CHARACTERIZATION OF SOME AGGREGATION FUNCTIONS ARISING FROM MCDM PROBLEMS

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ABSTRACT

This paper deals with some characterizations of two classes of non-conventional aggregation operators. The first class consists of the weighted averaging operators (OWA) introduced by Yager while the second class corresponds to the weighted maximum and related operators defined by Dubois and Prade. These characterizations are established via solutions of some functional equations.

1. Introduction

In fuzzy (valued) multicriteria decision making problems it is typical that we have quantitative judgments on the pairs of alternatives concerning each criterion. These judgments are very often expressed by the help of fuzzy preference relations. Synthesizing (aggregating) judgments is an important part in order to obtain an overall opinion (global preference relation) on the pairs of alternatives. For more details see Fodor and Roubens [6].

In addition to the classical aggregation operations (e.g. weighted arithmetic means, geometric means, root-power means, quasi-arithmetic means, etc), two new classes have been introduced in the eighties.

Dubois and Prade [4] defined and investigated the weighted maximum and minimum operators in 1986. The formal analogy with the weighted arithmetic mean is obvious.

Yager [16] introduced the ordered weighted averaging operators (OWA) in 1988. The basic idea of OWA is to associate weights with a particular ordered position rather than a particular element.

The same idea was used by Dubois et al. [5] to introduce ordered weighted maximum (OWMAX) and minimum for modelling soft partial matching.

The main difference between OWA and OWMAX (resp. OWMIN) is in the underlying non-ordered aggregation operation. OWA uses arithmetic mean while OWMAX (resp. OMIN) applies weighted maximum (resp. weighted minimum). At first glance, this does not seem to be an essential difference. However, Dubois and Prade [4] proved that OWMAX is equivalent to the median of the ordered values and some appropriately choosen additional numbers used instead of the original weights.

Although several papers have dealt with different aspects of these operations, their characterizations have not been known yet. The main aim of the present paper is to deliver these missing descriptions.

First we study the ordered weighted averaging operators in details. We formulate some natural properties which are obviously possessed by the OWA operators. Then we show that those conditions are sufficient to characterize the OWA family. Quasi-OWA aggregators are also introduced and a particular class is characterized.

Then we investigate the weighted maximum and minimum operators in the same spirit as in case of OWA. Finally, ordered weighted maximum and minimum are characterized. For

[1]Partially supported by OTKA I/3–2152 and I/6–14144.

more details and proofs see [8] and [7].

2. Ordered weighted averaging aggregation operators (OWA)

The ordered weighted averaging aggregation operator (OWA) was proposed by Yager [16] in 1988. Since its introduction, it has been applied to many fields as neural networks (Yager [14]), data base systems (Yager [15]), fuzzy logic controlers (Yager [17]) and group decision making (Yager [16], Cutello and Montero [2]). Its structural properties (Skala [13]) and its links with fuzzy integrals (Grabisch [9]) were also investigated.

We consider a vector $(x_1, \ldots, x_m) \in D^m$, $m > 1$, and we are willing to substitute to that vector a single value $M^{(m)}(x_1, \ldots, x_m) \in D$ where $D \subseteq \mathbb{R}$, using the aggregation operator (aggregator) M.

An OWA aggregator $M^{(m)}$ associated to the m non negative weights $(\omega_1^{(m)}, \ldots, \omega_m^{(m)})$ such that $\sum_{k=1}^{m} \omega_k^{(m)} = 1$ corresponds to

$$M^{(m)}(x_{(1)}, \ldots, x_{(m)}) = \sum_{i=1}^{m} \omega_i^{(m)} x_{(i)},$$

$$x_{(1)} \leq \cdots \leq x_{(i)} \leq \cdots \leq x_{(m)}.$$

$\omega_1^{(m)}$ is linked to the lowest value $x_{(1)}$, …, $\omega_m^{(m)}$ is linked to the greatest value $x_{(m)}$.

