Strong asymptotic independence on Wiener chaos

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Abstract

Let $F_n = (F_{1,n}, ..., F_{d,n})$, $n \geqslant 1$, be a sequence of random vectors such that, for every $j = 1, ..., d$, the random variable $F_{j,n}$ belongs to a fixed Wiener chaos of a Gaussian field. We show that, as $n \to \infty$, the components of F_n are asymptotically independent if and only if $\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) \to 0$ for every $i \neq j$. Our findings are based on a novel inequality for vectors of multiple Wiener-Itô integrals, and represent a substantial refining of criteria for asymptotic independence in the sense of moments recently established by Nourdin and Rosiński [9].

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1 Introduction

1.1 Overview

Let $X = \{X(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$ (see Section 1.2 and Section 2 for relevant definitions), and let $F_n = (F_{1,n}, ..., F_{d,n})$, $n \geqslant 1$, be a sequence of random vectors such that, for every $j = 1, ..., d$, the random variable $F_{j,n}$ belongs the q_jth Wiener chaos of X (the order $q_j \geqslant 1$ of the chaos being independent of n). The following result, proved by Nourdin and Rosiński in [9, Corollary 3.6], provides a useful criterion for the asymptotic independence of the components of F_n.

Theorem 1.1 (See [9]) *Assume that, as $n \to \infty$ and for every $i \neq j$,*

$$\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) \to 0 \quad and \quad F_{j,n} \xrightarrow{\mathrm{law}} U_j, \tag{1.1}$$

where each U_j is a moment-determinate* random variable. Then,

$$F_n \xrightarrow{\text{law}} (U_1, \ldots, U_d),$$

where the U_j's are assumed to be mutually stochastically independent.

In words, Theorem 1.1 allows one to deduce joint convergence from the component-wise convergence of the elements of F_n, provided the limit law of each sequence $\{F_{j,n}\}$ is moment-determinate and the covariances between the squares of the distinct components of F_n vanish asymptotically. This result and its generalisations have already led to some important applications, notably in connection with time-series analysis and with the asymptotic theory of homogeneous sums — see [1, 2, 9]. The aim of this paper is to study the following important question, which was left open in the reference [9]:

Question A. *In the statement of Theorem 1.1, is it possible to remove the moment-determinacy assumption for the random variables $U_1, ..., U_d$?*

Question A is indeed very natural. For instance, it is a well-known fact (see [14, Section 3]) that non-zero random variables living inside a fixed Wiener chaos of order $q \geqslant 3$ are not necessarily moment-determinate, so that Theorem 1.1 cannot be applied in several contexts where the limit random variables U_j have a chaotic nature. Until now, such a shortcoming has remarkably restricted the applicability of Theorem 1.1 — see for instance the discussion contained in [1, Section 3].

In what follows, we shall derive several new probabilistic estimates (stated in Section 1.3 below) for chaotic random variables, leading to a general positive answer to Question A. As opposed to the techniques applied in [9], our proof does not make use of combinatorial arguments. Instead, we shall heavily rely on the use of Malliavin calculus and Meyer inequalities (see the forthcoming formula (2.11)). This new approach will yield several quantitative extensions of Theorem 1.1, each having its own interest. Note that, in particular, our main results immediately confirm Conjecture 3.7 in [1].

The content of the present paper represents a further contribution to a recent and very active direction of research, revolving around the application of Malliavin-type techniques for deriving probabilistic approximations and limit theorems, with special emphasis on normal approximation results (see [11, 13] for two seminal contributions to the field, as well as [8] for recent developments).The reader is referred to the book [7] and the survey [4] for an overview of this area of research. One can also consult the constantly updated webpage [5] for literally hundreds of results related to the findings contained in [11, 13] and their many ramifications.

1.2 Some basic definitions and notation

We refer the reader to [7, 10] for any unexplained definition or result.

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*Recall that a random variable U with moments of all orders is said to be *moment-determinate* if $E[X^n] = E[U^n]$ for every $n = 1, 2, ...$ implies that X and U have the same distribution.

Let $\mathfrak{H}$ be a real separable infinite-dimensional Hilbert space. For any integer $q \geqslant 1$, let $\mathfrak{H}^{\otimes q}$ be the qth tensor product of $\mathfrak{H}$. Also, we denote by $\mathfrak{H}^{\odot q}$ the qth symmetric tensor product. From now on, the symbol $X = \{X(h) : h \in \mathfrak{H}\}$ will indicate an isonormal Gaussian process on $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. In particular, X is a centered Gaussian family with covariance given by $E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$. We will also assume that $\mathcal{F}$ is generated by X.

