The Node-Based Smoothed Finite Element Method in nonlinear elasticity

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Contents

1 Introduction

The idea of the stabilized conforming nodal integration introduced by Chen et al.[1] is to avoid the integration and evaluation of shape functions at the nodes in the mesh–free method because direct nodal integration leads to instability of numerical results. Liu et al.[3] extended this idea to finite element method (FEM) and called this "smoothed finite element method (SFEM)" with the divided smoothing cells in the elements.

The main feature of SFEM is that this method is suitable for heavily distorted meshes because this does not need the isoparametric mapping and does not require the derivatives of the shape functions. The computational cost is relatively lower than for the conventional FEM at the same accuracy level; moreover, n–sided polygonal elements can be used and the volumetric locking problem can be handled effectively.

Listed belows are some of the strengths and weaknesses of each SFEM technique: the node–based smoothed FEM (NS–FEM), the cell–based smoothed FEM (CS–FEM), and the edge–based smoothed FEM (ES–FEM).

- **Volumetric Locking** NS–FEM can handle effectively nearly incompressible materials where Poisson's ratio $\nu \approx 0.5$, while ES–FEM leads the volumetric locking. Combining NS– and ES–FEM method gives the so–called *the smoothing–domain–based selective ES/NS–FEM* which overcomes volumetric locking. In the case of CS–FEM, the volumetric locking can be avoided by separating the material property matrix for isotropic materials into two parts, one relating to the shearing modulus μ and one relating Lamé's parameter λ. Then the stiffness matrix can also be split into the respective two parts.

- **Upper and Lower Bound Properties** In the case of non–homogeneous Dirichlet boundary conditions and zero external forces, NS–FEM and FEM provide lower and upper bounds for the exact solution, respectively. If the problem is force driven, i.e. the Dirichlet boundary conditions are homogeneous, then NS–FEM and FEM provide the upper and lower bounds, respectively. In general, however, the order of their bounds is problem dependent. Regarding the solution obtained by ES–FEM, this lies this between those of the FEM and NS–FEM.

- **Static and Dynamic Analyses** ES–FEM gives accurate and stable results when solving either static or dynamic problems, because ES–FEM is not only spatially but also temporally stable. In contrast, although NS–FEM is spatially stable, it is temporally unstable. Therefore, to solve dynamic problems, NS–FEM needs stabilization techniques. CS–FEM can also be extended to solve dynamic problems.

- **Other features** In NS–FEM, the accuracy of displacement solutions is at the same level as for the standard FEM using the same mesh, whereas the accuracy of stress solutions in energy norm is much higher than for FEM. In terms of the computational time, in general, ES–FEM is more expensive than the conventional FEM with the same set of nodes.

2 Finite Element Method Approximation

2.1 Linear Elasticity

The Equilibrium equation in 2D is:

$$-\nabla\boldsymbol{\sigma} = \mathbf{f} \tag{2.1.1}$$

The variational form of Eq. (1) is:

$$-\int_\Omega \nabla\boldsymbol{\sigma}\cdot\mathbf{v}\mathrm{d}\Omega = \int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}\Omega \tag{2.1.2}$$

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \nabla \mathbf{v} \mathrm{d}\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \mathrm{d}\Gamma \tag{2.1.3}$$

where $\mathbf{f}$ is the vector of external body force, $\mathbf{g} = \boldsymbol{\sigma} \cdot \mathbf{n}$ is the prescribed traction vector on natural boundary Γ_N, and $\mathbf{v}$ is the test function.

The stress tensor $\boldsymbol{\sigma}$ is:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}\left(\boldsymbol{\varepsilon}\right) \mathbf{I} \tag{2.1.4}$$

where μ is the shear modulus and λ is Lamé's parameter, which can be expressed in terms of Young's modulus E and Poisson's ratio ν as follows:

$$\mu = \frac{E}{2\left(1+\nu\right)}, \quad \lambda = \frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)} \tag{2.1.5}$$

The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is:

$$\boldsymbol{\varepsilon} = \left\{\varepsilon_{ij}\right\}, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{2.1.6}$$

or equivalently:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right) \tag{2.1.7}$$

In Voigt notation, the stress tensor can be expressed as follows:

$$\left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right\} = \mathbb{C}\left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{array}\right\} \tag{2.1.8}$$

where $\mathbb{C}$ is the elasticity tensor:

$$\mathbb{C} = \left[\begin{array}{ccc} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & 2\mu \end{array}\right] \tag{2.1.9}$$

The discrete equation of FEM from the Galerkin weak form is:

$$\int_{\Omega} \mathbb{C}\boldsymbol{\varepsilon}\left(\mathbf{u}\right) \cdot \boldsymbol{\varepsilon}\left(\mathbf{v}\right) \mathrm{d}\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \mathrm{d}\Gamma \tag{2.1.10}$$

FEM uses the following trial and test functions, respectively:

$$\mathbf{u}^{\mathrm{h}}\left(\mathbf{x}\right) = \sum_{i=1}^{N} u_i \psi_i, \quad \mathbf{v}^{\mathrm{h}}\left(\mathbf{x}\right) = \sum_{i=1}^{N} v_i \psi_i \tag{2.1.11}$$

Then the standard discretised algebraic system of equations is:

$$\mathbf{K}\mathbf{u}^{\mathrm{h}} = \mathbf{b} \tag{2.1.12}$$

where $\mathbf{K}$ is the stiffness matrix and $\mathbf{b}$ is the element force vector, which have the following components, respectively:

$$K_{ij} = \int_{\Omega} \mathbb{C}\boldsymbol{\varepsilon}\left(\psi_i\right) \cdot \boldsymbol{\varepsilon}\left(\psi_j\right) \mathrm{d}\Omega, \tag{2.1.13}$$

$$b_i = \int_{\Omega} f\psi_i \mathrm{d}\Omega + \int_{\Gamma_N} g\psi_i \mathrm{d}\Gamma \tag{2.1.14}$$

2.2 Nonliear Elasticity

2.2.1 Hyperelastic material

In the nonlinear case,

$$\int_{\Omega} \frac{\partial \mathcal{W}}{\partial \mathbf{F}} \left(\mathbf{X}, \mathbf{F}\left(\mathbf{u}\right)\right) : \nabla \mathbf{v} \mathrm{d}\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d}V + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \mathrm{d}A \tag{2.2.1}$$

where the strain energy density function $\mathcal{W}$ for incompressible and compressible neo–Hookean materials are expressed respectively as follows:

$$\mathcal{W} = \frac{\mu}{2}\left(I_1 - 3\right) \tag{2.2.2}$$

and

$$\mathcal{W} = \frac{\mu}{2}\left(I_1 - 3\right) + \frac{\kappa}{2}\left(I_3 - 1\right) - \left(\frac{\mu}{2} + \frac{\kappa}{2}\right) \ln I_3 \tag{2.2.3}$$

where:

$$I_1 = \mathrm{tr}\left(\mathbf{C}\right), \quad I_2 = \frac{1}{2}\left(\mathrm{tr}\left(\mathbf{C}\right)^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right), \quad \text{and} \quad I_3 = \det\left(\mathbf{C}\right) \tag{2.2.4}$$

The deformation gradient $\mathbf{F}$ is:

$$\mathbf{F} = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^{\mathrm{T}} \quad \text{or} \quad F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{2.2.5}$$

To find an approximation solution to the eq. (2.2.1) in the displacement field $\mathbf{u}$, we employ Newton's method. An iteration *iter+1*, knowing the displacement $\mathbf{u}_{\mathrm{iter}}$ from iteration *iter*, find $\mathbf{r}_{\mathrm{iter}}$ that satisfies:

$$D\mathcal{R}\left(\mathbf{u}_{\mathrm{iter}}\right) \cdot \mathbf{r}_{\mathrm{iter}} = -\mathcal{R}\left(\mathbf{u}_{\mathrm{iter}}\right) \tag{2.2.6}$$

where:

$$\mathcal{R}\left(\mathbf{u}\right) = \int_{\Omega} \frac{\partial \mathcal{W}}{\partial F_{ij}} \left(\mathbf{X}, \mathbf{F}\left(\mathbf{u}\right)\right) \frac{\partial v_i}{\partial X_j} \mathrm{d}V - \int_{\Omega} f_i v_i \mathrm{d}V - \int_{\Gamma_N} g_i v_i \mathrm{d}A \tag{2.2.7}$$

$$D\mathcal{R}\left(\mathbf{u}\right) \cdot \mathbf{r} = \int_{\Omega} \frac{\partial^2 \mathcal{W}}{\partial F_{ij} \partial F_{kl}} \left(\mathbf{X}, \mathbf{F}\left(\mathbf{u}\right)\right) \frac{\partial r_k}{\partial X_l} \frac{\partial v_i}{\partial X_j} \mathrm{d}V \tag{2.2.8}$$

and $i, j, k, l = 1, 2$.

