mathcal{U} = \{u : \exists t \in \{0, 1, 2, \ldots\} \text{ such that } u = u^* + t(abc + cab)\}.$$

  We are going to define a preference function f as follows:

  $$f(u) = \begin{cases} \{abc, cab\} & \text{if } u \in \mathcal{U} \\ \mathcal{RP}(u) & \text{otherwise} \end{cases}$$

  Hence f corresponds to the Ranked Pairs rule, except for profiles belonging to $\mathcal{U}$. Let us note that if $u \in \mathcal{U}$, then

  $$\mathcal{RP}(u) = \{abc\} \subseteq \{abc, cab\} = f(u).$$

  Consequently we have that $\forall u, \mathcal{RP}(u) \subseteq f(u)$.

  It is easy to see that f verifies WCE. If u is a profile with an acyclic majority relation M, then $u \notin \mathcal{U}$, and, consequently, $f(u) = \mathcal{RP}(u) = \mathcal{E}(M)$.

  The preference function f also verifies WHOP. If u is odd then $u \notin \mathcal{U}$. This follows from the observation that the majority margins of an odd profile must be all odd (see Section 2.1). Hence:

  $$f(u) = \mathcal{RP}(u) = \mathcal{RP}(u + u) \subseteq f(u + u).$$

f also verifies WQMPC. Let u be a profile and let $u(M_\gamma)$ be a qualified majority profile corresponding to u. Either $u \in \mathcal{U}$ or $u \notin \mathcal{U}$.

In the first situation, either $u + u(M_\gamma) \in \mathcal{U}$, in which case $f(u) = f(u + u(M_\gamma))$, or $u + u(M_\gamma) \notin \mathcal{U}$, in which case

$$f(u + u(M_\gamma)) = \mathcal{RP}(u + u(M_\gamma)) \subseteq \mathcal{RP}(u) \subseteq f(u).$$

In the second situation, if $u \notin \mathcal{U}$, then $u + u(M_\gamma) \notin \mathcal{U}$. Let us suppose by contradiction that $u + u(M_\gamma) \in \mathcal{U}$. We would then have that there exists $t \in \{0, 1, 2, \dots\}$ such that

$$u + u(M_\gamma) = u^* + t(abc + cab).$$

In fact, this is only possible, due to the structure of u^*, if $u(M_\gamma) = (abc, cab)$ and $t = 1$. Consequently, $u = u^*$, and so $u \in \mathcal{U}$, which is a contradiction. Since $u \notin \mathcal{U}$ implies that $u + u(M_\gamma) \notin \mathcal{U}$, we have:

$$f(u + u(M_\gamma)) = \mathcal{RP}(u + u(M_\gamma)) \subseteq \mathcal{RP}(u) = f(u).$$

f also verifies MC. Either $u \in \mathcal{U}$ or $u \notin \mathcal{U}$. In the first situation, we have that $f(u) = \{abc, cab\}$. The following cases have to be considered, where in the first column we indicate the ordered pair which we will improve.

(a, b)	$f(u + abc + cab) = \{abc, cab\}$ (since $u + abc + cab \in \mathcal{U}$)
(b, c)	$f(u + bca + abc) = \{abc\}$
(a, c)	$f(u + acb + bac) = \{abc\}$
(c, a)	$f(u + cab + bca) = \{abc, cab\}$
(c, b)	$f(u + cba + acb) = \{abc, cab\}$

The reader can check that MC is verified in all these cases. In the second situation, if $u \notin \mathcal{U}$, then $u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j \notin \mathcal{U}$. In fact, if $u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j \in \mathcal{U}$, we would then have that there exists $t \in \{0, 1, 2, \dots\}$ such that

$$u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j = u^* + t(abc + cab).$$

This is only possible, due to the structure of u^*, if

$$(a_i a_j x_{-ij}, -x_{-ij} a_i a_j) = (abc, cba).$$

Consequently, $u = u^* + (t - 1)(abc + cab)$, and so $u \in \mathcal{U}$, which is a contradiction since we supposed that $u \notin \mathcal{U}$. Hence we have that if $O \in f(u) = \mathcal{RP}(u)$, then

$$O \in f(u + u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j) = \mathcal{RP}(u + u + a_i a_j x_{-ij} + -x_{-ij} a_i a_j).$$

However, f does not verify WEI. For instance, $f(u^*) = \{abc, cab\}$, but $f(u^* + abc + cba) = \mathcal{RP}(u^* + abc + cba) = \{abc\}$. Hence we have that $f(u^*) \nsubseteq f(u^* + u_E)$.

- *Weak Homogeneity for Odd Profiles*

We have defined the set of Slater orders $\mathcal{S}(u)$ in Section 2.2.2. Let us now consider the following preference function f:

$$f(u) = \begin{cases} \mathcal{RP}(u) & \text{if } u \text{ is even} \\ \mathcal{S}(u) & \text{if } u \text{ is odd} \end{cases}$$

Hence f corresponds to the Ranked Pairs ranking rule for even profiles and to Slater's rule for odd profiles. This ranking rule verifies WCE, WEI, MC since both the Ranked Pairs rule and Slater's rule verify these axioms and the modifications of the profiles involved in the axioms do not change their parity.

We also know that the Ranked Pairs ranking rule verifies WQMPC. This axiom is also verified for Slater's rule when the profile is odd. In fact, there cannot be any preference margins with value 0 since the profile is odd. That is why, the majority margin of any ordered pairs is either positive or negative. It is then easy to see that the strict majority relation of the profile u and the strict majority relation of profile $u + u(M_\gamma)$ are the same. Consequently, the set of Slater orders for these two profiles will be the same.

However, WHOP is not verified. Consider a profile u with the following majority margin matrix:

	a	b	c
a	.	5	–1
b	–5	.	3
c	1	–3	.

These majority margins correspond to an odd profile u and that is why $f(u) = \mathcal{S}(u) = \{abc, bca, cab\}$. On the other hand, $u + u$ is an even profile and so $f(u+u) = \mathcal{RP}(u+u) = \{abc\}$. Hence, $f(u) \not\subseteq f(u+u)$.

- *Weak Qualified Majority Consistency*

The lexicographic prudent order preference function $\mathcal{LPO}$ will be introduced in depth in Chapter 7. We now give the definition of this ranking rule. Let O be any linear order. For $k \in \{1, \ldots, \frac{n(n-1)}{2}\}$, we denote $B^{(k)}(O)$ the k^{th} smallest majority margin of the ordered pairs (a_i, a_j) belonging to O, with $a_i \neq a_j$.

The relation $\succeq^u_{lex}$ is a binary relation defined on $\mathcal{LO}$ as follows:

$$\forall O, \tilde{O} \in \mathcal{LO}, \qquad O \succeq^u_{lex} \tilde{O} \iff$$

$$\begin{array}{ll} \forall i \leq \frac{n(n-1)}{2}, & B^{(i)}(O) = B^{(i)}(\tilde{O}) \\ \text{or } \exists t \leq \frac{n(n-1)}{2} : & \left\{ \begin{array}{ll} \forall i < t, & B^{(i)}(O) = B^{(i)}(\tilde{O}) \\ B^{(t)}(O) > B^{(t)}(\tilde{O}) & \end{array} \right. \end{array}$$

The maximal elements of that weak order $\succeq^u_{lex}$ correspond to the set of lexicographic prudent orders.

$$\mathcal{LPO}(u) = \{O \in \mathcal{LO} : \forall \tilde{O} \in \mathcal{LO}, \quad O \succeq^u_{lex} \tilde{O}\}.$$

$\mathcal{LPO}$ verifies WCE, WEI, WHOP, MC but not WQMPC, as shown by the following counter-example. Let us consider the majority margin matrix:

	a	b	c	d
a	0	2	2	–2
b	–2	0	2	–2
c	–2	–2	0	2
d	2	2	–2	0

For any profile u with such majority margins, we have that $\mathcal{LPO}(u) = \{dabc\}$. Let us consider the following qualified majority profile at level $\gamma = 2$:

$$u(M_\gamma) = (cdab, bacd).$$

In fact, we just add two linear orders corresponding to the ordered pair (c, d). This leads to the following majority margin matrix:

	a	b	c	d
a	0	2	2	–2
b	–2	0	2	–2
c	–2	–2	0	4
d	2	2	–4	0

We now have that $\mathcal{LPO}(u + u(M_\gamma)) = \{cdab, acdb, abcd\}$. Hence $\mathcal{LPO}(u + u(M_\lambda)) \not\subseteq \mathcal{LPO}(u)$.

- *Monotone Consistency*

The prudent order preference function verifies WCE, WEI, WHOP, WQMPC but not MC.

6.6 An axiomatic framework for prudent ranking rules

We are now ready to summarize the axiomatic results which we have presented in the last two chapters. Let us recall that we have studied three prudent ranking rules:

- The prudent order preference function $\mathcal{PO}$.
- The extended prudent order preference function $\mathcal{XPO}$.
- The Ranked Pairs ranking rule $\mathcal{RP}$.

The set of compromise rankings obtained with these three rules are encapsulated.

$$\forall u, \quad \mathcal{RP}(u) \subseteq \mathcal{XPO}(u) \subseteq \mathcal{PO}(u).$$

We will see in Chapter 7 how the lexicographic prudent order preference function further refines the set of rankings obtained by the Ranked Pairs rule.

The axiomatic results obtained so far are recapitulated in Table 6.5. We developed two types of characterization results. Either, we showed that a particular prudent ranking rule is the *largest* preference function verifying a set of axioms. Axioms relative to this type of results are marked by a "X" in the column "Largest". Or, we showed that a particular prudent ranking rule is the *only* preference function verifying a set of axioms. Axioms relative to this type of results are marked by a "X" in the column "Only". Finally, axioms which are simply verified by a particular prudent ranking rule are marked with a "X" in the column "Verifies".

The axioms CE, EI and HOP (and their weaker versions) are rather standard conditions which are verified by most reasonable ranking rules, including the three ranking rules listed in this table. Consequently, these axioms do not allow us to distinguish between the ranking rules. That is why the most interesting information comes from the remaining axioms.

First of all it is worth noticing that WMPC is verified by all three ranking rules. Such an axiom thus seems to be particular for this kind of prudent ranking rules. In fact, we have shown that the largest preference function verifying WMPC (together with the three basic axioms WCE, WEI and WHOP) is the prudent order preference function. The Ranked Pairs rule is the largest preference function verifying these four axioms together with axiom MC, whereas the extended prudent order preference function is the largest preference function verifying these four axioms together with axiom

TC.

We have also proposed complete characterizations of the prudent order preference function and of the Ranked Pairs Rule. For the first ranking rule, axiom WMPC has been strengthened into MPC by replacing the inclusion by an equality, whereas for the second ranking rule, axiom WMPC has been strengthened into WQMPC by replacing the idea of a majority profile by the idea of a qualified majority profile.

Of course, this framework is not complete yet. Let us mention two important future research problems.

- We always presented two types of characterization results. The first one claims that a particular preference function is the largest function and the second one claims that a particular preference function is the only function verifying a set of axioms. For the extended prudent order preference function, we have only been able to prove a result of the first type. A result of the second type still needs to be done.

- We proposed a characterization of the prudent order preference function and of the Ranked Pairs rule. The missing link in this axiomatic framework is a characterization of the lexicographic prudent order preference function, which we will introduce in Chapter 7. This would bring together the three ranking rules based respectively on the min, the discrimin and the leximin. Although we have not been successful in developing such results, it still remains a feasible objective.

Axiom	$\mathcal{PO}$ Largest	Only	Verifies	$\mathcal{RP}$ Largest	Only	Verifies	$\mathcal{XPO}$ Largest	Verifies
Condorcet Extension		X	X			X		X
implies Weak Condorcet Extension	X		X	X	X	X	X	X
E-Invariance		X	X			X		X
implies Weak E-Invariance	X		X	X	X	X	X	X
Homog. for Odd Profiles		X	X			X		X
implies Weak Homog. for Odd Profiles	X		X	X	X	X	X	X
Majority Profile Consistency		X	X					
Weak Qualified Majority Profile Consistency			X		X	X		X
both imply Weak Majority Profile Consistency	X		X	X		X	X	X
Monotone Consistency				X	X	X		
Truchon Condorcet						X	X	X

Table 6.5: An axiomatic framework for prudent ranking rules.

Chapter 7

Lexicographic prudent orders

In this chapter, we introduce a new prudent ranking rule based on an underlying leximin relation. As a first attempt to understand this ranking rule, we focus on comparing the results to Kemeny orders, both theoretically and empirically. These results have been presented at the OSDA conference in Gent during June 2007.

The chapter is organized as follows. First, in Section 7.1, the ranking rule is formalized. We position in Section 7.2 the ranking rule in the prudent axiomatic framework introduced in the previous chapters. Section 7.3 compares the ranking rule with Kemeny orders. In Section 7.4, we present how lexicographic prudent orders can be computed. The results of some empirical simulations can be found in Section 7.5. Finally, we point out in Section 7.6 some future directions of research.

7.1 Definition

We have seen in Chapter 5 that prudent orders are the linear orders which are maximal according to an underlying "min" relation. The Ranked Pairs rule presented in Chapter 6 outputs those linear orders which are maximal according to an underlying "discrimin" relation. Often, the min relation and the discrimin relation are presented together with the leximin relation (see for instance [3, 45]), since the leximin relation refines the discrimin relation, which refines the min relation. That is why it seems natural to also define in our context a ranking rule based on an underlying leximin relation.

Intuitively, lexicographic prudent orders can be described as follows. Among all the linear orders, prudent orders are those that maximize the weakest link. Among all the prudent orders, select the ones that maximize the second weakest link. Among these, select the ones that maximize the third weakest link. Repeat that procedure $\frac{n(n-1)}{2}$ times, which is the number of non-reflexive pairs.

More formally, let $u = (O_1, O_2, \dots, O_q)$ be a profile with a majority margin matrix B. Let O be any linear order. For $k \in \{1, \dots, \frac{n(n-1)}{2}\}$, we denote $B^{(k)}(O)$ the k^{th} smallest majority margin of the ordered pairs (a_i, a_j) belonging to O, with $a_i \neq a_j$.

The relation $\succeq_{lex}^u$ is a binary relation defined on $\mathcal{LO}$ as follows:

$$\forall O, \tilde{O} \in \mathcal{LO}, \qquad O \succeq_{lex}^u \tilde{O} \iff$$

$$\begin{array}{ll} \forall i \leq \frac{n(n-1)}{2}, & B^{(i)}(O) = B^{(i)}(\tilde{O}) \\ \text{or } \exists t \leq \frac{n(n-1)}{2} : & \begin{cases} \forall i < t, & B^{(i)}(O) = B^{(i)}(\tilde{O}) \\ B^{(t)}(O) > B^{(t)}(\tilde{O}) \end{cases} \end{array}$$

We denote $\succ_{lex}^u$ and $\sim_{lex}^u$ the asymmetric and symmetric parts of relation $\succeq_{lex}$. The relation $\succeq_{lex}^u$ is transitive and complete, hence a weak order. By definition, the maximal elements of that weak order correspond to the set of lexicographic prudent orders.

Definition 13 *The set of lexicographic prudent orders of a profile u, denoted by $\mathcal{LPO}(u)$, corresponds to the maximal linear orders of $\succeq_{lex}$.*

$$\mathcal{LPO}(u) = \{O \in \mathcal{LO} : \forall \tilde{O} \in \mathcal{LO}, \quad O \succeq_{lex}^u \tilde{O}\}.$$

One may show that $\forall O, \tilde{O} \in \mathcal{LO}$, if $O \succ_{disc}^u \tilde{O}$, then $O \succ_{lex}^u \tilde{O}$. Consequently, the set of lexicographic prudent orders further refines the linear orders found by the Ranked Pairs rule. Since the Ranked Pairs rule refines the set of prudent orders, we have:

$$\forall u, \quad \mathcal{LPO}(u) \subseteq \mathcal{RP}(u) \subseteq \mathcal{PO}(u).$$

With respect to the Ranked Pairs rule presented in Chapter 6, $\mathcal{LPO}$ consists in lexicographically maximizing the number of pairs blocked in each equivalence class. Following the algorithm presented by Dubois and Fortemps [38], lexicographic prudent orders can be obtained the following way:

- Rank the ordered pairs according to their majority margins from the largest to the smallest into equivalence classes $E_1, E_2, \dots, E_r$.

- From $i = 1 : r$, do the following. Block in equivalence class E_i the maximal number of pairs such that no cycle is created with the pairs already blocked. In case of multiple optimal solutions, start to branch.

$\mathcal{PO}$	$\mathcal{RP}$	$\mathcal{LPO}$
dcba		
dcab		
dbca	X	
dacb		
dbac		
dabc	X	
cdba		
cdab	X	X
bdca		
bdac		
cbda		
bcda	X	

Table 7.1: The results of the prudent ranking rules $\mathcal{PO}, \mathcal{RP}$ and $\mathcal{LPO}$.

Among all the linear orders obtained that way, a final screening determines those which are leximin-optimal. In practice this approach means that for each equivalence class, an optimization problem has to be solved. In Section 7.4, we present another approach of computing lexicographic prudent orders.

Let us now illustrate $\mathcal{LPO}$ on an example with four alternatives. Let $u = (abcd, bcda, cdab, dabc, dcba)$. In Section 4.1, we have already computed the set of prudent orders for this profile. In fact, this set consists of the linear extensions of the relation $R_{>1} = \{(d, a)\}$. There are in total 12 prudent orders which are listed in the first column of Table 7.1. An "X" in the second column of that table indicates that the prudent order is also discrimin optimal.

There are 4 out of the 12 prudent orders which are discrimin optimal. These 4 linear orders are analyzed in Table 7.2. The majority margins of the ordered pairs belonging to these linear orders can be found in the second column of that table. For instance, for linear order *dbca*, we have that

$$B(d, b) = 1 \; B(d, c) = -1 \; B(d, a) = 3 \; B(b, c) = 1 \; B(b, a) = -1 \; B(c, a) = 1.$$

In the third column of that table, these majority margins are ordered from the smallest to the largest. It is then easy to see that:

$$cdab \succ^u_{lex} dbca \sim^u_{lex} dabc \sim^u_{lex} bcda.$$

Consequently, *cdab* is the unique lexicographic prudent order of profile u.

dbca	(1,–1,3,1,–1,1)	(–1,–1,1,1,1,3)
dabc	(3,1,–1,1,–1,1)	(–1,–1,1,1,1,3)
cdab	(1,–1,1,3,1,1)	(–1,1,1,1,1,3)
bcda	(1,–1,–1,1,1,3)	(–1,–1,1,1,1,3)

Table 7.2: Analysis of the four discrimin optimal linear orders.

