ON NONSTRICT MEANS

by

János FODOR
Eötvös Loránd University, Budapest

and

Jean-Luc MARICHAL
University of Liège

ORBEL 10, BRUXELLES
January 18-19, 1996

AGGREGATION OPERATORS

We consider m real numbers $x_1, \ldots, x_m$, $x_i \in [a, b]$, and we are willing to substitute to the vector $(x_1, \ldots, x_m)$ one simple number x using the aggregation operator M :

$$\begin{aligned} M \,:\, \Lambda = \bigcup_{m \in N_0} [a,b]^m &\rightarrow R \\ (x_1, \ldots, x_m) &\rightarrow x = M(x_1, \ldots, x_m). \end{aligned}$$

Synthesizing judgments is an important part of MCDM methods. The typical situation concerns individuals which form quantitative judgments about a measure. In order to obtain a consensus of these judgments, classical operators are proposed.

Some examples :

$M(x_1, \ldots, x_m) = \frac{1}{m} \Sigma_i \, x_i$ (arithmetic mean)

$M(x_1, \ldots, x_m) = (\Pi_i \, x_i)^{\frac{1}{m}}$ (geometric mean)

$M(x_1, \ldots, x_m) = \min_i x_i$ (minimum)

$M(x_1, \ldots, x_m) = \max_i x_i$ (maximum)

$M(x_1, \ldots, x_m) = \Sigma_i \, \omega_i^{(m)} x_i, \;\; \omega_i^{(m)} \geq 0, \; \Sigma_i \, \omega_i^{(m)} = 1$
(weighted arithmetic mean)

$M(x_1, \ldots, x_m) = \max_i \{\min(\omega_i^{(m)}, x_i)\}, \;\; \omega_i^{(m)} \geq 0, \; \max_i \omega_i^{(m)} = 1$
(weighted maximum)

etc.

AGGREGATION PROPERTIES
NATURAL PROPERTIES

An aggregation operator M can be

- **Continuous (Co)** :
  $\forall m \in N_0$, $M^{(m)}(x_1, \ldots, x_m)$ is a continuous function;

- **Symmetric (Sy)** :
  $\forall m \in N_0$, $M^{(m)}(x_1, \ldots, x_m)$ is a symmetric function;

- **Increasing (In)** :
  $\forall m \in N_0$, $M^{(m)}(x_1, \ldots, x_m)$ is increasing on each argument;

- **Strictly increasing (SIn)** :
  $\forall m \in N_0$, $M^{(m)}(x_1, \ldots, x_m)$ is strictly increasing on each argument;

- **Idempotent (Id)** :
  $\forall m \in N_0$, $M^{(m)}(x, \ldots, x) = x$.

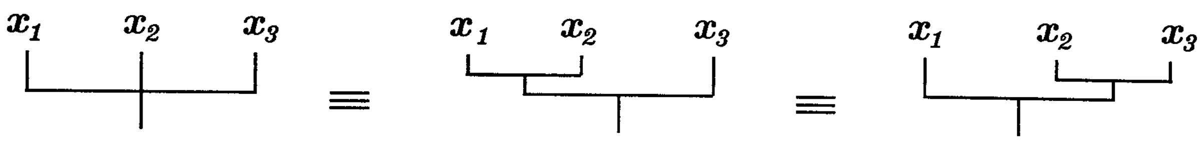
Proposition 1 *If* M *is* **In** *then*

$$M \text{ is } \mathbf{Id} \Leftrightarrow \min_i x_i \leq M(x_1, \ldots, x_m) \leq \max_i x_i.$$

AGGREGATION PROPERTIES
ITERATIVE PROPERTIES

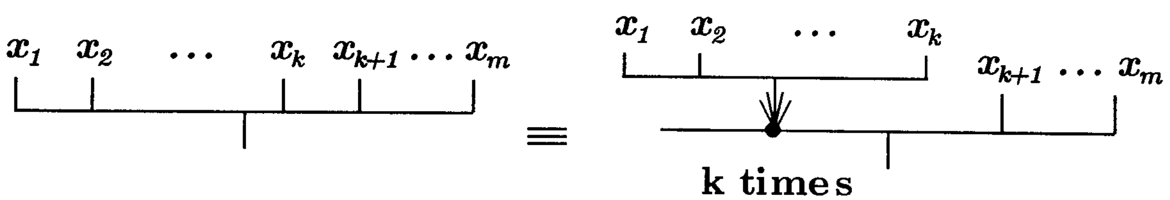
An aggregation operator M can be

- **Associative (As)** :
  $M^{(3)}(x_1, x_2, x_3) = M^{(2)}(x_1, M^{(2)}(x_2, x_3)) = M^{(2)}(M^{(2)}(x_1, x_2), x_3)$;
  $\forall m \in N_0,\ M^{(m)}(x_1, \ldots, x_m) = M^{(2)}(M^{(m-1)}(x_1, \ldots, x_{m-1}), x_m)$



Examples : $M(x_1, \ldots, x_m) = \Sigma_i\ x_i \ \vee\ \min_i x_i \ \vee\ \max_i x_i$.