For every integer $q \geqslant 1$, we let $\mathcal{H}_q$ be the qth *Wiener chaos* of X, that is, $\mathcal{H}_q$ is the closed linear subspace of $L^2(\Omega)$ generated by the class $\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{x^2/2} \frac{d^q}{dx^q} \big(e^{-x^2/2} \big).$$

We denote by $\mathcal{H}_0$ the space of constant random variables. For any $q \geqslant 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ can be extended a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$ norm). For $q = 0$, by convention $\mathcal{H}_0 = \mathbb{R}$, and I_0 is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$, that is: any square-integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{1.2}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. For every $q \geqslant 0$, we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (1.2), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

1.3 Main results

The main achievement of the present paper is the explicit estimate (1.3), appearing in the forthcoming Theorem 1.2. Note that, in order to obtain more readable formulae, we only consider multiple integrals with unit variance: one can deduce bounds in the general case by a standard rescaling procedure.

Remark on notation. Fix integers $m, q \geqslant 1$. Given a smooth function $\varphi : \mathbb{R}^m \to \mathbb{R}$, we shall use the notation

$$\|\varphi\|_q := \|\varphi\|_\infty + \sum \left\| \frac{\partial^k \varphi}{\partial x_{i_1}^{k_1} \cdots \partial x_{i_p}^{k_p}} \right\|_\infty,$$

where the sum runs over all $p = 1, ..., m$, all $\{i_1, ..., i_p\} \subset \{1, ..., m\}$, and all multi indices $(k_1, ..., k_p) \in \{1, 2, ...\}^p$ verifying $k_1 + \cdots + k_p := k \leqslant q$.

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Theorem 1.2 *Let $d \geqslant 2$ and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_d \geqslant 1$ be fixed integers. There exists a constant c, uniquely depending on d and $(q_1, ..., q_d)$, verifying the following bound for any d-dimensional vector*

$$F = (F_1, ..., F_d),$$

such that $F_j = I_{q_j}(f_j)$, $f_j \in \mathfrak{H}^{\odot q_j}$ $(j = 1, ..., d)$ and $E[F_j^2] = 1$ for $j = 1, ..., d - 1$, and for any collection of smooth test functions $\psi_1, ..., \psi_d : \mathbb{R} \to \mathbb{R}$,

$$\left| E\left[\prod_{j=1}^d \psi_j(F_j) \right] - \prod_{j=1}^d E[\psi_j(F_j)] \right| \leqslant c \, \|\psi_d'\|_\infty \prod_{j=1}^{d-1} \|\psi_j\|_{q_1} \sum_{1 \leqslant j < \ell \leqslant d} \mathrm{Cov}(F_j^2, F_\ell^2). \quad (1.3)$$

When applied to sequences of multiple stochastic integrals, Theorem 1.2 allows one to deduce the following strong generalization of [9, Theorem 3.4].

Theorem 1.3 *Let $d \geqslant 2$ and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_d \geqslant 1$ be fixed integers. For every $n \geqslant 1$, let $F_n = (F_{1,n}, ..., F_{d,n})$ be a d-dimensional random vector such that $F_{j,n} = I_{q_j}(f_{j,n})$, with $f_{j,n} \in \mathfrak{H}^{\odot q_j}$ and $E[F_{j,n}^2] = 1$ for all $1 \leqslant j \leqslant d$ and $n \geqslant 1$. Then, the following three conditions are equivalent, as $n \to \infty$:*

(1) $\mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) \to 0$ *for every* $1 \leqslant i \neq j \leqslant d$;

(2) $\|f_{i,n} \otimes_r f_{j,n}\| \to 0$ *for every* $1 \leqslant i \neq j \leqslant d$ *and* $1 \leqslant r \leqslant q_i \wedge q_j$;

(3) *The random variables $F_{1,n}, ..., F_{d,n}$ are asymptotically independent, that is, for every collection of smooth bounded test functions $\psi_1, ..., \psi_d : \mathbb{R} \to \mathbb{R}$,*

$$E\left[\prod_{j=1}^d \psi_j(F_{j,n}) \right] - \prod_{j=1}^d E[\psi_j(F_{j,n})] \longrightarrow 0.$$

We can now state the announced extension of Theorem 1.1 (see Section 1), in which the determinacy condition for the limit random variables U_j has been eventually removed.