This class of operators includes

- $\min(x_1, \ldots, x_m)$ if $\omega_1^{(m)} = 1$.
- $\max(x_1, \ldots, x_m)$ if $\omega_m^{(m)} = 1$.
- any order statistics $x_{(k)}$ if $\omega_k^{(m)} = 1$, $k = 1, \ldots, m$.
- the arithmetic mean if $\omega_1^{(m)} = \cdots = \omega_m^{(m)} = \frac{1}{m}$.
- the median $(x_{(m/2)} + x_{(m/2)+1})/2$ if $\omega_{(m/2)}^{(m)} = \omega_{(m/2)+1}^{(m)} = \frac{1}{2}$ and m is even.
- the median $x_{(m+1)/2}$ if $\omega_{(m+1)/2}^{(m)} = 1$ and m is odd.
- the arithmetic mean excluding the two extremes if $\omega_1^{(m)} = \omega_m^{(m)} = 0$ and $\omega_i^{(m)} = \frac{1}{m-2}$, $i \neq 1, m$.

Well-known and easy to prove properties of the OWA aggregators are summarized as follows (see also Yager [16], Cutello and Montero [2]).

Any OWA aggregator is

- *neutral* (or symmetric or commutative):

  $$M(x_1, \ldots, x_m) = M(x_{i_1}, \ldots, x_{i_m})$$

  for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, when $(i_1, \ldots, i_m) = \sigma(1, \ldots, m)$, where σ represents a permutation operation;

- *monotonic*:

  $$x_i' > x_i \text{ implies}$$

  $$M(x_1, \ldots, x_i', \ldots, x_m) \geq M(x_1, \ldots, x_i, \ldots, x_m);$$

- *idempotent*:

  $$M(x, \ldots, x) = x \quad , \quad \text{for all } x \in \mathbb{R};$$

- *compensative*:

  $$\min_{i=1,m} x_i \leq M^{(m)}(x_1, \ldots, x_m) \leq \max_{i=1,m} x_i.$$

Moreover, the following conditions, which are non-usual in the literature of MCDM, are also satisfied by any OWA aggregator.

- *ordered linkage property* (Marichal and Roubens [11]):

  For any given numbers $\{x_1, \ldots, x_{2m}\}$ ordered as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(2m)}$ we have

  $$M^{(m+1)}(y_1, y_2, \ldots, y_{m+1}) = M^{(m)}(z_1, z_2, \ldots, z_m)$$

  where $y_i = M^{(m)}(x_{(i)}, \ldots, x_{(m+i-1)})$ and $z_i = M^{(m+1)}(x_{(i)}, \ldots, x_{(m+i)})$.

- *stability for the same positive linear transformation*:

  $$M^{(m)}(rx_1 + t, \ldots, rx_m + t) = rM(x_1, \ldots, x_m) + t,$$

  for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$, all $r > 0$, all $t \in \mathbb{R}$.

- *ordered stability for positive linear transformations with the same unit and independent zeroes*:

$$M^{(m)}(rx_1+t_1,\ldots,rx_m+t_m) = rM(x_1,\ldots,x_m)+T(t_1,\ldots,t_m)$$

  holds for all $(x_1,\ldots,x_m)\in \mathbb{R}^m$, all $r>0$, all $(t_1,\ldots,t_m)\in\mathbb{R}^m$ and for the ordered values $x_{(1)}\le x_{(2)}\le\ldots\le x_{(m)}$, $t_{(1)}\le t_{(2)}\le\ldots\le t_{(m)}$.

Notice that, in general, OWA aggregators fail to satisfy

- *associativity*:

$$M(M(x_1,x_2),x_3) = M(x_1,M(x_2,x_3)),$$

$$M(M(x_1,\ldots,x_{m-1}),x_m) = M(x_1,M(x_2,\ldots,x_m))$$

  for all $(x_1,\ldots,x_m)\in\mathbb{R}^m$;

- *decomposability:* (Kolmogorov [10], Nagumo [12])

$$M^{(m)}(x_1,\ldots,x_k,x_{k+1},\ldots,x_m) = M^{(m)}(x,\ldots,x,x_{k+1},\ldots,x_m)$$

  when $x=M^{(k)}(x_1,\ldots,x_k)$, for all $(x_1,\ldots,x_m)\in\mathbb{R}^m$.