Then:

$$\mathbf{u}_{\mathrm{iter}+1} = \mathbf{u}_{\mathrm{iter}} + \mathbf{r}_{\mathrm{iter}} \tag{2.2.9}$$

Since $\frac{\partial \mathcal{W}}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \mathcal{W}}{\partial \mathbf{C}}$, the energy functional (2.2.7) and its derivatives (2.2.8) take the equivalent formulations, respectively:

$$\mathcal{R}\left(\mathbf{u}\right) = \int_{\Omega} 2\frac{\partial \mathcal{W}}{\partial C_{ij}} F_{ki} \frac{\partial v_k}{\partial X_j} \mathrm{d}V - \int_{\Omega} f_i v_i \mathrm{d}V - \int_{\Gamma_N} g_i v_i \mathrm{d}A \tag{2.2.10}$$

$$D\mathcal{R}\left(\mathbf{u}\right) \cdot \mathbf{r} = \int_{\Omega} \left\{ \frac{\partial^2 \mathcal{W}}{\partial C_{ij} \partial C_{kl}} F_{pi} \frac{\partial v_p}{\partial X_j} F_{sk} \frac{\partial r_s}{\partial X_l} + 2\frac{\partial \mathcal{W}}{\partial C_{ij}} \frac{\partial r_k}{\partial X_i} \frac{\partial v_k}{\partial X_j} \right\} \mathrm{d}V \tag{2.2.11}$$

where the right Cauchy–Green strain tensor $\mathbf{C}$ is:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{2.2.12}$$

The resulting algebraic system for the numerical approximation of eq. (2.2.6) is assembled from the block systems:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{2.2.13}$$

By taking $\mathbf{v} = \mathbf{N}$, we obtain the stiffness matrix $\mathbf{K}_{\mathrm{iter}}$ with the following entries:

$$K_{11} = \int_{\Omega} \left\{ \frac{\partial^2 \mathcal{W}}{\partial C_{ij} \partial C_{kl}} \left(\delta_{1i} + \frac{\partial u_1}{\partial X_j}\right) \frac{\partial N_1}{\partial X_j} \left(\delta_{1k} + \frac{\partial u_1}{\partial X_k}\right) \frac{\partial N_1}{\partial X_l} + 2\frac{\partial \mathcal{W}}{\partial C_{ij}} \frac{\partial N_1}{\partial X_i} \frac{\partial N_1}{\partial X_j} \right\} \mathrm{d}V \tag{2.2.14}$$

$$K_{12} = \int_\Omega 4 \frac{\partial^2 \mathcal{W}}{\partial C_{ij} \partial C_{kl}} \left(\delta_{1i} + \frac{\partial u_1}{\partial X_i} \right) \frac{\partial N_1}{\partial X_j} \left(\delta_{2k} + \frac{\partial u_2}{\partial X_k} \right) \frac{\partial N_2}{\partial X_l} \mathrm{d}V \tag{2.2.15}$$

$$K_{22} = \int_\Omega \left\{ \frac{\partial^2 \mathcal{W}}{\partial C_{ij} \partial C_{kl}} \left(\delta_{2i} + \frac{\partial u_2}{\partial X_j} \right) \frac{\partial N_2}{\partial X_j} \left(\delta_{2k} + \frac{\partial u_2}{\partial X_k} \right) \frac{\partial N_2}{\partial X_l} + 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \frac{\partial N_2}{\partial X_i} \frac{\partial N_2}{\partial X_j} \right\} \mathrm{d}V \tag{2.2.16}$$

Similarly, we obtain the load vector with following components:

$$r_1 = - \int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \left(\delta_{1i} + \frac{\partial u_1}{\partial X_i} \right) \frac{\partial N_1}{\partial X_j} \mathrm{d}V + \int_\Omega f_1 N_1 \mathrm{d}V + \int_{\Gamma_N} g_1 N_1 \mathrm{d}A \tag{2.2.17}$$

$$r_2 = - \int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \left(\delta_{2i} + \frac{\partial u_2}{\partial X_i} \right) \frac{\partial N_2}{\partial X_j} \mathrm{d}V + \int_\Omega f_2 N_2 \mathrm{d}V + \int_{\Gamma_N} g_2 N_2 \mathrm{d}A \tag{2.2.18}$$

The stiffness matrix is:

$$\begin{aligned}
\mathbf{K}^e = \int_\Omega \Bigg\{ & \begin{bmatrix} \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial X} & \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial Y} & \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial Y} & \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial X} \\ \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial X} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial Y} & \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial Y} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial X} \end{bmatrix} \\
& \begin{bmatrix} 4 \frac{\partial^2 \mathcal{W}}{\partial C_{11} \partial C_{11}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{11} \partial C_{22}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{11} \partial C_{12}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{11} \partial C_{21}} \\ 4 \frac{\partial^2 \mathcal{W}}{\partial C_{22} \partial C_{11}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{22} \partial C_{22}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{22} \partial C_{12}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{22} \partial C_{21}} \\ 4 \frac{\partial^2 \mathcal{W}}{\partial C_{12} \partial C_{11}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{12} \partial C_{22}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{12} \partial C_{12}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{12} \partial C_{21}} \\ 4 \frac{\partial^2 \mathcal{W}}{\partial C_{21} \partial C_{11}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{21} \partial C_{22}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{21} \partial C_{12}} & 4 \frac{\partial^2 \mathcal{W}}{\partial C_{21} \partial C_{21}} \end{bmatrix} \begin{bmatrix} \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_q}{\partial X} & \frac{\partial u_2}{\partial X} \frac{\partial N_q}{\partial X} \\ \frac{\partial u_1}{\partial X} \frac{\partial N_q}{\partial Y} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_q}{\partial Y} \\ \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_q}{\partial Y} & \frac{\partial u_2}{\partial X} \frac{\partial N_q}{\partial Y} \\ \frac{\partial u_1}{\partial Y} \frac{\partial N_q}{\partial X} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_q}{\partial X} \end{bmatrix} \\
& + \begin{bmatrix} \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} & 0 & 0 \\ 0 & 0 & \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} \end{bmatrix} \begin{bmatrix} 2 \frac{\partial \mathcal{W}}{\partial C_{11}} & 2 \frac{\partial \mathcal{W}}{\partial C_{12}} & 0 & 0 \\ 2 \frac{\partial \mathcal{W}}{\partial C_{21}} & 2 \frac{\partial \mathcal{W}}{\partial C_{22}} & 0 & 0 \\ 0 & 0 & 2 \frac{\partial \mathcal{W}}{\partial C_{11}} & 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \\ 0 & 0 & 2 \frac{\partial \mathcal{W}}{\partial C_{21}} & 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \end{bmatrix} \begin{bmatrix} \frac{\partial N_q}{\partial X} & 0 \\ \frac{\partial N_q}{\partial Y} & 0 \\ 0 & \frac{\partial N_q}{\partial X} \\ 0 & \frac{\partial N_q}{\partial Y} \end{bmatrix} \Bigg\} \mathrm{d}V
\end{aligned} \tag{2.2.19}$$

where $p, q = 1, 2, \ldots, \mathrm{ndof}$.

The residual force vector is:

$$\begin{aligned}
\mathbf{r}^e &= \int_\Omega \begin{bmatrix} \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial X} & \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial Y} & \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial Y} & \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial X} \\ \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial X} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial Y} & \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial Y} & \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial X} \end{bmatrix} \begin{bmatrix} 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \\ 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \\ 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \\ 2 \frac{\partial \mathcal{W}}{\partial C_{21}} \end{bmatrix} \mathrm{d}V \\
&= \int_\Omega \begin{bmatrix} 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial X} + 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial Y} + 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \left(1 + \frac{\partial u_1}{\partial X}\right) \frac{\partial N_p}{\partial Y} + 2 \frac{\partial \mathcal{W}}{\partial C_{21}} \frac{\partial u_1}{\partial Y} \frac{\partial N_p}{\partial X} \\ 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial X} + 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial Y} + 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \frac{\partial u_2}{\partial X} \frac{\partial N_p}{\partial Y} + 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \left(1 + \frac{\partial u_2}{\partial Y}\right) \frac{\partial N_p}{\partial X} \end{bmatrix} \mathrm{d}V
\end{aligned} \tag{2.2.20}$$

where $p = 1, 2, \ldots, \mathrm{ndof}$.

2.2.2 Numerical examples

Expressing the first derivative of all the static invariants $\mathcal{W}$ with respect to $\mathbf{C}$, by the chain rule, we obtain:

$$\begin{aligned}
\frac{\partial \mathcal{W}}{\partial \mathbf{C}} &= \frac{\partial \mathcal{W}}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \mathcal{W}}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \mathcal{W}}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{C}} \\
&= \frac{\partial \mathcal{W}}{\partial I_1} \mathbf{I} + \frac{\partial \mathcal{W}}{\partial I_2} \left(I_1 \mathbf{I} - \mathbf{C} \right) + \frac{\partial \mathcal{W}}{\partial I_3} I_3 \mathbf{C}^{-1}
\end{aligned} \tag{2.2.21}$$

where:

$$\begin{aligned}
\frac{\partial I_1}{\partial \mathbf{C}} &= \frac{\partial C_{kk}}{\partial C_{ij}} = \delta_{ki} \delta_{kj} \\
&= \sum_k \delta_{ki} \delta_{kj} = \sum_k \delta_{ik} \delta_{kj} = \mathbf{I} \cdot \mathbf{I} = \mathbf{I},
\end{aligned}$$

$$
\begin{aligned}
2\frac{\partial I_2}{\partial \mathbf{C}} &= \frac{\partial}{\partial C_{ij}}\left(C_{kk}^2 - C_{pq}C_{qp}\right) \\
&= 2\frac{\partial C_{kk}}{\partial C_{ij}}C_{kk} - \left(\frac{\partial C_{pq}}{\partial C_{ij}}C_{qp} + C_{pq}\frac{\partial C_{pq}}{\partial C_{ij}}\right) \\
&= 2\mathbf{I}\mathrm{tr}\left(\mathbf{C}\right) - \left(\delta_{pi}\delta_{qi}C_{qp} + C_{pq}\delta_{qi}\delta_{pi}\right) \\
&= 2\mathbf{I}\mathrm{tr}\left(\mathbf{C}\right) - \left(C_{ji} + C_{ji}\right)
\end{aligned}
$$

hence:

$$
\begin{aligned}
\frac{\partial I_2}{\partial \mathbf{C}} &= \mathbf{I}\mathrm{tr}\left(\mathbf{C}\right) - C_{ji} \\
&= \mathbf{I}\mathrm{tr}\left(\mathbf{C}\right) - \mathbf{C}^{\mathrm{T}} = \mathbf{I}\mathrm{tr}\left(\mathbf{C}\right) - \mathbf{C}
\end{aligned}
$$

and

$$
\frac{\partial I_3}{\partial \mathbf{C}} = \begin{bmatrix} C_{22}C_{33} - C_{32}C_{23} & C_{31}C_{23} - C_{21}C_{33} & C_{21}C_{32} - C_{31}C_{22} \\ & C_{11}C_{33} - C_{31}C_{13} & C_{31}C_{12} - C_{11}C_{32} \\ \text{symm.} & & C_{11}C_{22} - C_{21}C_{12} \end{bmatrix}
$$

since

$$
\begin{aligned}
\mathbf{C}^{-1} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{-1} \\
&= \frac{1}{\det\mathbf{C}} = \begin{bmatrix} C_{22}C_{33} - C_{32}C_{23} & C_{31}C_{23} - C_{21}C_{33} & C_{21}C_{32} - C_{31}C_{22} \\ & C_{11}C_{33} - C_{31}C_{13} & C_{31}C_{12} - C_{11}C_{32} \\ \text{symm.} & & C_{11}C_{22} - C_{21}C_{12} \end{bmatrix}
\end{aligned}
$$

it follows that:

$$
\frac{\partial I_3}{\partial \mathbf{C}} = \left(\det\mathbf{C}\right)\mathbf{C}^{-1} = I_3\mathbf{C}^{-1}
$$