Theorem 1.4 *Let $d \geqslant 2$ and let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_d \geqslant 1$ be fixed integers. For every $n \geqslant 1$, let $F_n = (F_{1,n}, ..., F_{d,n})$ be a d-dimensional random vector such that $F_{j,n} = I_{q_j}(f_{j,n})$, with $f_{j,n} \in \mathfrak{H}^{\odot q_j}$ and $E[F_{j,n}^2] = 1$ for all $1 \leqslant j \leqslant d$ and $n \geqslant 1$. Let $U_1, ..., U_d$ be independent random variables such that $F_{j,n} \overset{\mathrm{law}}{\to} U_j$ as $n \to \infty$ for every $1 \leqslant j \leqslant d$. Assume that either Condition (1) or Condition (2) of Theorem 1.3 holds. Then, as $n \to \infty$,*

$$F_n \overset{\mathrm{law}}{\to} (U_1, ..., U_d).$$

By considering linear combinations, one can also prove the following straightforward generalisations of Theorem 1.3 and Theorem 1.4 (which are potentially useful for applications), where each component of the vector F_n is replaced by a multidimensional object. The simple proofs are left to the reader.

Proposition 1.5 *Let $d \geqslant 2$, let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_d \geqslant 1$ and $m_1, ..., m_d \geqslant 1$ be fixed integers, and set $M := \sum_{j=1}^{d} m_j$. For every $j = 1, ..., d$, let*

$$\mathbf{F}_{j,n} = (F_{j,n}^{(1)}, ..., F_{j,n}^{(m_j)}) := \big(I_{q_j}(f_{j,n}^{(1)}), ..., I_{q_j}(f_{j,n}^{(m_j)})\big),$$

where, for $\ell = 1, ..., m_j$, $f_{j,n}^{(\ell)} \in \mathfrak{H}^{\odot q_j}$ and $E[(F_{j,n}^{(\ell)})^2] = q_j! \|f_{j,n}^{(\ell)}\|_{\mathfrak{H}^{\otimes q_j}}^2 = 1$. Finally, for every $n \geqslant 1$, write $\mathbf{F}_n$ to indicate the M-dimensional vector $(\mathbf{F}_{1,n}, ..., \mathbf{F}_{d,n})$. Then, the following three conditions are equivalent, as $n \to \infty$:

(1) $\mathrm{Cov}\big((F_{i,n}^{(\ell)})^2, (F_{j,n}^{(\ell')})^2\big) \to 0$ *for every* $1 \leqslant i \neq j \leqslant d$, *every* $\ell = 1, ..., m_i$ *and every* $\ell' = 1, ..., m_j$;

(2) $\|f_{i,n}^{(\ell)} \otimes_r f_{j,n}^{(\ell')}\| \to 0$ *for every* $1 \leqslant i \neq j \leqslant d$, *for every* $1 \leqslant r \leqslant q_i \wedge q_j$, *every* $\ell = 1, ..., m_i$ *and every* $\ell' = 1, ..., m_j$;

(3) *The random vectors* $\mathbf{F}_{1,n}, ..., \mathbf{F}_{d,n}$ *are asymptotically independent, that is: for every collection of smooth bounded test functions* $\psi_j : \mathbb{R}^{m_j} \to \mathbb{R}$, $j = 1, ..., d$,

$$E\left[\prod_{j=1}^{d} \psi_j(\mathbf{F}_{j,n})\right] - \prod_{j=1}^{d} E[\psi_j(\mathbf{F}_{j,n})] \longrightarrow 0.$$

Proposition 1.6 *Let the notation and assumptions of Proposition 1.5 prevail, and assume that either Condition (1) or Condition (2) therein is satisfied. Consider a collection $(\mathbf{U}_1, ..., \mathbf{U}_d)$ of independent random vectors such that, for $j = 1, ..., d$, $\mathbf{U}_j$ has dimension m_j. Then, if $\mathbf{F}_{j,n}$ converges in distribution to $\mathbf{U}_j$, as $n \to \infty$, one has also that*

$$\mathbf{F}_n \overset{\text{law}}{\to} (\mathbf{U}_1, \dots, \mathbf{U}_d).$$

The plan of the paper is as follows. Section 2 contains some further preliminaries related to Gaussian analysis and Malliavin calculus. The proofs of our main results are gathered in Section 3.

