On Nonstrict Means

János C. Fodor
Department of Mathematics
University of Agricultural Sciences, Gödöllő,
Páter K. u. 1, H-2103 Gödöllő, Hungary

Jean-Luc Marichal
Ecole d'Administration des Affaires, University of Liège
7, boulevard du Rectorat - B31, B-4000 Liège 1, Belgium
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Abstract The general form of continuous, symmetric, increasing, idempotent solutions of the bisymmetry equation is established and the family of sequences of functions which are continuous, symmetric, increasing, idempotent, decomposable is described.

Keywords: nonstrict mean values, bisymmetry equation, decomposability property, ordinal sums.

1 Introduction

Kolmogoroff [6] and Nagumo [8] established a fundamental result about mean values. In their definition a mean value is a sequence $(M^{(m)})_{m\in\mathbb{N}_0}$ of functions $M^{(m)} : [a,b]^m \to [a,b]$ (where $[a,b]$ is a closed real interval) which are continuous, symmetric, strictly increasing in each argument, and idempotent (that is $M^{(m)}(x,\dots,x) = x$ for all $x \in [a,b]$). These functions are also linked by a *pseudo-associativity* called the *decomposability* property by several authors (see e.g. [3, Chapter 5]):

$$\begin{aligned} & M^{(m)}(x_1,\dots,x_k,x_{k+1},\dots,x_m) \\ = \quad & M^{(m)}(M_k,\dots,M_k,x_{k+1},\dots,x_m) \end{aligned}$$

for all $m \in \mathbb{N}_0$, $k \in \{1,\dots,m\}$, $x_1,\dots,x_m \in [a,b]$, with $M_k = M^{(k)}(x_1,\dots,x_k)$.

The corresponding result of Kolmogoroff and Nagumo states that these conditions are necessary and sufficient for the existence of a continuous strictly monotonic real function f such that

$$M^{(m)}(x_1,\dots,x_m) = f^{-1}\left[\frac{1}{m}\sum_{i=1}^{m} f(x_i)\right]$$

for all $m \in \mathbb{N}_0$. Such an expression is called the *generalized mean.*

On the other hand, Aczél [1] (see also [2]) proved that a function M of two variables defined on $[a,b]$ is continuous, symmetric, strictly increasing in each argument, idempotent and fulfils the *bisymmetry equation*

$$\begin{aligned} & M[M(x_{11},x_{12}),M(x_{21},x_{22})] \\ & = M[M(x_{11},x_{21}),M(x_{12},x_{22})] \end{aligned} \tag{1}$$

if and only if

$$M(x,y) = f^{-1}\left[\frac{f(x)+f(y)}{2}\right], \quad x,y \in [a,b]$$

with some continuous strictly monotonic function f.

Note that Horváth [5] investigated the connection between the two concepts of bisymmetry and decomposability.

The aim of this paper is to study *nonstrict* means in an elementary way. That is, we investigate means satisfying either the conditions of Aczél's theorem or the conditions of Kolmogoroff and Nagumo's theorem above, except strict monotonicity. We describe the family of continuous, symmetric, increasing, idempotent, bisymmetric functions (Section 2) and also the family of sequences of continuous, symmetric, increasing, idempotent and decomposable functions (Section 3). The general families obtained are very similar to those ones well-known as *ordinal sums* in the theory of the associativity functional equation (see e.g. [7]). For space limitation, no proof of the results will be given.

2 Nonstrict solutions of the bisymmetry equation

The bisymmetry equation (1), which can be considered also as a generalization of simultaneous commutativity and associativity, has been investigated by several authors. For a list of references see [2].

A function $M : [a, b]^2 \to [a, b]$ is called

- *symmetric* if $M(x, y) = M(y, x)$ for all $x, y \in [a, b]$;
- *increasing* if $x \le x', y \le y'$ imply $M(x, y) \le M(x', y')$;
- *strictly increasing* if $x < x'$ implies $M(x, y) < M(x', y)$ and the same for $y < y'$;
- *idempotent* if $M(x, x) = x$ for all $x \in [a, b]$.

As said above, our goal is to describe the general form of continuous, symmetric, increasing, idempotent solutions of the bisymmetry equation (1). In their structures, these solutions are very similar to ordinal sums which are well-known in the theory of semigroups, see e.g. [7].

Aczél [1] proved the following result.

Theorem 1 *$M : [a, b]^2 \to [a, b]$ is a continuous, symmetric, strictly increasing, idempotent, bisymmetric function if and only if*

$$M(x, y) = f^{-1}\left[\frac{f(x) + f(y)}{2}\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a, b]$.

We also know that this result still holds for intervals of the form (a, b), $[a, b)$, $(a, b]$ or even for any unbounded interval of the real line (see [2], pp 250–251, 280).