In eq. (2.2.1), $\int_\Omega \mathbf{f}\cdot\mathbf{v}\mathrm{d}V$ and $\int_{\Gamma_N}\mathbf{g}\cdot\mathbf{v}\mathrm{d}A$ are zeros for the simple shear problem, therefore the left–hand side of eq. (2.2.1) can be expressed:

$$
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\left(\mathbf{I} + \nabla\mathbf{u}\right) : \nabla\mathbf{v}\mathrm{d}V = 0 \tag{2.2.22}
$$

$$
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\mathbf{I} : \nabla\mathbf{v}\mathrm{d}V + \int_\Omega \left(2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\nabla\mathbf{u}\right) : \nabla\mathbf{v}\mathrm{d}V = 0 \tag{2.2.23}
$$

$$
\int_\Omega \left(2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\nabla\mathbf{u}\right) : \nabla\mathbf{v}\mathrm{d}V = -\int_\Omega 2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\mathbf{I} : \nabla\mathbf{v}\mathrm{d}V \tag{2.2.24}
$$

For eq. (2.2.24), we take $\mathbf{v} = \mathbf{N}$, where $\mathbf{N}$ are the shape functions.

$$
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\nabla\mathbf{u} : \nabla\mathbf{N}\mathrm{d}V = -\int_\Omega 2\frac{\partial \mathcal{W}}{\partial \mathbf{C}}\mathbf{I} : \nabla\mathbf{N}\mathrm{d}V \tag{2.2.25}
$$

Using *Einstein summation*, eq. (2.2.25) can be expressed as follows:

$$
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{ij}}\frac{\partial \mathbf{u}_1}{\partial X_i}\frac{\partial \mathbf{N}}{\partial X_j}\mathrm{d}V = -\int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{ij}}\delta_{1i}\frac{\partial \mathbf{N}}{\partial X_j}\mathrm{d}V \tag{2.2.26}
$$

and

$$
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{ij}}\frac{\partial \mathbf{u}_2}{\partial X_i}\frac{\partial \mathbf{N}}{\partial X_j}\mathrm{d}V = -\int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{ij}}\delta_{2i}\frac{\partial \mathbf{N}}{\partial X_j}\mathrm{d}V \tag{2.2.27}
$$

Firstly we consider the left–hand side of eq. (2.2.26) with the displacement $\mathbf{u}$ in the horizontal (X_1) direction,

$$
\begin{aligned}
\int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{ij}}\frac{\partial \mathbf{u}_1}{\partial X_i}\frac{\partial N_q}{\partial X_j}\mathrm{d}V \quad &= \int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{11}}\frac{\partial \mathbf{u}_1}{\partial X_1}\frac{\partial N_q}{\partial X_1}\mathrm{d}V + \int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{12}}\frac{\partial \mathbf{u}_1}{\partial X_1}\frac{\partial N_q}{\partial X_2}\mathrm{d}V \\
&+ \int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{21}}\frac{\partial \mathbf{u}_1}{\partial X_2}\frac{\partial N_q}{\partial X_1}\mathrm{d}V + \int_\Omega 2\frac{\partial \mathcal{W}}{\partial C_{22}}\frac{\partial \mathbf{u}_1}{\partial X_2}\frac{\partial N_q}{\partial X_2}\mathrm{d}V
\end{aligned} \tag{2.2.28}
$$

where

$$\mathbf{u}_1 = \sum_{p=1}^{\text{ndof}} u_{1p} N_p \tag{2.2.29}$$

thus:

$$\frac{\partial \mathbf{u}_1}{\partial X_1} = \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_1}, \quad \frac{\partial \mathbf{u}_1}{\partial X_2} = \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_2} \tag{2.2.30}$$

and

$$\mathbf{u}_2 = \sum_{p=1}^{\text{ndof}} u_{2p} N_p \tag{2.2.31}$$

thus:

$$\frac{\partial \mathbf{u}_2}{\partial X_1} = \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_1}, \quad \frac{\partial \mathbf{u}_2}{\partial X_2} = \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_2} \tag{2.2.32}$$

Hence, the left–hand side of eq. (2.2.26) can be written as follows:

$$\begin{aligned}
&\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \frac{\partial \mathbf{u}_1}{\partial X_i} \frac{\partial N_q}{\partial X_j} \mathrm{d}V \\
&\quad = 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_1} \frac{\partial N_q}{\partial X_1} \mathrm{d}V + 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_1} \frac{\partial N_q}{\partial X_2} \mathrm{d}V \\
&\quad + 2 \frac{\partial \mathcal{W}}{\partial C_{21}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_2} \frac{\partial N_q}{\partial X_1} \mathrm{d}V + 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{1p} \frac{\partial N_p}{\partial X_2} \frac{\partial N_q}{\partial X_2} \mathrm{d}V
\end{aligned} \tag{2.2.33}$$

and similarly the left–hand side of eq. (2.2.27) is:

$$\begin{aligned}
&\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \frac{\partial \mathbf{u}_2}{\partial X_i} \frac{\partial N_q}{\partial X_j} \mathrm{d}V \\
&\quad = 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_1} \frac{\partial N_q}{\partial X_1} \mathrm{d}V + 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_1} \frac{\partial N_q}{\partial X_2} \mathrm{d}V \\
&\quad + 2 \frac{\partial \mathcal{W}}{\partial C_{21}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_2} \frac{\partial N_q}{\partial X_1} \mathrm{d}V + 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \int_\Omega \sum_{p=1}^{\text{ndof}} u_{2p} \frac{\partial N_p}{\partial X_2} \frac{\partial N_q}{\partial X_2} \mathrm{d}V
\end{aligned} \tag{2.2.34}$$

where $q = 1, \ldots, \text{ndof}$.

The right–hand side of eq. (2.2.26) can be expressed:

$$-\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \delta_{1i} \frac{\partial N_q}{\partial X_j} \mathrm{d}V = -\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{11}} \frac{\partial N_q}{\partial X_1} \mathrm{d}V - \int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{12}} \frac{\partial N_q}{\partial X_2} \mathrm{d}V \tag{2.2.35}$$

and similarly the right–hand side of eq. (2.2.27) is:

$$-\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{ij}} \delta_{2i} \frac{\partial N_q}{\partial X_j} \mathrm{d}V = -\int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{21}} \frac{\partial N_q}{\partial X_1} \mathrm{d}V - \int_\Omega 2 \frac{\partial \mathcal{W}}{\partial C_{22}} \frac{\partial N_q}{\partial X_2} \mathrm{d}V \tag{2.2.36}$$

where $q = 1, \ldots, \text{ndof}$.

Therefore we can obtain the matrix form:

$$\begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{u}_X \\ \mathbf{u}_Y \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_X \\ -\mathbf{f}_Y \end{bmatrix} \tag{2.2.37}$$

where $X = X_1$ and $Y = X_2$.

The stiffness matrix $\mathbf{K}$ (left–hand side of eq. (2.2.37)) is:

$$\mathbf{K} = 2\frac{\partial \mathcal{W}}{\partial C_{11}}\int_\Omega \frac{\partial N_p}{\partial X}\frac{\partial N_q}{\partial X}\mathrm{d}V + 2\frac{\partial \mathcal{W}}{\partial C_{12}}\int_\Omega \frac{\partial N_p}{\partial X}\frac{\partial N_q}{\partial Y}\mathrm{d}V + 2\frac{\partial \mathcal{W}}{\partial C_{21}}\int_\Omega \frac{\partial N_p}{\partial Y}\frac{\partial N_q}{\partial X}\mathrm{d}V + 2\frac{\partial \mathcal{W}}{\partial C_{22}}\int_\Omega \frac{\partial N_p}{\partial Y}\frac{\partial N_q}{\partial Y}\mathrm{d}V \tag{2.2.38}$$

and the right–hand side of eq. (2.2.37) is:

$$\mathbf{f}_X = -2\frac{\partial \mathcal{W}}{\partial C_{11}}\int_\Omega \frac{\partial N_q}{\partial X}\mathrm{d}V - 2\frac{\partial \mathcal{W}}{\partial C_{12}}\int_\Omega \frac{\partial N_q}{\partial Y}\mathrm{d}V \tag{2.2.39}$$

$$\mathbf{f}_Y = -2\frac{\partial \mathcal{W}}{\partial C_{21}}\int_\Omega \frac{\partial N_q}{\partial X}\mathrm{d}V - 2\frac{\partial \mathcal{W}}{\partial C_{22}}\int_\Omega \frac{\partial N_q}{\partial Y}\mathrm{d}V \tag{2.2.40}$$

Eq. (2.2.38) can be written in the following matrix form:

$$\mathbf{K} = \int_\Omega \begin{bmatrix} \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} \end{bmatrix} \begin{bmatrix} 2\frac{\partial \mathcal{W}}{\partial C_{11}} & 2\frac{\partial \mathcal{W}}{\partial C_{12}} \\ 2\frac{\partial \mathcal{W}}{\partial C_{21}} & 2\frac{\partial \mathcal{W}}{\partial C_{22}} \end{bmatrix} \begin{bmatrix} \frac{\partial N_q}{\partial X} \\ \frac{\partial N_q}{\partial Y} \end{bmatrix} \mathrm{d}V \tag{2.2.41}$$

Therefore the stiffness matrix can be expressed:

$$\mathbf{K}^e = \int_\Omega \begin{bmatrix} \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} & 0 & 0 \\ 0 & 0 & \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} \end{bmatrix} \begin{bmatrix} 2\frac{\partial \mathcal{W}}{\partial C_{11}} & 2\frac{\partial \mathcal{W}}{\partial C_{12}} & 0 & 0 \\ 2\frac{\partial \mathcal{W}}{\partial C_{21}} & 2\frac{\partial \mathcal{W}}{\partial C_{22}} & 0 & 0 \\ 0 & 0 & 2\frac{\partial \mathcal{W}}{\partial C_{11}} & 2\frac{\partial \mathcal{W}}{\partial C_{12}} \\ 0 & 0 & 2\frac{\partial \mathcal{W}}{\partial C_{21}} & 2\frac{\partial \mathcal{W}}{\partial C_{22}} \end{bmatrix} \begin{bmatrix} \frac{\partial N_q}{\partial X} & 0 \\ \frac{\partial N_q}{\partial Y} & 0 \\ 0 & \frac{\partial N_q}{\partial X} \\ 0 & \frac{\partial N_q}{\partial Y} \end{bmatrix} \mathrm{d}V \tag{2.2.42}$$

where $p, q = 1, 2, \ldots, \mathrm{ndof}$.