7.2 Positioning in the prudent axiomatic framework

The lexicographic prudent order preference function satisfies most of the axioms used in the characterizations presented in the previous chapters. More particularly, the reader can check that $\mathcal{LPO}$ verifies:

- Condorcet Extension
- Weak Condorcet Extension
- E-Invariance
- Weak E-Invariance
- Homogeneity for Odd Profiles
- Weak Homogeneity for Odd Profiles
- Weak Majority Profile Consistency
- Monotone Consistency
- Truchon Condorcet

We refer to Chapters 5 and 6 for a precise formulation of these axioms. It is worth highlighting that $\mathcal{LPO}$ verifies Weak Majority Profile Consistency. This axiom seems to be a crucial ingredient since it is verified by *all* the prudent ranking rules and by *no* non-prudent ranking we have so far investigated.

The only two axioms of our axiomatic framework which are not satisfied by the lexicographic prudent order preference function are:

- Majority Profile Consistency (which is used in the characterization of the prudent order preference function).
- Weak Qualified Majority Profile Consistency (which is used in the characterization of the Ranked Pairs rule).

This suggests that Majority Profile Consistency seems to be particular about the prudent order preference function, whereas Weak Qualified Majority Profile Consistency seems to be particular about the Ranked Pairs rule. Both these axioms are stronger versions of Weak Majority Profile Consistency, which is verified by $\mathcal{LPO}$. Although we have not been able so far to come up with a characterization for $\mathcal{LPO}$, the key for success lies probably in "adapting" the Weak Majority Profile Consistency axiom in some smart way.

We first present a counter example of **Majority Profile Consistency**. We assume that u is a profile with the following majority margins. According to Proposition 1 (see Section 2.1), we know that a profile of linear orders corresponding to these majority margins must exist.

	a	b	c	d
a	0	0	2	−2
b	0	0	2	−2
c	−2	−2	0	2
d	2	2	−2	0

It appears that $\mathcal{LPO}(u) = \{dabc, dbac\}$. We are now going to construct a majority profile u_M relative to u. For instance, one can consider the following linear orders:

$$u_M = (abcd, dcab, acdb, bdac, bcad, dabc, cdab, bacd, dabc, cbda, dbac, cadb).$$

In fact, we decided to break the indifference between a and b by adding the two linear orders $abcd$ and $dcab$. Furthermore, the preference strength of any two alternatives whose majority margins are strictly positive in u has been improved. The profile $u+u_M$ has then the following majority margins:

	a	b	c	d
a	0	2	4	−4
b	−2	0	4	−4
c	−4	−4	0	4
d	4	4	−4	0

We now have that $\mathcal{LPO}(u) = \{dabc\}$. Although the profile $u + u_M$ contains a majority cycle (there is a strict majority of a over b, of b over c, of c over d and of d over a), we do not have that $\mathcal{LPO}(u + u_M) = \mathcal{LPO}(u)$. Consequently, axiom Majority Profile Consistency is violated.

We now present a counter example of **Weak Qualified Majority Profile Consistency**. We assume that u is a profile with the following majority margins. According to Proposition 1 (see Section 2.1), we know that a profile of linear orders corresponding to these majority margins must exist.

	a	b	c	d
a	0	2	2	–2
b	–2	0	2	–2
c	–2	–2	0	2
d	2	2	–2	0

We can compute that $\mathcal{LPO}(u) = \{dabc\}$. We are now going to construct a qualified majority profile u_{M_λ} at level $\lambda = 2$. In fact, we are only improving the strength of c over d by considering the following two linear orders:

$$u_{M_\lambda} = (cdab, bacd)$$

This is in line with the definition of a qualified majority profile which allows to improve the strength of some (and not necessarily all) ordered pairs whose majority margins lie exactly on the chosen λ frontier. The profile $u + u_{M_\lambda}$ has then the following majority margins:

	a	b	c	d
a	0	2	2	–2
b	–2	0	2	–2
c	–2	–2	0	4
d	2	2	–4	0

After computation, we can conclude that $\mathcal{LPO}(u+u_{M_\lambda}) = \{cdab, acdb, abcd\}$. Consequently it is not true that $\mathcal{LPO}(u + u_{M_\lambda}) \subseteq \mathcal{LPO}(u)$, which shows that the lexicographic prudent order preference function can violate the Weak Qualified Majority Profile Consistency axiom.

7.3 Comparison with Kemeny orders

In this section, we analyze to what extend lexicographic prudent orders are linked with Kemeny orders. We refer to Section 2.2.2 for the definition of Kemeny's rule. In fact, both the lexicographic preference function and Kemeny's rule are intuitively rather similar ranking rules. Let $\mathcal{K}(u)$ denote the set of Kemeny orders of profile u.

It is interesting to note that for profiles with 3 alternatives, both rules give identical results.

Proposition 13 *If $n = 3$, then $\mathcal{LPO}(u) = \mathcal{K}(u)$.*

Proof: Let us consider the strict majority relation M. In case M is acyclic, we can show that the set of Kemeny orders and lexicographic prudent orders all consist of all the linear extensions of M. Hence,

$$\mathcal{K}(u) = \mathcal{LPO}(u) = \{O \in \mathcal{LO} : M \subseteq O\}.$$

Let us now suppose that M contains a cycle. Let us suppose that the alternatives are labeled such that $B_{12} > 0$, $B_{23} > 0$ and $B_{31} > 0$. Let us denote by p_1 the pair with the largest, p_2 with the second largest and p_3 with the third largest majority margin. We denote this as follows: $B(p_1) \geq B(p_2) \geq B(p_3) > 0$. We denote by $-p_1, -p_2, -p_3$ the reversed pairs. Every linear order can be seen as a triplet of pairs. Let us also recall that Saari and Merlin [96] mention that for profiles with three alternatives, no ties and with a cycle in the majority relation, the Kemeny order is obtained by reversing the pair in the cycle with the smallest majority margin. If this can be done in more than one way, we consider all the possibilities. We can thus easily compute the Kemeny orders and lexicographic prudent orders in the following four possible cases:

	$B(p_1) > B(p_2) > B(p_3)$	$B(p_1) > B(p_2) = B(p_3)$
$\mathcal{K}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$
$\mathcal{LPO}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$

	$B(p_1) = B(p_2) > B(p_3)$	$B(p_1) = B(p_2) = B(p_3)$
$\mathcal{K}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$ $(-p_1, p_2, p_3)$
$\mathcal{LPO}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$ $(-p_1, p_2, p_3)$

It is easy to see that in all situations we have that both rules give identical results.

□

The next proposition highlights a very close relationship between these two ranking rules, even for profiles with more than 3 alternatives.

Proposition 14 *$\forall u$, either $\mathcal{LPO}(u) \cap \mathcal{K}(u) = \emptyset$ or $\mathcal{LPO}(u) \subseteq \mathcal{K}(u)$.*

Proof: Let u be a profile. If $\mathcal{LPO}(u) \cap \mathcal{K}(u) = \emptyset$, then the proof is complete. If $\mathcal{LPO}(u) \cap \mathcal{K}(u) \neq \emptyset$, then let $O_{LK} \in \mathcal{LPO}(u) \cap \mathcal{K}(u)$. We need to show that for any lexicographic prudent order O of profile u, O is also a Kemeny order. Since $O_{LK} \in \mathcal{K}(u)$, we know that

$$\forall \tilde{O} \in \mathcal{LO}, \quad \sum_{(a_i,a_j) \in O_{LK}} B_{ij} \geq \sum_{(a_i,a_j) \in \tilde{O}} B_{ij}.$$

Since $O \in \mathcal{LPO}(u)$ and $O_{LK} \in \mathcal{LPO}(u)$, we also know that $O \sim^u_{lex} O_{LK}$. It follows from this that $\forall i \in \{1, \ldots, \frac{n(n-1)}{2}\}$, $B^{(i)}(O) = B^{(i)}(O_{LK})$. Hence we have that

$$\sum_{(a_i,a_j)\in O_{LK}} B_{ij} = \sum_{(a_i,a_j)\in O} B_{ij}.$$

We can thus conclude that:

$$\forall \tilde{O} \in \mathcal{LO}, \quad \sum_{(a_i,a_j)\in O} B_{ij} \geq \sum_{(a_i,a_j)\in \tilde{O}} B_{ij}.$$

Hence O is also a Kemeny order. This completes the proof. □

Proposition 14 tells us that for any profile, either all lexicographic prudent orders are as well Kemeny orders or none of them is a Kemeny order. In the first situation, the number of lexicographic prudent orders must be less or equal to the number of Kemeny orders. To illustrate this, let us consider the following majority margins of a profile with 6 alternatives $\{a, b, c, d, e, f\}$.

	a	b	c	d	e	f
a	0	4	–4	0	–4	–4
b	4	0	2	–4	–4	0
c	4	–2	0	–4	–2	4
d	0	4	4	0	–2	4
e	4	4	2	2	0	–4
f	4	0	–4	–4	4	0

For such a profile, the unique lexicographic prudent order is $dcfeab$, whereas there are three Kemeny orders: $dcfeab$, $edcfab$ and $edbcfa$. For an illustration of the situation where the intersection is empty, we refer the reader to Section 8.4, where we will show that the unique lexicographic prudent order can be "contradictory" to the unique Kemeny order.

7.4 Computation

In this section we build a 0-1 linear program which models the lexicographic prudent order preference function. This program is then solved using CPLEX 8, a popular optimization solver. More particularly, we adapt the formulation used to find Kemeny orders (see for instance Hudry [50]) to cope with lexicographic prudent orders. Let us recall that the problem of finding Kemeny orders is NP-complete (see Bartholdi & al. [5]).

A linear order can be modeled by means of n^2 binary variables x_{ij} $(1 \leq i \leq n, 1 \leq j \leq n)$. More particularly, the variable x_{ij} takes value 1 if the ordered pair (a_i, a_j) belongs to the linear order, and 0 if the ordered pair does not belong to the linear order. We denote x a n times n

dimensional matrix where the entry of row i and column j corresponds to x_{ij}.

In order to define the coefficients of the objective function of the optimization problem, we will need to do the following. First, let us order the pairs (a_i, a_j) with respect to their majority margins B_{ij}, with $1 \leq i \leq n, 1 \leq j \leq n, i \neq j$, from the largest to the smallest. As in Section 6.2, we denote E_1 the set of ordered pairs for which the majority margins are the largest, E_2 the set of ordered pairs for which the majority margins are the second largest and so on. Let us suppose that there are in total r such equivalence classes on the ordered pairs $E_1, E_2, \ldots, E_r$. We define "new" majority margins as follows:
$\forall a_i, a_j \in \mathcal{A}, a_i \neq a_j$, if $(a_i, a_j) \in E_t$, then

$$B_{ij}^{new} = (\frac{n(n-1)}{2} + 1)^{(r-t)}.$$

We then solve the following 0-1 linear program:

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}^{new} x_{ij} \tag{7.1}$$

$$s.t. \quad \forall i, j, k \quad x_{ij} + x_{jk} \leq x_{ik} + 1 \tag{7.2}$$

$$\forall i, j \quad x_{ij} + x_{ji} = 1 \tag{7.3}$$

$$\forall i \quad x_{ii} = 0 \tag{7.4}$$

$$\forall i, j \quad x_{ij} \in \{0, 1\} \tag{7.5}$$

In this formulation, constraints (7.2) guarantee the transitivity and constraints (7.3) the completeness of the linear order. Constraints (7.4) model the fact that the reflexive pairs do not matter. Let us note that an optimal solution of this linear program can be seen as a Kemeny order of a fictitious profile with a majority margin matrix B^{new}. We now show that an optimal solution of this program corresponds in fact to a lexicographic prudent order of profile u.

The constraints (7.2), (7.3), (7.4) and (7.5) define a set of feasible solutions denoted by T. Let $x^* \in T$ be an optimal solution of this program. Burkard and Rendl [22] showed that (see Theorem 1, page 304):
$\forall x \in T$,

$$\sum_{(i,j):x_{ij}^*=1} B_{ij}^{new} \geq \sum_{(i,j):x_{ij}=1} B_{ij}^{new}$$

$\iff \forall x \in T$,

$$(B_{ij} : x_{ij}^* = 1) \geq_{lex} (B_{ij} : x_{ij} = 1),$$

where $(B_{ij} : x^*_{ij} = 1)$ is a vector containing all the values B_{ij} such that $x^*_{ij} = 1$, $(B_{ij} : x_{ij} = 1)$ is a vector containing all the values B_{ij} such that $x_{ij} = 1$ and $\geq_{lex}$ is the usual leximin relation defined between these two vectors. In fact, if x^* models the linear order O^* and x models the linear order O, then this is equivalent to stating that

$$\forall O \in \mathcal{LO}, \quad O^* \succeq^u_{lex} O.$$

Consequently, the optimal solutions of the linear program correspond to the lexicographic prudent orders of profile u.

Furthermore, we can easily compute the relation $R_{>\beta}$ of profile u by applying Kohler's algorithm on the majority margin matrix B (see Section 4.1). Since every prudent order, and consequently every lexicographic prudent order, must contain this relation, we can directly fix some variables. That is why we add the following constraints to simplify our program:

$$\forall i, j : (a_i, a_j) \in R_{>\beta}, \qquad x_{ij} = 1 \tag{7.6}$$

In case we are looking for all the lexicographic prudent orders, we use the following strategy. First, we solve the linear program and find one optimal linear order denoted by O. The following constraint depending on O will then be added to the program:

$$\sum_{\substack{(a_i,a_j)\in O \\ i\neq j}} x_{ij} \leq \frac{n(n-1)}{2} - 1 \tag{7.7}$$

Let O^* be an optimal solution of the program with this additional constraint. If $O \sim^u_{lex} O^*$, then this means that O^* is also a lexicographic prudent order. Constraint (7.7) relative to O^* will be added and the search for other lexicographic prudent order continues by resolving the linear program with this new constraint. If $O \succ^u_{lex} O^*$, then O^* is not a lexicographic prudent order. In fact, this means that all the lexicographic prudent order have been found.

There are two major difficulties with solving this linear program. On the one hand, the value of the objective function (7.1) can get very large. This may lead to numerical problems. On the other hand, working with discrete instead of continuous variables further complicates the search for the optimum. Although other algorithmic solutions should be studied in the future, this approach seems however adequate for our purposes.

7.5 Empirical results

In this section, we present and discuss the results obtained in some empirical simulations. In Section 7.5.1, the random profile generation on which the

n	q	observed frequency	Gehrlein's frequency
5	5	0.78	0.8
5	7	0.77	0.78
5	15	0.79	0.76
7	5	0.69	0.7
7	7	0.69	0.68
7	15	0.67	0.66
15	5	0.5	0.46
15	7	0.47	0.46
15	15	0.42	0.42

Table 7.3: Comparing the observed frequencies with Gehrlein's frequencies of a Condorcet winner under the IC assumption.

simulations are based is introduced. In Section 7.5.2, we analyze the number of lexicographic prudent orders. In Section 7.5.3, we check how often a lexicographic prudent order is at the same time a Kemeny order.

7.5.1 Random profile generation

In this section, we focus on the random generation of profiles. Often, in empirical research in social choice theory, the impartial culture (IC) is assumed. This means that each linear order has equal probability to be chosen. Profiles with different values for the number of alternatives and the number of linear orders are generated that way and can subsequently be analyzed with respect to some particular property.

First, we verify that our random profile generator behaves sufficiently close to the impartial culture assumption. To test this indirectly, we generate profiles with 5, 7 and 15 alternatives and with 5, 7 and 15 linear orders. For each profile, we check if a Condorcet winner exists. Let us recall that a Condorcet winner is an alternative $a_i \in \mathcal{A}$ such that $\forall a_j \in \mathcal{A}, a_j \neq a_i$, we have $(a_i, a_j) \in M$. The observed frequency of the existence of a Condorcet winner can then be compared to the known frequency of a Condorcet winner under the IC assumption computed by Gehrlein [47]. For each combination of number of alternatives and number of linear orders, 1000 repetitions where performed. The results can be found in Table 7.3.

Working under the IC assumption is a defensible choice. However, the IC assumption is not fully satisfactory when we want to empirically study Condorcet ranking consistent (CRC) rules (see Section 2.3), such as for instance Kemeny's rule or the lexicographic prudent order preference function. It still happens relatively frequently for small profiles generated under the IC assumption that the majority relation contains few cycles (see already the relatively high frequencies of the existence of a Condorcet winner shown in Table 7.3). Consequently a lexicographic prudent order will be very close to this majority relation. For such profiles, the result is not very specific of the lexicographic prudent order preference function, since it is almost equal to that majority relation. This can be uninspiring, especially since we want to compare the result obtained by this rule to Kemeny's rule. Since a Kemeny order will also be very close to this majority relation, both the Kemeny and lexicographic prudent order will trivially be very close or even coincide.

In order to generate profiles which are more difficult to tackle for CRC rules, we chose to develop another profile generation model which makes profiles more cyclic. More particularly, we will add to a profile generated under the IC assumption a list of linear orders obtained by rotation. For instance, if $a_1a_2\dots a_n$ is the first linear order added, then $a_2a_3\dots a_na_1$ will be the second linear order added, and so. In all, n linear orders can be constructed with this rotation procedure.

We now formalize our "IC spiced up" model:

- Generate q linear orders of n alternatives under the IC assumption.
- Generate one linear order uniformly.
- Compute the corresponding $n-1$ linear orders by rotation.
- Consider every linear order obtained by rotation twice.

A profile constructed this way consists of $q + 2n$ linear orders. It is strongly inspired by Saari's profile decomposition (see for instance [96]), and more particularly by the so-called Condorcet portion of this profile decomposition.

We chose to work with profiles with a number of alternatives n equal to 5, 6, 7, 10, 15 and 20 and with a number of linear orders q equal to 5, 6, 7, 10, 15 and 20. For each combination (n, q), we generated 1000 profiles under the IC assumption with n alternatives and q linear orders and 1000 profiles under the IC spiced up assumption presented in the previous section with n alternatives and $q + 2n$ linear orders.

