Proximity-sensitive individual deprivation measures*

Walter Bossert
Department of Economics and CIREQ, University of Montreal
walter.bossert@umontreal.ca

Conchita D'Ambrosio
INSIDE, Université du Luxembourg
conchita.dambrosio@uni.lu

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Abstract. We propose and characterize a generalization of the classical linear index of individual deprivation based on income shortfalls. Unlike the original measure, our class allows for increases in the income of a higher-income individual to have a stronger impact on a person's deprivation the closer they occur to the income of the individual whose deprivation is being assessed. The subclass of our measures with this property is axiomatized in our second result. *Journal of Economic Literature* Classification No.: D63.

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1 Introduction

Relative deprivation has attracted increasing attention in the past decades when the measurement of individual well-being gained importance not only in the academic context but also in the public discourse and in policy-making circles. The main reason for this is the characteristic at the basis of the concept: the observation that, since individuals do not live in isolation, they determine their well-being also from comparisons with others.

The pioneering contribution in the economics literature on the measurement of relative deprivation is Yitzhaki (1979). He proposes to consider income as the object of relative deprivation. Individual comparisons, which are at the basis of all indices of relative deprivation proposed since then, are absent in Yitzhaki's (1979) contribution, as he himself acknowledges in Yitzhaki (1980).

The *income shortfall* approach to individual deprivation measurement is due to Hey and Lambert (1980). They obtain the index proposed by Yitzhaki (1979) by providing an alternative individual-based interpretation of it: each individual compares herself to better-off individuals and the sentiment of deprivation felt with respect to each of them is quantifiable by how short of income she is with respect to the richer individual. The individual index of relative deprivation is the sum of these sentiments divided by the population size. This interpretation has become one of this measure's distinguishing features.

Some authors who deal with individual deprivation focus on the task of capturing the *intensity* of deprivation felt by an individual in the comparison to those who are better off by enriching measures that are based on income shortfalls. Among other features, their contributions can be viewed as addressing the *feasibility* aspect of deprivation underlined by Runciman (1966). According to Runciman (1966, p.10), "[t]he qualification of 'feasibility' is obviously imprecise, but it is necessary in order to exclude fantasy wishes. A man may say with perfect truth that he wants to be as rich as the Aga Khan [...] but to include these under the heading of relative deprivation would rob the term of its value." See also Gurr (1968, p.1104) for a discussion.

The question of how to deal with the feasibility aspect is a subtle issue. One possible response is to simply reduce Yitzhaki's (1979) proposed comparison group of *all* richer individuals by eliminating individuals who are 'much richer' (such as the Aga Khan in the above Runciman quote) altogether. However, such a rather drastic move would seem to have problems of its own. First of all, it is by no means obvious how the term 'much richer' can be defined properly and it seems doubtful that an unambiguous and widely acceptable definition of that notion can be formulated. Moreover, even if we may not want to accord the same standing to individuals who are 'considerably' richer, it seems unwarranted to exclude them entirely from consideration. It seems desirable to us that there be at least *some* increase in individual deprivation if, *ceteris paribus,* the income of a richer person increases; we consider this an essential monotonicity requirement, as do the other authors dealing with this issue. A more adequate response that we (along with all other relevant contributions that we are aware of) endorse is to find a way of assigning more significance to a richer individual depending on how close her income is to that of the person under consideration. However, as will become clear in the following section, we depart significantly from the earlier literature by retaining a structure that is based on income shortfalls.

Contributions that are close to our own as far as the feasibility issue is concerned include those of Paul (1991), Chakravarty and Chattopadhyay (1994), Podder (1996) and Esposito (2010). Among these, Esposito (2010) is the only one that provides a characterization of the individual deprivation index that is being proposed. All of these authors abandon the income shortfall approach in the sense that they either operate within a *utility* shortfall framework as that mentioned in Hey and Lambert (1980) or focus on income *ratios* rather than income differences. One of the objectives of the present paper is to show that these modifications are not necessary in order to address the feasibility problem: to ensure that higher incomes have a higher impact on individual deprivation the closer they are to the income of the individual in question, the income shortfall approach can be retained. We provide a characterization of a class of individual indices with this property in addition to axiomatizing a more general class.

2 Generalized income shortfall deprivation measures

We consider a fixed population of $n+1$ individuals and analyze the deprivation of a fixed individual (for convenience, individual $n+1$) where n can be any positive integer greater than one. Our results remain true if the case in which n equals one is included. However, this case is somewhat degenerate and axioms such as anonymity and additive decomposability are redundant if $n = 1$. The set of individuals other than the person under consideration is $N = \{1, \ldots, n\}$.