Note: $\mathbf{u}^e = (\mathbf{u})^{\mathrm{T}}$

As the same way, the right–hand side of eq. (2.2.37) also can be presented as:

$$\mathbf{f}^e = -\int_\Omega \begin{bmatrix} \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} & 0 & 0 \\ 0 & 0 & \frac{\partial N_p}{\partial X} & \frac{\partial N_p}{\partial Y} \end{bmatrix} \begin{bmatrix} 2\frac{\partial \mathcal{W}}{\partial C_{11}} \\ 2\frac{\partial \mathcal{W}}{\partial C_{12}} \\ 2\frac{\partial \mathcal{W}}{\partial C_{21}} \\ 2\frac{\partial \mathcal{W}}{\partial C_{22}} \end{bmatrix} \mathrm{d}V, \quad p = 1, 2, \ldots, \mathrm{ndof} \tag{2.2.43}$$

Simple shear: substituting eq. (2.2.3) into eq. (2.2.21), we obtain:

$$\frac{\partial \mathcal{W}}{\partial I_1} = \frac{\mu}{2}, \quad \frac{\partial \mathcal{W}}{\partial I_2} = 0, \quad \text{and} \quad \frac{\partial \mathcal{W}}{\partial I_3} = \frac{\kappa}{2} - \left(\frac{\mu}{2} + \frac{\kappa}{2}\right)\frac{1}{I_3} \tag{2.2.44}$$

Therefore, eq. (2.2.21) can be expressed equivalently as:

$$\frac{\partial \mathcal{W}}{\partial \mathbf{C}} = \frac{\mu}{2}\mathbf{I} + \left(\frac{\kappa}{2} - \left(\frac{\mu}{2} + \frac{\kappa}{2}\right)\frac{1}{I_3}\right) I_3 \mathbf{C}^{-1} \tag{2.2.45}$$

where $I_1 = 3 + k^2 = I_2$ and $I_3 = 1$, and the inverse of the right Cauchy–Green strain tensor is

$$\mathbf{C}^{-1} = \begin{bmatrix} 1 + k^2 & -k \\ -k & 1 \end{bmatrix} \tag{2.2.46}$$

and then we obtain:

$$\frac{\partial \mathcal{W}}{\partial \mathbf{C}} = \frac{\mu}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\mu}{2}\begin{bmatrix} 1 + k^2 & -k \\ -k & 1 \end{bmatrix} = \frac{\mu}{2}\begin{bmatrix} -k^2 & k \\ k & 0 \end{bmatrix} \tag{2.2.47}$$

So we obtain the first Piola–Kirchhoff stress:

$$\mathbf{P} = 2\mathbf{F}\frac{\partial \mathcal{W}}{\partial \mathbf{C}} = 2\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\frac{\mu}{2}\begin{bmatrix} -k^2 & k \\ k & 0 \end{bmatrix} = \mu\begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \tag{2.2.48}$$

Thus, the Cauchy stress is:

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{\mathrm{T}} = \mu \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \mu \begin{bmatrix} k^2 & k \\ k & 0 \end{bmatrix} \tag{2.2.49}$$

We assume prescribed displacement on all boundaries, and the deformation gradient and the right Cauchy–Green strain tensor are respectively:

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \tag{2.2.50}$$

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ k & k^2+1 \end{bmatrix} \tag{2.2.51}$$

We use the prescribed displacement $k = 1.0$ and then the deformation gradient $\mathbf{F}$ is

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{2.2.52}$$

Simple tension: the deformation gradient, the left and right Cauchy–Green strain tensors for simple tension are

$$\mathbf{F} = \begin{bmatrix} f_a & 0 \\ 0 & f_b \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} f_a^2 & 0 \\ 0 & f_b^2 \end{bmatrix} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C} \tag{2.2.53}$$

and the inverse of the right Cauchy–Green strain tensor is:

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{f_a^2} & 0 \\ 0 & \frac{1}{f_b^2} \end{bmatrix} \tag{2.2.54}$$

The strain invariants are given by:

$$I_1 = 2b_1 + b_2, \quad I_2 = b_1^2 + 2b_1b_2, \quad \text{and} \quad I_3 = b_1^2 b_2$$

Hence, eq. (2.2.45) can be expressed as follows:

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \mathbf{C}} &= \frac{\mu}{2}\mathbf{I} + \left(\frac{\kappa}{2} - \left(\frac{\mu}{2} + \frac{\kappa}{2}\right)\frac{1}{I_3}\right) I_3 \mathbf{C}^{-1} \\ &= \frac{\mu}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{\kappa}{2} - \left(\frac{\mu}{2} + \frac{\kappa}{2}\right)\frac{1}{f_a^4 f_b^2}\right) f_a^4 f_b^2 \begin{bmatrix} \frac{1}{f_a^2} & 0 \\ 0 & \frac{1}{f_b^2} \end{bmatrix} \end{aligned} \tag{2.2.55}$$

We use Lamé's constant, $\mu = 0.6$ and $\kappa = 1.95$ and the deformation gradient $\mathbf{F}$ is:

$$\mathbf{F} = \begin{bmatrix} 0.8944 & 0 \\ 0 & 1.2879 \end{bmatrix} \tag{2.2.56}$$

3 Smoothed Finite Element Method Approximation

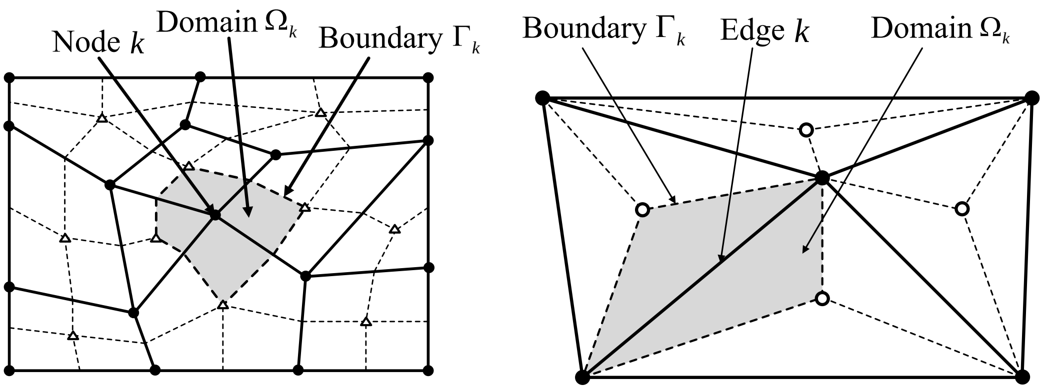
3.1 Linear Elasticity

The infinitesimal strain tensor from Eq. (6) is assumed to be the *smoothed strain* on the smoothing domain Ω_k associated with node k:

$$\boldsymbol{\varepsilon}\left(\mathbf{u}^{\mathrm{h}}\right) \approx \tilde{\boldsymbol{\varepsilon}}^{\mathrm{h}}\left(\mathbf{x}_k\right) = \int_{\Omega_k} \boldsymbol{\varepsilon}\left(\mathbf{x}\right)\Phi\left(\mathbf{x}_k\right)\mathrm{d}\Omega, \quad \forall \mathbf{x} \in \Omega_k \tag{3.1.1}$$

or

$$\tilde{\varepsilon}_{ij}^{\mathrm{h}}\left(\mathbf{x}_k\right) = \int_{\Omega_k} \frac{1}{2}\left(\frac{\partial u_i^{\mathrm{h}}}{\partial x_j} + \frac{\partial u_j^{\mathrm{h}}}{\partial x_j}\right)\Phi\left(\mathbf{x}_k\right)\mathrm{d}\Omega \tag{3.1.2}$$



(a) NS–FEM

(b) ES–FEM

Figure 1: (a) n–sided polygonal and triangular elements, and the smoothing domains associated with node k for NS–FEM, (b) triangular element with the smoothing domains associated with edge k for ES–FEM

where $\Phi\left(\mathbf{x}_k\right)$ is the smoothed gradient operator and satisfies following properties:

$$\int_{\Omega_k} \Phi\left(\mathbf{x}_k\right) \mathrm{d}\Omega = 1 \tag{3.1.3}$$

$$\Phi\left(\mathbf{x}_k\right) = \begin{cases} 1/A_k & \mathbf{x} \in \Omega_k \\ 0 & \mathbf{x} \notin \Omega_k \end{cases} \tag{3.1.4}$$

where $A_k = \int_{\Omega_k} \mathrm{d}\Omega$ is the area of smoothing domain Ω_k and applying the divergence theorem, the smoothed strain is obtained as follows:

$$\tilde{\boldsymbol{\varepsilon}}^{\mathrm{h}}\left(\mathbf{x}_k\right) = \frac{1}{A_k}\int_{\Omega_k} \boldsymbol{\varepsilon}\left(\mathbf{x}\right) \mathrm{d}\Omega = \frac{1}{A_k}\int_{\Gamma_k^s} \mathbf{n}\left(\mathbf{x}_k\right) \mathbf{u}^{\mathrm{h}}\left(\mathbf{x}\right) \mathrm{d}\Gamma \tag{3.1.5}$$

or

$$\tilde{\varepsilon}_{ij}^{\mathrm{h}}\left(\mathbf{x}_k\right) = \frac{1}{2A_k}\int_{\Gamma_k} \left(u_i^{\mathrm{h}} n_j\left(\mathbf{x}_k\right) + u_j^{\mathrm{h}} n_i\left(\mathbf{x}_k\right)\right) \mathrm{d}\Gamma \tag{3.1.6}$$

where Γ_L is the boundary of the smoothing domain Ω_k and $\mathbf{n}\left(\mathbf{x}_k\right)$ is the outward normal vector matrix on the boundary Γ_k.