2 Further notation and results from Malliavin calculus

Let $\{e_k, \, k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \dots, p \wedge q\}$, the rth *contraction* of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,...,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.4}$$

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$,

$f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. In the particular case $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q}$ can be identified with the space $L_s^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ of μ^q-almost everywhere symmetric and square-integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral of order q of f with respect to X and (2.4) can be written as

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) = \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r)$$
$$\times \ g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).$$

We will now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X (see again [7, 10] for any unexplained notion or result). Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right), \tag{2.5}$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF \ = \ \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(X(\phi_1), \ldots, X(\phi_n)\right)\phi_i.$$

By iteration, one can define the qth derivative $D^q F$ for every $q \geqslant 2$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$.

For $q \geqslant 1$ and $p \geqslant 1$, $\mathbb{D}^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\| \cdot \|_{\mathbb{D}^{q,p}}$, defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p \ = \ E\left[|F|^p\right] + \sum_{i=1}^q E\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

The *Malliavin derivative D* verifies the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F)DF_i. \tag{2.6}$$

Note also that a random variable F as in (1.2) is in $\mathbb{D}^{1,2}$ if and only if

$$\sum_{q=1}^{\infty} q q! \|f_q\|_{\mathfrak{H}^{\otimes q}}^2 < \infty,$$

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and, in this case, $E\left[\|DF\|_{\mathfrak{H}}^2\right] = \sum_{q \geqslant 1} qq!\|f_q\|_{\mathfrak{H}^{\otimes q}}^2$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (1.2) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}\left(f_q(\cdot, a)\right), \quad a \in A. \tag{2.7}$$

We denote by δ the adjoint of the operator D, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, noted $\mathrm{Dom}\,\delta$, if and only if it verifies

$$\left|E\left[\langle DF, u\rangle_{\mathfrak{H}}\right]\right| \leqslant c_u \sqrt{E[F^2]}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (customarily called 'integration by parts formula'):

$$E[F\delta(u)] = E\left[\langle DF, u\rangle_{\mathfrak{H}}\right], \tag{2.8}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The formula (2.8) extends to the multiple Skorohod integral δ^q, and we have

$$E[F\delta^q(u)] = E\left[\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\right] \tag{2.9}$$

for any element u in the domain of δ^q and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$.

The following property, corresponding to [6, Lemma 2.1], will be used in the paper. Let $q \geqslant 1$ be an integer, suppose that $F \in \mathbb{D}^{q,2}$, and let u be a symmetric element in $\mathrm{Dom}\,\delta^q$. Assume that $\langle D^r F, \delta^j(u)\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q-r-j})$ for any $0 \leqslant r + j \leqslant q$. Then $\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of δ^{q-r} for any $r = 0, \ldots, q-1$, and we have

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\left(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}\right). \tag{2.10}$$

(We use the convention that $\delta^0(v) = v$, $v \in \mathbb{R}$, and $D^0 F = F$, $F \in L^2(\Omega)$.)

For any Hilbert space V, we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables (see [10, page 31]). The operator δ^q is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$, for any $p > 1$ and any integers $k \geq q \geq 1$, and one has the estimate

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leqslant c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})} \tag{2.11}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$, and some constant $c_{k,p} > 0$. These inequalities are direct consequences of the so-called *Meyer inequalities* (see [10, Proposition 1.5.7]). In particular, these estimates imply that $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\,\delta^q$ for any integer $q \geq 1$.

The operator L is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. The domain of this operator in $L^2(\Omega)$ is the set

$$\mathrm{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \, \|J_q F\|^2_{L^2(\Omega)} < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relationship between the operators D, δ and L. A random variable F belongs to the domain of L if and only if $F \in \mathrm{Dom}\,(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\,\delta$), and in this case

$$\delta DF = -LF. \tag{2.12}$$

We also define the operator L^{-1}, which is the *pseudo-inverse* of L, as follows: for every $F \in L^2(\Omega)$ with zero mean, we set $L^{-1}F = \sum_{q \geq 1} -\frac{1}{q}J_q(F)$. We note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and that $LL^{-1}F = E - E[F]$ for any $F \in L^2(\Omega)$.

3 Proofs of the results stated in Section 1.3

3.1 Proof of Theorem 1.2

The proof of Theorem 1.2 is based on a recursive application of the following quantitative result, whose proof has been inspired by the pioneering work of Üstünel and Zakai on the characterization of the independence on Wiener chaos (see [15]).