An income distribution is an $(n+1)$-dimensional vector $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ where $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}_+^n$ is the income vector of the members of society other than individual $n+1$ and $x \in \mathbb{R}_+$ is the income of this person $n+1$. For $n \in \mathbb{N}$, $\mathbf{1}_n$ is the vector consisting of n ones. For $\mathbf{y}, \mathbf{y}' \in \mathbb{R}_+^n$ and a subset M of N, the vector $\mathbf{z} = (\mathbf{y}|_M, \mathbf{y}'|_{N \setminus M})$ is defined as follows. For all $i \in N$,

$$z_i = \begin{cases} y_i & \text{if } i \in M, \\ y'_i & \text{if } i \in N \setminus M. \end{cases}$$

An individual measure of deprivation is a function $D\colon \mathbb{R}_+^{n+1} \to \mathbb{R}_+$. $D(\mathbf{y}; x)$ is interpreted as the deprivation suffered by the person under consideration with income x when the remaining members of the society have the incomes described by $\mathbf{y}$ in the distribution $(\mathbf{y}; x)$. We let $B(\mathbf{y}; x) = \{j \in N \mid y_j > x\}$ denote the set of those with a higher income than person $n+1$.

The *linear income shortfall deprivation measure* D^L proposed by Hey and Lambert (1980) and inspired by Yitzhaki (1979) is defined as follows. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^L(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} \frac{1}{n} (y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

This index can be generalized in an intuitive manner. For any increasing function $F\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$, the corresponding individual deprivation index D^F is defined by letting, for all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$,

$$D^F(\mathbf{y}; x) = \begin{cases} 0 & \text{if } B(\mathbf{y}; x) = \emptyset, \\ \sum_{j \in B(\mathbf{y}; x)} F(y_j - x) & \text{if } B(\mathbf{y}; x) \neq \emptyset. \end{cases}$$

The measure D^L is the special case obtained by choosing $F(t) = t/n$ for all $t \in \mathbb{R}_{++}$ in the definition of D^F. Consequently, we refer to the members of this class as *generalized income shortfall deprivation measures.*

In contrast to the earlier contributions mentioned in the introduction, the measures we advocate—the subclass of D^F corresponding to increasing and strictly concave functions F—retain the traditional reliance on income shortfalls, thereby illustrating that the desire to incorporate feasibility issues does not require the income shortfall approach to be abandoned altogether. We now turn to our axioms.

Normalization requires individual deprivation to be equal to zero whenever there is no one in society with a higher income than the individual under consideration.

Normalization. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, if $B(\mathbf{y}; x) = \emptyset$, then

$$D(\mathbf{y}; x) = 0.$$

The next axiom is a focus axiom, requiring that the income levels of those who are at or below person $(n+1)$'s income level are irrelevant.

Focus. For all $\mathbf{y}, \mathbf{y}' \in \mathbb{R}_+^n$ and for all $x \in \mathbb{R}_+$, if $B(\mathbf{y}; x) = B(\mathbf{y}'; x)$ and $y_j = y'_j$ for all $j \in B(\mathbf{y}; x)$, then

$$D(\mathbf{y}; x) = D(\mathbf{y}'; x).$$

Anonymity demands that the index treat all individuals other than the person whose deprivation is being measured equally, paying no attention to their identities.

Anonymity. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ and for all permutations $\pi\colon N \to N$,

$$D\left((y_{\pi(1)}, \ldots, y_{\pi(n)}); x\right) = D(\mathbf{y}; x).$$

We require D to be increasing in the incomes of those with higher incomes than the individual under consideration. The axiom only applies in situations such that the set $B(\mathbf{y}; x)$ is non-empty; if this set is empty, the property is vacuously satisfied.

Increasingness. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$, for all $j \in B(\mathbf{y}; x)$ and for all $\varepsilon \in \mathbb{R}_{++}$,

$$D\left(((\mathbf{y} + \varepsilon \mathbf{1}_n)|_{\{j\}}, \mathbf{y}|_{N \setminus \{j\}}); x\right) > D(\mathbf{y}; x).$$

Translation invariance requires the index to be absolute, that is, invariant with respect to equal absolute changes in all incomes.