7.5.2 Number of lexicographic prudent orders

Given a profile, we enumerate all the corresponding lexicographic prudent orders. For practical reasons, we have decided to stop the enumeration after

	$q = 5$	$q = 6$	$q = 7$	$q = 10$	$q = 15$	$q = 20$	All q
$n = 5$	1	3	1	3	1	2	1
$n = 6$	1	6	1	3	1	2	2
$n = 7$	1	9	1	5	1	3	2
$n = 10$	2	$\geq$11	2	8	1	4	3
$n = 15$	2	$\geq$11	2	8	1	3	3
$n = 20$	2	$\geq$ 11	2	1	1	2	2

Table 7.4: The median of the number of lexicographic prudent orders for profiles generated under the IC assumption.

	$q = 5$	$q = 6$	$q = 7$	$q = 10$	$q = 15$	$q = 20$	All q
$n = 5$	1	2	1	1	1	1	1
$n = 6$	1	3	1	2	1	2	1
$n = 7$	1	2	1	2	1	2	1
$n = 10$	1	2	1	2	1	2	1
$n = 15$	1	2	1	2	1	2	1
$n = 20$	1	2	1	3	3	4	2

Table 7.5: The median of the number of lexicographic prudent orders for profiles generated under the IC spiced up assumption.

at most 11 lexicographic prudent orders have been found.

The results are summarized, using the median, in Table 7.4 (for profiles generated under the IC assumption) and in Table 7.5 (for profiles generated under the IC spiced up assumption). The column "All q" contains the median of the number of lexicographic prudent orders over 6000 profiles generated for a given value n and for the 6 different values for q.

The following observations can be made:

- If the number of alternatives n increases, the number of lexicographic prudent orders usually increases as well. This seems natural since increasing the number of alternatives dramatically increases the size of the set of linear orders. Let us however note the drop from 8 (in the case $n = 15$, $q = 10$) to 1 (in the case $n = 20$, $q = 10$) for profiles generated under the IC assumption.

- The impact of increasing the number of linear orders q in the profile remains unclear in both tables. However, one may notice that the number of lexicographic prudent orders seems to be higher for even profiles than for odd profiles. This is most striking for profiles with 6 linear orders generated under the IC assumption. In fact, even pro-

	$q = 5$	$q = 6$	$q = 7$	$q = 10$	$q = 15$	$q = 20$	All q
$n = 5$	100.00%	100.00%	100.00%	100.00%	99.90%	99.90%	99.97%
$n = 6$	100.00%	100.00%	99.90%	100.00%	100.00%	99.90%	99.97%
$n = 7$	100.00%	100.00%	99.70%	100.00%	99.50%	99.30%	99.75%
$n = 10$	100.00%	100.00%	98.40%	99.90%	96.40%	97.50%	98.70%
$n = 15$	97.20%	100.00%	92.30%	97.70%	86.60%	87.50%	93.55%
$n = 20$	94.60%	99.60%	81.60%	91.40%	71.00%	70.60%	84.80%

Table 7.6: The frequency of a profile u generated under the IC assumption with $\mathcal{LPO}(u) \subseteq \mathcal{K}(u)$.

files can lead to pairs with majority margins equal to zero, which may increase the number of equivalent (according to the leximin relation) optimal solutions.

- There is some evidence that the number of lexicographic prudent orders seems to be higher for profiles generated under the IC assumption than for profiles generated under the IC spiced up assumption. Our intuitive interpretation of this observation is that, since profiles generated under the IC spiced up assumption are more "cyclic", hence more complicated, the ranking rule can more easily discriminate and determine a "best" linear order.

In both tables, the number of lexicographic prudent orders remains reasonably low. Our aim with this empirical study is to show that the lexicographic prudent order preference function is useful in situations where the goal is to construct only a few solution rankings. This is in sharp contrast to the prudent order preference function, which may output a very large number of solution rankings (see the result of Debord's simulations presented in the appendix).

7.5.3 Intersection with Kemeny orders

Following the result stated in Proposition 14, we study in this section the frequency of the situation where $\mathcal{LPO}(u) \subseteq \mathcal{K}(u)$ for a given combination (n, q), both under the IC assumption and under the IC spiced up assumption. The results can be found in Tables 7.6 and 7.7.

The following observations can be made:

- If the number of alternatives increases, the frequency decreases. This can be explained by the fact that increasing the number of alternatives

	$q = 5$	$q = 6$	$q = 7$	$q = 10$	$q = 15$	$q = 20$	All q
$n = 5$	99.50%	99.40%	98.40%	98.80%	99.10%	98.90%	99.02%
$n = 6$	96.10%	97.50%	95.50%	98.10%	96.80%	97.20%	96.87%
$n = 7$	88.60%	91.90%	89.40%	95.10%	93.30%	94.10%	92.07%
$n = 10$	71.30%	76.30%	71.10%	77.40%	74.50%	81.10%	75.28%
$n = 15$	33.80%	38.60%	34.30%	36.60%	34.70%	42.20%	36.70%
$n = 20$	12.10%	14.10%	10.20%	13.50%	10.60%	15.00%	12.58%

Table 7.7: The frequency of a profile u generated under the IC spiced up assumption with $\mathcal{LPO}(u) \subseteq \mathcal{K}(u)$.

dramatically increases the number of linear orders, which may increase the likelihood that two ranking rules will lead to two different results.

- The frequencies for profiles generated under the IC assumption are considerably higher than the frequencies generated under the IC spiced up assumption. This is due to the fact that under the IC spiced up assumption we deliberately increase the cyclic part of a profile. As formalized by Saari [96] in his profile decomposition, it is precisely this part which is responsible for the differences between CRC rules.

- For profiles up to 10 alternatives generated under the IC assumption, the frequency is astonishingly close to 100 %. For profiles with 20 alternatives generated under the IC spiced up assumption, the frequency drops down to almost 10 %.

These simulations confirm our idea that the lexicographic prudent order preference function and Kemeny's rule are in general rather similar ranking rules.

7.6 Future directions of research

In this chapter we have introduced lexicographic prudent orders as a natural refinement of the linear orders obtained by the Ranked Pairs rule. We will come back to lexicographic prudent orders in Chapter 10 where we will argue that the use of weights is more transparent in the lexicographic prudent order preference function than in Kemeny's rule. Apart from this issue of weights, there are many interesting open questions about these lexicographic prudent orders which still deserve further investigation:

- The lexicographic prudent order preference function satisfies most of the axioms used in the characterizations presented in Chapter 5 and

Chapter 6. That is why, it should be realistic to characterize lexicographic prudent orders in this same axiomatic framework. This would be a significant step forward in building an axiomatic theory of prudent ranking rules.

- Kemeny's rule is one of the most widely studied and understood pairwise based ranking rules. That is why, it is important that the relationship between lexicographic prudent orders and the well known Kemeny orders should be further clarified. We believe that the lexicographic prudent order preference function can become a very tough competitor for Kemeny's rule.

- So far we computed lexicographic prudent orders by solving a particular linear optimization problem. This formulation is not ideal from a numerical point of view, since it has to handle very large numbers. More research should be spent on developing and testing more efficient algorithmic solutions. The complexity of the ranking rule could also be investigated.

Chapter 8

Paradoxical results compared to non-prudent ranking rules

The aim of this chapter is to compare prudent orders to the solutions obtained by four well-known ranking rules: Borda's rule, Copeland's rule, Slater's rule and Kemeny's rule. In fact, we show that the prudence principle is not compatible, at least not for all profiles, with these non-prudent ranking rules. The content of this chapter has been presented at the 8th International Conference of the Society for Social Choice and Welfare in Istanbul during July 2006 and will be published in [66]. I would like to thank Denis Bouyssou and Christian Klamler for their help and comments concerning this chapter.

The chapter is organized as follows. Section 8.1 contains some references to similar type of studies. In Section 8.2, we present some preliminaries which will be useful in the proofs later on. Section 8.3 is devoted to the comparison with Borda's and Copeland's rule, whereas Section 8.4 contains the comparisons with Slater's and Kemeny's rule. All the results are summarized in Section 8.5

8.1 Literature review

Comparing ranking rules can be done in a variety of ways. Often, two rules are compared using a set of properties that are each verified either for both rules or for one of the two rules. For instance, Lansdowne [69] compared the properties of prudent orders to other ranking rules, including Borda's rule. In this chapter however, we concentrate on the existence of profiles which give "contradictory" results. Although the ranking rules all seem, a priori, *reasonable*, their results may be rather different.

Rules	$\mathbf{n_{min}}$	**Type of result**	**Reference**
Dodgson and Kemeny	4	W/P	[90]
Copeland and Dodgson	4	W/P	[57]
Kemeny and Slater	4	W/L	[58]
Kemeny and Dodgson	3	W/L	[59]
Slater and Dodgson	3	W/L	
Borda and Dodgson	4	W. Opp.	[60]
Borda and Maximin	4	W. Opp.	[61]
Copeland and Maximin	4	W. Opp.	

Table 8.1: Overview of some papers dealing with paradoxical results between two ranking rules.

This approach has already been adopted to compare various pairs of ranking rules. We say that the weak order O_1 is weakly opposite to the weak order O_2 if:
$\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n,$

$$(a_i, a_j) \in O_1 \iff (a_j, a_i) \in O_2$$

and there exists at least one pair (a_k, a_l) such that

$$(a_k, a_l) \notin O_1 \iff (a_l, a_k) \notin O_2.$$

In Table 8.1, we list several of such results valid for profiles with a minimum number of n_{min} alternatives. "W. Opp." means that the ranking obtained with one rule can be weakly opposite to the ranking obtained with the second rule. "W/L" stands for the fact that there exists a profile for which the unique winner of one rule is the unique loser of the other rule. More generally, "W/P" means that there exists a profile for which the winner of one rule can be found at any position in the ranking of the other rule.

This chapter can be seen as a contribution to this type of comparisons between ranking rules. Although paradoxes related to prudent orders have been studied by Durand [39], the particular issue of comparing contradictory results has not been addressed yet.

8.2 Preliminaries

In this section, we make some preliminary remarks concerning the ranking rules which we will be addressing in this chapter. These remarks are useful in order to simplify the proofs.

We are mainly concerned with profiles such that the prudent order is unique, i.e. profiles u such that $|\mathcal{PO}(u)| = 1$. That is why, the following lemma dealing with a unique prudent order will be of help.

Lemma 5 *Let u be a profile of linear orders on n alternatives and let B_{ij} ($1 \leq i \leq n, 1 \leq j \leq n$) denote the majority margin between alternative a_i and a_j. If*

$$\min\{B_{12}, B_{23}, \ldots, B_{n-1n}\} > \max_{\substack{i,j:\\ i>j}} B_{ij},$$

then $a_1 a_2 a_3 \ldots a_n$ is the unique prudent order.

Proof: Let us define a relation T as follows:

$$T = \{(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)\}.$$

Let γ be defined as follows:

$$\gamma = \min\{B_{12}, B_{23}, \ldots, B_{n-1n}\} - 2.$$

Then $T \subseteq R_{>\gamma}$. Furthermore, $R_{>\gamma}$ is acyclic, since it contains only pairs (a_i, a_j) with $i < j$. In fact all the pairs (a_i, a_j) with $i > j$ have majority margins which are less or equal to γ:

$$\gamma = \min\{B_{12}, B_{23}, \ldots, B_{n-1n}\} - 2 \geq \max_{\substack{i,j:\\ i>j}} B_{ij}.$$

This implies that $\beta \leq \gamma$. Hence $R_{>\gamma} \subseteq R_{>\beta}$. If O is a prudent order, then we must have that $R_{>\beta} \subseteq O \Rightarrow R_{>\gamma} \subseteq O \Rightarrow T \subseteq O \Rightarrow t(T) \subseteq O$, where $t(.)$ denotes the transitive closure of a relation. Since the transitive closure of T is the linear order $a_1 a_2 a_3 \ldots a_n$, this must be the unique prudent order.

□

We define a prudent order winner (resp. loser) as an alternative which is ranked first (resp. last) in at least one linear order in $\mathcal{PO}(u)$.

According to definition 8 (see Section 2.2.1), a prudent ranking rule is a preference function such that $\forall u$, $f(u) \subseteq \mathcal{PO}(u)$. We have presented several prudent ranking rules in the last chapters. However, if the prudent order is unique, every prudent ranking rule must coincide and must exhibit exactly this unique prudent order. That is why we can replace in Theorems 9, 10, 11 and in Proposition 15 "... the unique prudent order ..." by "... the unique linear order found by any prudent ranking rule ..." and in Proposition 16, 17 and in Theorem 11 "... the unique prudent order winner ..." by "...the unique winner of any prudent ranking rule...". With this respect, our results can be applied to any prudent ranking rule.

In Section 2.2.2, we have presented Borda's rule, Copeland's rule, Slater's rule and Kemeny's rule. We will denote $\succeq_B$ and $\succeq_C$ the weak order obtained with Borda's rule or with Copeland's rule. A Borda or Copeland winner (resp. loser) is an alternative ranked first (resp. last) in the weak order $\succeq_B$ or in the weak order $\succeq_C$. Furthermore, we denote $\mathcal{S}(u)$ the set of Slater orders and $\mathcal{K}(u)$ the set of Kemeny orders of a profile u. A Slater or Kemeny winner (resp. loser) is an alternative ranked first (resp. last) in at least one linear order of $\mathcal{S}(u)$ or $\mathcal{K}(u)$.

In order to simplify the proofs, we rely in this chapter on different from the original but still equivalent definitions of Borda's and Kemeny's rule. Let us now present these two definitions.

First, we do not use the original Borda scores but rather compute a score based on the majority margins (see for instance Young [108]). This score, denoted by b_i, is defined for an alternative a_i as follows:
$\forall i : 1 \leq i \leq n$,

$$b_i = \sum_{k=1}^{n} B_{ik}.$$

The Borda ranking is then the weak order $\succeq_B$ defined as follows:
$\forall i, j : 1 \leq i \leq n, 1 \leq j \leq n$,

$$(a_i, a_j) \in \succeq_B \iff b_i \geq b_j.$$

Second, we use a definition proposed by Saari and Merlin [96] to define a Kemeny order. Let us evaluate a linear order O as follows:

$$g(O) = \sum_{(a_i,a_j) \in O : B_{ij} < 0} |B_{ij}|.$$

Then O_K is a linear order found by Kemeny's rule if and only if $g(O_K)$ is minimal.

When computing Slater orders, we often rely on the following lemma dealing with two three-cycles which have an ordered pair in common.

Lemma 6 *Let M be the strict majority relation of a profile u with at least four alternatives x, y, z and w. If*

$$(z, x) \in M \quad (y, z) \in M \quad (w, x) \in M \quad (y, w) \in M,$$

then $\forall O \in \mathcal{LO}$ with $(x, y) \in O$, we have that $\delta(O, M) \geq 2$.

Proof: If $(x,y) \in O$, then:

$$((x,z) \in O \text{ or } (z,y) \in O) \text{ and } ((x,w) \in O \text{ or } (w,y) \in O).$$

Hence O must differ from M by at least two ordered pairs.

In the same line, the following lemma deals with the function $g(.)$ used to determine Kemeny orders.

Lemma 7 *Let us consider a profile with at least four alternatives x, y, z and w. Let r and s be two positive integers. Let us suppose that the majority margins of the profile are such that:*

$$B_{y,z} \in \{r,s\} \quad B_{z,x} \in \{r,s\} \quad B_{y,w} \in \{r,s\} \quad B_{w,x} \in \{r,s\}.$$

Then $\forall O \in \mathcal{LO} : (x,y) \in O$, we have that

$$g(O) \geq \min\{s+r, 2s, 2r\}.$$

The proof of this lemma directly follows from the result recalled in Lemma 6. In fact, we know that any linear order O such that $(x,y) \in O$ will differ from the majority relation by at least two ordered pairs. Since the majority margin of these two ordered pairs is either r or s, $g(O)$ must at least be greater or equal to the minimum of $2s$, $2r$ and $r+s$.

Finally, we will often refer to Proposition 1 introduced in Section 2.1.

8.3 Prudent order vs. Borda's rule and Copeland's rule

We say that a linear order O_1 is the opposite of the linear order O_2 if: $\forall i,j : 1 \leq i \leq n, 1 \leq j \leq n$,

$$(a_i, a_j) \in O_1 \iff (a_j, a_i) \in O_2.$$

We are interested to construct profiles where the unique prudent order is the opposite of the ranking obtained with another ranking rule. To illustrate this idea, let us consider a profile with 46 linear orders and 4 alternatives which can be found in Table 8.2. The number in front of each linear order stands for the number of times that this particular linear order appears in the profile. The majority margins for this profile are depicted in Table 8.3.

Since the scores used to compute the Borda ranking for this example are –4 for a, –2 for b, 2 for c and 4 for d, the Borda ranking will be the linear order $dcba$. On the other hand,

$$\underbrace{\min\{B_{a,b}, B_{b,c}, B_{c,d}\}}_{=8} > \underbrace{\max\{B_{b,a}, B_{c,a}, B_{c,b}, B_{d,a}, B_{d,b}, B_{d,c}\}}_{=6}.$$

8	*abcd*	3	*cabd*
4	*dcab*	6	*dbca*
9	*bcda*	3	*dacb*
6	*adbc*	3	*acdb*
4	*cdba*		

Table 8.2: The profile.

B	a	b	c	d
a	.	8	–6	–6
b	–8	.	12	–6
c	6	–12	.	8
d	6	6	–8	.

Table 8.3: The majority margins.

According to Lemma 5, the unique prudent order of this example is *abcd*. Consequently, we have found an example with four alternatives where the ranking found by Borda's rule and the prudent order are opposite.

In this section, we prove that, also when n is larger than four, there exists a profile such that the unique prudent order of this profile is the opposite of Borda's ranking. The same result holds for Copeland's rule for profiles with at least five alternatives.

Theorem 9 *If $n \geq 5$, then there exists a profile of linear orders for which the unique prudent order is the opposite of the ranking obtained with Copeland's rule. If $n \geq 4$, then there exists a profile of linear orders for which the unique prudent order is the opposite of the ranking obtained with Borda's rule.*

Proof: The case for $n = 4$ for Borda's rule has been proved in the introductory example. Let us suppose from now on that $n \geq 5$. Let $r > 0$ and $s > 0$ be even numbers such that $2s > r > s$. It is of course possible to chose r and s such that these conditions are verified. Let the majority margins be defined as follows:
$\forall i, j : 1 \leq i \leq n, i < j \leq n,$

$$B_{ij} = \begin{cases} r & \text{if } j = i + 1 \\ 0 & \text{if } i = 2 \text{ and } j = n - 1 \\ -s & \text{otherwise} \end{cases}$$

According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile.

