Global energy minimization for multiple fracture growth

Danas Sutula

November 25, 2013

1 Formulation

The energy release rate with respect to a fracture growth direction θ_i can be obtained by differentiation of the potential energy Π of the system:

$$Gs_i = -\frac{\partial \Pi}{\partial \theta_i} \tag{1}$$

Considering a general case of multiple fractures, the rate of the energy release rate can be obtained as:

$$Hs_{i,j} = \frac{\partial Gs_i}{\partial \theta_j} = -\frac{\partial^2 \Pi}{\partial \theta_i \partial \theta_j} \tag{2}$$

In a discrete setting, the potential energy of a static system can be written as:

$$\Pi = \frac{1}{2} u' K u - u' f \tag{3}$$

where u, K, and f are the displacement vector, the stiffness matrix, and the applied force vector. The energy release rate with respect to some arbitrary crack incitement angle θ_i is defined as the negative variation of the potential energy:

$$Gs_i = -\frac{1}{2} u' \delta_i K u + u' \delta_i f - \delta_i u' (Ku - f) \tag{4}$$

in which case the last term in (4) can be presumed zero due to equilibrium of the discrete system i.e. $Ku = f$. Hence, the expression for the energy release rate becomes:

$$Gs_i = -\frac{1}{2} u' \delta_i K u + u' \delta_i f \tag{5}$$

where $\delta_i f$ only needs to be accounted for if the applied loads influence the virtual crack rotation, e.g. due to crack face tractions and body-type loads. Also, it is worth observing that, non-zero contributions to the variations $\delta_i K$, $\delta_i f$ occur only in those elements that experience the virtual crack rotation. The rates of the energy release rate, Hs_{ij} are obtained by differentiating Gs_i in (5) with respect to θ_j:

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta^2_{ij}Ku - u'\delta^2_{ij}f\right) - \delta_j u'\left(\delta_i Ku - \delta_i f\right) \tag{6}$$

The variations of displacements $\delta_j u$ in (6) are not local but global, and can be determined from the condition of equilibrium of the discrete system by considering its variation i.e. $\delta_j(Ku - f) = 0$:

$$\delta u = -K^{-1}(\delta Ku - \delta f) \tag{7}$$

Therefore, substitution of (7) in (6) leads to:

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta^2_{ij}Ku - u'\delta^2_{ij}f\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f) \tag{8}$$

It is worth highlighting that second order cross derivatives $\delta^2_{ij}K$, $\delta^2_{ij}f$ in (8) capture the interaction between the rotations of different crack increments when $i \neq j$ and as such can be disregarded since the variations of both K and f are local and generally disjoint from one another. Hence, in keeping only the self-interaction i.e. $\delta^2_{ii}K$, $\delta^2_{ii}f$, equation (8) turns to:

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta^2_{ii}Ku - u'\delta^2_{ii}F\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f) \tag{9}$$

The global stiffness matrix is obtained by summing the element level contributions:

$$\mathbf{K} = \sum_{i=1}^{n_{\text{el}}} \int_{\bar{\Omega}_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\,\det(\mathbf{J})\,\mathrm{d}\xi\mathrm{d}\eta \tag{10}$$

The global force vector e.g. for fracture surface tractions, is computed in a similar way, by summing the element level contributions of those elements that are cut by the interface:

$$\mathbf{f} = \sum_{i=1}^{n_{\text{el}}^{\text{cut}}} \int_{-1}^{+1} [\![\mathbf{N}]\!]^T \begin{bmatrix} -m & l \\ l & m \end{bmatrix} \begin{bmatrix} p \\ \tau \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\zeta}(s)\,\mathrm{d}\zeta \tag{11}$$

where $[\![\cdot]\!]$ denotes a jump in the shape functions across the interface and p and τ are the local crack surface tractions, namely: pressure (positive when induces opening) and shear (positive when induces top face sliding relative to bottom face, in the direction of the crack crack). Variations of the element level stiffness matrix can be obtained as follows (note: $\mathrm{d}\bar{\Omega} := \mathrm{d}\xi\mathrm{d}\eta$):

$$\begin{aligned}
\delta\mathbf{K}_{\text{e}} &= \int_{\Omega_e} (\delta\mathbf{B}^T\mathbf{D}\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta\mathbf{B})\det(\mathbf{J})\,\mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\,\delta\det(\mathbf{J})\,\mathrm{d}\bar{\Omega} && (12)\\
\delta^2\mathbf{K}_{\text{e}} &= \int_{\Omega_e} (\delta^2\mathbf{B}^T\mathbf{D}\mathbf{B} + 2\delta\mathbf{B}^T\mathbf{D}\delta\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta^2\mathbf{B})\det(\mathbf{J})\,\mathrm{d}\bar{\Omega} + \\
&+ \int_{\Omega_e} 2\,(\delta\mathbf{B}^T\mathbf{D}\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta\mathbf{B})\,\delta\det(\mathbf{J})\,\mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\,\delta^2\det(\mathbf{J})\,\mathrm{d}\bar{\Omega} && (13)
\end{aligned}$$