- **Decomposable (De)** : $\forall 1 \leq k \leq m$,
  $M^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = M^{(m)}(M_k, \ldots, M_k, x_{k+1}, \ldots, x_m)$
  where $M_k = M^{(k)}(x_1, \ldots, x_k)$.



Examples: $M(x_1, \ldots, x_m) = \frac{1}{m}\ \Sigma_i\ x_i \ \vee\ \min_i x_i \ \vee\ \max_i x_i$.

Proposition 2	**As & Id $\Rightarrow$ De.**

THE GENERALIZED MEAN

Theorem 1 *M is defined on Λ ($\Lambda = \cup_{m \in N_0}[a,b]^m$) and fulfils* **Co, Sy, <u>SIn</u>, Id, De** $\Leftrightarrow \forall m \in N_0$,

$$M(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\ \Sigma_i\ f(x_i)\right]$$

where f is any continuous strictly monotonic function on $[a,b]$.

Kolmogoroff (1930), Sur la notion de la moyenne, *Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez.*, **12** : 388-391.

Examples :

Generator $f(x)$	Mean $M(x_1, \ldots, x_m)$	Name
x	$\frac{1}{m}\ \Sigma\ x_i$	arithmetic
x^2	$\sqrt{\frac{1}{m}\ \Sigma\ x_i^2}$	quadratic
x^{-1}	$\frac{1}{\frac{1}{m}\ \Sigma\ \frac{1}{x_i}}$	harmonic
x^α $(\alpha \neq 0)$	$\left(\frac{1}{m}\ \Sigma\ x_i^\alpha\right)^{\frac{1}{\alpha}}$	root-power
$\log x$	$\sqrt[m]{\Pi\ x_i}$	geometric

THE MAIN PURPOSE

To describe the class $\mathcal{D}$ of operators M defined on Λ and fulfilling **Co, Sy, <u>In</u>, Id, De**.

NONSTRICT MEANS

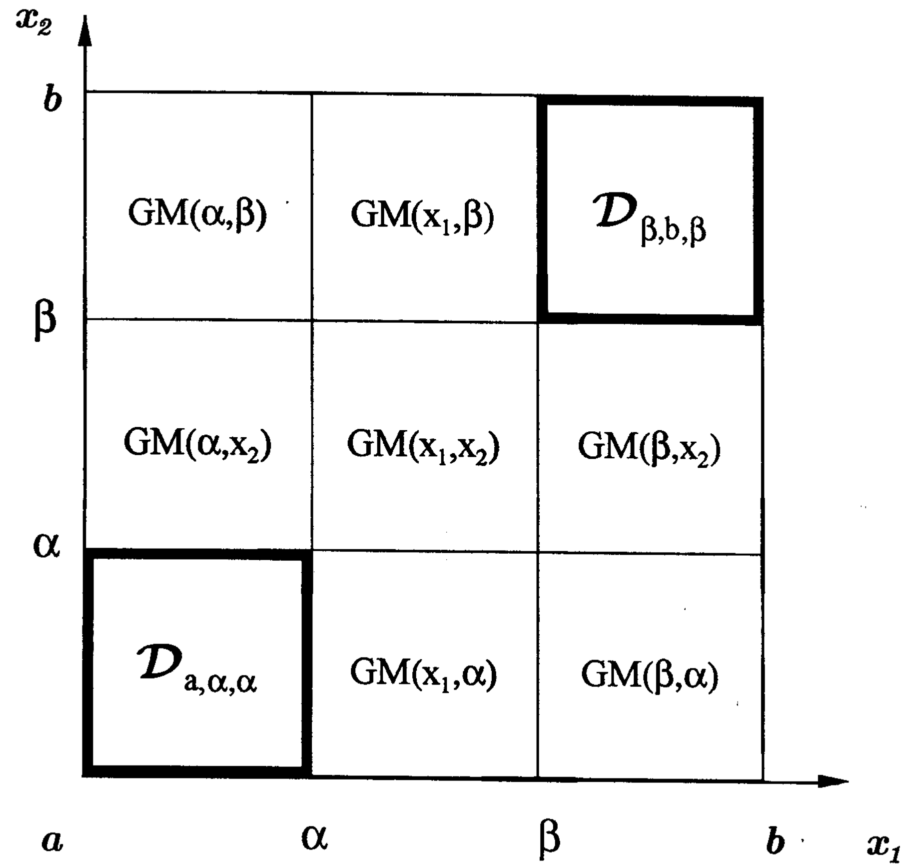
Toward a description of $\mathcal{D}$: **Co, Sy, <u>In</u>, Id, De**.