In addition, associativity and decomposability together imply the ordered linkage property, while the converse is not true in general.

Now we turn to the characterization of OWA operators, i.e., we choose two sets of sufficient conditions from the above list of necessary conditions.

3. Characterization of the OWA aggregator

A foundational paper of Aczél, Roberts and Rosenbaum [1] shows that the general solution of a functional equation related to the stability for positive linear transformations (case ♯5) :

$$M(rx_1+t,\ldots,rx_m+t) = rM(x_1,\ldots,x_m)+t, \quad r>0,$$

where M is a mapping from $\mathbb{R}^m\to\mathbb{R}$ given by

$$M(x_1,\ldots,x_m) = S(x)f\left(\tfrac{x_1-A(x)}{S(x)},\ldots,\tfrac{x_m-A(x)}{S(x)}\right)+A(x)$$

if $S(x)\neq 0$ and

$$M(x_1,\ldots,x_m)=x$$

if $S(x)=0$ ($\Leftrightarrow x_1=x_2=\cdots=x_m=x$), where $S^2(x)=\sum_i(x_i-A(x))^2$ and $A(x)$ represents the arithmetic mean; f is an arbitrary function from $\mathbb{R}^m$ to $\mathbb{R}$.

It is also true that the weighted mean corresponds to monotonic and idempotent aggregators which satisfy the (SPLU)-property (see [1], case ♯ 9).

From results obtained by Marichal and Roubens [11], we know that neutral, continuous, stable for the same positive linear transformations and associative (resp. decomposable) operators are characterized by the min or max operators (resp. min or max or $A(x)$).

Weaker property than associativity or decomposability is needed to be able to characterize the OWA operators which include min, max and the arithmetic means. This intermediate property is related to the ordered linkage property.

Theorem 1 *The class of ordered weighted averaging aggregators corresponds to the operators which satisfy the properties of neutrality, monotonicity, stability for the same positive linear transformations and ordered linkage.*

Another characterization of OWA operators corresponds to the following proposition.

Theorem 2 *The class of ordered weighted averaging operators corresponds to the aggregators which satisfy the properties of neutrality, monotonicity, idempotency and stability for positive linear transformations with the same unit, independent zeroes and ordered values.*

Proofs of these results and some related issues can be found in [8].

4. Decomposable quasi-OWA aggregators

The quasi-arithmetic mean was first considered and characterized by Kolmogorov [10] and Nagumo [12]. It corresponds to the aggregator

$$M(x_1,\ldots,x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

where f is a continuous strictly monotonic function.

It is natural to consider the quasi-OWA operators

$$M(x_1,\ldots,x_m) = f^{-1}\left[\sum_i \omega_i^{(m)} f(x_{(i)})\right].$$

These aggregators have still to be characterized but one can prove the following proposition.

Proposition 1 *Any decomposable quasi-OWA operator corresponds to the* min *or* max *or the quasi-arithmetic mean.*

5. Weighted maximum and minimum

Using the concept of possibility and necessity of fuzzy events [18, 3], one can evaluate the possibility that a relevant goal is attained, and the necessity that all the relevant goals are attained by the help of the following formulas (see [4] for more details) *weighted maximum:*

$$\max_{i=1,m}\{\min(w_i,x_i)\},\quad w_i\in[0,1],\quad \max_{i=1,m} w_i = 1 \tag{1}$$

and

weighted minimum:

$$\min_{i=1,m}\{\max(w_i,x_i)\},\quad w_i\in[0,1],\quad \min_{i=1,m} w_i = 0. \tag{2}$$

The analogy between the weighted arithmetic mean and the weighted maximum is obvious: product corresponds to minimum, sum does to maximum. It is emphasized in [4] that weighted maximum and minimum operators can be calculated as medians, i.e., the qualitative counterparts of means. More formally, the following result is true (only the weighted maximum is recalled).