Let us compute the Copeland scores for this profile.

$$\begin{aligned} c_1 &= 2 \\ c_2 &= 3 \\ 2 < i < n-1\colon\ c_i &= 2(i-1) = 2i-2 \\ c_{n-1} &= 2(n-3)+1 = 2n-5 \\ c_n &= 2(n-2) = 2n-4 \end{aligned}$$

It is clear that $c_1 < c_2$ and $c_{n-1} < c_n$. Furthermore $c_2 = 3 < c_3 = 4$ and $c_{n-2} = 2n-6 < c_{n-1} = 2n-5$. Finally, $c_i < c_{i+1}$ $(2 < i < n-1)$ since $i > 0$. We can thus conclude that the Copeland ranking is the linear order $a_n a_{n-1} \dots a_2 a_1$.

Let us now consider the scores used to compute the Borda ranking for this same profile .

$$\begin{aligned} b_1 &= r-(n-2)s \\ b_2 &= (4-n)s \\ 2 < i < n-1\colon\ b_i &= (2i-1-n)s \\ b_{n-1} &= (n-4)s \\ b_n &= (n-2)s-r \end{aligned}$$

$b_1 < b_2$ and $b_{n-1} < b_n$ since we supposed that $r < 2s$. Furthermore, $b_2 < b_3$ and $b_{n-2} < b_{n-1}$ since $s > 0$. Finally, $b_i < b_{i+1}$ $(2 < i < n-1)$ since $s > 0$. We can thus conclude that the Borda ranking is the linear order $a_n a_{n-1} \dots a_2 a_1$.

Using Lemma 5, we know that $a_1 a_2 \dots a_n$ is the unique prudent order. This prudent order is the opposite of the ranking found by Copeland's rule or by Borda's rule. $\square$

Let us comment on this theorem. A similar result has already been obtained by Klamler [61] who showed that the maximin ranking can be opposite to the ranking found by Borda's rule and by Copeland's rule. Let us recall that the maximin ranking rule outputs the weak order $\succeq_{MM}$ defined as follows:
$\forall i, j : 1 \le i \le n, 1 \le j \le n,$

$$(a_i, a_j) \in \succeq_{MM} \iff \min_{k \neq i} B_{ik} \ge \min_{k \neq j} B_{jk}.$$

Although Kohler's rule and the maximin rule are two different ranking rules, Kohler's rule is closely related to the maximin rule in the sense that it simply applies the maximin choice function sequentially on the set of alternatives that have not been ranked yet (see Section 2.2.1). However, our result is stronger than Klamler's result since we consider linear orders that are opposite and not weak orders that are weakly opposite.

Moreover, let us also refer to the profile decomposition of Saari and Merlin (see Theorem 5 in [96]), which gives a further insight into discrepancies between the ranking obtained with Borda's rule and any ranking rule based on pairwise comparisons, such as for instance a prudent ranking rule.

Finally, one can show that this result of opposite linear orders cannot be obtained for profiles with $n \leq 4$ alternatives (for Copeland's rule) and $n \leq 3$ (for Borda's rule).

Proposition 15 *If $n \leq 4$, then the Copeland ranking cannot be the opposite of the unique prudent order.*

Proof: Let us suppose that $a_1a_2a_3a_4$ is the unique prudent order of a profile u. Then the majority margins of this profile u must be such that $B_{12} > 0, B_{23} > 0$ and $B_{34} > 0$. Consequently $B_{21} < 0$, $B_{32} < 0$ and $B_{43} < 0$. This implies that $\forall i \in \{1, 2, 3\}$ $c_i \geq 2$ and that $c_4 \leq 4$. Let us suppose by contradiction that the linear order $a_4a_3a_2a_1$ is the Copeland ranking. Then we must have that $c_1 < c_2 < c_3 < c_4$. Since $c_1 \geq 2$, this implies that $c_2 \geq 3$, which implies that $c_3 \geq 4$, which implies that $c_4 \geq 5$. This is not possible, since $c_4 \leq 4$. This proves that the Copeland ranking cannot be the opposite of the unique prudent order for profiles with 4 alternatives. Similarly, let us suppose that $a_1a_2a_3$ is the unique prudent order of a profile u. Then the majority margins of this profile u must be such that $B_{12} > 0$ and $B_{23} > 0$. Consequently $B_{21} < 0$ and $B_{32} < 0$. This implies that $c_1 \geq 2$, $c_2 \geq 2$ and $c_3 \leq 2$. Let us suppose by contradiction that the linear order $a_3a_2a_1$ is the Copeland ranking. Then we must have that $c_1 < c_2 < c_3$. Since $c_1 \geq 2$, this implies that $c_2 \geq 3$, which implies that $c_3 \geq 4$. This is not possible, since $c_3 \leq 2$. This proves that the Copeland ranking cannot be the opposite of the unique prudent order for profiles with 3 alternatives.

□

Proposition 16 *For $n = 4$, the unique prudent order winner can be the unique Copeland loser.*

Proof: Let us consider the following majority margins, where s and r are positive even integers such that $r > s$:

B	a_1	a_2	a_3	a_4
a_1	.	r	–s	–s
a_2	–r	.	r	0
a_3	s	–r	.	r
a_4	s	0	–r	.

According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. Using Lemma 5 we know that $a_1a_2a_3a_4$ is the unique prudent order. Since $c_1 = 2, c_2 = 3, c_3 = 4$ and $c_4 = 3$, a_1 is the unique Copeland loser. Consequently, we have constructed a profile where the unique prudent order winner is the unique Copeland loser. □

Proposition 17 *If $n = 3$, then the unique prudent order winner cannot be a Borda loser.*

Proof: Let us suppose that $a_1a_2a_3$ is the unique prudent order of a profile u. Then the majority margins of this profile u must be such that:

$$B_{12} > \max\{B_{21}, B_{32}, B_{31}\} \text{ and } B_{23} > \max\{B_{21}, B_{32}, B_{31}\}.$$

The first inequality implies that $B_{12} > B_{31}$ and the second inequality implies that $B_{23} > B_{31}$. Because of the constant-sum property, this last inequality implies that $B_{32} < B_{13}$. We now have that:

$$b_1 = B_{12} + B_{13} > B_{31} + B_{32} = b_3.$$

Consequently, a_1, the alternative ranked first in the unique prudent order, cannot be ranked last in the Borda ranking. □

8.4 Prudent order vs. Slater's rule and Kemeny's rule

Before focusing on possible inconsistencies between the prudent order preference function and Slater's rule and Kemeny's rule, let us note that it is not possible for any order found by Slater's rule or by Kemeny's rule to be opposite to the unique prudent order. More generally, it is not possible for any ranking rule that verifies a certain Condorcet property to be opposite to the unique prudent order.

Following Young and Levenglick's terminology [109], we say that a ranking rule f, which associates to a profile of linear orders u a set of linear orders $f(u)$, verifies the Young-Condorcet property if the following holds: if $B_{ij} > 0$, then a_j cannot directly precede a_i in any linear order $O \in f(u)$, i.e. it is not possible that $(a_j, a_i) \in O$ with no a_k such that $(a_j, a_k) \in O$ and $(a_k, a_i) \in O$. In particular, both Kemeny's rule and Slater's rule verify this Young-Condorcet property. Indeed, if a_j directly precedes a_i and $B_{ij} > 0$, permuting the two alternatives a_j and a_i would yield a ranking that is closer to the initial profile both in the Kemeny and Slater sense.

In order to prove the impossibility of a ranking obtained by a rule that verifies the Young-Condorcet property to be the exact opposite of the unique prudent order, we need the following lemma.

Lemma 8 *If O_P is the unique prudent order, then $\forall (a_i, a_j) \in O_P$ with a_i directly precedes a_j, we must have $B_{ij} > 0$.*

Proof: Let us suppose that $a_1 a_2 \ldots a_n$ is the unique prudent order. Then this means that $t(R_{>\beta})$ must be equal to this linear order, where $t(.)$ still denotes the transitive closure operator. Consequently, the chain

$$(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)$$

must be contained in $R_{>\beta}$. Let us suppose that there exists a pair (a_i, a_{i+1}) $(1 \leq i \leq n-1)$ belonging to this chain with $B_{i,i+1} \leq 0$. Consequently $B_{i+1i} \geq 0$ since $B_{ii+1} + B_{i+1i} = 0$. Since in such a case $\beta < B_{ii+1} \leq B_{i+1i}$, both pairs (a_i, a_{i+1}) and (a_{i+1}, a_i) belong to $R_{>\beta}$, which is impossible since $R_{>\beta}$ is an acyclic relation. That is why we must have that $B_{ii+1} > 0$. □

Theorem 10 *There does not exist a profile of linear orders with a unique prudent order such that this unique prudent order is the opposite of the result of a ranking rule that verifies the Young-Condorcet property.*

Proof: Let us suppose that $a_1 a_2 \ldots a_n$ is the unique prudent order. Following Lemma 8:

$$\forall i \in \{1, \ldots, n-1\} \quad B_{ii+1} > 0.$$

Let us now suppose that $a_n a_{n-1} \ldots a_2 a_1$ is the result obtained with a ranking rule that verifies the Young-Condorcet property. This would however contradict the Young-Condorcet property since, for instance, $B_{n-1n} > 0$ and a_{n-1} precedes a_n directly. □

Nevertheless, Arrow and Raynaud [2] (page 96) noticed already that the order found by Kemeny's rule is not necessarily a prudent order. In fact, the prudent order preference function and Slater's and Kemeny's rule are not connected in the following sense:

Theorem 11 *Let $n \geq 4$. Then there exists a profile of linear orders such that:*

- *the unique prudent order winner can be found at any position in the corresponding unique order found by Slater's rule and in the corresponding unique order found by Kemeny's rule.*
- *the unique Slater winner and the unique Kemeny winner can be found at any position in the corresponding unique prudent order.*

Proof: Let r and s be two even integers with $0 < s < r < 2s$.

First we show that if a_1 is the unique prudent order winner then we can always construct a profile u of linear orders such that a_1 can be found at any position in the corresponding unique order found by Slater's rule and by Kemeny's rule. Let us denote by ρ the rank of the alternative a_1 in the order found by Slater's rule and by Kemeny's rule. We will consider four cases, according to the value of ρ.

- $\rho = 1$
  Let us consider a profile with one unique linear order O. Trivially, $\mathcal{PO}(u) = \mathcal{S}(u) = \mathcal{K}(u) = \{O\}$, and so the unique prudent order winner is also the unique Slater winner and the unique Kemeny winner.

- $\rho = 2$
  Let us consider the following majority margins:
  $\forall i, j : 1 \leq i \leq n, i+1 \leq j \leq n$:

  $$B_{ij} = \begin{cases} -s & \text{if } i = 1, j = 3 \text{ or } i = 2, j = 4 \\ r & \text{otherwise.} \end{cases}$$

  According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. In fact, the majority margins are such that $\forall i \in \{1, \ldots, n-1\}$, $B_{ii+1} = r$, and $\max_{i,j:i>j} B_{ij} = s$. Since $s < r$, according to Lemma 5, $a_1 a_2 \ldots a_n$ is the unique prudent order and a_1 is the unique prudent order winner.

  We show that $O_S = a_3 a_1 a_4 a_2 a_5 \ldots a_n$ is the unique order found by Slater's rule. Let M denote the strict majority relation of u. Then $\delta(M, O_S) = 1 : (a_3, a_2) \in O_S$ and $(a_3, a_2) \notin M$. Any linear order O where $(a_3, a_2) \in O$ and $O \neq O_S$ is thus at least at distance 2 from M. Since there are two three-cycles (see Figure 8.1), using Lemma 6, the same holds for any linear order O where $(a_2, a_3) \in O$. Similarly, we show that $O_K = a_3 a_1 a_4 a_2 a_5 \ldots a_n$ is the unique order found by Kemeny's rule: $g(O_K) = r$. Any linear order O where $(a_3, a_2) \in O$ and $O \neq O_K$ has a larger value $g(O)$ since r or s will be added. Using Lemma 7 (see Figure 8.1), any linear order O such that $(a_2, a_3) \in O$ is such that $g(O) \geq \min\{2s, 2r, r+s\}$. Since we supposed that $r < 2s$, we must have that $g(O_K) = r < \min\{2s, 2r, r+s\} \leq g(O)$.

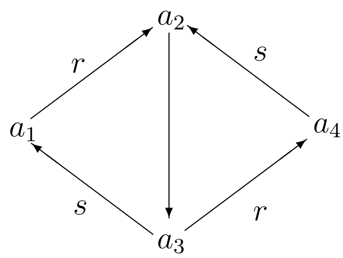


Figure 8.1: Two three-cycles in the case $\rho = 2$.

- $\rho = 3$
  Let us consider the following majority margins:
  $\forall i, j : 1 \leq i \leq n, i+1 \leq j \leq n,$

  $$B_{ij} = \begin{cases} -s & \text{if } i = 1, j = 3 \text{ or } i = 1, j = 4 \text{ or } i = 2, j = 4 \\ r & \text{otherwise.} \end{cases}$$

  According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. Similarly to case $\rho = 2$, we can show that $a_1 a_2 \dots a_n$ is the unique prudent order and consequently a_1 is the unique prudent order winner. Using the same strategy as in case $\rho = 2$, we can show that $a_3 a_4 a_1 a_2 a_5 a_6 \dots a_{n-1} a_n$ is the unique order found by Slater's rule and the unique order found by Kemeny's rule.

- $\rho \geq 4$
  Let us consider the following majority margins:
  $\forall i, j : 1 \leq i \leq n, i+1 \leq j \leq n,$

  $$B_{ij} = \begin{cases} -s & \text{if } i = 1 \text{ and } j \in \{3...\rho\} \\ r & \text{otherwise.} \end{cases}$$

  According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. Similarly to case $\rho = 2$, we can show that $a_1 a_2 \dots a_n$ is the unique prudent order and consequently a_1 is the unique prudent order winner. Using the same strategy as in case $\rho = 2$, we can show that $a_2 a_3 a_4 \dots a_1 \dots a_n$ is the unique order found by Slater's rule and the unique order found by Kemeny's rule.

We now show that then we can construct a profile u of linear orders such that $a_1 a_2 a_3 \dots a_n$ is the unique prudent order and a_t ($t \in \{1, \dots, n\}$) is the unique Slater winner or the unique Kemeny winner. This will prove that the unique Slater or Kemeny winner can be found at any position in the corresponding unique prudent order. We will consider four cases, according to the value of t.

- $t = 1$
  Let us consider a profile with one unique linear order O. Trivially, $\mathcal{PO}(u) = \mathcal{S}(u) = \mathcal{K}(u) = \{O\}$, and so the unique Slater and Kemeny winner a_1 is ranked first in the unique prudent order.

- $t = 2$
  Let us consider the majority margins treated previously in this proof under case $\rho \geq 4$. We arbitrarily fix $\rho = 4$. We have shown that $a_1 a_2 \dots a_n$ is the unique prudent order and $a_2 a_3 a_4 a_1 \dots a_n$ is the unique order found by Slater's rule and by Kemeny's rule. Hence a_2 is the unique Slater winner and the unique Kemeny winner.

- $t = 3$
  Let us consider the majority margins treated previously in this proof under case $\rho = 3$. We have shown that $a_1 a_2 \dots a_n$ is the unique prudent order and $a_3 a_4 a_1 a_2 \dots a_n$ is the unique order found by Slaters' rule and by Kemeny's rule. Hence a_3 is the unique Slater winner and the unique Kemeny winner.

- $t \geq 4$
  Let us consider the following majority margins:
  $\forall i, j : 1 \leq i \leq n, i + 1 \leq j \leq n,$

  $$B_{ij} = \begin{cases} -s & \text{if } i \in \{1, ..., t-2\} \text{ and } j = t \\ r & \text{otherwise.} \end{cases}$$

  According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. Similarly to case $\rho = 2$, we can show that $a_1 a_2 \dots a_n$ is the unique prudent order and consequently a_1 is the unique prudent order winner. Using the same strategy as in case $\rho = 2$, we can show that $a_t a_1 a_2 a_3 \dots a_{t-1} a_{t+1} \dots a_n$ is the unique order found by Slater's rule and the unique order found by Kemeny's rule.

A particular case of this result is that, for $n \geq 4$, there exists a profile such that the unique prudent order winner is the unique Slater loser or the unique Kemeny loser. Similarly, there exists a profile such that the unique Kemeny winner or the unique Slater winner is the unique prudent order loser.

Let us now analyze in further details the more general situation where the prudent order is not necessarily unique, which means that the set of prudent orders can possibly contain more than one linear order. First of all, for profiles defined on 3 alternatives, the sets $\mathcal{PO}(u), \mathcal{S}(u)$ and $\mathcal{K}(u)$ are closely linked.

Proposition 18 *For every profile u defined on 3 alternatives, we have:*

$$\mathcal{S}(u) \cap \mathcal{PO}(u) \neq \emptyset$$

and

$$\mathcal{K}(u) \subseteq \mathcal{PO}(u).$$

Proof: Let us consider the strict majority relation M. In case M is acyclic, we can show that the set of Kemeny orders, Slater orders and prudent orders all consist of all the linear extensions of M. Hence $\mathcal{K}(u) = \mathcal{S}(u) = \mathcal{PO}(u) = \{O \in \mathcal{LO} : M \subseteq O\}$.