Similarly, the variations of the element level force vector due to crack surface tractions are given as (assuming constant values, and noting that for an anti-clockwise rotation convention we have: $\delta l = -m$, $\delta m = l$):

$$\delta \mathbf{f}_{\mathrm{e}} = \int_{\bar{\Gamma}_{\mathrm{crk}}} [\![\mathbf{N}]\!]^T \left(\begin{bmatrix} -l & -m \\ -m & l \end{bmatrix} \begin{bmatrix} p \\ \tau \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\zeta}(s) + \begin{bmatrix} -m & l \\ l & m \end{bmatrix} \begin{bmatrix} p \\ \tau \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\zeta}(\delta s) \right) \mathrm{d}\zeta \tag{14}$$

The variations of various components of equations (12), (13) are given below:

$$\delta \mathbf{J} = \sum_{I=1}^{n_{\mathrm{nd}}} \begin{bmatrix} \frac{\partial N_I}{\partial \xi} \\ \frac{\partial N_I}{\partial \eta} \end{bmatrix} \begin{bmatrix} \delta x_I & \delta y_I \end{bmatrix} \tag{15}$$

$$\delta^2 \mathbf{J} = \sum_{I=1}^{n_{\mathrm{nd}}} \begin{bmatrix} \frac{\partial N_I}{\partial \xi} \\ \frac{\partial N_I}{\partial \eta} \end{bmatrix} \begin{bmatrix} \delta^2 x_I & \delta^2 y_I \end{bmatrix} \tag{16}$$

$$\det(\mathbf{J}) = \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) y_J \tag{17}$$

$$\begin{aligned} \delta \det(\mathbf{J}) = & \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} \delta x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) y_J \; + \\ & + \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) \delta y_J \end{aligned} \tag{18}$$

$$\begin{aligned} \delta^2 \det(\mathbf{J}) = & \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} \delta^2 x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) y_J \; + \\ & + \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} 2\, \delta x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) \delta y_J \; + \\ & + \sum_{I=1}^{n_{\mathrm{nd}}} \sum_{J=1}^{n_{\mathrm{nd}}} x_I \left(\frac{\partial N_I}{\partial \xi} \frac{\partial N_J}{\partial \eta} - \frac{\partial N_I}{\partial \eta} \frac{\partial N_J}{\partial \xi} \right) \delta^2 y_J \end{aligned} \tag{19}$$

The variations of the inverse of the Jackobian matrix are (note: $\mathbf{J}^{-1}\mathbf{J} = \mathbf{I}$):

$$\delta \mathbf{J}^{-1} = -\, \mathbf{J}^{-1}\, \delta \mathbf{J}\, \mathbf{J}^{-1} \tag{20}$$

$$\delta^2 \mathbf{J}^{-1} = -\, \mathbf{J}^{-1}\, \delta^2 \mathbf{J}\, \mathbf{J}^{-1} + 2\, \mathbf{J}^{-1}\, \delta \mathbf{J}\, \mathbf{J}^{-1}\, \delta \mathbf{J}\, \mathbf{J}^{-1} \tag{21}$$

The variation of the Cartesian derivatives are then:

$$\delta \begin{bmatrix} \frac{\partial N_I}{\partial x} \\ \frac{\partial N_I}{\partial y} \end{bmatrix} = \delta \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_I}{\partial \xi} \\ \frac{\partial N_I}{\partial \eta} \end{bmatrix} \tag{22}$$

$$\delta^2 \begin{bmatrix} \frac{\partial N_I}{\partial x} \\ \frac{\partial N_I}{\partial y} \end{bmatrix} = \delta^2 \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_I}{\partial \xi} \\ \frac{\partial N_I}{\partial \eta} \end{bmatrix} \tag{23}$$

The variations of the strain matrix are:

$$\delta \mathbf{B}_I = \begin{bmatrix} \delta \frac{\partial N_I}{\partial x} & 0 & \delta \frac{\partial N_I}{\partial y} \\ 0 & \delta \frac{\partial N_I}{\partial y} & \delta \frac{\partial N_I}{\partial x} \end{bmatrix} \tag{24}$$

$$\delta^2 \mathbf{B}_I = \begin{bmatrix} \delta^2 \frac{\partial N_I}{\partial x} & 0 & \delta^2 \frac{\partial N_I}{\partial y} \\ 0 & \delta^2 \frac{\partial N_I}{\partial y} & \delta^2 \frac{\partial N_I}{\partial x} \end{bmatrix} \tag{25}$$