Proposition 2 *Let* $(a_1,\ldots,a_m)\in[0,1]^m$ *and* $(b_1,\ldots,b_m)\in[0,1]^m$ *be such that* $a_1\le a_2\le\ldots\le a_m$ *and* $1=b_1\ge b_2\ge\ldots\ge b_m$. *Then*

$$\max_{i=1,m}\{\min(a_i,b_i)\} = \mathit{median}(a_1,\ldots,a_m,b_1,\ldots,b_m).$$

It is easy to see that weighted maximum satisfies idempotency and monotonicity. Moreover, it fulfils also (with $T^{(m)}=M^{(m)}$)

- *stability for maximum* (SMAX):

  $$M^{(m)}(x_1\vee t_1,\ldots,x_m\vee t_m) = M^{(m)}(x_1,\ldots,x_m)\vee T^{(m)}(t_1,\ldots,t_m)$$

  for all $(x_1,\ldots,x_m)\in[0,1]^m$, $(t_1,\ldots,t_m)\in[0,1]^m$.

- *stability for minimum with the same unit* (SMINU):

  $$M^{(m)}(r\wedge x_1,\ldots,r\wedge x_m) = r\wedge M^{(m)}(x_1,\ldots,x_m)$$

  for all $(x_1,\ldots,x_m)\in[0,1]^m$, $r\in[0,1]$.

In a sense, the converse is also true, as we state in the following theorem.

Theorem 3 *Suppose that M is a nondecreasing function from $[0,1]^m$ to $[0,1]$ such that $M(0,\ldots,0)=0$ and $M(1,\ldots,1)=1$. Then M satisfies SMAX and SMINU if and only if there exist weights $w_1,\ldots,w_m\ge 0$ with $\max w_i = 1$ such that*

$$M(x_1,\ldots,x_m) = \max_{i=1,\ldots,m}\{\min(w_i,x_i)\}.$$

By duality, we can introduce the corresponding stability conditions in the case of the weighted minimum as follows:

- *stability for minimum* (SMIN):

  $$M^{(m)}(x_1\wedge t_1,\ldots,x_m\wedge t_m) = M^{(m)}(x_1,\ldots,x_m)\wedge T^{(m)}(t_1,\ldots,t_m)$$

  for all $(x_1,\ldots,x_m)\in[0,1]^m$, $(t_1,\ldots,t_m)\in[0,1]^m$.

- *stability for maximum with the same unit* (SMAXU):

$$M^{(m)}(r \vee x_1, \ldots, r \vee x_m) = r \vee M^{(m)}(x_1, \ldots, x_m)$$

for all $(x_1, \ldots, x_m) \in [0,1]^m$, $r \in [0,1]$.

Obviously, the weighted minimum (2) satisfies both conditions. We state that the converse is also true in the following sense.

Theorem 4 *Suppose that M is a nondecreasing function from $[0,1]^m$ to $[0,1]$ such that $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$. Then M satisfies SMIN and SMAXU if and only if there exist weights $w_1, \ldots, w_m \geq 0$ with $\max w_i = 1$ such that*

$$M(x_1, \ldots, x_m) = \min_{i=1,\ldots,m} \{\max(w_i, x_i)\}.$$

6. Ordered weighted minimum and maximum

Suppose that $(x_1, \ldots, x_m) \in [0,1]^m$ and order these numbers increasingly: $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(m)}$. The ordered weighted maximum (OWMAX) operator associated to the m nonnegative weights $(w_1, \ldots, w_m)$ with $\max w_i = 1$ corresponds to

$$M(x_1, \ldots, x_m) = \max_{i=1\ldots,m} \{\min(w_i, x_{(i)})\}, \quad (3)$$

see [5].

The weight w_1 is linked to the lowest value $x_{(1)}, \ldots, w_m$ is linked to the greatest value $x_{(m)}$.

This class of operators includes

- $\min(x_1, \ldots, x_m)$ if $w_1 = 1$ and $w_i = 0$ for $i \geq 2$;
- $\max(x_1, \ldots, x_m)$ if $w_m = 1$;
- any order statistics x_k if $w_k = 1$ and $w_i = 0$ for $i > k$;

Obviously, any OWMAX operator is neutral, nondecreasing, idempotent and compensative. In addition, SMAX and SMINU are also satisfied for the ordered values $x_{(1)} \leq \ldots \leq x_{(m)}$, $t_{(1)} \leq \ldots \leq t_{(m)}$ as follows:

$$M(x_{(1)} \vee t_{(1)}, \ldots, x_{(m)} \vee t_{(m)}) = M(x_{(1)}, \ldots, x_{(m)}) \vee T(t_{(1)}, \ldots, t_{(m)}),$$

$$M(r \wedge x_{(1)}, \ldots, r \wedge x_{(m)}) = r \wedge M(x_{(1)}, \ldots, x_{(m)}).$$

Fortunately, the converse is also true in the following form.