The 2D outward normal vector matrix is:

$$\mathbf{n}\left(\mathbf{x}_k\right) = \begin{bmatrix} n_1\left(\mathbf{x}_k\right) & 0 \\ 0 & n_2\left(\mathbf{x}_k\right) \\ n_2\left(\mathbf{x}_k\right) & n_1\left(\mathbf{x}_k\right) \end{bmatrix} \tag{3.1.7}$$

where $n_x = n_1$ and $n_y = n_2$.

In NS–FEM, the trial function $\mathbf{u}^{\mathrm{h}}\left(\mathbf{x}\right)$ and the force vector $\mathbf{b}$ are calculated as for FEM. Substituting eq. (11) into eq. (20), the smoothed strain can be written in terms of the nodal displacements as follows:

$$\tilde{\boldsymbol{\varepsilon}}^{\mathrm{h}}\left(\mathbf{x}_k\right) = \sum_{I \in G_k} \tilde{\mathbf{B}}_I\left(\mathbf{x}_k\right) \mathbf{u}_I^{\mathrm{h}} \tag{3.1.8}$$

where G_k is a set of nodes in which the associated smoothing domain covers node k,

$$\tilde{\boldsymbol{\varepsilon}}^{\mathrm{h}^{\mathrm{T}}} = \left[\tilde{\varepsilon}_{11}^{\mathrm{h}}, \tilde{\varepsilon}_{22}^{\mathrm{h}}, 2\tilde{\varepsilon}_{12}^{\mathrm{h}}\right], \quad \mathbf{u}_I^{\mathrm{T}} = \left[u_{1I}, u_{2I}\right] \tag{3.1.9}$$

and the smoothed displacement–strain matrix $\tilde{\mathbf{B}}\left(\mathbf{x}_L\right)$ in 2D can be expressed as follows:

$$\tilde{\mathbf{B}}_I\left(\mathbf{x}_k\right) = \begin{bmatrix} \tilde{B}_{I1}\left(\mathbf{x}_k\right) & 0 \\ 0 & \tilde{B}_{I2}\left(\mathbf{x}_k\right) \\ \tilde{B}_{I2}\left(\mathbf{x}_k\right) & \tilde{B}_{I1}\left(\mathbf{x}_k\right) \end{bmatrix} \tag{3.1.10}$$

where

$$\tilde{B}_{Ii}\left(\mathbf{x}_k\right) = \frac{1}{A_k}\int_{\Gamma_k}\psi_I\left(\mathbf{x}\right)n_i\left(\mathbf{x}\right)\mathrm{d}\Gamma \tag{3.1.11}$$

where I is the set of all interior nodes such that $I\colon \operatorname{supp}\left(\psi_I\right)\cap\Gamma=\emptyset$.

The linear system to solve is:

$$\tilde{\mathbf{K}}\mathbf{u}^{\mathrm{h}} = \mathbf{b} \tag{3.1.12}$$

where the smoothed stiffness matrix $\tilde{\mathbf{K}}$ is assembled by a similar process as in FEM:

$$\tilde{K}_{ij} = \sum_{k=1}^{N_n}\left(\tilde{\mathbf{B}}_i^{\mathrm{T}}\left(\mathbf{x}_k\right)\mathbb{C}\tilde{\mathbf{B}}_j\left(\mathbf{x}_k\right)\right)A_k \tag{3.1.13}$$

$$b_i = \sum_{k=1}^{N_n}\left(\psi_i\left(\mathbf{x}\right)f\left(\mathbf{x}\right)\right)A_k + \sum_{k=1}^{N_{nb}}\left(\psi_i\left(\mathbf{x}\right)g\left(\mathbf{x}\right)\right)s_k \tag{3.1.14}$$

where N_n is the number of nodes, N_{nb} is the number of nodes on the natural boundary, and s_k are the weights associated with the boundary point.

3.2 Geometric nonlinearity

The nonlinear system to solve is:

$$\tilde{\mathbf{K}}^{\tan}\mathbf{u}^{\mathrm{h}} = \mathbf{b} - \tilde{\mathbf{R}} \tag{3.2.1}$$

where the smoothed tangent stiffness matrix is $\tilde{\mathbf{K}}^{\tan} = \tilde{\mathbf{K}}^{\mathrm{mat}} + \tilde{\mathbf{K}}^{\mathrm{geo}}$, and the material stiffness matrix $\tilde{\mathbf{K}}^{\mathrm{mat}}$ can be expressed as follows:

$$\tilde{\mathbf{K}}^{\mathrm{mat}} = \int_{\Omega}\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbb{C}\tilde{\mathbf{B}}_0\mathrm{d}\Omega = \sum_{k=1}^{N_n}\int_{\Omega_k}\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbb{C}\tilde{\mathbf{B}}_0\mathrm{d}\Omega = \sum_{k=1}^{N_n}\tilde{\mathbf{B}}_0^{\mathrm{T}}\mathbb{C}\tilde{\mathbf{B}}_0 A_k \tag{3.2.2}$$

where $\mathbb{C}$ is the elasticity tensor from eq. (2.1.9), N_n is the number of nodes and the area of subcell A_k is given by:

$$A_k = \int_{\Omega_k}\mathrm{d}\Omega = \frac{1}{3}\sum_{j=1}^{n_k^e}A_j^e \tag{3.2.3}$$

The smoothed strain–displacement matrix $\tilde{\mathbf{B}}_0$ is:

$$\tilde{\mathbf{B}}_0 = \frac{1}{A_k}\sum_{j=1}^{n_k^e}\frac{1}{3}A_j^e\mathbf{B}_{0,j}^e \tag{3.2.4}$$

where n_k^e is the number of elements sharing target node k and matrix $\mathbf{B}_0$ for the linear triangular element Ω_i^e in 2D problem is given by:

$$B_{0,i}^e = \begin{bmatrix} F_{11}b_1 & F_{21}b_1 & F_{11}b_2 & F_{21}b_2 & F_{11}b_3 & F_{21}b_3 \\ F_{12}c_1 & F_{22}c_1 & F_{12}c_2 & F_{22}c_2 & F_{12}c_3 & F_{22}c_3 \\ F_{11}c_1+F_{12}b_1 & F_{22}b_1+F_{21}c_1 & F_{11}c_2+F_{12}b_2 & F_{22}b_2+F_{21}c_2 & F_{11}c_3+F_{12}b_3 & F_{22}b_3+F_{21}c_3 \end{bmatrix} \tag{3.2.5}$$

where b_j and c_j are

$$b_j = \frac{1}{2A_i^e}\left(y_k - y_l\right), \quad c_j = \frac{1}{2A_i^e}\left(x_l - x_k\right), \quad j=1,2,3, \tag{3.2.6}$$

and F_{ij} is the deformation gradient:

$$\mathbf{F}^e = \left(\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\right)^{\mathrm{T}} = \begin{bmatrix} \left(1+\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right) & \frac{\partial\mathbf{u}}{\partial\mathbf{Y}} \\ \frac{\partial\mathbf{v}}{\partial\mathbf{X}} & \left(1+\frac{\partial\mathbf{v}}{\partial\mathbf{Y}}\right) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{3.2.7}$$

where subscript j varies from 1 to 3, and k and l are determined by cyclic permutation in the order of j, k, l. For example, if $j = 1$, then $k = 2$, $l = 3$ or if $j = 2$, then $k = 3$, $l = 1$. A_i^e, the area of the linear triangular element Ω_i^e, is:

$$A_i^e = \frac{1}{2}\det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{3.2.8}$$

Similarly, the geometric stiffness matrix $\mathbf{K}^{\text{geo}}$ is:

$$\tilde{\mathbf{K}}^{\text{geo}} = \int_{\Omega} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} \mathrm{d}\Omega = \sum_{k=1}^{N_n} \int_{\Omega_k} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} \mathrm{d}\Omega = \sum_{k=1}^{N_n} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} A_k \tag{3.2.9}$$

where smoothed strain–displacement matrix $\tilde{\mathcal{B}}$ is:

$$\tilde{\mathcal{B}} = \frac{1}{A_k} \sum_{j=1}^{n_k^e} \frac{1}{3} A_j^e \mathcal{B}_j^e \tag{3.2.10}$$

and matrix $\mathcal{B}$ is given by:

$$\mathcal{B}_i^e = \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ c_1 & 0 & c_2 & 0 & c_3 & 0 \\ 0 & b_1 & 0 & b_2 & 0 & b_3 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \end{bmatrix} \tag{3.2.11}$$

and matrix $\tilde{\mathbf{S}}$ for the node-based smoothing domain is:

$$\tilde{\mathbf{S}} = \frac{1}{A_k} \sum_{j=1}^{n_k^e} \frac{1}{3} A_j^e \mathbf{S}_j^e \quad \text{and} \quad \mathbf{S}^e = \begin{bmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{22} & 0 & 0 \\ 0 & 0 & S_{11} & S_{12} \\ 0 & 0 & S_{12} & S_{22} \end{bmatrix} \tag{3.2.12}$$