Now define $\mathcal{B}_{a,b,\theta}$ as the set of functions $M : [a, b]^2 \to [a, b]$ which are continuous, symmetric, increasing, idempotent, bisymmetric and such that $M(a, b) = \theta$, θ being a given number in $[a, b]$. A general element of a class $\mathcal{B}_{a,b,\theta}$ is usually denoted by $M_{a,b,\theta}$ in the sequel. Then we have the following result.

Theorem 2 *$M : [a, b]^2 \to [a, b]$ is a continuous, symmetric, increasing, idempotent, bisymmetric function if and only if there exists two numbers α and β fulfilling $a \le \alpha \le \beta \le b$ such that*

i) $M(x, y) = M_{a,\alpha,\alpha}(x, y)$ *if* $x, y \in [a, \alpha]$*;*

ii) $M(x, y) = M_{\beta,b,\beta}(x, y)$ *if* $x, y \in [\beta, b]$*;*

iii) $M(x, y) = f^{-1}\left[\frac{f[median(\alpha, x, \beta)] + f[median(\alpha, y, \beta)]}{2}\right]$ *otherwise,*

with some $M_{a,\alpha,\alpha} \in \mathcal{B}_{a,\alpha,\alpha}$, $M_{\beta,b,\beta} \in \mathcal{B}_{\beta,b,\beta}$, and f is any continuous strictly monotonic function on $[\alpha, \beta]$.

Now, describe the two families $\mathcal{B}_{a,b,a}$ and $\mathcal{B}_{a,b,b}$.

Theorem 3 *We have $M \in \mathcal{B}_{a,b,a}$ if and only if*

- *either*
$$M(x, y) = \min(x, y),$$
- *or*
$$M(x, y) = g^{-1}\sqrt{g(x)g(y)},$$
where g is any continuous strictly increasing function on $[a, b]$, with $g(a) = 0$,
- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a, b]$ such that*
$$M(x, y) = \begin{cases} g_k^{-1}\sqrt{g_k[\min(x, b_k)]g_k[\min(y, b_k)]} & \textit{if there exists } k \in K \textit{ such that } \min(x, y) \in (a_k, b_k), \\ \min(x, y) & \textit{otherwise,} \end{cases}$$
where g_k is any continuous strictly increasing function on $[a_k, b_k]$, with $g_k(a_k) = 0$.

Theorem 4 *We have $M \in \mathcal{B}_{a,b,b}$ if and only if*

- *either*
$$M(x, y) = \max(x, y),$$
- *or*
$$M(x, y) = g^{-1}\sqrt{g(x)g(y)},$$
where g is any continuous strictly decreasing function on $[a, b]$, with $g(b) = 0$,
- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a, b]$ such that*
$$M(x, y) = \begin{cases} g_k^{-1}\sqrt{g_k[\max(a_k, x)]g_k[\max(a_k, y)]} & \textit{if there exists } k \in K \textit{ such that } \max(x, y) \in (a_k, b_k), \\ \max(x, y) & \textit{otherwise,} \end{cases}$$
where g_k is any continuous strictly decreasing function on $[a_k, b_k]$, with $g_k(b_k) = 0$.

3 Extended Kolmogoroff-means

We show now that the results obtained in the previous section can be extend to the mean values by replacing bisymmetry by decomposability. According to Fodor and Roubens [3], we can see any mean value M as an aggregation operator:

$$\begin{array}{rcl} M : \Lambda = \bigcup_{m=1}^{\infty} [a,b]^m & \to & [a,b] \\ (x_1, \ldots, x_m) & \to & x = M^{(m)}(x_1, \ldots, x_m)^1 \end{array}$$

Such an operator M is said to be

- *continuous* if for all $m \in \mathbb{N}_0$, $M(x_1, \ldots, x_m)$ is a continuous function on $[a,b]^m$;
- *symmetric* if for all $m \in \mathbb{N}_0$, $M(x_1, \ldots, x_m)$ is a symmetric function on $[a,b]^m$:

  $$M(x_1, \ldots, x_m) = M(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

  where σ is a permutation of $\{1, \ldots, m\}$;
- *increasing* if for all $m \in \mathbb{N}_0$, $M(x_1, \ldots, x_m)$ is increasing in each argument:

  $$\begin{array}{rcl} x_i < x_i' & \Rightarrow & M(x_1, \ldots, x_i, \ldots, x_m) \\ & & \leq M(x_1, \ldots, x_i', \ldots, x_m), \end{array}$$

  for $i = 1, \ldots, m$;
- *strictly increasing* if for all $m \in \mathbb{N}_0$, $M(x_1, \ldots, x_m)$ is strictly increasing in each argument:

  $$\begin{array}{rcl} x_i < x_i' & \Rightarrow & M(x_1, \ldots, x_i, \ldots, x_m) \\ & & < M(x_1, \ldots, x_i', \ldots, x_m), \end{array}$$

  for $i = 1, \ldots, m$;
- *idempotent* if for all $m \in \mathbb{N}_0$, M has to satisfy

  $$M^{(m)}(x, \ldots, x) = x, \quad \forall x \in [a,b];$$
- *decomposable* if for all $m \in \mathbb{N}_0$ and all $k \in \{1, \ldots, m\}$, M has to satisfy

  $$\begin{array}{rl} & M^{(m)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) \\ = & M^{(m)}(M_k, \ldots, M_k, x_{k+1}, \ldots, x_m) \end{array}$$

  where $M_k = M^{(k)}(x_1, \ldots, x_k)$.