Let us now suppose that M contains a cycle. Let us suppose that the alternatives are labelled such that $B_{12} > 0$, $B_{23} > 0$ and $B_{31} > 0$. Then $\mathcal{S}(u) = \{a_1a_2a_3, a_2a_3a_1, a_3a_1a_2\}$. Let us denote by p_1 the pair with the largest, p_2 with the second largest and p_3 with the third largest majority margin. We denote this as follows: $B(p_1) \geq B(p_2) \geq B(p_3) > 0$. We denote by $-p_1, -p_2, -p_3$ the reversed pairs. Every linear order can be seen as a triplet of pairs. For instance, the set of Slater orders can always be rewritten as follows: $\mathcal{S}(u) = \{(-p_1, p_2, p_3), (p_1, -p_2, p_3), (p_1, p_2, -p_3)\}$. Let us also recall that [96] mention that for profiles with three alternatives, no ties and with a cycle in the majority relation, the Kemeny order is obtained by reversing the pair in the cycle with the smallest majority margin. If this can be done in more than one way, we consider all the possibilities. We can thus easily compute the Kemeny orders and prudent orders in the following four possible cases:

	$B(p_1) > B(p_2) > B(p_3)$	$B(p_1) > B(p_2) = B(p_3)$
$\mathcal{K}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$
$\mathcal{PO}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$ $(p_1, -p_2, -p_3)$

	$B(p_1) = B(p_2) > B(p_3)$	$B(p_1) = B(p_2) = B(p_3)$
$\mathcal{K}(u)$	$(p_1, p_2, -p_3)$	$(p_1, p_2, -p_3)$ $(p_1, -p_2, p_3)$ $(-p_1, p_2, p_3)$
$\mathcal{PO}(u)$	$(p_1, p_2, -p_3)$	every lin. order with 3 alt.

The reader can check that the set relations stated in the proposition are all verified in this table.

□

Let us highlight that for profiles with three alternatives, the set of prudent order winners and Kemeny winners is always the same. The situation with respect to Slater's rule is more ambiguous. One may notice that for profiles with preference margins such that $B_{12} > B_{23} = B_{31} > 0$, $a_1a_3a_2$ is a prudent order and $a_2a_3a_1$ is an order found using Slater's rule. Hence, if the prudent order is not unique, then there can be profiles where one order found by Slater's rule is the opposite of one prudent order. A similar situation arises for profiles with preference margins such that $B_{12} = B_{23} = B_{31} > 0$. On the one hand, every linear order is a prudent order. Consequently $a_3a_2a_1$ is a prudent order. On the other hand, there are three linear orders found by Slater's rule and by Kemeny's rule, among which the linear order $a_1a_2a_3$.

We show in the next theorem that we can find profiles with at least 4 alternatives where the unique order found by Slater's rule or the unique order found by Kemeny's rule is the exact opposite of one prudent order.

Theorem 12 *Let $n \geq 4$. Then there exists a profile of linear orders such that the unique order found by Slater's rule or by Kemeny's rule is the opposite of a prudent order.*

Proof: Let r and s are two positive integers such that $s < r < 2s$. It is always possible to choose r and s such that these conditions are satisfied. Let us consider the following majority margins:
$\forall i, j : 1 \leq i \leq n, i+1 \leq j \leq n,$

$$B_{ij} = \begin{cases} -r & \text{if } i = 1 \text{ and } j = n \\ s & \text{otherwise.} \end{cases}$$

According to Proposition 1, there exists a profile u of linear orders such that B is the majority margin matrix of this profile. For such a profile, $\beta = s$, since $R_{>s} = \{(a_n, a_1)\}$ is an acyclic relation, whereas $\forall \lambda < s$, $R_{>\lambda}$ contains a cycle since $\{(a_1, a_2), (a_2, a_n), (a_n, a_1)\} \subseteq R_{>\lambda}$. The set of prudent orders thus consists of all the linear extensions of the relation $\{(a_n, a_1)\}$. In particular $a_na_{n-1} \dots a_2a_1$ is a prudent order. Using the same strategy as in the proof of Theorem 11 (case $\rho = 2$), we can show that $a_1a_2a_3 \dots a_n$ is the unique order found by Slater's rule and the unique order found by Kemeny's rule. We thus have found a profile u where the unique order found by Slater's or Kemeny's rule is the exact opposite of one prudent order.

□

	$n = 3$	$n = 4$	$n \geq 5$
Borda	NOT(W/L)	opp.	opp.
Copeland	NOT(opp.)	W/L NOT(opp.)	opp.
Slater	$\mathcal{S}(u) \cap \mathcal{PO}(u) \neq \emptyset$	W/L one opp. NOT(opp.)	W/L one opp. NOT(opp.)
Kemeny	$\mathcal{K}(u) \subseteq \mathcal{PO}(u)$	W/L one opp. NOT(opp.)	W/L one opp. NOT(opp.)

Table 8.4: Summary of paradoxical results.

It is not possible to state a corresponding theorem where the roles of Kemeny and Slater's rule have been switched with the prudent order preference function. In fact, we know already from Theorem 10 that a similar paradox cannot occur for profiles where the prudent order is unique but Slater's rule and Kemeny's rule lead to more than one linear order.

8.5 Summary

We now summarize the results that have been presented in this chapter in Table 8.5, where we use the following notation:

- W/L: the unique prudent order winner can be the unique loser of the other ranking rule.
- opp.: the unique prudent order can be the opposite of the unique order of the other ranking rule.
- one opp.: one prudent order can be the opposite of the unique order of the other ranking rule.

If paradox "opp." is possible, then paradox "W/L" and paradox "one opp." are also possible. We use the notation NOT(*) to state that the given paradox cannot occur.

Part III

Prudent ranking rules: applications

Chapter 9

The group ranking problem

In this chapter, we consider the problem of supporting a group in agreeing on a common compromise ranking. More particularly, we suggest to use the concept of robustness in order to manage the diversity of prudent orders. The content of this chapter has been presented at the Meeting of the EURO Working Group on Multicriteria Decision Aid held in Porto, Portugal, during March 2006 and has been published in [67]. I would like to thank Luis Dias for his help and comments concerning this part of the thesis.

The chapter is organized as follows. Motivations for the group ranking problem are given in Section 9.1. The methodological framework of our approach is motivated more in depth in Section 9.2. In Section 9.3, we compute robust conclusions on prudent orders. The strength of these robust conclusions is analyzed in Section 9.4, using simulations. In Section 9.5, we briefly address the issue of mutual preference probabilities and rank frequencies. We discuss an adaptation step in Section 9.6. The concepts introduced are illustrated on an example in Section 9.7.

9.1 Introduction to the group ranking problem

A group decision happens when a group of people have a common problem and they want to work together to reach a solution for this problem. In general, the members of this group are experts or decision makers belonging to the same organization (e.g. the same company). Despite the fact that they all have a common goal, finding a solution for their organization or their company, each individual has his own perception about the way to tackle the problem. Inevitably, conflicting preferences will emerge.

Bui [21] defines such a co-operative group decision-making situation as a process in which (i) there are two or more persons, each characterized by his or her own perceptions, attitudes, and personalities, (ii), who have recognized the existence of a common problem, and (iii), who attempt to use a system to reach a collective decision. In order to build this collective decision, Jelassi [51] notes that a crucial part of a group decision problem is the reduction of different individual preferences into a single collective preference.

Although involving multiple decision makers increases the complexity of a decision problem, it can contribute to enrich the decision process. For instance, the combined knowledge and experience of several people usually outranks the knowledge and experience of a unique decision maker. The interaction between the group members can also help to discover new points of view and arguments which a unique decision maker may not have thought of.

The group ranking problem can be approached in a variety of ways, ranging from a simple ranking solely based on, let's say, cost to complex multicriteria group ranking techniques. De Keyser & al. [56] differentiate between different types of multicriteria multidecision maker models. We follow here their taxonomy by presenting three distinct architectures:

1. In the first architecture, the group needs to agree on a multicriteria decision aid (MCDA) method, on the data (i.e. the evaluations of the alternatives on the criteria) and on the parameters of the selected MCDA method (weights, thresholds,...). The multicriteria problem is simply solved as in the single decision maker case by considering the group as one entity.

   Dias and Clìmaco [36] also refer to this scheme as "input aggregation". One drawback of this approach is that some group members may feel that there opinion is badly represented by the model. They may also be skeptical with respect to the MCDA method selected by the group.

2. In the second architecture, the group has also to agree on a MCDA method but each group member now defines his own multicriteria model by individually fixing the data and the parameters of the selected MCDA method. The MCDA method is applied separately for each group member up to a given stage where the results obtained (e.g. the net flows, a valued outranking relation ...) are aggregated into a common multicriteria group model which is subsequentially solved under the selected MCDA method. The fact that all the group members use the same method is thus crucial, since the combination of the individual models into a collective group model is specifically based on the particular MCDA method. In fact, it is very hard to combine in a

meaningful way preferential information obtained with distinct MCDA methods.

Dias and Clìmaco [36] also refer to this scheme as "output aggregation". This way, many multicriteria methods have been extended to handle group situations. This approach is for instance advocated in [74] for a group extension of the PROMETHEE method or in [72] for a group extension of the ELECTRE III method.

Although this second architecture values the individual opinions more than the first architecture, some group members may still not feel at ease by being confined to a particular MCDA method. Furthermore, the way the individual models are aggregated into a group model can sometimes be intransparent.

3. In the third architecture, no assumptions are made with respect to a particular MCDA method. In fact, every group member is free to use the MCDA method of his choice with the data of his choice. The only thing that matters is the final result, i.e. the ranking of the alternatives, that each group member will eventually obtain. In fact, a group member can use no MCDA method at all and simply randomly rank the alternatives, if he wishes to do so. In a second stage of the decision process, the individual rankings have then to be aggregated into a common group ranking.

We are going to adopt the last view, i.e. we suppose that each group member explicitly and honestly states his individual ranking. To support the group to reach a decision, we then suggest to use an ordinal ranking rule.

- An ordinal ranking rule, by opposition to simple informal group discussion, can avoid that in the end, it is not always the opinion of the hierarchical superior or simply of the one with the strongest personality that prevails. In fact, by relying on a formal aggregation mechanism, the preferences of each group member are somehow taken into account.

- An ordinal ranking rule can also reduce the time spent until a final decision has been taken, which in the end can also help reducing costs (see for instance the study of [82]).

The solving of a group ranking problem can take place in the framework of a Group Decision Support System (GDSS). As defined by Desanctis and Gallupe [97], a GDSS *combines communication, computing and decision support technologies to facilitate formulation and solution of unstructured problems by a group of people working together*. Hence, the whole system can be found at the intersection of computer supports (hardware), decision aid techniques and group-collaborative tools (software).

As soon as the group working in a GDSS environment has

1. clearly defined the set of alternatives,

2. agreed on the fact that a ranking of these alternatives should be obtained,

3. agreed on the fact that every one has to submit a complete ranking, despite the cognitive effort that this may put on some group members,

the use of an ordinal ranking rule can be considered. When working in such a computerized environment, apart from the choice of a particular rule, the following considerations, which have been discussed by Gavish and Gerdes [46], can influence the result:

- It is often possible that group members submit their preferences in an anonymous way. In some situations, this can be useful since it reduces people influencing each other and encourage them stating more honestly their preferences.

- Although traditionally the group meets in a specially designed conference room, meetings can nowadays be distributed both in time and in space. This means that the group members do not necessarily meet physically, but interact with other despite being in different locations or working at different moments.

- A group member's ranking could be annotated with some comments. This may help clarify his position or can be useful when interpreting results.

- Intermediate results can be made available to all or to only some group members at various moments during the decision process. By reacting to such an insight, the group ranking may be pushed into a particular direction.

Throughout this chapter, we assume that the input data provided by the group members consists of linear orders. This can be criticized, especially from an operational point of view. However, in the theoretical research in Part II of this thesis, we always assumed that the profile consists of linear orders. Moreover, the approach presented in the next sections can also be seen as a first step toward designing a similar decision aid tool which would allow for more complex preference structures.

9.2 A robust framework for using prudent orders

The basic assumption in this work is that the group agrees on the fact that the solution should be a prudent order. Although prudent orders are, in general, not unique, they depict however a whole range of possible, potentially interesting, compromise solutions. In case a group actually accepts to use such a preference function, we feel that there is a need to actually exploit the existence of multiple compromise rankings and to support the group in selecting the "right" compromise ranking.

More particularly, we suggest that the possible diversity of prudent orders should be handled by computing so-called robust conclusions, which, in the end, will support the group to select one prudent order. Following the terminology of Roy [92], a robust conclusion is an assertion valid for all the prudent orders.

Initially, the concept of robustness has been presented as a way to handle decision problems with imprecise or uncertain parameters (see Bisdorff [10] and Dias and Clìmaco [34, 35, 36]). In our setting however, the robustness issue arises conceptually because of the non-uniqueness of a compromise ranking, which can be seen as a consequence of the difficulty and ambiguity of aggregating ordinal data.

More generally, Dias [33] distinguishes between three roles of robustness in decision aid. A first approach sees robustness as an ex-ante concern, where a robustness criterion, such as for instance the maximin criterion, has to be optimized, or where a robustness condition has to be satisfied (see Kalaï and Lamboray [52] for two examples of such conditions). In a second approach, robustness is seen as an ex-post concern, where the various solutions that can be obtained from the various versions of the decision aid problem are analyzed, but not aggregated anymore. This is in line with the ideas of robust conclusions of Bernard Roy. Finally, in a third approach, robustness is used as a tool to progress in a decision aid process, for instance by helping to refine some parameters of the decision model. This work can be seen as a contribution that combines the last two roles and uses robustness both as an ex-post exploitation procedure and as a refinement tool in a decision aid process.

Concerning the ex-post exploitation, we compute the intersection of the prudent orders, the best and worst rank that an alternative can occupy in all the prudent orders and the maximal (or minimal) rank differences between any two alternatives.

One potential benefit of this approach relies in the fact that the information contained in the set of prudent orders is captured while reducing the cognitive load. The solutions are explicitly delimited, which can help the group to better understand the possibilities of comprise.

Furthermore, the fact that different prudent orders can be contradictory is not perceived as a problem anymore. The quality of the robust conclusions obtained are inversely proportional to the degree of contradictions contained in the initial profile. Incomparabilities in the intersection, different best and worst ranks and large maximal rank differences point out problematic alternatives or parts of the compromise ranking. Hence, the group has to concentrate on these parts in order to reach a final compromise ranking.

Following the ideas of Dias [33], the robustness concept is also used as a tool to progressively refine a decision model. In fact, by agreeing on some parts and by adapting accordingly their individual rankings, the group gradually moves toward a compromise ranking. The following four steps can thus be considered:

1. *Data collection step*
   First every group member proposes his individual ranking. This ranking can be obtained, for instance, by a multicriteria method of his choice.

2. *Aggregation step*
   The rankings provided by the group define a set of prudent orders. If requested, within this set of rankings, an automatic procedure can select the "best" one, according to some predefined criterion, such as for instance a lexicographic prudent order. This ranking is then proposed to the group. Either the group accepts this solution, and the procedure stops, or the group proceeds to the next step.

3. *Analysis and discussion step*
   The set of prudent orders can be described and analyzed by various robust conclusions. Such information will help the group to understand

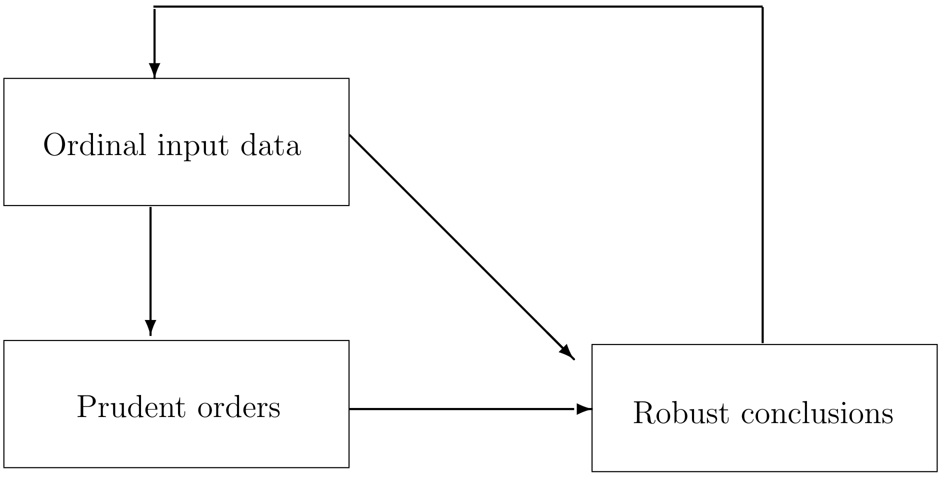


Figure 9.1: Computing robust conclusions.

the current possible compromise solutions.

4. *Adaptation step*
   Through the information learned from the previous step, each decision maker has the possibility to adapt his individual ranking in order to converge toward one compromise ranking. The group then proceeds back to step 1.

It is important to note that during the whole process, no solution is imposed to the group. The robust conclusions that are presented to the group can be considered as a guidance.

9.3 Computing robust conclusions

Instead of explicitly considering all the prudent orders belonging to $\mathcal{PO}(u)$, we are going to describe them by their intersection (Section 9.3.1), by the possible ranks that an alternative can occupy (Section 9.3.2) and by the maximal rank differences between two alternatives (Section 9.3.3). Although rank ranges and rank differences can be easily understood by a group, their cardinal character should be handled with care, especially in view of the ordinal character of the prudent order model.

Since the number of prudent orders can increase dramatically as the number of alternatives increases and as the contradictions in the profile increases, complete enumeration soon becomes infeasible from a computational point of view. Fortunately, for all the conclusions that we will consider in this section, no complete enumeration of all the prudent orders is necessary. This means that the robust conclusions can be directly deduced from the profile (see Figure 9.1). In fact, the computations are solely based on the equivalence between prudent orders and linear extensions.

9.3.1 Intersection

A natural robust conclusion is to state that an alternative a_i is preferred to an alternative a_j in all the prudent orders of $\mathcal{PO}(u)$:

$$\forall O \in \mathcal{PO}(u), \quad (a_i, a_j) \in O.$$

This is equivalent to stating that (a_i, a_j) belongs to the intersection of all the prudent orders. We know from Proposition 3 (see Section 4.1) that the set $\mathcal{PO}(u)$ is equivalent to all the linear extensions of the partial order $t(R_{>\beta})$. Since the intersection of all the linear extensions of a partial order is the partial order itself (see Dushnik and Miller [40]), we have :

$$\bigcap_{O \in \mathcal{PO}(u)} O = t(R_{>\beta}) = P$$

Hence, we do not need to explicitly enumerate all the prudent orders of $\mathcal{PO}(u)$ in order to know their intersection. Instead it suffices to compute one transitive closure. The transitive closure can be computed by using for example Roy and Warschall's algorithm [107].