The spatial variations used in the equations above can be written solely in terms of the variation of the polar angle itself since the radius is constant due to pure rotation (note the polar-Cartesian relationship: $x = r\cos(\theta)$, $y = r\sin(\theta)$):

$$\delta x = -y\,\delta\theta_i, \quad \delta y = \ \ x\,\delta\theta_i \tag{26}$$

$$\delta^2 x = -x\,\delta\theta_i, \quad \delta^2 y = -y\,\delta\theta_i \tag{27}$$

2 Implementation

The first and second order global variations of the stiffness matrix, K_g and the force vector f_g are determined by assembling the element level contributions K_e and f_e of those elements affected by the infinitesimal crack tip rotation. Figure 1, illustrates a general instance of a finite length crack tip segment undergoing a clockwise rotation, in turn subjecting the elements in the vicinity to changes in geometry, which brings about the variations in K_e and f_e. In order to carry out the geometrical differentiation of elements due to the crack tip rotation, it is first useful to separate elements into ones that rotate purely rigidly, and into those that experience some change in shape. From the point of view of XFEM it is computationally efficient to consider the enriched elements in the vicinity of the crack tip as purely rotational elements since the derivatives become computationally cheap to compute. The elements surrounding the enriched patch are the non-enriched or standard elements, therefore it is chosen that these elements undergo the conformal changes in geometry that are coherent with the rotating patch of enriched elements. As seen in figure 1, the standard elements lasso the enriched elements.

2.1 Updating of enrichment ?

3 Results

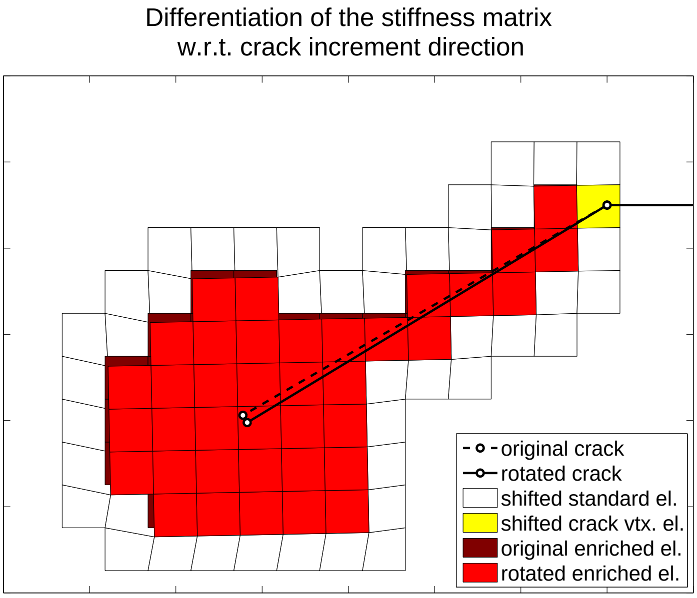

Figure 1: variations of the stiffness matrix

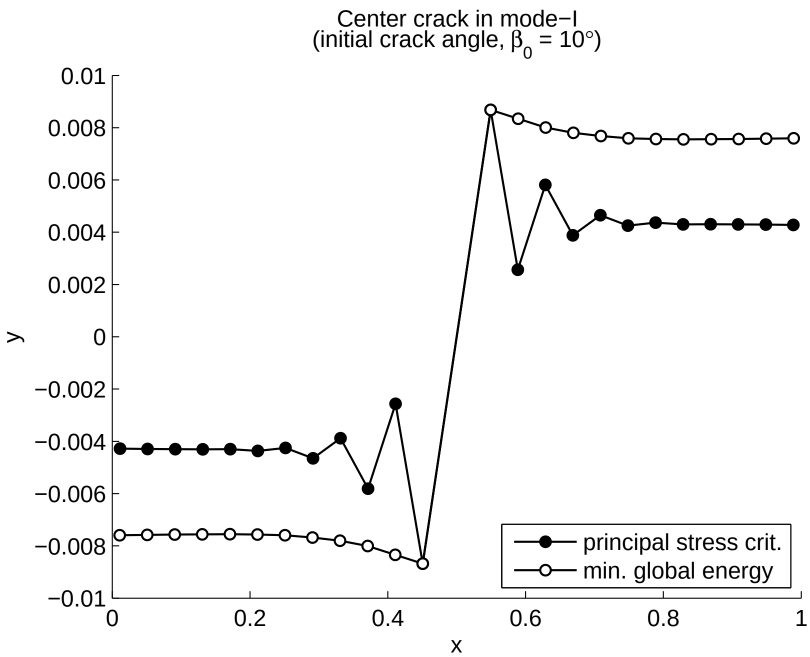

Figure 2: RMS roughness vs. percentage cracked for different damage depths