The smoothed internal force vector $\tilde{\mathbf{R}}$ can be expressed as follows:

$$\tilde{\mathbf{R}} = \sum_{k=1}^{N_n} \tilde{\mathbf{B}}_0 \left\{ \tilde{\mathbf{S}} \right\} A_k \tag{3.2.13}$$

where

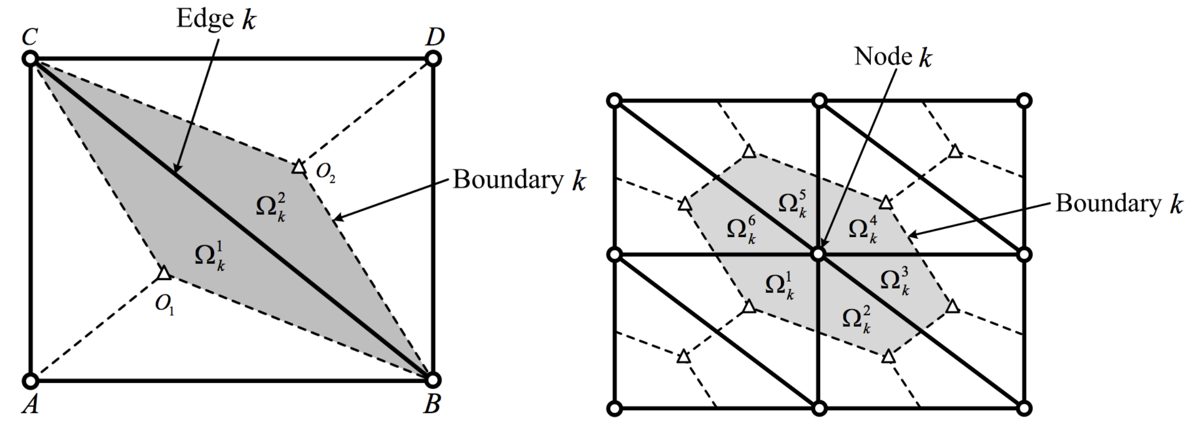
$$\left\{ \tilde{\mathbf{S}} \right\} = \frac{1}{A_k} \sum_{j=1}^{n_k^e} \frac{1}{3} A_j^e \left\{ \mathbf{S} \right\}_j^e \tag{3.2.14}$$

The entries S_{IJ} of matrix $\mathbf{S}^e$ in eq. (3.2.12) are derived from the second Piola–Kirchhoff stress tensor $\{\mathbf{S}^e\}$, and the second Piola–Kirchhoff stress tensor of the element is:

$$\{\mathbf{S}\}^e = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{Bmatrix} = \mathbb{C} \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix} \tag{3.2.15}$$

where the entries of E_{IJ} are derived from the Green–Lagrange strain tensor $\mathbf{E}^e$ of the element:

$$\mathbf{E}^e = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \frac{1}{2} \left((\mathbf{F}^e)^{\mathrm{T}} \mathbf{F}^e - \mathbf{I} \right) \tag{3.2.16}$$



(a) ES–FEM

(b) NS–FEM

Figure 2: (a) the smoothing domains for the smoothed deformation gradient $\tilde{\mathbf{F}}$ for ES–FEM, (b) the smoothing domains for the smoothed deformation gradient $\tilde{\mathbf{F}}$ for NS–FEM

3.3 Material Nonlinearity

For the material nonlinearity, the smoothed deformation gradient $\tilde{\mathbf{F}}$ in ES–FEM is given by Appendix B:

$$\tilde{\mathbf{F}}\left(\mathbf{x}_k\right) = \frac{1}{A_k}\int_{\Omega_k} \bar{\mathbf{F}}\left(\mathbf{x}_k\right)\Phi\left(\mathbf{x}_k\right)\mathrm{d}\Omega \tag{3.3.1}$$

Eq. (2.2.6) can be expressed as the same way in standard FEM:

$$D\mathcal{R}\left(\mathbf{u}_{\mathrm{iter}}\right)\cdot\mathbf{r}_{\mathrm{iter}} = -\mathcal{R}\left(\mathbf{u}_{\mathrm{iter}}\right) \tag{3.3.2}$$

where

$$\mathcal{R}\left(\mathbf{u}\right) = \int_{\Omega}\frac{\partial\mathcal{W}}{\partial\tilde{F}_{ij}}\left(\mathbf{X},\tilde{\mathbf{F}}\left(\mathbf{u}\right)\right)\frac{\partial v_i}{\partial X_j}\mathrm{d}\Omega - \int_{\Omega} f_i v_i \mathrm{d}\Omega - \int_{\Gamma_N} g_i v_i \mathrm{d}\Gamma \tag{3.3.3}$$

$$D\mathcal{R}\left(\mathbf{u}\right)\cdot\mathbf{r} = \int_{\Omega}\frac{\partial^2\mathcal{W}}{\partial\tilde{F}_{ij}\partial\tilde{F}_{kl}}\left(\mathbf{X},\tilde{\mathbf{F}}\left(\mathbf{u}\right)\right)\frac{\partial r_k}{\partial X_l}\frac{\partial v_i}{\partial X_j}\mathrm{d}\Omega \tag{3.3.4}$$

where $i,j,k,l = 1,2$, and $\mathbf{u}_{\mathrm{iter}+1} = \mathbf{u}_{\mathrm{iter}} + \mathbf{r}_{\mathrm{iter}}$.

The energy functional and its derivatives can be taken the equivalent formulations:

$$\mathcal{R}\left(\mathbf{u}\right) = \int_{\Omega} 2\frac{\partial\mathcal{W}}{\partial\tilde{C}_{ij}}\widetilde{\left(F_{ki}\frac{\partial v_k}{\partial X_j}\right)}\mathrm{d}\Omega - \int_{\Omega} f_i v_i \mathrm{d}\Omega - \int_{\Gamma_N} g_i v_i \mathrm{d}\Gamma \tag{3.3.5}$$

$$D\mathcal{R}\left(\mathbf{u}\right)\cdot\mathbf{r} = \int_{\Omega} 4\frac{\partial^2\mathcal{W}}{\partial\tilde{C}_{ij}\tilde{C}_{kl}}\widetilde{\left(F_{pi}\frac{\partial v_p}{\partial X_j}\right)}\widetilde{\left(F_{sk}\frac{\partial r_s}{\partial X_l}\right)} + 2\frac{\partial\mathcal{W}}{\partial\tilde{C}_{ij}}\widetilde{\left(\frac{\partial r_k}{\partial X_i}\right)}\widetilde{\left(\frac{\partial v_k}{\partial X_j}\right)}\mathrm{d}\Omega \tag{3.3.6}$$

where the smoothed right Cauchy–Green tensors $\tilde{\mathbf{C}}$ is:

$$\tilde{\mathbf{C}} = \tilde{\mathbf{F}}^{\mathrm{T}}\tilde{\mathbf{F}} \tag{3.3.7}$$

From eqs. (3.2.1), (3.3.5) and (3.3.6), the smoothed material stiffness matrix $\tilde{\mathbf{K}}^{\mathrm{mat}}$ is

$$\tilde{\mathbf{K}}^{\mathrm{mat}} = \int_{\Omega}\tilde{\mathbf{B}}_0^{\mathrm{T}}\tilde{\mathbb{C}}\tilde{\mathbf{B}}_0\mathrm{d}\Omega = \sum_{k=1}^{N_n}\int_{\Omega_k}\tilde{\mathbf{B}}_0^{\mathrm{T}}\tilde{\mathbb{C}}\tilde{\mathbf{B}}_0\mathrm{d}\Omega = \sum_{k=1}^{N_n}\tilde{\mathbf{B}}_0^{\mathrm{T}}\tilde{\mathbb{C}}\tilde{\mathbf{B}}_0 A_k, \tag{3.3.8}$$

where the smoothed neo–Hookean model $\tilde{\mathbb{C}}$ is

$$\begin{aligned} 4\frac{\partial^2\mathcal{W}}{\partial\tilde{C}_{ij}\tilde{C}_{kl}} &= \mu\left(\delta_{ik}\tilde{B}_{jl} + \tilde{B}_{il}\delta_{jk} - \frac{2}{3}\left(\tilde{B}_{ij}\delta_{kl} + \delta_{ij}\tilde{B}_{kl}\right) + \frac{2}{9}\mathrm{tr}\tilde{\mathbf{B}}\,\delta_{ij}\delta_{kl}\right)\frac{1}{\tilde{J}^{\frac{2}{3}}} \\ &\quad + \kappa\left(2\tilde{J} - 1\right)\tilde{J}\delta_{ij}\delta_{kl}, \end{aligned} \tag{3.3.9}$$

where the left Cauchy–Green tensor $\tilde{\mathbf{B}} = \tilde{\mathbf{F}}\tilde{\mathbf{F}}^{\mathrm{T}}$ and $\tilde{J} = \det\tilde{\mathbf{F}}$.