Moreover, the P relation can be interpreted as the minimal compromise that can be reached with a given profile. The richness or strength of this relation will be further analyzed in Section 9.4. In case, the group members are willing to achieve a stronger compromise, they should bring closer their individual rankings with this P relation. This idea will be further formalized in Section 9.6.

9.3.2 Rank range

We now concentrate on the possible ranks that an alternative can occupy in the compromise ranking. By convention, if an alternative is ranked first in a linear order O, it has rank 1, if it is ranked second, it has rank 2 and so on. We denote by $\rho_O(a_i)$ the rank of alternative a_i in the linear order O. Let ρ_i^+ and ρ_i^- be the best and the worst rank a given alternative a_i occupies: $\rho_i^+ = \min_{O \in \mathcal{PO}(u)} \rho_O(a_i)$ and $\rho_i^- = \max_{O \in \mathcal{PO}(u)} \rho_O(a_i)$. Computing maximal and minimal ranks that an alternative occupies in the solutions of a preference function has already been suggested by Guénoche [48] in the context of determining median orders of difficult valued tournaments. However his motivation was rather algorithmic.

Since the set $\mathcal{PO}(u)$ consists of all the linear extensions of the partial order $t(R_{>\beta})$, one can show (see for instance Bruggemann [19]) that the best rank and the worst rank can be computed as follows:
$\forall a_i \in \mathcal{A}$,

$$\rho_i^+ = |N_i^+| + 1 \quad \text{where } N_i^+ = \{a_j \in \mathcal{A} : (a_j, a_i) \in t(R_{>\beta})\} \tag{9.1}$$
$$\rho_i^- = n - |N_i^-| \quad \text{where } N_i^- = \{a_j \in \mathcal{A} : (a_i, a_j) \in t(R_{>\beta})\}. \tag{9.2}$$

Once ρ_i^+ and ρ_i^- are computed, a robust conclusion would be to state that the rank of alternative a_i is higher or equal than ρ_i^+ and smaller or equal than ρ_i^-. Furthermore, for each r such that $\rho_i^+ \leq r \leq \rho_i^-$, we know that there exists at least one prudent order where alternative a_i has rank r. Hence, apart from extreme rank values, the whole rank range is covered. The difference $\rho_i^- - \rho_i^+$, which is called the variability of a_i by Bruggemann [19], gives an indication about the degree of contradictions or uncertainties concerning the alternative a_i. In fact, the bigger this difference is, the more unclear it is assigning a rank in the compromise ranking to a_i.

9.3.3 Maximal rank differences

We are interested in the largest possible rank difference between alternative a_j and alternative a_i. This difference quantifies the possible advantage of an alternative a_i over an alternative a_j. Formally, we compute the following quantity.
$\forall a_i, a_j \in \mathcal{A}$,

$$\Delta_{ij}^{max} = \max_{O \in \mathcal{PO}(u)} (\rho_O(a_j) - \rho_O(a_i)).$$

More precisely, if $\Delta_{ij}^{max} \geq 0$, then this means that, at best, alternative a_i is ranked Δ_{ij}^{max} positions ahead of alternative a_j. If $\Delta_{ij}^{max} < 0$, then this means that a_i will always be at least Δ_{ij}^{max} ranks below a_j.

These quantities can be computed as follows:

Proposition 19 $\forall a_i, a_j \in \mathcal{A}$,

$$\Delta_{ij}^{max} = \begin{cases} n-1-(|N_j^-|+|N_i^+|) & \textit{if } (a_j,a_i) \notin t(R_{>\beta}) \\ |N_i^- \cap N_j^+|+1 & \textit{if } (a_j,a_i) \in t(R_{>\beta}) \end{cases}$$

Proof:

- $(a_j, a_i) \notin t(R_{>\beta})$
  Since $n-1-(|N_j^-|+|N_i^+|) = \rho_j^- - \rho_i^+$, we know that

  $$\Delta_{ij}^{max} \leq n-1-(|N_j^-|+|N_i^+|).$$

  We prove that there exists one linear order $O \in \mathcal{PO}(u)$ (i.e. one linear extension of $t(R_{>\beta})$) such that this upper bound can be reached.

  A linear extension of a partial order can be constructed sequentially as follows: remove any maximal element from the partial order and rank it below the already ranked alternatives in the linear extension; stop the procedure when all the alternatives have been ranked. In our case, this procedure can be applied to the partial order $t(R_{>\beta})$. Let N_i^+ and N_j^- be defined as in (9.1) and (9.2) and let $\bar{N}_{ij} = \mathcal{A} \setminus (N_i^+ \cup N_j^-)$. First, we rank all the alternatives belonging to N_i^+, then we rank all the alternatives from $\bar{N}_{ij}$ (ranking a_i first and ranking a_j last) and finally we rank all the alternatives belonging to N_j^-. One can check that at each step it is possible to find a maximal element in the relevant subset of alternatives. Let us also note that $N_i^+ \cap N_j^- = \emptyset$, because otherwise there would exist an alternative a_t such that $(a_t, a_i) \in t(R_{>\beta})$ and $(a_j, a_t) \in t(R_{>\beta})$, which contradicts the fact that $(a_j, a_i) \notin t(R_{>\beta})$. Consequently, the rank of a_i in this order is $|N_i^+|+1$, the rank of a_j is $n-|N_j^-|$ and so the rank difference between a_i and a_j is $n-1-(|N_j^-|+|N_i^+|)$.

- $(a_j, a_i) \in t(R_{>\beta})$
  Since $(a_j, a_i) \in t(R_{>\beta})$, this implies that a_j is always above a_i in all the prudent orders of $\mathcal{PO}(u)$. In order to maximize the quantity $\rho_O(a_j) - \rho_O(a_i)$, we want a_i to be as close as possible to a_j. Since the alternatives of $N_i^- \cap N_j^+$ have to be anyway between a_j and a_i, we must have that $\Delta_{ij}^{max} \leq |N_i^- \cap N_j^+|+1$. We prove that there exists one linear order $O \in \mathcal{PO}(u)$ (i.e. one linear extension of $t(R_{>\beta})$) such that this upper bound can be reached by using the procedure described in the previous paragraph. First, we rank the alternatives belonging to $N_j^+ \setminus N_i^-$, then we rank a_j, then we rank the alternatives belonging to $N_i^- \cap N_j^+$, then we rank a_i and finally we rank the remaining alternatives. □

9.4 Strength of the conclusions obtained

We have seen that all the robust conclusions introduced in the preceding section depend on the relation $P = t(R_{>\beta})$. On the one hand, this relation should only be as rich as the ordinal data contained in the profile allows it to be since the strength of this relation should reflect the level of difficulty to construct a compromise ranking. On the other hand, we do not want this relation to be too empty, otherwise the decision aid resulting from the robust conclusions is too weak. In fact, the poorer this relation, the more undetermined the robust conclusions will be.

In order to evaluate the strength of this P relation, we adopt an empirical simulation approach. We chose to evaluate the solution of a profile u as follows:

$$\mu(u) = \frac{|t(R_{>\beta})|}{\frac{n(n-1)}{n}}.$$

An ordered pair that belongs to $t(R_{>\beta})$ actually belongs to all the prudent orders. For such a pair (a_i, a_j), the aggregation problem has been solved, since all the prudent orders agree on the preference direction between a_i and a_j. The size of the relation $t(R_{>\beta})$ is thus equal to the number of pairs for which the aggregation problem has been solved. We divide this by $\frac{n(n-1)}{n}$, which is the number of pairs of a linear order with n alternatives.

If, on the one hand $t(R_{>\beta})$ is a linear order, then this is the unique prudent order. In that case, the aggregation problem has been unambiguously solved and $\mu(u) = 1$. If, on the other hand, $t(R_{>\beta})$ is empty, then every linear order is a prudent order. In that case, no decision aid (with respect to aggregating) has been provided and $\mu(u) = 0$. Hence μ is an indicator between 0 and 1: the closer it is to 0, the poorer the result is, and the closer it is to 1 the richer the result is.

Let us note that $\mu(u)$ measures more accurately the determinateness of the prudent order model than for instance $|\mathcal{PO}(u)|$, the number of prudent orders of profile u. In fact, the number of prudent orders does not tell us anything about how these prudent orders are correlated.

As in Chapter 7, we chose to perform tests with n (the number of alternatives) being equal to 5, 6, 7, 10, 15 and 20 and q (the number of linear orders or group members) being equal to 5, 6, 7, 10, 15 and 20. Such values for q and n seem reasonable in the context of the group ranking problem. For each combination n and q, 10000 simulations have been performed under the Impartial Culture. The results can be found in Table 9.1.

The following observations can be made:

	$q=5$	$q=6$	$q=7$	$q=10$	$q=15$	$q=20$
$n=5$	0.72	0.73	0.75	0.77	0.82	0.81
$n=6$	0.62	0.73	0.66	0.74	0.76	0.78
$n=7$	0.54	0.71	0.60	0.70	0.72	0.74
$n=10$	0.43	0.61	0.54	0.56	0.64	0.64
$n=15$	0.44	0.39	0.56	0.44	0.51	0.56
$n=20$	0.46	0.27	0.56	0.44	0.42	0.48

Table 9.1: The average μ-value for profiles with n alternatives and q linear orders.

- For q fixed, in general, if the number of alternatives n goes up, then there is a tendency that μ goes down. However, for $q = 5$, the quality goes down until profiles with 10 alternatives before increasing again thereafter.

- For n fixed, if the number of linear orders q goes up, then there is a tendency that μ usually goes up as well.

- Following the last two remarks, the best quality is obtained for the case $n = 5$ and $q = 20$ (average $\mu = 0.81$), whereas the worst quality is obtained with $n = 20$ and $q = 6$ (average $\mu = 0.27$).

- 27 out of the 36 possible combinations of cases have an average quality larger than 0.5. This means that, in average, such profiles give a result that is at least as rich as the information contained in “half” a linear order.

9.5 Mutual preference probabilities and rank frequencies

A set of linear extensions can be described with mutual preference probabilities or with rank frequencies. Such computations could also be considered in our framework if one wishes to enrich the robust information obtained so far.

Instead of simply knowing that a_i is not preferred to a_j in all the prudent orders, a richer information consists in actually knowing in how many prudent orders a_i is nevertheless preferred to a_j. Formally, the mutual preference probability π_{ij} between a_i and a_j is defined as follows:

$$\forall a_i, a_j \in \mathcal{A}, \quad \pi_{ij} = \frac{|\{O \in \mathcal{PO}(u) : (a_i, a_j) \in O\}|}{|\mathcal{PO}(u)|}$$

In a similar way, the rank frequency f_{ir} indicates how often an alternative a_i actually occupies a given rank r.

$$\forall a_i \in \mathcal{A}, \forall r \in \{1, \ldots, n\}, \quad f_{ir} = \frac{|\{O \in \mathcal{PO}(u) : \rho_O(a_i) = r\}|}{|\mathcal{PO}(u)|}$$

An interesting property concerning rank frequencies is that they are log-concave (see Daykin & al [27]). This means that the rank frequencies for "extreme" ranks (e.g. for ρ_i^+ and ρ_i^-) are smaller than the rank frequencies for the ranks in between ρ_i^+ and ρ_i^-. Hence we might want to tighten the rank range by ignoring some extreme rank possibilities.

Although it looks tempting to use such statistics in order to further discriminate between the prudent orders, these results should however handled with care and their interpretation can be ambiguous.

First of all, mutual preference probabilities should not be confused with preference intensities, which are given by the majority margin matrix B. Usually the information contained in B is different from the information given by the mutual preference probabilities.

Furthermore, mutual preference probabilities can create new intransitivities. For instance, it can happen that the mutual preference probability between a_1 and a_2, a_2 and a_3, and a_3 and a_1 is always strictly larger than $\frac{1}{2}$ (see Fishburn [43] for such an example). This is especially disturbing, since the aim of prudent orders is exactly to achieve a transitive result from a profile that can contains intransitivities.

Another problem related to these statistics concerns their practical computation. Counting linear extensions of a partial order is shown by Brightwell and Winkler [18] to be a #P-complete problem. If explicit enumeration, using for instance the algorithm of Pruesse and Ruskey [89], becomes out of reach, other strategies have to be considered. For instance, De Loof & al [73] proposed an approach that does not explicitly enumerate all the linear extensions, but achieves to count linear extensions, mutual preference probabilities and rank frequencies by exploiting the lattice of ideals. This is more efficient than complete enumeration since the number of ideals is less than the number of linear extensions. Nevertheless, the number of ideals can still be exponential.

An alternative approach to enumeration is to consider randomized approximation algorithms. For instance, Bubley and Dyer [20] proposed a polynomial time algorithm that approximates the number of linear extensions within a given tolerance. Let us also mention Lerche and Sorensen [71], who proposed to compute the rank frequencies by randomly generating linear extensions.

9.6 Adaptation step

Let us suppose that the profile $u = (O_1, O_2, \ldots, O_q)$ provided by the group members have been aggregated into the set of prudent orders $\mathcal{PO}(u)$. We now assume that a unique final compromise ranking has not been reached so far. The robust information introduced in the previous section should help the group to understand the current possibilities of compromise. We then consider the following two possibilities:

1. Several or all the group members agree to adapt their rankings. This new profile leads to a new set of prudent orders. The analysis explained in the previous section can then be reapplied to that new set of prudent orders.

2. The group agrees on an ordered block partition of the alternatives. For instance, the group agrees on the fact that a and b occupy the first two positions of the compromise ranking, whereas the remaining alternatives have at least rank 3. This is an example of a 2-block partition, but, more generally, we can consider any ordered block partition $C_1, C_2, \ldots C_r$ of $\mathcal{A}$. Each block can then be reexamined separately.

We analyze now the possible convergence of the set of prudent orders in these two possibilities. Let us recall that P is the intersection of all the prudent orders of the initial profile u. This relation P will be crucial when analyzing the possible convergence of the compromise.

We denote $u^{new} = (O_1^{new}, O_2^{new}, \ldots, O_q^{new})$ the new rankings of the group members.

Definition 14 *We say that the adaptation from O_k into O_k^{new} ($k \in \{1, \ldots, q\}$) is not against the compromise if:*

$$\forall a_i, a_j : \quad (a_i, a_j) \in O_k^{new} \wedge (a_i, a_j) \notin O_k \Rightarrow (a_i, a_j) \in P.$$

Proposition 20 *If $\forall k \in \{1, \ldots, q\}$ the adaptation from O_k into O_k^{new} is not against the compromise, then $\mathcal{PO}(u^{new}) \subseteq \mathcal{PO}(u)$.*

Proof:

Let $R_{>\lambda}$ be the cut-relation at level λ and β be the optimal cut-value for profile u. Let $R_{>\lambda}^{new}$ be the cut-relation at level λ and β^{new} be the optimal cut-value for profile u^{new}. Since the transformation $\forall k$ from O_k into O_k^{new} is not against the compromise, we have that:

$$\begin{array}{rl} & R_{>\beta} \subseteq \; R_{>\beta}^{new} \subseteq R_{>\beta} \cup \{(a_i,a_j) : \exists k \text{ with } (a_i,a_j) \in O_k^{new} \text{ and } (a_i,a_j) \notin O_k\} \\ \Rightarrow & R_{>\beta} \subseteq \; R_{>\beta}^{new} \subseteq R_{>\beta} \cup \{(a_i,a_j) : (a_i,a_j) \in t(R_{>\beta})\} \\ \Rightarrow & R_{>\beta} \subseteq \; R_{>\beta}^{new} \subseteq t(R_{>\beta}). \end{array}$$

Since $t(R_{>\beta})$ is acyclic, $R_{>\beta}^{new}$ is acyclic, which implies that $\beta^{new} \leq \beta$. Hence:

$$R_{>\beta} \subseteq R_{>\beta}^{new} \subseteq R_{>\beta^{new}}^{new}.$$

This means that $\mathcal{PO}(u^{new}) \subseteq \mathcal{PO}(u)$. □

Hence if the adaptations of all the group members are in favor of the compromise, then convergence is ensured. It is thus reasonable to encourage the group to agree with P as much as possible. However, the group should not be obliged to stick to this type of adaptations and the possibility should be given to slightly shift the focus of the set of prudent orders. Hence, it may happen that, after adapting individually, the new rankings in u^{new} may yield new contradictions and cycles, avoiding thus a clear convergence.

Second, let us suppose that the group applies the ordered block partition approach. Let $C_1, \ldots, C_r$ be an ordered block partition of the alternatives.

Definition 15 *We say that this partition is compatible with the compromise if:*

$$\forall a_i, a_j \in \mathcal{A} \textit{ with } a_i \in C_k \textit{ and } a_j \in C_l \textit{ and } k < l \Rightarrow (a_j, a_i) \notin P.$$

When examining the blocks separately, the next proposition says that the set of compromise rankings can possibly converge. Let u_{C_i} be the profile restricted to the alternatives belonging to a block C_i. $\mathcal{PO}(u_{C_i})$ thus corresponds to the prudent orders of that restricted profile. Furthermore, let $(\mathcal{PO}(u))_{C_i}$ be the set of prudent orders of the profile u, but restricted to the alternatives of block C_i.

Proposition 21 *Let $C_1, C_2, \ldots, C_r$ be an ordered block partition compatible with the compromise. Then*

$$\forall C_l \quad (l \in \{1, \ldots, r\}), \quad \mathcal{PO}(u_{C_l}) \subseteq (\mathcal{PO}(u))_{C_l}.$$

Proof: We show that a block C_l is P-compatible, according to the definition introduced in Section 4.4. Let us suppose that $a_i \in C_l$ and $a_j \in C_l$, and there exists a_k such that $(a_i, a_k) \in C_l$ and $(a_k, a_j) \in C_l$. We are going to show that a_k must also belong to C_l. Suppose (by contradiction) that a_k belongs to higher block $C_{<l}$. This contradicts the fact that the block

partition is compatible with the compromise since $(a_i, a_k) \in P$, but a_i is in a lower block than a_k. Suppose that a_k belongs to lower block $C_{>l}$. This contradicts the fact that the block partition is compatible with the compromise since $(a_k, a_j) \in P$, but a_j is in a higher block than a_k. Hence a_k must belong to bloc C_l. Consequently, we can apply Proposition 6 (see Section 4.4) and conclude that $\mathcal{PO}(u_{C_l}) \subseteq (\mathcal{PO}(u))_{C_l}$.