Two families :

1. $\mathcal{D}_{a,b,a} \subset \mathcal{D}$ with $M(a,b) = a$ $(\min \in \mathcal{D}_{a,b,a})$
2. $\mathcal{D}_{a,b,b} \subset \mathcal{D}$ with $M(a,b) = b$ $(\max \in \mathcal{D}_{a,b,b})$

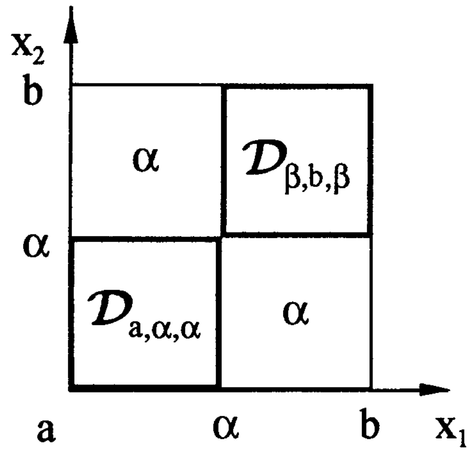
Theorem 2 $M \in \mathcal{D} \Leftrightarrow \exists\, a \le \alpha \le \beta \le b$ *such that* $\forall m \in N_0$,

- $M \in \mathcal{D}_{a,\alpha,\alpha}$ *on* $[a,\alpha]^m$ $(M(a,\alpha) = \alpha)$
- $M \in \mathcal{D}_{\beta,b,\beta}$ *on* $[\beta,b]^m$ $(M(\beta,b) = \beta)$
- $M(x_1,\ldots,x_m) = f^{-1}\left[\frac{1}{m}\, \Sigma\, f[median(\alpha, x_i, \beta)]\right]$ *everywhere else, where* f *is any continuous strictly monotonic function on* $[\alpha,\beta]$.



Three observations

1. $[\alpha, \beta] = [a, b] \;\Leftrightarrow\; M$ is **SIn** (generalized mean of Kolmogoroff)
2. If $\alpha = \beta$, we have



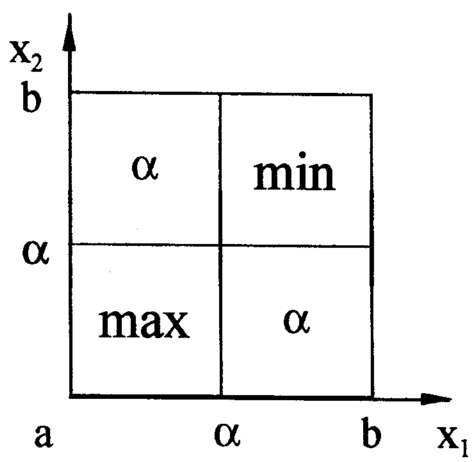
close to :

Theorem 3 *M is defined on* Λ *and fulfils* **Co, Sy, In, Id, As** $\Leftrightarrow$ (**Id, As** $\Rightarrow$ **De**)

$\exists\ a \le \alpha \le b$ *such that* $\forall m \in N_0$,

$$M(x_1, \ldots, x_m) = median(\max_i x_i, \alpha, \min_i x_i).$$

Fung and Fu (1975)



3. $\mathcal{D}_{a,\alpha,\alpha}$ and $\mathcal{D}_{\beta,b,\beta}$, or equivalently, $\mathcal{D}_{a,b,b}$ and $\mathcal{D}_{a,b,a}$ are yet to be described.