Similarly the smoothed geometric stiffness matrix $\tilde{\mathbf{K}}^{\mathrm{geo}}$ and the smoothed internal force vector $\tilde{\mathbf{R}}$ are

$$\tilde{\mathbf{K}}^{\mathrm{geo}} = \int_{\Omega} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} \mathrm{d}\Omega = \sum_{k=1}^{N_n} \int_{\Omega_k} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} \mathrm{d}\Omega = \sum_{k=1}^{N_n} \tilde{\mathcal{B}}^{\mathrm{T}} \tilde{\mathbf{S}} \tilde{\mathcal{B}} A_k \tag{3.3.10}$$

and

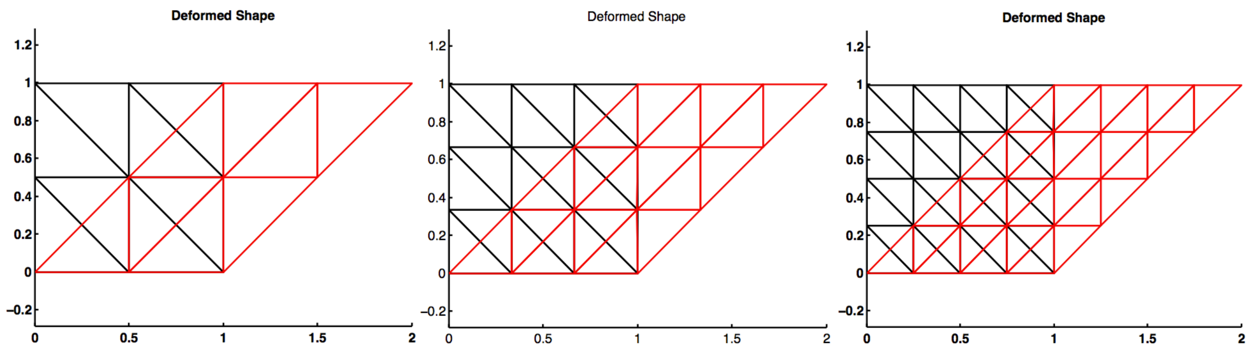
$$\tilde{\mathbf{R}} = \sum_{k=1}^{N_n} \tilde{\mathbf{B}}_0 \left\{ \tilde{\mathbf{S}} \right\} A_k. \tag{3.3.11}$$

3.4 Numerical examples

We represent numerical results of Dirchlet and Neumann BCs for simple shear and simple tension problems in the neo–Hookean material.

Simple shear: as the same former numerical example of FEM, we use the same prescribed displacement $k = 1$ for the deformation gradient. Fig. (3) shows the numerical results of triangular 2×2, 3×3 and 4×4 elements for simple shear with Dirichlet boundary conditions in nonlinear elasticity.

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



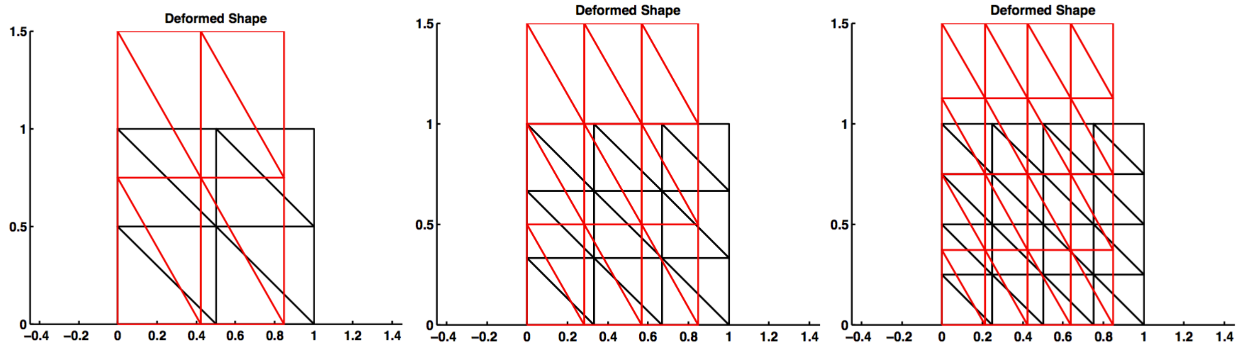
(a) 2×2 element (b) 3×3 element (c) 4×4 element

Figure 3: Numerical results of triangular elements for simple shear with Dirichlet BCs in nonlinear elasticity

Simple tension: for this numerical example, we use Lamé's constant, $\mu = 0.6$ and $\kappa = 1.95$, and the prescribed deformation gradient $\mathbf{F}$ as the same for FEM:

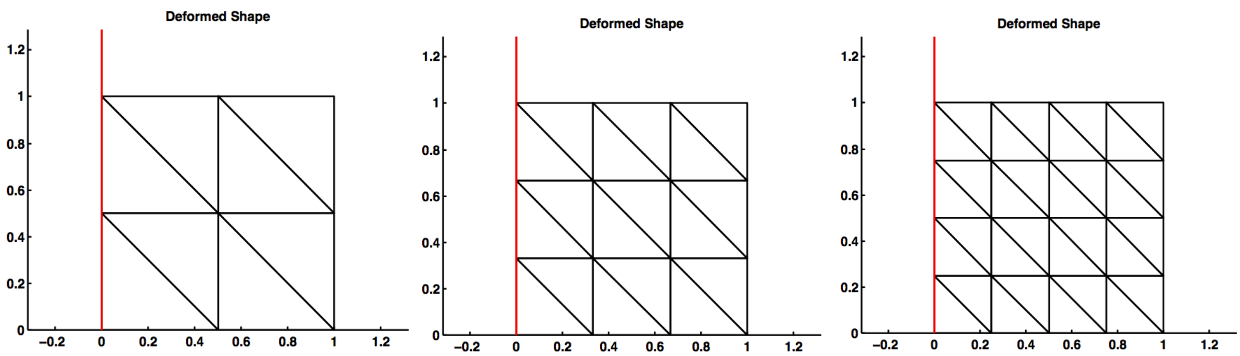
$$\mathbf{F} = \begin{bmatrix} 0.8944 & 0 \\ 0 & 1.2879 \end{bmatrix}$$

Figs. (4) and (5) describe the numerical results of triangular 2×2, 3×3 and 4×4 elements for simple tension with Dirichlet and Neumann boundary conditions in nonlinear elasticity.



(a) 2×2 element (b) 3×3 element (c) 4×4 element

Figure 4: Numerical results of triangular elements for simple tension with Dirichlet BCs in nonlinear elasticity



(a) 2×2 element (b) 3×3 element (c) 4×4 element

Figure 5: Numerical results of triangular elements for simple tension with Neumann BCs in nonlinear elasticity

A Imposing Dirichlet boundary conditions

A.1 Theorem

Implementing the Dirichlet boundary conditions (BCs) involves modifying the assembled stiffness matrix and right–hand vector of nodal forces by three operations [5]:

1. Move the know products to the right–hand column of the matrix equation;
2. Replace the columns and rows of the stiffness matrix corresponding to the known displacements by zeros, and set the coefficient on the main diagonal to one;
3. Replace the corresponding component of the right–hand column by the specified value of the displacements.

Consider the following n algebraic equations in the full matrix form:

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ k_{31} & k_{32} & k_{33} & \cdots & k_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} \hat{f}_1 \\ \bar{u}_2 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_n \end{Bmatrix} \tag{A.1.1}$$

where u_i are the global displacement degrees of freedom, f_i are the corresponding nodal forces, and k_{ij} are the assembled coefficients. Suppose that $u_s = \bar{u}_s$ is specified. Recall that when the displacement at a node is known, the corresponding nodal force is unknown, and vice versa. Set $k_{ss} = 1$ and $f_s = \bar{u}_s$; further, set $k_{is} = k_{si} = 0$ for $i = 1, 2, \ldots, n$ and $i \neq s$. For $s = 2$, the modified equations are:

$$\begin{bmatrix} k_{11} & 0 & k_{13} & \cdots & k_{1n} \\ 0 & 1 & 0 & \cdots & 0 \\ k_{31} & 0 & k_{33} & \cdots & k_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n1} & 0 & k_{n3} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} \hat{f}_1 \\ \bar{u}_2 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_n \end{Bmatrix} \tag{A.1.2}$$

where

$$\hat{f}_i = f_i - k_{i2}\bar{u}_2 \qquad (i = 1, 3, 4, \ldots, n \quad \text{and} \quad i \neq 2) \tag{A.1.3}$$

Thus, in general, if $u_s = \bar{u}_2$ is known, we have:

$$k_{ss} = 1, \quad f_s = \bar{u}_s, \quad \hat{f}_i = f_i - k_{is}\bar{u}_s, \quad \text{and} \quad k_{is} = k_{si} = 0 \tag{A.1.4}$$

where $i = 1, 2, \ldots, s-1, s+1, \ldots, n \quad (i \neq 2)$. This procedure is respected for every specified displacement. Then, the modification for the stiffness equation in eq. (A.1.1) for displacement BCs procedures the modified system:

$$\hat{\mathbf{K}}\mathbf{u} = \hat{\mathbf{f}} \tag{A.1.5}$$

System (A.1.5) is solved for the unknown nodal displacements.

A.2 Implementation

In this section, we present how to impose and solve the Dirichlet BCs in the numerical code. Fig. (6) shows the simple example of the imposing Dirichlet BCs. Stiffness matrix $\mathbf{K}$, displacements $\mathbf{u}$, and force vector $\mathbf{f}$ are:

$$\begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{A.2.1}$$

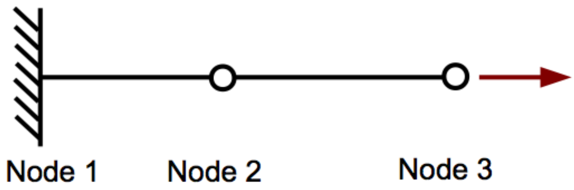


Figure 6: Simple example for Dirichlet BCs

According to fig. (6), we know as $u_1 = 0$ and u_3 is prescribed displacement. Thus, eq. (A.2.1) can be expanded as:

$$\begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & K_{23} \\ 0 & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ u_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{A.2.2}$$

where $\bar{u}_1$ and $\bar{u}_3$ are knowns. Hence,

$$K_{11}\bar{u}_1 + K_{12}u_2 = f_1$$

$$K_{12}\bar{u}_1 + K_{22}u_2 + K_{23}\bar{u}_3 = f_2$$

$$K_{32}\bar{u}_2 + K_{33}u_3 = f_3 \tag{A.2.3}$$

Left–hand side of equation with known displacements is moved to right–hand side of equation to solve unknown,u_2:

$$K_{12}u_2 = f_1 - K_{11}\bar{u}_1$$

$$K_{22}u_2 = f_2 - K_{12}\bar{u}_1 - K_{23}\bar{u}_3 \tag{A.2.4}$$

$$K_{32}u_2 = f_3 - K_{33}\bar{u}_3 \tag{A.2.5}$$

This process can be written in the MATLAB as follows:

$$\texttt{r}\,(\texttt{iNonFixed}) = \texttt{r}\,(\texttt{iNonFixed}) - \texttt{k}\,(\texttt{iNonFixed}, \texttt{iFixDof}) * \texttt{iFixVal}$$

where `k` is the stiffness matrix, `r` is the residual vector, `iFixVal` is the vector containing the value which the dofs should be fixed in the right order, `iFixDof` is vector containing the number of each dofs with non–zero Dirichlet conditions, and `iNonFixed` is vector containing the number of each dofs whose value is not fixed to a non–zero value.