Kolmogoroff [6] established the following result.

Theorem 5 *An aggregation operator M, defined on Λ, is continuous, symmetric, strictly increasing, idempotent and decomposable if and only if for all $m \in \mathbb{N}_0$,*

$$M^{(m)}(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

(generalized mean) where f is any continuous strictly monotonic function on $[a,b]$.

Theorem 5 still holds for sets Λ of the form $\bigcup_{m=1}^{\infty}(a,b)^m$, $\bigcup_{m=1}^{\infty}[a,b)^m$, $\bigcup_{m=1}^{\infty}(a,b]^m$, even if $a = -\infty$ and/or $b = +\infty$.

Define $\mathcal{D}_{a,b,\theta}$ as the set of aggregation operators $M : \Lambda = \bigcup_{m=1}^{\infty}[a,b]^m \to [a,b]$ which are continuous, symmetric, increasing, idempotent, decomposable and such that $M(a,b) = \theta$, θ being a given number in $[a,b]$. Then we have the following result:

Theorem 6 *An aggregation operator M, defined on Λ, is continuous, symmetric, increasing, idempotent and decomposable if and only if there exists two numbers α and β fulfilling $a \leq \alpha \leq \beta \leq b$, such that, for all $m \in \mathbb{N}_0$,*

i) $M(x_1, \ldots, x_m) = M_{a,\alpha,\alpha}(x_1, \ldots, x_m)$ *if* $\max_i x_i \in [a, \alpha]$;

ii) $M(x_1, \ldots, x_m) = M_{\beta,b,\beta}(x_1, \ldots, x_m)$ *if* $\min_i x_i \in [\beta, b]$;

iii) $M(x_1, \ldots, x_m) = f^{-1}\left[\frac{1}{m}\sum_i f[median(\alpha, x_i, \beta)]\right]$ *otherwise,*

where $M_{a,\alpha,\alpha} \in \mathcal{D}_{a,\alpha,\alpha}$, $M_{\beta,b,\beta} \in \mathcal{D}_{\beta,b,\beta}$ and where f is any continuous strictly monotonic function on $[\alpha, \beta]$.

Now, describe the two families $\mathcal{D}_{a,b,a}$ and $\mathcal{D}_{a,b,b}$.

Theorem 7 *We have $M \in \mathcal{D}_{a,b,a}$ if and only if for all $m \in \mathbb{N}_0$,*

- *either*

  $$M(x_1, \ldots, x_m) = \min_i x_i,$$
- *or*

  $$M(x_1, \ldots, x_m) = g^{-1}\sqrt[m]{\prod_i g(x_i)},$$

  where g is any continuous strictly increasing function on $[a,b]$, with $g(a) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a,b]$ such that*

$$M(x_1,\ldots,x_m) = \begin{cases} g_k^{-1}\sqrt[m]{\prod_i g_k[\min(x_i,b_k)]} \\ \quad \textit{if there exists } k \in K \\ \quad \textit{such that} \\ \quad \min_i x_i \in (a_k,b_k), \\ \\ \min_i x_i \ \textit{otherwise,} \end{cases}$$

where g_k is any continuous strictly increasing function on $[a_k, b_k]$, with $g_k(a_k) = 0$.

Theorem 8 *We have $M \in \mathcal{D}_{a,b,b}$ if and only if for all $m \in \mathbb{N}_0$,*

- *either*

$$M(x_1,\ldots,x_m) = \max_i x_i,$$

- *or*

$$M(x_1,\ldots,x_m) = g^{-1}\sqrt[m]{\prod_i g(x_i)},$$

where g is any continuous strictly decreasing function on $[a,b]$, with $g(b) = 0$,

- *or there exists a countable index set K and a family of disjoint subintervals $\{(a_k, b_k) : k \in K\}$ of $[a,b]$ such that*

$$M(x_1,\ldots,x_m) = \begin{cases} g_k^{-1}\sqrt[m]{\prod_i g_k[\max(a_k,x_i)]} \\ \quad \textit{if there exists } k \in K \\ \quad \textit{such that} \\ \quad \max_i x_i \in (a_k,b_k), \\ \\ \max_i x_i \ \textit{otherwise,} \end{cases}$$

where g_k is any continuous strictly decreasing function on $[a_k, b_k]$, with $g_k(b_k) = 0$.

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