Translation invariance. For all $(\mathbf{y}; x) \in \mathbb{R}_+^{n+1}$ and for all $\delta \in \mathbb{R}$ such that $(\mathbf{y} + \delta \mathbf{1}_n; x + \delta) \in \mathbb{R}_+^{n+1}$,

$$D(\mathbf{y} + \delta \mathbf{1}_n; x + \delta) = D(\mathbf{y}; x).$$

Additive decomposability is a separability property. The following axiom is a weakening of the version employed by Bossert and D'Ambrosio (2006).

Additive decomposability. For all $(\mathbf{y}; x) \in \mathbb{R}^{n+1}_+$ such that $B(\mathbf{y}; x) \neq \emptyset$ and for all $B^1, B^2 \subseteq B(\mathbf{y}; x)$ such that $B^1 \cap B^2 = \emptyset$ and $B^1 \cup B^2 = B(\mathbf{y}; x)$,

$$D(\mathbf{y}; x) = D\left((\mathbf{y}|_{B^1}, x\mathbf{1}_n|_{N\setminus B^1}); x\right) + D\left((\mathbf{y}|_{B^2}, x\mathbf{1}_n|_{N\setminus B^2}); x\right).$$

As far as we are aware, Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) are the only contributions that provide characterizations of D^L, a special case of the class of measures advocated here. Thus, we consider it essential to illustrate that our system of axioms differs substantially from those employed in these earlier articles.

As is the case for the present contribution, Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) employ the focus axiom and translation invariance. Unlike the present paper, both of these earlier articles use linear homogeneity which is essential in order to obtain the linear structure of D^L.

The normalization properties in these earlier contributions are considerably stronger than ours because they impose specific positive values of individual deprivation for specific distributions. This move is required to obtain the result that the sum of income shortfalls from those with higher incomes is divided by the total number of individuals in the population, as is the case for D^L.

Ebert and Moyes (2000) employ an anonymity condition just like ours. This requirement is not imposed by Bossert and D'Ambrosio (2006) because the normalization condition used in that contribution (in conjunction with the remaining axioms) is sufficient to ensure that the resulting index is anonymous.

Both Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006) impose an additive decomposability property. The two versions differ in that the additive-decomposition axiom in Ebert and Moyes (2000) can only be used in their specific setting where the reference group is permitted to vary independently from the income distribution under consideration. Thus, this axiom cannot be formulated in the framework considered in Bossert and D'Ambrosio (2006) and in the present contribution. A more detailed discussion of this crucial issue can be found in Bossert and D'Ambrosio (2006). Additive decomposability as defined here is a weakening of the axiom that is used by Bossert and D'Ambrosio (2006); the version employed here only applies to situations in which the set of individuals $B(\mathbf{y}; x)$ is non-empty.

The independence condition of Ebert and Moyes (2000) cannot be formulated in the setting used here and in Bossert and D'Ambrosio (2006) because it rests on the assumption made by Ebert and Moyes (2000) that the composition and the size of the reference group may vary in a way that is independent of the income distribution. Consequently, the axiom does not appear in the present contribution and neither is it used in Bossert and D'Ambrosio (2006).

The only property we use that does not play a role in these earlier articles is increasingness. It is not required in Ebert and Moyes (2000) and in Bossert and D'Ambrosio (2006) because degenerate measures such as that assigning an individual deprivation value of zero to all income distributions are ruled out by their other axioms, notably their normalization properties that are also responsible for features other than the assignment of a zero value to situations with no individual deprivation.

The above axioms characterize the class D^F of individual deprivation measures.

Theorem 1 *An individual deprivation index D satisfies normalization, focus, anonymity, increasingness, translation invariance and additive decomposability if and only if there exists an increasing function $F\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $D = D^F$.*

Proof. That D^F satisfies the requisite axioms for any increasing function $F\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ is straightforward to verify.

Conversely, suppose D is an individual deprivation index satisfying the axioms. Consider first all distributions of the form $\left((y_j\mathbf{1}_n|_{\{j\}}, x\mathbf{1}_n|_{N\setminus\{j\}}); x\right)$ with $y_j > x$. Thus, for each of these distributions, there exists an individual $j \in N$ such that

$$B\left((y_j\mathbf{1}_n|_{\{j\}}, x\mathbf{1}_n|_{N\setminus\{j\}}); x\right) = \{j\} \neq \emptyset$$

and everyone in N other than j has the same income as person $n+1$. Using translation invariance with $\delta = -x$ implies

$$D\left((y_j\mathbf{1}_n|_{\{j\}}, x\mathbf{1}_n|_{N\setminus\{j\}}); x\right) = D\left(((y_j - x)\mathbf{1}_n|_{\{j\}}, 0\mathbf{1}_n|_{N\setminus\{j\}}); 0\right). \tag{1}$$