□

Apart from these theoretical convergence results, more efforts should be spend in developing tools which support the group members during this adaption step. For instance, a tool may be needed that supports the group members in adapting their rankings in an efficient way or that suggest to the group "good" block partitions.

9.7 Illustration: Ranking FNR research domains

In order to illustrate our approach, we present an example where a group of junior researchers, mainly working in the field of information technologies in various research institutions in Luxembourg, were asked to rank a set of research domains. The problem took place in the framework of the Foresight exercise [29] organized by the Luxembourg FNR (Fonds National de la Recherche). The aim of this project is to identify socio-economic needs in order to decide on scientific research domains for Luxembourg in the medium and long term.

To achieve this goal, stakeholders were involved and, in particular, a one day workshop was organized for a group of junior researchers. During that day, 40 different research domains were put forward, amongst which 11 were finally selected as the most pertinent by the group of participants. They are listed in Figure 9.2. The actual workshop ended at this stage.

The participants were asked after the workshop to rank these research domains. This ranking should represent their view on the prioritization of the different research domains. Let us stress that this is not an application, but only an illustration of the methodology presented in this chapter. After having submitted their individual rankings, the researchers were not confronted with the results presented in this section.

Since it can be difficult for a participant of the workshop to quantify the difference of importance between two research domains, it is reasonable to use an ordinal scale. Furthermore, working with rankings as input avoids to fix a common evaluation method for the whole group.

a_1	Knowledge management technology
a_2	E-Government
a_3	IT security
a_4	Electronic cooperation networks
a_5	Mobile communications
a_6	Innovative materials and techniques in construction
a_7	Finance and banking sector
a_8	Business improvement research
a_9	Image processing
a_{10}	Human-Machine interface
a_{11}	Artificial intelligence, multi-agent systems

Table 9.2: The 11 research domains.

uni1	$a_4a_5a_3a_2a_8a_7a_1a_{11}a_{10}a_9a_6$
uni2	$a_3a_9a_4a_5a_2a_1a_6a_{11}a_7a_8a_{10}$
uni3	$a_4a_5a_2a_3a_1a_6a_7a_8a_{11}a_{10}a_9$
uni4	$a_1a_6a_8a_7a_{11}a_{10}a_3a_2a_5a_9a_4$
uni5	$a_{11}a_1a_9a_5a_7a_{10}a_6a_4a_3a_2a_8$
iee	$a_6a_5a_7a_4a_2a_3a_8a_1a_9a_{11}a_{10}$
cvce	$a_1a_{10}a_3a_4a_5a_2a_8a_7a_{11}a_9a_6$
tud1	$a_1a_4a_5a_8a_7a_3a_2a_9a_{10}a_{11}a_6$
tud2	$a_7a_8a_1a_4a_3a_9a_{10}a_{11}a_5a_6a_2$
lip	$a_4a_1a_2a_5a_6a_7a_{10}a_9a_3a_{11}a_8$

Table 9.3: The rankings of the 10 researchers.

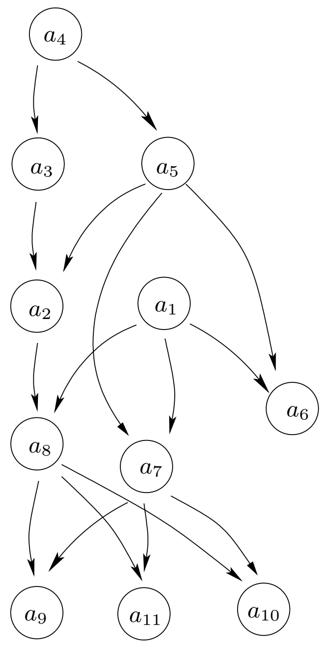


Figure 9.2: The intersection of all the prudent orders.

The rankings of the 10 researchers are represented in Table 9.3. This profile of 10 linear orders with 11 alternatives leads to 1350 prudent orders, which means that $\frac{1350}{11!} = 0.0034\%$ of all the linear orders on 11 alternatives are prudent orders. A straightforward choice of one prudent order seems to be practically impossible. In order to go on with the decision process, several robust informations are computed.

First of all, the intersection of all the prudent orders is represented in Figure 9.2 (the transitivity arcs are omitted). We can learn from this figure that, for instance, a_4 (Electronic cooperation networks) is preferred to a_7 (Finance and banking sector) in all the prudent orders.

The rank ranges are depicted in Table 9.4. For instance the rank of a_4 (Electronic cooperation networks) will be either 1 or 2. On the other hand, the rank of a_6 (Innovative materials) seems to be unclear since, at best this alternative has rank 4 but at worst it is ranked last.

Finally, the maximal rank differences are depicted in Table 9.5. For instance, at best, a_1 (Knowledge management technology) can be six ranks ahead of a_2 (E-Government), whereas a_2 can only be, at best, one rank ahead of a_1. There are also negative maximal rank differences. For instance, a_{11}(Artificial intelligence) will always be at least 3 ranks below a_1.

alt.	ρ^+	ρ^-
a_1	1	7
a_2	3	7
a_3	2	6
a_4	1	2
a_5	2	4
a_6	4	11
a_7	4	5
a_8	6	9
a_9	7	11
a_{10}	7	11
a_{11}	7	11

Table 9.4: The rank ranges of the prudent orders.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
a_1	0	6	5	1	3	10	7	7	10	10	10
a_2	1	0	−1	−3	−1	7	4	4	7	7	7
a_3	3	5	0	−1	2	9	6	6	9	9	9
a_4	4	6	5	0	3	10	7	7	10	10	10
a_5	3	5	4	−1	0	9	6	6	9	9	9
a_6	−1	3	2	−2	−1	0	4	4	7	7	7
a_7	−1	3	2	−2	−1	7	0	4	7	7	7
a_8	−1	−1	−2	−4	−2	5	2	0	5	5	5
a_9	−3	−2	−3	−6	−4	3	−1	−1	0	3	3
a_{10}	−3	−2	−3	−6	−4	3	−1	−1	3	0	3
a_{11}	−3	−2	−3	−6	−4	3	−1	−1	3	3	0

Table 9.5: Maximal rank differences.

Among the 1350 prudent orders, there are 18 lexicographic prudent orders. One of these rankings, could be automatically proposed to the group. Another possibility is to perform robust conclusions on these 18 rankings. For instance, the best and worst rank of each alternative in all these lexicographic prudent orders are indicated in Table 9.6. This information is of course consistent with the information in Table 9.4, since lexicographic orders are a refinement of prudent orders. However, unlike for prudent orders, the extreme ranks of lexicographic prudent orders do not determine rank ranges: for instance, we know that there exists at least one lexicographic prudent order where a_6 has rank 7 and at least one where a_6 has rank 11, but we do not have any guarantee that there also exists at least one lexico-

alt.	ρ^+	ρ^-
a_1	1	2
a_2	6	6
a_3	5	5
a_4	1	2
a_5	3	3
a_6	7	11
a_7	4	4
a_8	7	8
a_9	8	11
a_{10}	9	11
a_{11}	8	10

Table 9.6: Best and worst rank of the lexicographic prudent orders.

graphic prudent order where a_6 has rank 8, 9 or 10.

When analyzing the rank ranges of prudent orders, apart from the more problematic alternative a_6 (Innovative materials), let us assume that the group agrees to adopt the block partition approach in order to move toward a compromise. Let us suppose that three blocks are identified. A first block could contain a_1, a_3, a_4 and a_5, a second block could contain a_2, a_7 and a_8 and a third block could contain a_9, a_{10} and a_{11}.

Furthermore, let us assume that the group agrees on the fact that the more problematic alternative a_6 belongs to the middle block. Since the best rank of a_6 is 4, a_6 does not really belong to the first block. Taking a closer look at the rank differences, the ability of a_6 to be much higher ranked than a_9, a_{10} and a_{11} does not make it a very bad alternative neither. Consequently, a_6 fits best in the middle part of the global ranking. Let us note that this 3 block partition is compatible with the P relation.

Given the 3 block partition, the new rankings of the 10 participants can be found in Table 9.7. We then compute the set of prudent orders separately in each block. Given this new data, the intersection of all the prudent orders can be found in Figure 9.3 (the transitivity arcs are omitted). The final ranking is now almost complete. In fact, the still incomparable pairs are pairs such that there are always five group members who prefer the first over the second alternative and five group members who prefer the second over the first alternative. In order to achieve a complete ranking, the group now has to concentrate on these pairs.

As a conclusion, the 10 group members can agree on the following com-

	a_1, a_3, a_4, a_5	a_2, a_6, a_7, a_8	$a_9, a_{10}a_{11}$
uni1	$a_4a_5a_3a_1$	$a_2a_8a_7a_6$	$a_{11}a_{10}a_9$
uni2	$a_3a_4a_5a_1$	$a_2a_6a_7a_8$	$a_9a_{11}a_{10}$
uni3	$a_4a_5a_3a_1$	$a_2a_6a_7a_8$	$a_{11}a_{10}a_9$
uni4	$a_1a_3a_5a_4$	$a_6a_8a_7a_2$	$a_{11}a_{10}a_9$
uni5	$a_1a_5a_4a_3$	$a_7a_6a_2a_8$	$a_{11}a_9a_{10}$
iee	$a_5a_4a_3a_1$	$a_6a_7a_2a_8$	$a_9a_{11}a_{10}$
cvce	$a_1a_3a_4a_5$	$a_2a_8a_7a_6$	$a_{10}a_{11}a_9$
tud1	$a_1a_4a_5a_3$	$a_8a_7a_2a_6$	$a_9a_{10}a_{11}$
tud2	$a_1a_4a_3a_5$	$a_7a_8a_6a_2$	$a_9a_{10}a_{11}$
lip	$a_4a_1a_5a_3$	$a_2a_6a_7a_8$	$a_{10}a_9a_{11}$

Table 9.7: The adapted rankings of the 10 researchers.

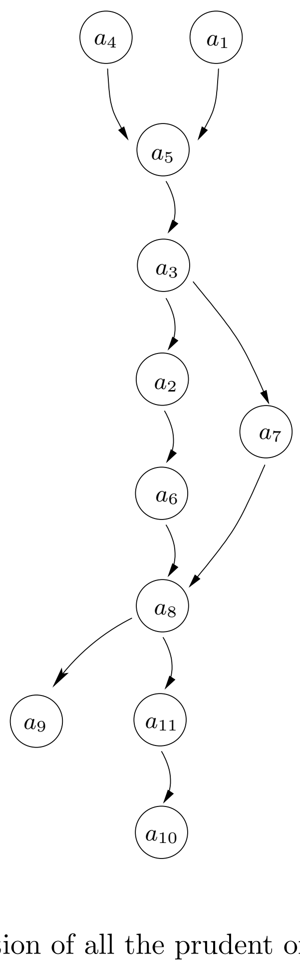


Figure 9.3: The intersection of all the prudent orders after the adaptation.

promise ranking. The most important research domains are Knowledge management technologies and Electronic cooperation networks. These two domains are then followed by Mobile communications and IT security. Although E-Government is perceived as more important than Innovative materials, the precise rank of Finance remains unclear in the middle of the ranking. It follows Business improvement research. Finally, Artificial intelligence followed by Human-Machine interface research can be found at the bottom of the compromise ranking. The final position of Image processing has still to be discussed in this last part of the ranking.

Chapter 10

The composite indicator problem

This chapter presents another possible application of prudent ranking rules, namely the problem of aggregating sub-indicators. The chapter will be organized as follows. In Section 10.1, we introduce the composite indicator problem in order to highlight the motivations for using ordinal aggregation techniques in such a context. Section 10.2 deals with the formalization of the model, whereas in Section 10.3 the model is illustrated on real data.

10.1 Introduction to the composite indicator problem

Socio-economic composite indicators have become a popular tool to evaluate complex objects such as countries, companies, universities etc. with respect to some particular issue. They are nowadays built and used by many organizations and governments, covering almost every topic of life. In Section 1.2.2, we presented already two examples of such indicators.

The aim of an indicator is to measure a certain reality or idea, such as for instance the state of development of a country or the quality of an university. Such an approach can be interesting from various points of view.

First of all, some indicators have a pedagogical role of translating a complex reality into a single figure. More ambitiously, some indicators are used by decision makers, such as governments or companies, to assist them in making the right choices. For instance, some economic indicators may influence an investment decision of a company. Finally, since indicators are often computed at regular intervals, they are also useful in monitoring the evolution over time. They can for instance tell if the quality of an university has improved or not over the last couple of years.

A composite indicator is usually a combination of various sub-indicators which all capture a particular aspect of the reality which we would like to evaluate. However, combining different sub-indicators will inevitably lead to the delicate question of how to aggregate them. Often, the non-commensurability of the different sub-indicators is overcome by means of a particular rescaling technique. The final indicator is then obtained by either a simple average, assuming equal weights for all the sub-indicators, or by means of a weighted sum, interpreting weights as trade-offs. Eventually a score is obtained for each object.

Let us note that the choice of a particular rescaling technique or a set of weights is far from trivial and can dramatically affect the outcome. These, and other issues such as monotonicity or meaningfulness are discussed more in depth in Chapter 4 in [15].

Since the designers of an indicator have to select i) a set of formally defined sub-indicators and ii) a particular aggregation method that combines these sub-indicators, a composite indicator cannot be designed in an objective way. The reality that it tries to reflect is thus only the reality as perceived by the designers. Paradoxically, those who use the indicators to make a decision will not necessarily be the designers, and they consequently do not have the same preferences as the designers. This makes indicators at least a controversial matter.

Let us mention the following remark that can be found in a working paper of the OECD: "...it is hard to imagine that the debate on the use of composite indicators will ever be settled. Official statisticians may tend to resent composite indicators, whereby a lot of work in data collection and editing is wasted or hidden behind a single number of dubious significance. On the other hand, the temptation of stakeholders and practitioners to summarize complex and sometime elusive process (e.g. sustainability, single market policies, etc.) into a single figure to benchmark country performance for policy consumption seems likewise irresistible."[1]

Another approach to combine sub-indicators consists in ignoring the evaluations of the objects on each sub-indicator and considering only the underlying order. These sub-indicator rankings have then to be combined into a global ranking by means of a ranking rule. In Section 1.3, we argued

[1] M. Nardo, M. Saisana, A. Saltelli, S. Tarantola, A. Hoffman and E. Giovannini (2005), *Handbook on constructing composite indicators: methodology and user guide*, OECD Working Paper.

that the nature of the scale of a dimension and the non-commensurability between the dimensions can be two motivations to only work with ordinal data as input. In fact, these two reasons are relevant in a composite indicator context:

- The nature of the scale of each sub-indicator is far from obvious. A sub-indicator often consists of a combination of various input variables, and often the result is further normalized. In order to avoid making exaggerated interpretations of differences of evaluations between two objects on a sub-indicator and to avoid the use of hazardous rescaling techniques, assuming an ordinal scale makes perfect sense.

- We have also highlighted that working with ordinal data as input is useful when the various dimensions which need to be combined are non-commensurable. This is precisely the case in the composite indicator problem. Since it is already difficult to assess the meaning of the numerical value of an object on one sub-indicator, it seems out of reach to correctly combine the numerical values of an object on two different sub-indicators.

That is why we propose in this chapter to only assume an ordinal scale on each sub-indicator. The different sub-indicators are then aggregated by means of an ordinal ranking rule. There are also some drawbacks associated with such an ordinal approach.

- Most composite indicators construct a global score for each object. This implicitly defines a weak order, which ranks the objects from the best to the worst. When using an ordinal ranking rule, we do not obtain such a score, but we can solely propose one (or several) rankings instead.

- The choice of a particular ranking rule often seems rather suspicious. As we have seen in this thesis that choice is crucial and can tremendously influence the result. Taking the average of the sub-indicators appears to be much more familiar, simple and reliable.

Despite these difficulties, ordinal aggregation techniques in a composite indicator context have been suggested for instance by Munda and Nardo [81] or by Patil & al [83]. In Munda and Nardo's methodological argumentation, the authors moreover concluded that the ordinal ranking rule should be what we called in Section 2.3 "Condorcet ranking consistent". More particularly, they proposed to use in a composite indicator framework a "weighted" extension of Kemeny's ranking rule (see Section 2.2.2 for a definition of Kemeny's rule).

Munda and Nardo [81] highlight the fact that, in their model, weights have to be interpreted as importance coefficients, in opposition to the weighted average aggregation model, where weights have to be interpreted as trade-offs. Working with trade-offs implies that a poor performance on one sub-indicator can be compensated by a good performance on another sub-indicator. However, such an interpretation is not always desirable. Consider for instance the example of an environmental indicator where fresh air can compensate polluted water. Furthermore, fixing properly trade-offs assumes a proper understanding of the scales of the sub-indicators since trade-offs should be fixed by answering questions such as "how many units on sub-indicator x are you willing to lose to gain one unit on sub-indicator y ?".

Although we encourage the use of weights as importance coefficients, the elicitation of these weights in Munda and Nardo's model remains difficult. One may also criticize the fact that the authors did not take into account the possible multiplicity of Kemeny-optimal solutions. We will show in the next section we can avoid these two limitations.

10.2 A prudent composite indicator

We suppose that a set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ of n objects (countries, universities,...) has been evaluated on q sub-indicators, where $s_k(a_i)$ denotes the evaluation of object a_i on the sub indicator k, with $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, q\}$. Without any loss of generalization, we suppose that all the sub-indicators have to be maximized. We will only exploit the underlying order of the sub-indicators. Formally, we are going to construct q rankings, denoted by $O_1, O_2, \ldots, O_q$ defined as follows:

$$\forall k \in \{1, \ldots, q\}, \forall a_i, a_j \in \mathcal{A}, \quad (a_i, a_j) \in O_k \text{ if } s_k(a_i) \geq s_k(a_j).$$

We suppose in this chapter that these rankings are linear orders, which implies that on any sub-indicator no two objects have the same evaluation. This will be for instance the case in the example of the competitivity indicator which we will introduce in the next section. As in the previous chapters, we denote $u = (O_1, O_2, \ldots, O_q)$ the ordered list containing the q linear orders.