Proposition 3.1 *Let $m \geq 1$ and $p_1, ..., p_m, q$ be integers such that $p_j \geq q$ for every $j = 1, ..., m$. There exists a constant c, uniquely depending on m and $p_1, ..., p_m, q$, such that one has the bound*

$$|E[\varphi(F)\psi(G)] - E[\varphi(F)]E[\psi(G)]| \leq c\|\psi'\|_\infty \|\varphi\|_q \sum_{j=1}^{m} \mathrm{Cov}(F_j^2, G^2),$$

for every vector $F = (F_1, ..., F_m)$ such that $F_j = I_{p_j}(f_j)$, $f_j \in \mathfrak{H}^{\odot p_j}$ and $E[F_j^2] = 1$ $(j = 1, ..., m)$, for every random variable $G = I_q(g)$, $g \in \mathfrak{H}^{\odot q}$, and for every pair of smooth test functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$.

Proof. Throughout the proof, the symbol c will denote a positive finite constant uniquely depending on m and $p_1, ..., p_m, q$, whose value may change from line to line. Using the chain rule (2.6) together with the relation $-DL^{-1} = (I - L)^{-1}D$ (see, e.g., [12]), one has

$$\varphi(F) - E[\varphi(F)] = LL^{-1}\varphi(F) = -\delta(DL^{-1}\varphi(F)) = \sum_{j=1}^{m} \delta((I - L)^{-1}\partial_j\varphi(F)DF_j),$$

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from which one deduces that

$$E[\varphi(F)\psi(G)] - E[\varphi(F)]E[\psi(G)] = \sum_{j=1}^{m} E[\langle(I-L)^{-1}\partial_j\varphi(F)DF_j, DG\rangle_{\mathfrak{H}}\psi'(G)]$$

$$\leq \|\psi'\|_{\infty} \sum_{j=1}^{m} E\big[|\langle(I-L)^{-1}\partial_j\varphi(F)DF_j, DG\rangle_{\mathfrak{H}}|\big].$$

We shall now fix $j = 1, ..., m$, and consider separately every addend appearing in the previous sum. As it is standard, without loss of generality, we can assume that the underlying Hilbert space $\mathfrak{H}$ is of the form $L^2(A, \mathcal{A}, \mu)$, where μ is a σ-finite measure without atoms. It follows that

$$\langle(I-L)^{-1}\partial_j\varphi(F)DF_j, DG\rangle_{\mathfrak{H}} = p_j q \int_A [(I-L)^{-1}\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, \theta))]I_{q-1}(g(\cdot, \theta))\mu(d\theta).$$

$$(3.13)$$

Now we apply the formula (2.10) to $u = g(\cdot, \theta)$ and $F = (I-L)^{-1}\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, \theta))$ and we obtain, using $D^r(I-L)^{-1} = ((r+1)I-L)^{-1}D^r$ as well (see, e.g., [12]),

$$[(I-L)^{-1}\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, \theta))]I_{q-1}(g(\cdot, \theta)) \qquad (3.14)$$

$$= \sum_{r=0}^{q-1} \binom{q-1}{r} \delta^{q-1-r} \left(\langle g(\cdot, \theta), D^r[(I-L)^{-1}\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, \theta))]\rangle_{\mathfrak{H}^{\otimes r}} \right)$$

$$= \sum_{r=0}^{q-1} \binom{q-1}{r} \delta^{q-1-r} \left(\langle g(\cdot, \theta), ((r+1)I-L)^{-1}D^r[\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, \theta))]\rangle_{\mathfrak{H}^{\otimes r}} \right).$$

Now, substituting (3.14) into (3.13) yields

$$\langle(I-L)^{-1}\partial_j\varphi(F)DF_j, DG\rangle_{\mathfrak{H}} = p_j q \sum_{r=0}^{q-1} \binom{q-1}{r}$$

$$\times \delta^{q-1-r} \left(\int_{A^{r+1}} g(\cdot, \mathbf{s}^{r+1})((r+1)I-L)^{-1}D^r_{s_1,...,s_r}[\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, s_{r+1}))]\mu(d\mathbf{s}^{r+1}) \right),$$