DESCRIPTION OF $\mathcal{D}_{a,b,a}$ $\quad (M(a,b) = a)$

Theorem 4 $M \in \mathcal{D}_{a,b,a} \;\Leftrightarrow\; \forall m \in N_0$,

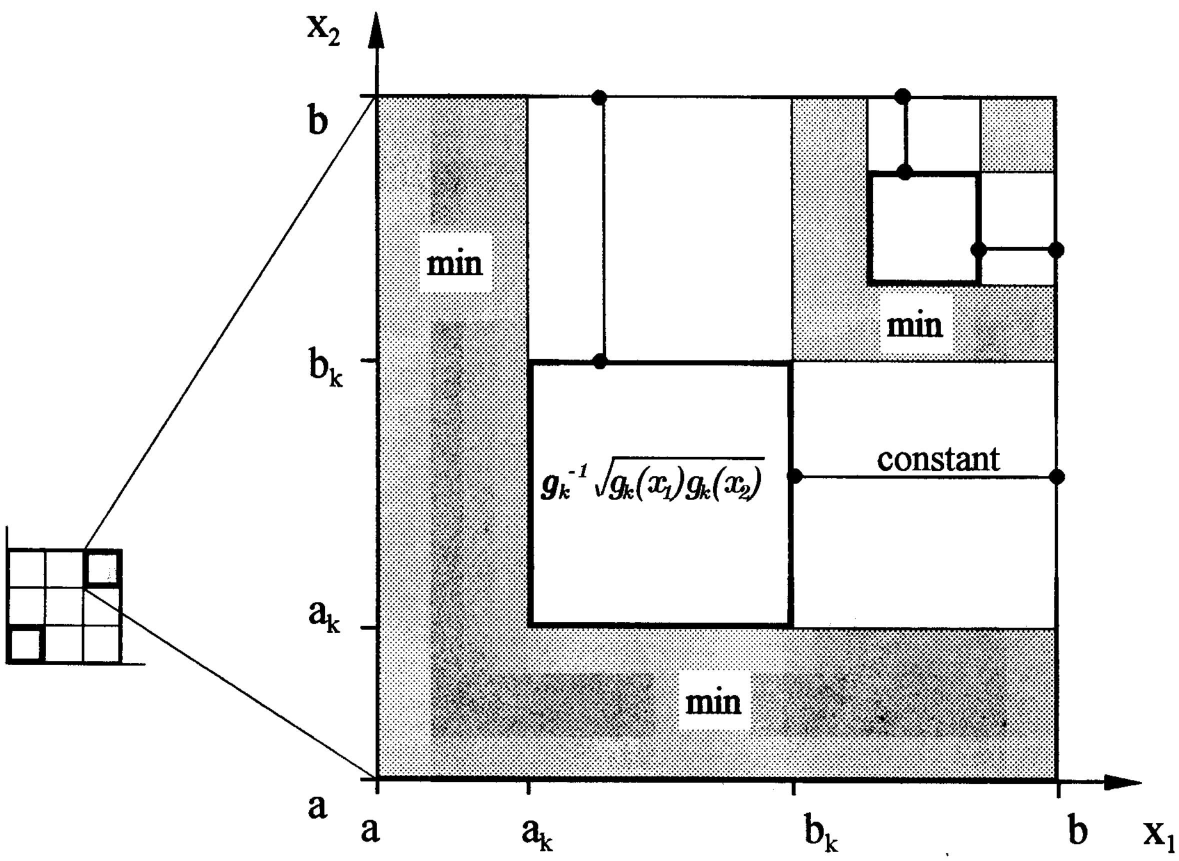
- *either* $\quad M(x_1, \ldots, x_m) = \min_i x_i,$
- *or* $\quad M(x_1, \ldots, x_m) = g^{-1} \sqrt[m]{\Pi\, g(x_i)}$

  where g is any continuous strictly increasing function on $[a,b]$ with $g(a) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) | k \in K\}$ of $[a,b]$ such that*

$$M(x_1, \ldots, x_m) = \begin{cases} g_k^{-1} \sqrt[m]{\Pi_i\, g_k[\min(x_i, b_k)]} & \text{if } \exists\, k \in K \text{ such that} \\ & \min_i x_i \in (a_k, b_k) \\ \min_i x_i & \text{otherwise,} \end{cases}$$

where g_k is any continuous strictly increasing function on $[a_k, b_k]$, with $g_k(a_k) = 0$.