B The Smoothed Deformation Gradient

B.1 ES–FEM

If the deformation gradient $\mathbf{F}$ is homogeneous on element, the displacement field on single element can be explained as following [4]:

$$\mathbf{u}\,(\mathbf{X}) = \begin{bmatrix} u_1\,(\mathbf{X}) \\ u_2\,(\mathbf{X}) \end{bmatrix} = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 + b_1 \\ a_{21}X_1 + a_{22}X_2 + b_2 \end{bmatrix} \tag{B.1.1}$$

where the undetermined coefficients a_{ij} and b_i, for $i, j = 1, 2$, are constant.

The deformation gradient on a triangle $\triangle ABC$ for the standard FEM in Figure 7 is

$$\mathbf{F} = \begin{bmatrix} a_{11}+1 & a_{12} \\ a_{21} & a_{22}+1 \end{bmatrix} = \begin{bmatrix} \left(u_1^B - u_1^A\right)/h + 1 & \left(u_1^C - u_1^A\right)/h \\ \left(u_2^B - u_2^A\right)/h & \left(u_2^C - u_2^A\right)/h + 1 \end{bmatrix} \tag{B.1.2}$$

For the smoothed deformation gradient $\tilde{\mathbf{F}}$ in the smoothing domain Ω_k in Figure 7, the deformation gradient in the smoothing domain Ω_k^1 can be expressed as following:

$$u_1\,(O_1) = \frac{1}{3}\left(u_1^A + u_1^B + u_1^C\right) \qquad \text{and} \qquad u_2\,(O_1) = \frac{1}{3}\left(u_2^A + u_2^B + u_2^C\right) \tag{B.1.3}$$

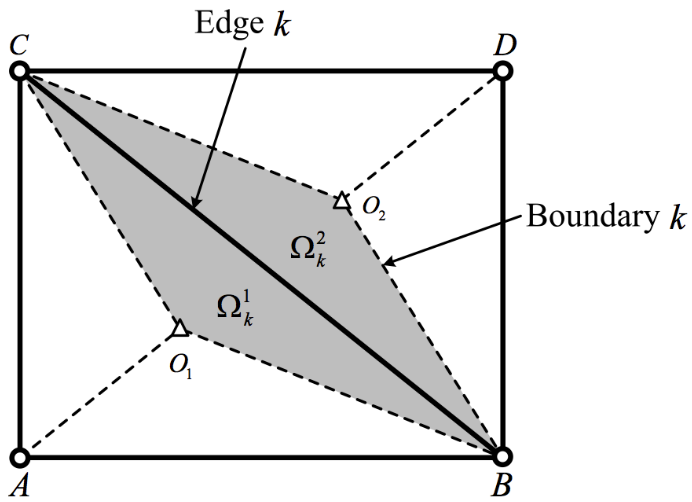


Figure 7: The smoothing domain $\Omega_k = \Omega_k^1 + \Omega_k^2$ associated edge k for ES–FEM

Substituting eq. (B.1.3) into eq. (B.1.1), the displacement field on mid–point O_1 is given by

$$\frac{1}{3}\left(u_1^A + u_1^B + u_1^C\right) = a_{11}\frac{h}{3} + a_{12}\frac{h}{3} + b_1$$

$$\frac{1}{3}\left(u_2^A + u_2^B + u_2^C\right) = a_{21}\frac{h}{3} + a_{22}\frac{h}{3} + b_2 \tag{B.1.4}$$

Similarly, the displacement fields on node B and C can be written as:

$$u_1^B = a_{11}h + b_1 \qquad \text{and} \qquad u_2^B = a_{21}h + b_2 \tag{B.1.5}$$

and

$$u_1^C = a_{12}h + b_1 \qquad \text{and} \qquad u_2^C = a_{22}h + b_2 \tag{B.1.6}$$

Substituting eq. (B.1.6) into eq. (B.1.5),

$$a_{11} - a_{12} = \frac{u_1^B - u_1^C}{h} \qquad \text{and} \qquad a_{21} - a_{22} = \frac{u_2^B - u_2^C}{h} \tag{B.1.7}$$

Hence, the displacements on the mid–point O_1 are given by

$$u_1^A + u_1^B + u_1^C = a_{11}h + a_{12}h + 3\left(u_1^C - a_{12}h\right) \tag{B.1.8a}$$
$$u_2^A + u_2^B + u_2^C = a_{21}h + a_{22}h + 3\left(u_2^C - a_{22}h\right) \tag{B.1.8b}$$

From eq. B.1.8, the undetermined coefficient a_{ij} are defined as follows:

$$a_{11} = \frac{u_1^B - u_1^A}{h}, \quad a_{12} = \frac{u_1^C - u_1^A}{h}, \quad a_{21} = \frac{u_2^B - u_2^A}{h}, \quad a_{22} = \frac{u_2^C - u_2^A}{h} \tag{B.1.9}$$

Similarly, the undetermined coefficient a_{ij} for triangle $\triangle DCB$ in Figure 7 are given by

$$a_{11} = \frac{u_1^C - u_1^D}{h}, \quad a_{12} = \frac{u_1^B - u_1^D}{h}, \quad a_{21} = \frac{u_2^C - u_2^D}{h}, \quad a_{22} = \frac{u_2^B - u_2^D}{h} \tag{B.1.10}$$

The smoothed deformation gradient is given by [2]:

$$\begin{aligned} \tilde{F}_{ij}(\mathbf{x}_k) &= \frac{1}{A_k}\int_{\Omega_k} \bar{F}_{ij}(\mathbf{x}_k)\,\bar{\Phi}(\mathbf{x}_k)\,\mathrm{d}\Omega \\ &= \frac{1}{A_k}\int_{\Omega_k} \left(\frac{\partial u_i^{\mathrm{h}}}{\partial X_j}\right)\bar{\Phi}(\mathbf{x}_k)\,\mathrm{d}\Omega + \delta_{ij} \end{aligned} \tag{B.1.11}$$

where $\bar{\Phi}$ is:

$$\bar{\Phi} = \begin{cases} 1 & \mathbf{x} \in \Omega_k \\ 0 & \text{otherwise} \end{cases} \tag{B.1.12}$$

and then,

$$\tilde{F}_{11} = \frac{1}{A_k}\left\{\int_{\Omega_k^1} \frac{\partial u_1^{\mathrm{h}}}{\partial X_1}\mathrm{d}\Omega + \int_{\Omega_k^2} \frac{\partial u_1^{\mathrm{h}}}{\partial X_1}\mathrm{d}\Omega\right\} + 1 = \frac{3}{h^2}\left(a_{11}^1\frac{h^2}{6} + a_{11}^2\frac{h^2}{6}\right) + 1 \tag{B.1.13a}$$

$$\tilde{F}_{12} = \frac{1}{A_k}\left\{\int_{\Omega_k^1} \frac{\partial u_1^{\mathrm{h}}}{\partial X_2}\mathrm{d}\Omega + \int_{\Omega_k^2} \frac{\partial u_1^{\mathrm{h}}}{\partial X_2}\mathrm{d}\Omega\right\} = \frac{3}{h^2}\left(a_{12}^1\frac{h^2}{6} + a_{12}^2\frac{h^2}{6}\right) \tag{B.1.13b}$$

$$\tilde{F}_{21} = \frac{1}{A_k}\left\{\int_{\Omega_k^1} \frac{\partial u_2^{\mathrm{h}}}{\partial X_1}\mathrm{d}\Omega + \int_{\Omega_k^2} \frac{\partial u_2^{\mathrm{h}}}{\partial X_1}\mathrm{d}\Omega\right\} = \frac{3}{h^2}\left(a_{21}^1\frac{h^2}{6} + a_{21}^2\frac{h^2}{6}\right) \tag{B.1.13c}$$

$$\tilde{F}_{22} = \frac{1}{A_k}\left\{\int_{\Omega_k^1} \frac{\partial u_2^{\mathrm{h}}}{\partial X_2}\mathrm{d}\Omega + \int_{\Omega_k^2} \frac{\partial u_2^{\mathrm{h}}}{\partial X_2}\mathrm{d}\Omega\right\} + 1 = \frac{3}{h^2}\left(a_{22}^1\frac{h^2}{6} + a_{22}^2\frac{h^2}{6}\right) + 1 \tag{B.1.13d}$$

where $A_k = A_k^1 + A_K^2 = \frac{h^2}{6} + \frac{h^2}{6} = \frac{h^2}{3}$, and the matrix form is:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{1}{2}\left(\frac{u_1^B - u_1^A}{h} + \frac{u_1^C - u_1^D}{h}\right) + 1 & \frac{1}{2}\left(\frac{u_1^C - u_1^A}{h} + \frac{u_1^B - u_1^D}{h}\right) \\ \frac{1}{2}\left(\frac{u_2^B - u_2^A}{h} + \frac{u_2^C - u_2^D}{h}\right) & \frac{1}{2}\left(\frac{u_2^C - u_2^A}{h} + \frac{u_2^B - u_2^D}{h}\right) + 1 \end{bmatrix} \tag{B.1.14}$$

In case the edge is on the boundary, the smoothed deformation gradient $\tilde{\mathbf{F}}$ can be described as following:

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{1}{2}\left(\frac{u_1^B - u_1^A}{h}\right) + 1 & \frac{1}{2}\left(\frac{u_1^C - u_1^A}{h}\right) \\ \frac{1}{2}\left(\frac{u_2^B - u_2^A}{h}\right) & \frac{1}{2}\left(\frac{u_2^C - u_2^A}{h}\right) + 1 \end{bmatrix} \tag{B.1.15}$$

B.2 NS–FEM

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