where $\mathbf{s}^{r+1} = (s_1, \ldots, s_{r+1})$. We have, by the Leibniz rule,

$$D^r_{s_1,...,s_r}[\partial_j\varphi(F)I_{p_j-1}(f_j(\cdot, s_{r+1}))] = \sum_{\alpha=0}^{r} \binom{r}{\alpha} D^\alpha_{s_1,...,s_\alpha}[\partial_j\varphi(F)]D^{r-\alpha}_{s_{\alpha+1},...,s_r}[I_{p_j-1}(f_j(\cdot, s_{r+1}))]$$

$$= \sum_{\alpha=0}^{r} \binom{r}{\alpha} \frac{(p_j-1)!}{(p_j-r+\alpha-1)!} D^\alpha_{s_1,...,s_\alpha}[\partial_j\varphi(F)]I_{p_j-r+\alpha-1}(f_j(\cdot, s_{\alpha+1}, \ldots, s_{r+1})).$$

Fix $0 \leq r \leq q-1$ and $0 \leq \alpha \leq r$. It suffices to estimate the following expectation

$$E\left| \delta^{q-1-r} \left(\int_{A^{r+1}} g(\cdot, \mathbf{s}^\alpha, \mathbf{t}^{r-\alpha+1}) \right. \right. \qquad (3.15)$$

$$\left. \left. \times ((r+1)I-L)^{-1}D^\alpha_{\mathbf{s}^\alpha}[\partial_j\varphi(F)]I_{p_j-r+\alpha-1}(f_j(\cdot, \mathbf{t}^{r-\alpha+1}))\mu(d\mathbf{s}^\alpha)\mu(d\mathbf{t}^{r-\alpha+1}) \right) \right|.$$

9

Note that, in the previous formula, the symbol '·' inside the argument of the kernel g represents variables that are integrated with respect to the multiple Skorohod integral δ^{q-1-r}, whereas the '·' inside the argument of f_j stands for variables that are integrated with respect to the multiple Wiener-Itô integral $I_{p_j-r+\alpha-1}$. By Meyer's inequalities (2.11), we can estimate the expectation (3.15), up to a universal constant, by the sum over $0 \leqslant \beta \leqslant q-r-1$ of the quantities

$$\left(\int_{A^{q-r-1+\beta}} E\left(\int_{A^{r+1}} g(\mathbf{v}^{q-r-1}, \mathbf{s}^\alpha, \mathbf{t}^{r-\alpha+1}) D_{\mathbf{u}^\beta}^\beta \big\{ ((r+1)I - L)^{-1} D_{\mathbf{s}^\alpha}^\alpha [\partial_j \varphi(F)] \right.$$

$$\times I_{p_j-r+\alpha-1}(f_j(\cdot, \mathbf{t}^{r-\alpha+1})) \big\} \mu(d\mathbf{s}^\alpha) \mu(d\mathbf{t}^{r-\alpha+1}) \bigg)^2 \mu(d\mathbf{v}^{q-r-1}) \mu(d\mathbf{u}^\beta) \bigg)^{\frac{1}{2}}$$

$$= \left(\int_{A^{q-r-1+\beta}} E\left(((\beta+r+1)I - L)^{-1} \int_{A^{r+1}} g(\mathbf{v}^{q-r-1}, \mathbf{s}^\alpha, \mathbf{t}^{r-\alpha+1}) D_{\mathbf{u}^\beta}^\beta \big\{ D_{\mathbf{s}^\alpha}^\alpha [\partial_j \varphi(F)] \right.$$

$$\times I_{p_j-r+\alpha-1}(f_j(\cdot, \mathbf{t}^{r-\alpha+1})) \big\} \mu(d\mathbf{s}^\alpha) \mu(d\mathbf{t}^{r-\alpha+1})) \bigg)^2 \mu(d\mathbf{v}^{q-r-1}) \mu(d\mathbf{u}^\beta) \bigg)^{\frac{1}{2}}$$

$$\leq c \left(\int_{A^{q-r-1+\beta}} E\left(\int_{A^{r+1}} g(\mathbf{v}^{q-r-1}, \mathbf{s}^\alpha, \mathbf{t}^{r-\alpha+1}) D_{\mathbf{u}^\beta}^\beta \big\{ D_{\mathbf{s}^\alpha}^\alpha [\partial_j \varphi(F)] \right.$$

$$\times I_{p_j-r+\alpha-1}(f_j(\cdot, \mathbf{t}^{r-\alpha+1})) \big\} \mu(d\mathbf{s}^\alpha) \mu(d\mathbf{t}^{r-\alpha+1})) \bigg)^2 \mu(d\mathbf{v}^{q-r-1}) \mu(d\mathbf{u}^\beta) \bigg)^{\frac{1}{2}}.$$