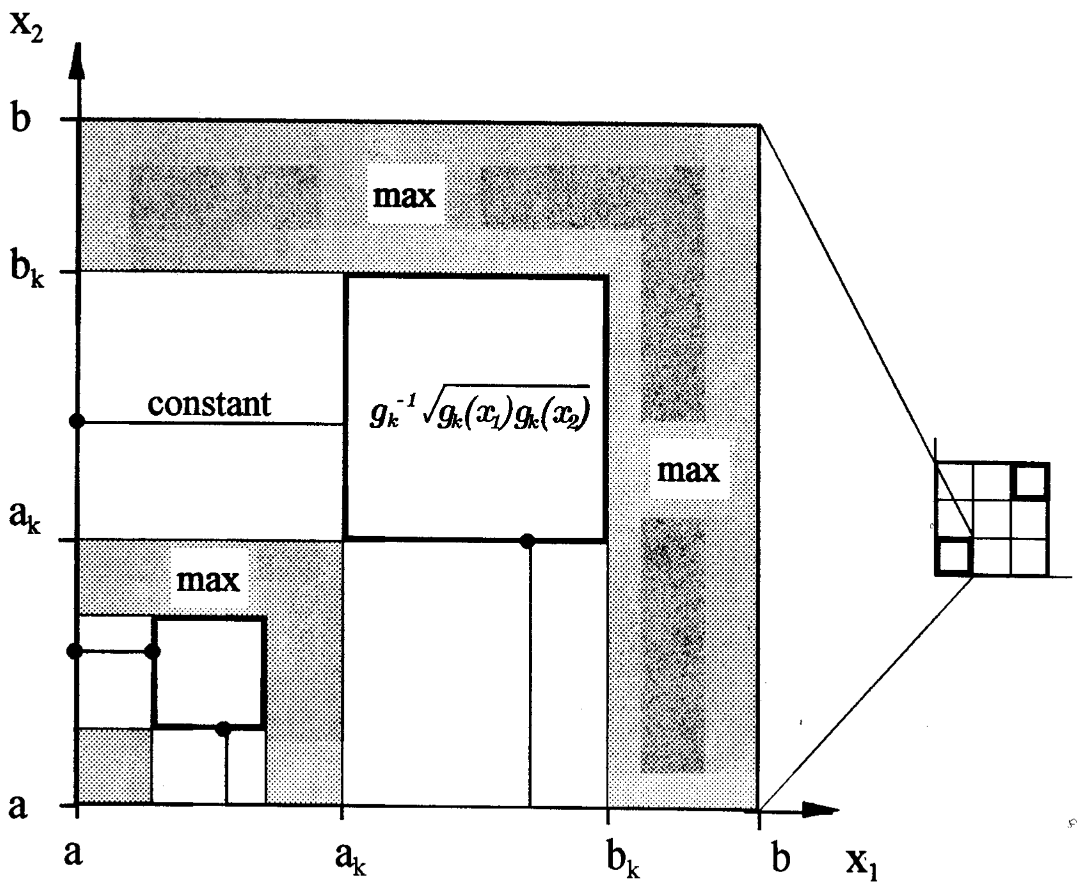
DESCRIPTION OF $\mathcal{D}_{a,b,b}$ $(M(a,b) = b)$

Theorem 5 $M \in \mathcal{D}_{a,b,b} \Leftrightarrow \forall m \in N_0$,

- *either* $$M(x_1, \ldots, x_m) = \max_i x_i,$$
- *or* $$M(x_1, \ldots, x_m) = g^{-1} \sqrt[m]{\Pi\, g(x_i)}$$
  where g is any continuous strictly decreasing function on $[a,b]$ with $g(b) = 0$,
- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) | k \in K\}$ of $[a,b]$ such that*

$$M(x_1, \ldots, x_m) = \begin{cases} g_k^{-1} \sqrt[m]{\Pi_i\, g_k[\max(x_i, b_k)]} & \text{if } \exists\, k \in K \text{ such that } \max_i x_i \in (a_k, b_k) \\ \max_i x_i & \text{otherwise,} \end{cases}$$

where g_k is any continuous strictly decreasing function on $[a_k, b_k]$, with $g_k(b_k) = 0$.



BISYMMETRY EQUATION

A function $M(x_1, x_2)$ of two variables is said to be **bisymmetric (Bi)** if it satisfies the following equation

$$M[M(x_{11}, x_{12}), M(x_{21}, x_{22})] = M[M(x_{11}, x_{21}), M(x_{12}, x_{22})].$$

Theorem 6 *M is defined on $[a, b]^2$ and fulfils* **Co, Sy, SIn, Id, Bi** $\Leftrightarrow$

$$M(x_1, x_2) = f^{-1}\left[\frac{f(x_1) + f(x_2)}{2}\right]$$

where f is any continuous monotonic function on $[a, b]$.

Aczél (1948), On mean values, *Bulletin of the American Math. Society*, **54** : 392-400.

Theorem 7 *Let $\mathcal{B}$ be the class of functions M defined on $[a, b]^2$ and fulfilling* **Co, Sy, In, Id, Bi**.

To obtain a description of $\mathcal{B}$, it suffices to consider the case $m = 2$ in the description of $\mathcal{D}$ presented before.