It may happen that the designers of a composite indicator would like to assign different importances to the sub-indicators. In our setting, $\forall k \in \{1, \ldots, q\}$, we denote $w_k \in \mathbb{N}$ the importance coefficients of sub-indicator k. Let $w = (w_1, \ldots, w_q)$ be a weight vector. As in Munda and Nardo [81], we shall interpret these weights as importance coefficients. Given the profile $u = (O_1, \ldots, O_q)$ containing the underlying order of the sub-indicators and given a weight vector $w = (w_1, \ldots, w_q)$, we define a weighted profile, denoted by u_w, as a profile containing w_i times the linear order O_i $\forall i \in \{1, \ldots, q\}$.

A weighted profile u_w can then be aggregated into the set of prudent orders $\mathcal{PO}(u_w)$ and the set of lexicographic prudent orders $\mathcal{LPO}(u_w)$. It is clear that with equal weights, we have that $\mathcal{PO}(u_w) = \mathcal{PO}(u)$ and $\mathcal{LPO}(u_w) = \mathcal{LPO}(u)$.

The idea behind using two prudent ranking rules is that $\mathcal{LPO}$ can be used to establish an almost complete ranking of the alternatives from the best to the worst, whereas $\mathcal{PO}$ can be used to assess the quality of that ranking. In fact, we know that the sets $\mathcal{PO}(u_w)$ and $\mathcal{LPO}(u_w)$ may contain more than one linear order. Whereas the number of prudent orders can be rather high, we concluded in Section 7.5.2 that the number of lexicographic prudent orders is significantly smaller.

Instead of looking at all these solution rankings one by one, we propose to represent the information contained in these sets by looking at the best and at the worst rank that an object can occupy in any prudent ranking. More formally we denote $\forall a_i \in \mathcal{A}$:

$$\rho_i^+ = \min\{\rho_O(a_i) : O \in \mathcal{PO}(u_w)\} \quad \rho_i^- = \max\{\rho_O(a_i) : O \in \mathcal{PO}(u_w)\}.$$

$$\tilde{\rho}_i^+ = \min\{\rho_O(a_i) : O \in \mathcal{LPO}(u_w)\} \quad \tilde{\rho}_i^- = \max\{\rho_O(a_i) : O \in \mathcal{LPO}(u_w)\}.$$

In fact, such rank information can be conveniently represented. Furthermore, it is easily understood by the users of composite indicators. One can argue that most people who refer to a composite indicator rather look at the rank obtained by a country, a company or a university than at the precise numerical score.

Despite the absence of final scores, our ordinal model outputs that way a richer information than solely a ranking. In fact, the difference $\rho_i^- - \rho_i^+$, which can be rather large due to the high number of prudent orders, is an indication for the difficulty and ambiguity of assigning a precise rank to object a_i. Since the difference $\tilde{\rho}_i^- - \tilde{\rho}_i^+$ is usually small, the lexicographic prudent order model can be used to rank the objects from the best to the worst, which is our main goal.

Furthermore, since $\mathcal{LPO}(u_w) \subseteq \mathcal{PO}(u_w)$, all these maximal and minimal ranks remain consistent.

Proposition 22 *$\forall a_i \in \mathcal{A}$,*

$$\rho_i^+ \leq \tilde{\rho}_i^+ \leq \tilde{\rho}_i^- \leq \rho_i^-.$$

Because of the constant-sum property, the ranks ρ_i^+ and ρ_i^- can be computed without enumerating the whole set $\mathcal{PO}(u_w)$. It suffices to apply Kohler's algorithm to find the optimal strict cut value, compute the transitive closure of the corresponding strict cut relation and apply the formulas presented in Section 9.3.2. The ranks $\tilde{\rho}_i^+$ and $\tilde{\rho}_i^-$ are more cumbersome to obtain since we have to rely, as in Section 7.4, on complete enumeration.

Another particularity of these two models with respect to the model presented by Munda and Nardo [81] is that only the order of the importance of coalitions of sub-indicators matters, and not the precise numerical values of the importance coefficients.

Proposition 23 *Let $w = (w_1, \dots, w_q)$ and $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_q)$ be two weight vectors. If, for every coalition of sub-indicators $K \subseteq \{1, \dots, q\}$ and $K' \subseteq \{1, \dots, q\}$, we have*

$$\sum_{i \in K} w_i > \sum_{i \in K'} w_i \iff \sum_{i \in K} \tilde{w}_i > \sum_{i \in K'} \tilde{w}_i$$

$$\sum_{i \in K} w_i = \sum_{i \in K'} w_i \iff \sum_{i \in K} \tilde{w}_i = \sum_{i \in K'} \tilde{w}_i,$$

then

$$\mathcal{PO}(u_w) = \mathcal{PO}(u_{\tilde{w}})$$

and

$$\mathcal{LPO}(u_w) = \mathcal{LPO}(u_{\tilde{w}}).$$

This result further clarifies the meaning and elicitation of the weights. In fact, the designers of a composite indicator have rather to agree on a ranking of the coalitions than on precise numerical values for the importance coefficients. We refer for instance to [42] for a practical elicitation technique of such preferences.

In the ranking model proposed by Munda and Nardo [81], weights are also interpreted as importance coefficients and not as trade-offs. However, their model does not verify this invariance property since the precise numerical value of the importance coefficients do matter in the computation of the final ranking.

In order to illustrate Proposition 23, we suppose that there are three sub-indicators. In the first scenario, the weight vector is $(5, 4, 2)$, whereas in the second scenario the weight vector is $(8, 5, 4)$. Let us note that in both scenarios, the order of the importance of the coalitions is the same:

Coalition	$\{1,2,3\}$		$\{1,2\}$		$\{1,3\}$		$\{2,3\}$		$\{1\}$		$\{2\}$		$\{3\}$
Scenario 1	11	>	9	>	7	>	6	>	5	>	4	>	2
Scenario 2	17	>	13	>	12	>	9	>	8	>	5	>	4

We suppose that there are 6 objects $\{a, b, c, d, e, f\}$ and the ranking for the first sub-indicator is $ebfadc$, for the second sub-indicator is $dbefac$ and for the third sub-indicator is $acedfb$. We obtain the following results:

	Kemeny	Lex-prudent
Scenario 1	$edbfac$	$ebfadc$
Scenario 2	$ebfadc$	$ebfadc$

Hence the ranking under the Kemeny model as suggested by Munda and Nardo is different, depending on the precise choice for the weights, whereas the result in the lexicographic prudent order model remains invariant.

10.3 Illustration: Building a competitivity indicator

In this section, we illustrate the application of the prudent model on real data. The *Observatoire de la compétitivité* of the Luxembourg government was interested in evaluating the competitivity of European countries with respect to the agenda set by the Lisbon strategy. To do so, they constructed a composite indicator consisting of 10 sub-indicators. These 10 sub-indicators, inspired both from the goals defined in the Lisbon agenda and from a recent external report delivered by Professor Fontagné, are listed below:

1. Macro-economic performance
2. Employment
3. Productivity and cost of employment
4. Market operations
5. Institutional context
6. Entrepreneurship
7. Education
8. Knowledge Economy
9. Social Cohesion
10. Environment

For a detailed description of these indicators, we refer to the official report [28]. Each of the 25 countries of the European Union has been evaluated according to these 10 sub-indicators. The final indicator was then obtained by summing the 10 sub-indicators together, hence assuming equal weights

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
Germany	25	10	14	23	25	19	12	6	10	5
Austria	15	7	9	1	24	17	13	9	5	1
Belgium	19	18	15	17	23	21	11	8	1	20
Cyprus	12	5	22	25	22	8	16	25	19	16
Denmark	6	2	2	5	21	22	4	3	3	2
Spain	5	13	20	16	20	9	23	12	18	12
Estonia	4	15	1	2	19	15	10	14	25	23
Finland	10	9	23	13	18	24	6	2	2	13
France	18	17	13	18	17	18	14	11	8	14
Greece	20	23	17	9	16	2	24	21	17	15
Hungary	21	21	6	22	15	6	20	18	12	4
Ireland	2	4	19	24	14	4	21	17	21	6
Italy	23	22	24	20	13	5	22	13	13	11
Latvia	11	14	5	6	12	1	8	24	20	7
Lithuania	3	19	11	8	11	13	7	23	16	24
Luxembourg	1	11	4	11	10	11	2	5	9	8
Malta	17	20	7	21	9	16	17	19	7	19
Netherlands	14	1	18	3	8	23	5	4	6	17
Poland	16	25	16	14	7	7	18	22	14	21
Portugal	22	8	25	19	6	3	25	20	24	22
Slovakia	24	24	21	10	5	10	19	15	22	18
Czech Republic	7	12	3	12	4	12	15	16	15	25
UK	13	6	10	4	3	14	9	7	23	10
Slovenia	8	16	12	15	2	20	3	10	11	9
Sweden	9	3	8	7	1	25	1	1	4	3

Table 10.1: The ranks of the 25 countries on the 10 sub-indicators.

in the trade-off sense.

Instead of working with the numerical evaluations of the countries on each sub-indicator, we are going to simply take into account the underlying order. The rank of each country on these 10 sub-indicators can be found in Table 10.1.

We shall aggregate these 10 linear orders into the set of prudent orders and into the set of lexicographic prudent orders by assuming also equal weights. In fact, we are mostly interested in computing for each country the best and worst ranks ρ^+, ρ^-, $\tilde{\rho}^+$ and $\tilde{\rho}^-$. The results obtained can be found in the first four columns in Table 10.2. We listed the countries in the order provided by the best rank $\tilde{\rho}^+$ obtained by the lexicographic prudent orders.

Country	$\tilde{\rho}^+$	$\tilde{\rho}^-$	ρ^+	ρ^-	CI	Best Kemeny	Worst Kemeny
Denmark	1	1	1	1	2	1	1
Sweden	2	2	2	2	1	2	2
Luxembourg	3	3	3	5	3	3	3
UK	4	4	4	13	6	4	4
Austria	5	5	3	12	5	5	5
Slovenia	6	6	5	14	8	6	6
Netherlands	7	7	3	11	4	7	7
Czech Republic	8	8	4	18	10	8	8
Latvia	9	10	4	15	7	9	10
Lithuania	9	14	5	21	14	9	14
Ireland	10	11	5	21	11	10	11
Finland	11	12	4	15	9	11	12
Estonia	12	13	5	18	13	12	13
Malta	13	16	12	21	18	13	16
France	14	15	10	20	17	14	15
Belgium	15	16	11	23	16	15	16
Hungary	17	17	9	21	15	17	17
Germany	18	18	8	25	12	18	18
Poland	19	21	11	25	22	19	21
Italy	19	21	13	25	23	19	21
Spain	20	22	9	22	19	20	22
Greece	20	22	16	22	20	20	22
Cyprus	23	23	19	25	21	23	23
Slovakia	24	24	19	24	24	24	24
Portugal	25	25	21	25	25	25	25

Table 10.2: The results of the competitivity indicator.

For comparative purposes, we give in the fifth column of Table 10.2 the ranking of the composite indicator presented in the report. Since this ranking has been obtained by summing the values of the ten sub-indicators, it cannot be obtained with a ranking rule, but requires the knowledge of the exact values of the 10 sub-indicators. Finally, in the sixth and seventh column, we list the best and worst rank obtained in all the Kemeny orders. This is in fact the ranking which would have been obtained in the model proposed by Munda and Nardo. Let us however stress again that Munda and Nardo did not explicitly take into account the multiplicity of optimal solutions. In this example for instance, there have been 96 different Kemeny orders, only one of which would have been randomly chosen in Munda and Nardo's model.

Let us now comment on these results:

- According to lexicographic prudent orders, apart from Lithuania ($\tilde{\rho}^+ = 9$, $\tilde{\rho}^- = 14$) and Malta ($\tilde{\rho}^+ = 13$, $\tilde{\rho}^- = 16$), the difference between $\tilde{\rho}^-$ and $\tilde{\rho}^-$ is never more than 2. For 12 out of 25 countries, the best and the worst rank are even equal. This confirms the idea that the lexicographic prudent orders are useful when objects have to be completely ranked from the best to the worst.

- According to the prudent orders, the smallest differences between the worst rank and the best rank is achieved by Sweden and Denmark. In fact, for both countries there is no ambiguity in assigning a rank: Denmark is always put in the first position and Sweden is always put in the second position. This is also confirmed by the Kemeny orders.

- According to prudent orders, the largest difference between the worst rank and the best rank is achieved by Germany. In fact there are in total 18 different ranks which we could assign to this country. This indicates that Germany is a country difficult to rank since in 5 out of the 10 indicators they are ranked in the upper half (i.e. below rank 12.5) and in 5 out of the 10 indicators they are ranked in the lower half (i.e. above rank 12.5).

- For 9 out of the 25 countries, the original composite indicator is compatible with the rank ranges provided by the lexicographic prudent orders. The largest discrepancy is obtained for Germany, which was ranked in the composite indicator on the 12th position, whereas lexicographic prudent orders unambiguously rank this country on the 18th position. This confirms again the lesson learned from the prudent orders (rank difference of 18) which suggest that establishing a clear position for Germany is difficult.

- The rank ranges of the set of prudent orders is compatible with the original composite indicator, except for Denmark (CI=2, $\rho^+ = 1$, $\rho^- = 1$) and Sweden (CI=1, $\rho^+ = 2$, $\rho^- = 2$). Let us however emphasize that Denmark is preferred to Sweden in 7 out of the 10 indicators. From a pairwise comparison perspective, there are thus good reasons to put Denmark before Sweden.

- For this data, both Kemeny's rule and the lexicographic prudent order preference function give identical results, and that is why both ranking rules lead to the same best rank and to the same worst rank for all countries. Since the ranks of the lexicographic prudent orders are always compatible with the ranks of the prudent orders (see Proposition 22), it is worth noticing that in this example the ranks of the Kemeny orders are also compatible with the ranks of the prudent orders. We refer the reader to Section 7.3 for a more detailed discussion on the link between lexicographic prudent orders and Kemeny orders. Let us also recall that in some situations, Kemeny orders and lexicographic prudent orders can nevertheless be very contradictory (see Section 8.4).

This example illustrates the applicability of prudent ranking rules to the composite indicator problem. Compared to the model introduced by Munda and Nardo, our model outputs a richer result, taking explicitly into account the multiplicity of prudent orders in order to assess the quality of the rank of an object. Furthermore, lexicographic prudent orders can be used to discriminate between the objects while always remaining consistent with the results obtained by the prudent order preference function. Finally, our model can handle importance coefficients in a more transparent way in the sense that only the ranking of the importance coefficients matter and not the precise numerical values of these coefficients.

Conclusion

At the beginning of this thesis we have postulated that the prudence principle can be an appropriate concept in a decision aid context. We have chosen to illustrate this on the group ranking problem and on the composite indicator problem.

Following these motivations, we mainly focused on gaining a better understanding of the family of prudent ranking rules. Studying such a complex mechanism as an ordinal ranking rule, and, more particularly, its implications on profiles in all kinds of different situations is however far from trivial. It is particularly tricky since as soon as you think you get a grip on the ranking rule, an even more perplexing situation pops up which may turn things around all over again. Although no easy answer can be provided, it is precisely this complexity that fascinates many researchers.

First of all, it seems important to us that we could delimit prudent ranking rules from non-prudent ranking rules. The "paradoxical" results which we have established clearly show that the prudence principle may lead to very different solutions than those obtained by more traditional approaches.

In order to push these ideas further and to distinguish between the different prudent ranking rules, we chose to spend a lot of efforts on axiomatic characterizations. This choice can be criticized and some may not be fully convinced because of the technical nature of the results. This being said, we still believe that these results give a first insight into the behavior of prudent ranking rules and may help an analyst to be more comfortable next time he decides to use such a tool. Moreover, the differences between these prudent ranking rules can more easily be recognized since we characterized the rules in a common axiomatic framework.

Finally, in our research we discovered a new prudent ranking rule, namely the lexicographic prudent order preference function. It is surprising how close this ranking rule is linked to Kemeny's rule which is much more known and popular than prudent ranking rules.

However, there are still many open questions or problems which deserve further attention. For instance, the prudent axiomatic framework which we have built in this thesis should be further enriched by characterizing other prudent ranking rules. More time should also be spent on studying the lexicographic prudent order preference function, especially in comparison to Kemeny's rule.

Besides, we would like to stress the following, more general, directions of future research:

- Saari suggests to analyze a ranking rule by relying on geometric representations. The main benefit of such an approach relies in its simplicity to identify all the profiles that will lead to a certain result. Such an insight into a ranking rule can help to better understand its properties, anomalies and paradoxes. Although we have not had time to pursue this idea, we strongly believe that analyzing prudent ranking rules from a geometric point of view will be enlightening.

- The models which we analyzed can be generalized. In multicriteria decision aid, the so-called outranking methods usually consist of two steps. First, the alternatives are compared pairwise in order to build a valued outranking relation. This outranking relation is then exploited in order to come up with a ranking for the decision maker. Some research should deal with extending and applying the prudence principle to the problem of exploiting such valued outranking relations.

- We concentrated on prudent ranking rules. A promising line of research is to take on the problem of prudent choice rules. A unifying framework for such prudent choice rules has not been established yet. There is a large potential for choice problems and one of the challenges could be to discuss the appropriateness of a prudent choice for multicriteria decision aid problems.

We hope that we have convinced the reader that research on prudent decision models is stimulating and we encourage anyone to work on one of the problems that we have just mentioned.

Appendix

The number of prudent orders

Debord [30] randomly generated profiles of q linear orders and n alternatives under the IC assumption, with q ranging from 3 to 17 and n ranging from 3 to 7. For each combination (n, q), 1000 repetitions were performed. The average number of prudent orders for odd profiles can be found in the following table:

(n, q)	3	5	7	9	11	13	15	17
3	1.305	1.301	1.297	1.255	1.207	1.193	1.153	1.187
4	3.635	3.232	2.704	2.478	2.508	2.038	1.943	1.961
5	20.190	14.345	10.280	7.955	5.968	6.155	5.556	4.800
6	107.141	57.855	37.053	26.026	20.057	19.470	19.190	15.699
7	581.456	237.815	119.454	80.346	74.214	79.803	76.753	65.460

The average number of prudent orders for even profiles can be found in the following table:

(n, q)	4	6	8	10	12	14	16
3	2.211	1.903	1.801	1.765	1.664	1.635	1.592
4	3.855	3.353	3.080	2.727	2.630	2.478	1.418
5	7.846	7.641	6.424	7.738	7.539	5.908	5.644
6	19.502	30.651	30.442	25.821	26.035	18.977	16.620
7	56.571	190.166	176.312	134.336	109.803	77.326	67.123

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