Thanks to the Leibniz formula, the last bound implies that we need to estimate, for any $0 \leq \eta \leq \beta \leq q-r-1$, the following quantity

$$\left(\int_{A^{q-r-1+\beta}} E\left(\int_{A^{r+1}} g(\mathbf{v}^{q-r-1}, \mathbf{s}^\alpha, \mathbf{t}^{r-\alpha+1}) D_{\mathbf{s}^\alpha, \mathbf{w}^\eta}^{\alpha+\eta} [\partial_j \varphi(F)] \right.$$

$$\times I_{p_j-r+\alpha-1+\eta-\beta}(f_j(\cdot, \mathbf{t}^{r-\alpha+1}, \mathbf{y}^{\beta-\eta})) \mu(d\mathbf{s}^\alpha) \mu(d\mathbf{t}^{r-\alpha+1}) \bigg)^2 \mu(d\mathbf{v}^{q-r-1}) \mu(d\mathbf{w}^\eta) \mu(d\mathbf{y}^{\beta-\eta}) \bigg)^{\frac{1}{2}}.$$

We can rewrite this quantity as

$$\left(\int_{A^{q-r-1-\beta}} E\left((I_{p_j-r+\alpha-1+\eta-\beta}(f_j(\cdot, \mathbf{y}^{\beta-\eta})) \otimes_{r-\alpha+1} g(\cdot, \mathbf{v}^{q-r-1})) \otimes_\alpha D^{\alpha+\eta}[\partial_j \varphi(F)](\mathbf{w}^\eta) \right)^2 \right.$$

$$\times \mu(d\mathbf{y}^{\beta-\eta}) \mu(d\mathbf{w}^\eta) \mu(d\mathbf{v}^{q-r-1}) \bigg)^{\frac{1}{2}}.$$

Applying the Cauchy-Schwarz inequality yields that such a quantity is bounded by

$$\left(E\left[\| I_{p_j-r+\alpha-1+\eta-\beta}(f_j) \otimes_{r-\alpha+1} g \|^2 \| D^{\alpha+\eta}[\partial_j \varphi(F)] \|^2 \right] \right)^{\frac{1}{2}}$$

$$\leq \left(E \| I_{p_j-r+\alpha-1+\eta-\beta}(f_j) \otimes_{r-\alpha+1} g \|^4 \right)^{\frac{1}{4}} \left(E \| D^{\alpha+\eta}[\partial_j \varphi(F)] \|^4 \right)^{\frac{1}{4}}.$$

Set $\gamma = \alpha + \eta$. Applying the generalized Faá di Bruno's formula (see, e.g., [3]) we deduce that

$$D^\gamma[\partial_j\varphi(F)] = \sum \frac{\gamma!}{\prod_{i=1}^\gamma i!^{k_i} \prod_{i=1}^\gamma \prod_{j=1}^m q_{ij}!} \frac{\partial^k \partial_j\varphi(F)}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}} \prod_{i=1}^{\gamma \wedge p^*} (D^i F_1)^{\otimes q_{i1}} \otimes \cdots \otimes (D^i F_m)^{\otimes q_{im}},$$

where $p^* = \min\{p_1, ..., p_m\}$, and the sum runs over all nonnegative integer solutions of the system of $\gamma + 1$ equations

$$k_1 + 2k_2 + \cdots + \gamma k_\gamma = \gamma,$$
$$q_{11} + q_{12} + \cdots + q_{1m} = k_1,$$
$$q_{21} + q_{22} + \cdots + q_{2m} = k_2,$$
$$\cdots \cdots$$
$$q_{\gamma 1} + q_{\gamma 2} + \cdots + q_{\gamma m} = k_\gamma,$$

and we have moreover set $p_j = q_{1j} + \cdots + q_{\gamma j}$, $j = 1, ..., r$, and $k = p_1 + \cdots + p_m = k_1 + \cdots + k_\gamma$. This expression yields immediately that

$$\|D^\gamma[\partial_j\varphi(F)]\| \le c\|\varphi\|_q \sum \prod_{i=1}^{\gamma \wedge p^*} \|D^i F_1\|^{q_{i1}} \cdots \|D^i F_m\|^{q_{im}}$$

and using the facts that all $\mathbb{D}^{k,p}$ norms $(k, p \ge 1)$ are equivalent on a fixed Wiener chaos and that the elements of the vector F have unit variance by assumption, we infer that

$$\left(E\|D^\gamma[\partial_j\varphi(F)]\|^4\right)^{\frac{1}{4}} \le c\|\varphi\|_q.$$