Let, for all $t \in \mathbb{R}_{++}$,

$$F(t) = D\left((t\mathbf{1}_n|_{\{j\}}, 0\mathbf{1}_n|_{N\setminus\{j\}}); 0\right). \tag{2}$$

The function F thus defined clearly has the domain $\mathbb{R}_{++}$. Increasingness and the definition of F together imply that $F(y_j - x) > F((y_j + x)/2 - x)$ and the term on the right side of this inequality is non-negative because D cannot assume negative values. Thus, F maps into R_{++}. Using increasingness and the definition of F again, it follows that F must be an increasing function. By anonymity, F can be chosen to be independent of j.

Now let $(\mathbf{y}; x) \in \mathbb{R}^{n+1}_{+}$ be arbitrary.

If $B(\mathbf{y}; x) = \emptyset$, normalization implies that $D(\mathbf{y}; x) = 0$.

If $B(\mathbf{y}; x) \neq \emptyset$, the focus axiom allows us to assume, without loss of generality, that $y_i = x$ for all $i \in N \setminus B(\mathbf{y}; x)$. Focus and repeated application of additive decomposability together imply

$$D(\mathbf{y}; x) = \sum_{j \in B(\mathbf{y};x)} D\left((y_j\mathbf{1}_n|_{\{j\}}, x\mathbf{1}_n|_{N\setminus\{j\}}); x\right) = \sum_{j \in B(\mathbf{y};x)} F(y_j - x)$$

because of (1) and (2). ■

The axioms used in the above characterization result are independent; a proof of this claim can be obtained from the authors on request.

3 Proximity-sensitive deprivation measures

The following axiom expresses what we think is a plausible aspect of individual deprivation. Loosely speaking, it requires that, *ceteris paribus,* an individual's deprivation is more affected by increases in higher incomes that occur closer to her own.

Proximity sensitivity. For all $(\mathbf{y};x) \in \mathbb{R}^{n+1}_{+}$, for all $j \in B(\mathbf{y};x)$ and for all $y'_j, y''_j \in \mathbb{R}_{++}$ such that $y_j < y'_j < y''_j$,

$$\begin{aligned} & \frac{D\left((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N\setminus\{j\}});x\right) - D\left((y_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N\setminus\{j\}});x\right)}{y'_j - y_j} \\ > \; & \frac{D\left((y''_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N\setminus\{j\}});x\right) - D\left((y'_j \mathbf{1}_n|_{\{j\}}, \mathbf{y}|_{N\setminus\{j\}});x\right)}{y''_j - y'_j}. \end{aligned}$$

If proximity sensitivity is added to the list of axioms in Theorem 1, the subclass of our measures that are associated with a function F that is strictly concave in addition to being increasing is characterized. In the presence of proximity sensitivity, the axiom increasingness is redundant and can therefore be dropped from the list.

We are well aware that proximity sensitivity is, by definition, closely linked to concavity and, thus, the additional step when moving from Theorem 1 to the following result is relatively small. For this reason, we do not provide a proof. As is the case for Theorem 1, the axioms employed are independent. Again, details are available from the authors on request.

Theorem 2 *An individual deprivation index D satisfies normalization, focus, anonymity, translation invariance, additive decomposability and proximity sensitivity if and only if there exists an increasing and strictly concave function $F\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $D = D^F$.*

An interesting observation is that the presence of proximity sensitivity rather than increasingness in the list of axioms implies that the function F must be continuous; this is an immediate consequence of the well-known result that (strict) concavity implies continuity for functions whose domain is an open subset of $\mathbb{R}$. Note that no such implication is obtained in Theorem 1.

Clearly, any strictly concave function $F\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ can be used to generate an individual deprivation index that belongs to the subclass characterized in Theorem 2. Prominent examples include functions of the form $F(t) = t^\alpha/n$ for all $t \in \mathbb{R}_{++}$ where the power α is in the interval $(0,1)$ to ensure that the resulting function is strictly concave. For instance, the square root multiplied by $1/n$ is obtained for $\alpha = 1/2$.

An interesting task for future research consists of extending the analysis by incorporating *intertemporal* aspects. Allowing past experiences and future expectations to matter may enhance our understanding of the impact of including feasibility considerations when assessing individual deprivation.

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