Theorem 5 *A nondecreasing function $M : [0,1]^m \to [0,1]$ with $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$ satisfies SMAX and SMINU for ordered elements if and only if there exist weights $1 = w_1 \geq \ldots \geq w_m \geq 0$ such that*

$$M(x_1, \ldots, x_m) = \mathit{median}\{w_2, \ldots, w_m, x_{(1)}, \ldots x_{(m)}\}.$$

Notice that we can obtain similar characterization when using SMIN and SMAXU for ordered values. We formulate the statement without proof as follows.

Theorem 6 *A nondecreasing function $M : [0,1]^m \to [0,1]$ with $M(0,\ldots,0) = 0$ and $M(1,\ldots,1) = 1$ satisfies SMIN and SMAXU for ordered elements if and only if there exist weights $1 \geq w_1 \geq \ldots \geq w_m = 0$ such that*

$$M(x_1, \ldots, x_m) = \mathit{median}\{w_1, \ldots, w_{m-1}, x_{(1)}, \ldots x_{(m)}\}.$$

References

[1] J. Aczél, F.S. Roberts and Z. Rosenbaum, On the scientific laws without dimensional constants, *Journal of Math. Analysis and Appl.*, **119** (1986), 389–416.

[2] V. Cutello and J. Montero, Hierarchies of aggregation operators, Working paper (1993), unpublished.

[3] D. Dubois and H. Prade, *Théorie des Possibilités. Application à la Représentation des Connaissances en Informatique*, Masson, Paris, 1985.

[4] D. Dubois and H. Prade, Weighted minimum and maximum operations in fuzzy set theory, *Inf. Sci.* **39** (1986) 205–210.

[5] D. Dubois, H. Prade and C. Testemale, Weighted fuzzy pattern-matching, *Fuzzy Sets and Systems* **28** (1988) 313–331.

[6] J. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Aid*, to be published by Kluwer.

[7] J. Fodor and M. Roubens, Characterization of weighted maximum and some related operations (submitted, 1994).

[8] J. Fodor, J.-L. Marichal and M. Roubens, Characterization of the ordered weighted averaging operators, Prépublication n^o 93.011, Institut de Mathématique, Université de Liège, Belgium, 1993.

[9] M. Grabisch, On the use of fuzzy integral as a fuzzy connective, Proceedings of the Second IEEE International Conference on Fuzzy Systems, San Francisco (1993) 213-218.

[10] A.N. Kolmogoroff, Sur la notion de la moyenne, *Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez.,* **12** (1930) 388-391.

[11] J.L. Marichal and M. Roubens, Characterization of some stable aggregation functions, in : Proc. Intern. Conf. on Industrial Engineering and Production Management, Mons, Belgium (1993) 187–196.

[12] M. Nagumo, Über eine Klasse der Mittelwerte, *Japanese Journal of Mathematics*, **6** (1930) 71-79.

[13] H.J. Skala, Concerning ordered weighted averaging aggregation operators, Statistical Papers **32** (1991), 35–44.

[14] R.R. Yager, On the aggregation of processing units in neural networks, Proc. 1st IEEE Int. Conference on Neural Networks, San Diego, Vol. II (1987) 927–933.

[15] R.R. Yager, A note on weighted queries in information retrieval systems, *J. Amer. Soc. Information Sciences*, **28** (1987) 23–24.

[16] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, *IEEE Trans. on Systems, Man and Cybernetics*, **18** (1988), 183–190.

[17] R. R. Yager, Connectives and quantifiers in fuzzy sets, *Fuzzy Sets and Systems* **40** (1991) 39–75.

[18] L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, **1** (1978), 3–28.