On the other hand, using hypercontractivity one has that

$$\left(E\|I_{p_j - r + \alpha - 1 + \eta - \beta}(f_j) \otimes_{r-\alpha+1} g\|^4\right)^{\frac{1}{4}} \le c \left(E\|I_{p_j - r + \alpha - 1 + \eta - \beta}(f_j) \otimes_{r-\alpha+1} g\|^2\right)^{\frac{1}{2}}.$$

Since

$$E\|I_{p_j - r + \alpha - 1 + \eta - \beta}(f_j) \otimes_{r-\alpha+1} g\|^2 = (p_j - r + \alpha - 1 + \eta - \beta)!\|f_j \otimes_{r-\alpha+1} g\|^2,$$

and

$$\max_{1 \le r \le q} \|f_j \otimes_r g\| \le \mathrm{Cov}(F_j^2, G^2) \quad (\text{see, e.g., } [9, \text{inequality } (3.26)]),$$

we finally obtain

$$E|\langle (I - L)^{-1} \partial_j\varphi(F) DF_j, DG \rangle_{\mathfrak{H}}| \le c\|\varphi\|_q \mathrm{Cov}(F_j^2, G^2),$$

thus concluding the proof. $\blacksquare$

Proof of Theorem 1.2. Just observe that

$$
\left| E\left[\prod_{j=1}^{d} \psi_j(F_j) \right] - \prod_{j=1}^{d} E[\psi_j(F_j)] \right|
$$

$$
\leqslant \sum_{j=2}^{d} \Big| E[\psi_1(F_1)\cdots\psi_{j-1}(F_{j-1})]\, E[\psi_j(F_j)]\, \cdots\, E[\psi_d(F_d)]
$$

$$
- E[\psi_1(F_1)\cdots\psi_j(F_j)]\, E[\psi_{j+1}(F_{j+1})]\, \cdots\, E[\psi_d(F_d)] \Big|,
$$

so that the conclusion is achieved (after some routine computations) by applying Proposition 3.1 (in the case $m = j - 1$, $p_i = q_i$, $i = 1, ..., j - 1$, and $q = q_j$) to each summand on the right-hand side of the previous estimate. ∎

3.2 Proof of Theorem 1.3

The equivalence between (1) and (2) follows from [9, Theorem 3.4]. That (3) implies (1) would have been immediate if the square function $x \mapsto x^2$ were bounded. To overcome this slight difficulty, it suffices to combine the hypercontractivity property of chaotic random variables (from which it follows that our sequence (F_n) is bounded in $L^p(\Omega)$ for any $p \geqslant 1$) with a standard approximation argument. Finally, the implication (1) $\Rightarrow$ (3) is a direct consequence of (1.3).

3.3 Proof of Theorem 1.4

Assume that there exists a subsequence of $\{F_n\}$ converging in distribution to some limit $(V_1, \ldots, V_d)$. For any collection of smooth test functions $\psi_1, ..., \psi_d : \mathbb{R} \to \mathbb{R}$, one can then write

$$
E\left[\prod_{j=1}^{d} \psi_j(V_j) \right] = \prod_{j=1}^{d} E\left[\psi_j(V_j) \right] = \prod_{j=1}^{d} E\left[\psi_j(U_j) \right] = E\left[\prod_{j=1}^{d} \psi_j(U_j) \right]. \tag{3.16}
$$

Indeed, the first equality in (3.16) is a direct consequence of Theorem 1.3, the second one follows from the fact that $V_j \overset{\text{law}}{=} U_j$ for any j by assumption, and the last one follows from the independence of the U_j. Thus, we deduce from (3.16) that $(U_1, \ldots, U_d)$ is the only possible limit in law for any converging subsequence extracted from $\{F_n\}$. Since the sequence $\{F_n\}$ is tight (indeed, it is bounded in $L^2(\Omega)$), one deduces that $F_n \overset{\text{law}}{\to} (U_1, \ldots, U_d)$, which completes the proof of Theorem 1.4.

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